Regularization of an Ill-Posed Partial Differential Equation Problem

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Abstract

We consider the problem $K(x)u_{xx} = u_{tt}$, $0 < x < 1$, $t \geq 0$, with boundary condition $u(0,t) = g(t) \in L^2$ and $u_x(0,t) = 0$, where $K(x)$ is continuous and does not come close to zero. This is an ill-posed problem in the sense that, if the solution exists, it does not depend continuously on g. Considering the existence of a solution $u(x,\cdot)$ belonging to the Sobolev space $H^1(R)$ and using a wavelet Galerkin method with Meyer multiresolution analysis, we regularize the ill-posedness of the problem approaching it by well-posed problems in the scaling spaces.

1 Introduction

In a previous work [2], we studied the following parabolic partial differential equation problem with variable coefficients:

$$\begin{gathered} K(x)u_{xx}(x,t) = u_t(x,t), \quad t \geq 0, \quad 0 < x < 1 \\ u(0,\cdot) = g, \quad u_x(0,\cdot) = 0 \\ 0 < \alpha \leq K(x) < +\infty, \quad K \textit{ continuous.} \end{gathered}$$

Under the hypothesis of the existence of a solution for this problem, using a wavelet Galerkin method, we constructed a sequence of well-posed approximating problems in the scaling spaces of the Meyer multiresolution analysis, which has the property to filter away the high frequencies. We had shown the convergence of the method, applied to this problem, and we gave an estimate of the solution error.

In this work, we will extend those results to the problem:

$$\begin{gathered} K(x)u_{xx}(x,t) = u_{tt}(x,t), \quad t \geq 0,\ 0 < x < 1 \\ u(0,\cdot) = g, \quad u_x(0,\cdot) = 0 \\ 0 < \alpha \leq |K(x)| < +\infty, \quad \text{K } \textit{continuous.} \end{gathered} \tag{1.1}$$

This problem will be hyperbolic when $K(x) > 0$ and elliptic when $K(x) < 0$. We assume $g \in L^2(R)$, when it is extended as vanishing for $t < 0$, and the problem to have a solution $u(x,\cdot)$ in the Sobolev space[1]

$$H^1(R) = \left\{ f \in L^2(R) \quad / \quad \frac{d}{dx} f \in L^2(R) \right\}$$

when it is extended as vanishing for $t < 0$.

Our approach follows quite closely to the one used in and [2].

In note 1 we show that (1.1) is an ill-posed problem in the sense that a small disturbance on the boundary specification g, can produce a big alteration on its solution, if it exists.

We consider the Meyer multiresolution analysis. The advantage in making use of Meyer's wavelets is its good localization in the frequency domain, since its Fourier transform has compact support. Orthogonal projections onto Meyer's scaling spaces, can be considered as low pass filters, cutting off the high frequencies.

[1] where the derivate is in the distribution sense

From the variational formulation of the approximating problem in the scaling space V_j, we get an infinite-dimensional system of second order ordinary differential equations with variable coefficients. An estimate obtained for the solution of this evolution problem, is used to regularize the ill-posed problem approaching it by well-posed problems. Using an estimate obtained for the difference between the exact solution of problem (1.1) and its orthogonal projection onto V_j, we get an estimate for the difference between the exact solution of problem (1.1) and the orthogonal projection, onto V_j, of the solution of the approximating problem defined in the scaling space V_{j-1}.

In section 2, we construct the Meyer multiresolution analysis. In section 3, we get the estimates of the numerical stability and in section 4 we regularize (1.1).

For a function $h \in L^1(R) \bigcap L^2(R)$ its Fourier Transform is given by $\widehat{h}(\xi) := \int_{\mathbb{R}} h(x)e^{-ix\xi}dx$. We use the notation e^x and $\exp x$ indistinctly.

2 Meyer Multiresolution analysis

Definition A *Multiresolution analysis*, as defined in [1], is a sequence of closed subspaces V_j in $L^2(\mathbb{R})$, called *scaling spaces*, satisfying:

(M1) $V_j \subseteq V_{j-1}$ for all $j \in \mathbb{Z}$

(M2) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$

(M3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

(M4) $f \in V_j$ if and only if $f(2^j\cdot) \in V_0$

(M5) $f \in V_0$ if and only if $f(\cdot - k) \in V_0$ for all $k \in \mathbb{Z}$

(M6) There exists $\phi \in V_0$ such that $\{\phi_{0k} : k \in \mathbb{Z}\}$ is an orthonormal basis in V_0, where $\phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x - k)$ for all $j, k \in \mathbb{Z}$. The function ϕ is called the *scaling function* of the Multiresolution analysis.

The scaling function of the Meyer Multiresolution Analysis is the function φ defined by its Fourier Transform:

$$\widehat{\varphi}(\xi) := \begin{cases} 1, & |\xi| \leq \frac{2\pi}{3} \\ \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi| - 1\right)\right], & \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3} \\ 0, & |\xi| > \frac{4\pi}{3} \end{cases}$$

where ν is a differentiable function satisfying

$$\nu(x) = \begin{cases} 0 & if \quad x \leq 0 \\ 1 & if \quad x \geq 1 \end{cases}$$

and

$$\nu(x) + \nu(1-x) = 1$$

The associated mother wavelet ψ, called Meyer's Wavelet, is given by (see [1])

$$\widehat{\psi}(\xi) = \begin{cases} e^{i\xi/2}\sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi| - 1\right)\right], & \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3} \\ e^{i\xi/2}\cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\xi| - 1\right)\right], & \frac{4\pi}{3} \leq |\xi| \leq \frac{8\pi}{3} \\ 0, & |\xi| > \frac{8\pi}{3} \end{cases}$$

We will consider the Meyer Multiresolution Analysis with scaling function φ. We have

$$\begin{aligned} \widehat{\psi_{jk}}(\xi) &= \int_{\mathbb{R}} \psi_{jk}(x)e^{-ix\xi}dx \\ &= \int_{\mathbb{R}} 2^{-\frac{j}{2}}\psi(2^{-j}x - k)e^{-ix\xi}dx \\ &= \int_{\mathbb{R}} 2^{j/2}\psi(y-k)e^{-i2^j y\xi}dy \\ &= 2^{j/2}\int_{\mathbb{R}} \psi(t)e^{-i2^j(t+k)\xi}dt \\ &= 2^{j/2}\int_{\mathbb{R}} \psi(t)e^{-i2^j t\xi - i2^j k\xi}dt = 2^{j/2}e^{-i2^j k\xi}\widehat{\psi}(2^j\xi) \end{aligned}$$

Since $\text{supp}(\widehat{\psi}) = \left\{\xi : \frac{2}{3}\pi \leq |\xi| \leq \frac{8}{3}\pi\right\}$ we have that

$$\text{supp}(\widehat{\psi_{jk}}) = \left\{\xi; \frac{2}{3}\pi 2^{-j} \leq |\xi| \leq \frac{8}{3}\pi 2^{-j}\right\} \quad \forall k \in \mathbb{Z} \tag{2.1}$$

Furthermore,

$$\operatorname{supp}(\widehat{\varphi_{jk}}) = \left\{\xi; |\xi| \leq \frac{4}{3}\pi 2^{-j}\right\} \quad \forall k \in \mathbb{Z} \tag{2.2}$$

Now we consider the orthogonal projection onto V_j, $P_j : L^2(R) \to V_j$,

$$P_j f(t) = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{jk}\rangle \varphi_{jk}(t)$$

The hypothesis M1 and M2 imply that $\lim_{j\to-\infty} P_j f = f$, for all $f \in L^2(R)$. This means that from a representation of f in a given scale, we can get f by adding details which are given at higher frequencies. From (2.2), we see that P_j filters away the frequencies higher than $\frac{4}{3}\pi 2^{-j}$ (low pass filter).

We have, for all $f \in L^2(R)$,

$$\begin{aligned} f &= P_j f - P_j f + f = P_j f + (I - P_j) f \\ &= \sum_{k\in\mathbb{Z}} \langle f, \varphi_{jk}\rangle \varphi_{jk} + \sum_{l\leq j} \sum_{k\in\mathbb{Z}} \langle f, \psi_{lk}\rangle \psi_{lk} \end{aligned}$$

Since, by (2.1), $\widehat{\psi}_{lk}(\xi) = 0$ for all $l \leq j$ and $|\xi| \leq \frac{2}{3}\pi 2^{-j}$, this implies

$$\widehat{P_j f}(\xi) = \widehat{f}(\xi) \quad \text{for } |\xi| \leq \frac{2}{3}\pi 2^{-j} \tag{2.3}$$

Considering the corresponding orthogonal projections in the frequency space, $\widehat{P}_j : L^2(R) \to \widehat{V}_j = \overline{\operatorname{span}\{\widehat{\varphi_{jk}}\}}_{k\in\mathbb{Z}}$,

$$\widehat{P}_j f = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi} \langle f, \widehat{\varphi_{jk}}\rangle \widehat{\varphi_{jk}}$$

we have

$$\widehat{P}_j \widehat{f} = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi} \langle \widehat{f}, \widehat{\varphi_{jk}}\rangle \widehat{\varphi_{jk}} = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{jk}\rangle \widehat{\varphi_{jk}} = \widehat{P_j f}$$

Then (2.3) implies that

$$\begin{aligned} \|(I - P_j) f\| &= \frac{1}{\sqrt{2\pi}} \|[(I - P_j) f]^\wedge\| = \frac{1}{\sqrt{2\pi}} \|(I - \widehat{P}_j)\widehat{f}\| \\ &= \frac{1}{\sqrt{2\pi}} \|(I - \widehat{P}_j)\chi_j \widehat{f}\| \leq \|\chi_j \widehat{f}\| \end{aligned} \tag{2.4}$$

where χ_j is the characteristic function in $(-\infty, -\frac{2}{3}\pi 2^{-j}] \cup [\frac{2}{3}\pi 2^{-j}, +\infty)$.

3 Stability

In this section we prove that the approximating problems, in the scaling spaces, are well-posed, and we get the estimates of the numerical stability. The next lemma is given in [2].

Lemma 3.1. *Let u and v be positive continuous functions, $x \geq a$ and $c > 0$. If $u(x) \leq c + \int_a^x \int_a^s v(\tau)u(\tau)\, d\tau ds$ then*

$$u(x) \leq c \exp\Big(\int_a^x \int_a^s v(\tau)\, d\tau ds \Big).$$

Applying the Fourier Transform with respect to time in Problem (1.1), we obtain the following problem in the frequency space:

$$\begin{gathered} \widehat{u}_{xx}(x,\xi) = \frac{-\xi^2}{K(x)} \widehat{u}(x,\xi), \quad 0 < x < 1,\ \xi \in R \\ \widehat{u}(0,\xi) = \widehat{g}(\xi), \quad \widehat{u}_x(0,\cdot) = 0 \end{gathered}$$

whose solution satisfies

$$\widehat{u}(x,\xi) \leq |\widehat{g}(\xi)| + \int_0^x \int_0^s \frac{\xi^2}{|K(\tau)|} \widehat{u}(\tau,\xi)\, d\tau ds$$

Then, by lemma 3.1, for $\widehat{g}(\xi) \neq 0$, we have

$$|\widehat{u}(x,\xi)| \leq |\widehat{g}(\xi)| \exp\Big[\xi^2 \int_0^x \int_0^s \frac{1}{|K(\tau)|}\, d\tau ds \Big] \tag{3.1}$$

Lemma 3.2. *The operator $D_j(x)$ defined by*

$$[(D_j)_{lk}(x)]_{l\in\mathbb{Z},\ k\in\mathbb{Z}} = \Big[\frac{1}{K(x)} \langle \varphi_{jl}'', \varphi_{jk} \rangle \Big]_{l\in\mathbb{Z}, k\in\mathbb{Z}}$$

satisfies the following three conditions:

1) $(D_j)_{lk}(x) = (D_j)_{kl}(x)$

2) $(D_j)_{lk}(x) = (D_j)_{(l-k)0}(x)$. Hence, $(D_j)_{lk}(x)$ is a Töplitz matrix.

3) $\|D_j(x)\| \leq \frac{\pi^2 4^{-j+1}}{|K(x)|}$

Proof. 1) Since φ and φ' are reals and $\varphi_{jk}(x) \to 0$, $\varphi'_{jk}(x) \to 0$, when $x \to \pm\infty$, two integrations by parts give the result.

2) Since $\phi_{jm}(t) = 2^{-j/2}\phi(2^{-j}t - m)$, the substitution $2^{-j}s = 2^{-j}t - k$ in $(D_j)_{lk}(x)$ gives:

$$\begin{aligned}(D_j)_{lk}(x) &= \frac{1}{K(x)}\int_{\mathbb{R}} \varphi''_{jl}(t)\varphi_{jk}(t)dt = \frac{1}{K(x)}\int_{\mathbb{R}} \varphi''_{j(l-k)}(s)\varphi_{j0}(s)ds \\ &= (D_j)_{(l-k)0}(x)\end{aligned}$$

3) We have

$$\|D_j(x)\| = \Big\|\frac{1}{K(x)}B_j\Big\| = \frac{1}{|K(x)|}\|B_j\|$$

where $(B_j)_{lk} = \langle \varphi''_{jl}, \varphi_{jk}\rangle$. From results 1) and 2), we have $(B_j)_{lk} = (B_j)_{kl}$, $(B_j)_{lk} = -\frac{1}{2\pi}\int_{\mathbb{R}} \xi^2 e^{-i(l-k)\xi 2^j}|\widehat{\varphi_{j0}}(\xi)|^2 d\xi = (B_j)_{(l-k)0}$ and $(B_j)_{lk}$ is a Töplitz matrix. We will show that $\|B_j\| \le \pi^2 4^{-j+1}$. Thus, we will have

$$\|D_j(x)\| \le \frac{\pi^2 4^{-j+1}}{|K(x)|}$$

For $|t| \le \pi 2^{-j}$,

$$\begin{aligned}\Gamma_j(t) = &-2^{-j}\big[(t - 2^{-j+1}\pi)^2|\widehat{\varphi_{j0}}(t - 2^{-j+1}\pi)|^2 + t^2|\widehat{\varphi_{j0}}(t)|^2 \\ &+ (t + 2^{-j+1}\pi)^2|\widehat{\varphi_{j0}}(t + 2^{-j+1}\pi)|^2\big]\end{aligned}$$

Extend Γ_j periodically to $\mathbb{R}$ and expand it in Fourier series as

$$\Gamma_j(t) = \sum_{k\in\mathbb{Z}} \gamma_k e^{ikt2^j}$$

We have $\gamma_k = b_k$ for all k, where b_k is the element in diagonal k of B_j. In fact, since $\widehat{\varphi_{j0}}(t) = 0$ for $|t| \ge \frac{4}{3}\pi 2^{-j}$, it follows that

$$\begin{aligned}\gamma_k &= \frac{1}{2^{-j+1}\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} \Gamma_j(t)e^{-ikt2^j}dt \\ &= -\frac{1}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} (t - 2^{-j+1}\pi)|\widehat{\varphi_{j0}}(t - 2^{-j+1}\pi)|^2 e^{-ikt2^j}dt \\ &\quad -\frac{1}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}dt \\ &\quad -\frac{1}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} (t + 2^{-j+1}\pi)|\widehat{\varphi_{j0}}(t + 2^{-j+1}\pi)|^2 e^{-ikt2^j}dt\end{aligned}$$

Making a change of variable, we obtain:

$$\begin{aligned}
\gamma_k &= -\frac{1}{2\pi}\int_{-3\pi 2^{-j}}^{-\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt - \frac{1}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt \\
&\quad -\frac{1}{2\pi}\int_{\pi 2^{-j}}^{3\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt \\
&= -\frac{1}{2\pi}\int_{-3\pi 2^{-j}}^{3\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt \\
&= -\frac{1}{2\pi}\int_{\mathbb{R}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}\,dt = b_k
\end{aligned}$$

Now, $\|B_j\| = \sup_{\|f\|=1}\|B_j f\|$ where $\|f\|^2 = \sum_{k\in\mathbb{Z}}|f_k|^2$. Let $F(t) = \sum_{k\in\mathbb{Z}} f_k e^{ikt2^j}$ and define $W(t) = \Gamma_j(t)F(t)$. We have

$$W(t) = \sum_{k\in\mathbb{Z}} \omega_k e^{ikt2^j} \quad \text{and} \quad \omega_k = \sum_{l\in\mathbb{Z}} b_{k-l} f_l = (B_j f)_k$$

Hence

$$\begin{aligned}
\|\omega\|^2 = \sum_{k\in\mathbb{Z}} |\omega_k|^2 &= \frac{1}{2\pi 2^{-j}}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} |W(t)|^2 dt \\
&= \frac{1}{2\pi 2^{-j}}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} |\Gamma_j(t)F(t)|^2 dt \\
&\leq \sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|^2 \frac{1}{2\pi 2^{-j}}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} |F(t)|^2 dt \\
&= \sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|^2 \|f\|^2
\end{aligned}$$

Then

$$\|B_j\| \leq \sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|$$

On the other hand, Γ_j is an odd function. Hence

$$\sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)| = \sup_{0\leq t\leq \pi 2^{-j}} |\Gamma_j(t)|$$

But, for $0 \leq t \leq \pi 2^{-j}$, we have $t+\pi 2^{-j+1} \geq \pi 2^{-j+1}$ and $t-\pi 2^{-j+1} \leq -\pi 2^{-j}$. Hence

$$\widehat{\varphi_{j0}}(t+\pi 2^{-j+1}) = 0 \quad \text{and} \quad |\widehat{\varphi_{j0}}(t-\pi 2^{-j+1})|^2 \leq |\widehat{\varphi_{j0}}(t)|^2$$

for $t \in [0, \pi 2^{-j}]$. Thus

$$\begin{aligned}
\sup_{0 \le t \le \pi 2^{-j}} |\Gamma_j(t)| &\le 2^{-j} \sup_{0 \le t \le \pi 2^{-j}} [t^2 + (t - \pi 2^{-j+1})^2] |\widehat{\varphi_{j0}}(t)|^2 \\
&\le 2^{-j} \pi^2 4^{-j+1} \sup_{0 \le t \le \pi 2^{-j}} |\widehat{\varphi_{j0}}(t)|^2 \\
&= \pi 4^{-j+1} \sup_{0 \le t \le \pi 2^{-j}} |\widehat{\varphi}(2^j t)|^2 \\
&= \pi^2 4^{-j+1} \sup_{0 \le s \le \pi} |\widehat{\varphi}(s)|^2
\end{aligned}$$

By definition of $\widehat{\varphi}$ we have $|\widehat{\varphi}(s)|^2 \le 1$ for $0 \le s \le \pi$. Then

$$\sup_{0 \le t \le \pi 2^{-j}} |\Gamma_j(t)| \le \pi^2 4^{-j+1}$$

Thus

$$\|D_j(x)\| = \frac{1}{|K(x)|} \|B_j\| \le \frac{1}{|K(x)|} \sup_{|t| \le \pi 2^{-j}} |\Gamma_j(t)| \le \frac{\pi^2 4^{-j+1}}{|K(x)|}$$

which completes the proof of lemma 3.2. □

Let us now consider the following approximating problem[2] in V_j:

$$\left\{ \begin{array}{l}
K(x)\, u_{xx}(x,t) = P_j u_{tt}(x,t), \qquad t \ge 0, \qquad 0 < x < 1 \\
u(0, \cdot) = P_j g \\
u_x(0, \cdot) = 0 \\
u(x,t) \in V_j
\end{array} \right. \tag{3.2}$$

Its variational formulation is

$$\left\{ \begin{array}{l}
\langle K\,(x)\; u_{xx} - u_{tt} \;,\; \varphi_{jk} \rangle = 0 \\
\langle u(0, \cdot) \;,\; \varphi_{jk} \rangle = \langle P_j g \,,\, \varphi_{jk} \rangle \,, \qquad \langle u_x(0, \cdot) \;,\; \varphi_{jk} \rangle = \langle 0 \,,\, \varphi_{jk} \rangle \,, \qquad k \in Z
\end{array} \right.$$

where φ_{jk} is the orthonormal basis of V_j given by the scaling function φ. Consider u_j a solution of the approximating problem (3.2), given by

[2]The projection in the first equation of (3.2) is needed because we can have $\varphi \in V_j$ with $\varphi'' \notin V_j$ (see note 2 below).

$u_j(x,t) = \sum_{l\in\mathbb{Z}} w_l(x)\varphi_{jl}(t)$. Then, we have $(u_j)_{tt}(x,t) = \sum_{l\in\mathbb{Z}} w_l(x)\varphi''_{jl}(t)$ and $(u_j)_{xx}(x,t) = \sum_{l\in\mathbb{Z}} w''_l(x)\varphi_{jl}(t)$. Therefore,

$$K(x)(u_j)_{xx}(x,t) - (u_j)_{tt}(x,t) = K(x)\sum_{l\in\mathbb{Z}} w''_l(x)\varphi_{jl}(t) - \sum_{l\in\mathbb{Z}} w_l(x)\varphi''_{jl}(t)$$

Hence

$$\begin{aligned}
&\langle K(x)(u_j)_{xx} - (u_j)_{tt}, \varphi_{jk}\rangle = 0 \Longleftrightarrow \langle\sum_{l\in\mathbb{Z}} K(x)w''_l\varphi_{jl} - \sum_{l\in\mathbb{Z}} w_l\varphi''_{jl}, \varphi_{jk}\rangle = 0\\
&\Longleftrightarrow \sum_{l\in\mathbb{Z}} K(x)w''_l\langle\varphi_{jl}, \varphi_{jk}\rangle = \sum_{l\in\mathbb{Z}} w_l\langle\varphi''_{jl}, \varphi_{jk}\rangle\\
&\Longleftrightarrow K(x)w''_k = \sum_{l\in\mathbb{Z}} w_l\langle\varphi''_{jl}, \varphi_{jk}\rangle \quad k\in\mathbb{Z}\\
&\Longleftrightarrow \frac{d^2}{dx^2}w_k = \sum_{l\in\mathbb{Z}} w_l\frac{1}{K(x)}\langle\varphi''_{jl}, \varphi_{jk}\rangle \quad \Longleftrightarrow \quad \frac{d^2}{dx^2}w_k = \sum_{l\in\mathbb{Z}} w_l(D_j)_{lk}(x).
\end{aligned}$$

where, as defined before, $(D_j)_{lk}(x) = \frac{1}{K(x)}\langle\varphi''_{jl}, \varphi_{jk}\rangle$. Thus, we get an infinite-dimensional system of ordinary differential equations:

$$\begin{cases} \frac{d^2}{dx^2}w = D_j(x)w \\ w(0) = \gamma, \\ w'(0) = 0 \end{cases} \tag{3.3}$$

where γ is given by

$$P_jg = \sum_{z\in\mathbb{Z}}\gamma_z\varphi_{jz} = \sum_{z\in\mathbb{Z}}\langle g, \varphi_{jz}\rangle\varphi_{jz}$$

Lemma 3.3. *If w is a solution of the evolution problem of second order (3.3), then*

$$\|w(x)\| \le \|\gamma\|\exp\left(4^{-j+1}\pi^2\int_0^x\int_0^s \frac{1}{|K(\tau)|}\,d\tau\,ds\right)$$

Proof. Since $w(x) = \gamma + \int_0^x \int_0^s (D_j)(\tau) w(\tau)\, d\tau ds$,

$$\|w(x)\| \leq \|\gamma\| + \int_0^x \int_0^s \|D_j(\tau)\| \|w(\tau)\|\, d\tau\, ds$$

By lemma 3.2 this implies

$$\|w(x)\| \leq \|\gamma\| + \int_0^x \int_0^s \frac{4^{-j+1}\pi^2}{|K(x)|} \|w(\tau)\|\, d\tau\, ds.$$

Then by lemma 3.1 we have

$$\|w(x)\| \leq \|\gamma\| \exp\Big(4^{-j+1}\pi^2 \int_0^x \int_0^s \frac{1}{|K(\tau)|}\, d\tau ds\Big)$$

which completes the proof. □

Theorem 3.4 (Stability of the wavelet Galerkin method). *Let u_j and v_j be solutions in V_j of the approximating problems (3.2) for the boundary specifications g and $\widetilde{g}$, respectively. If $\|g - \widetilde{g}\| \leq \epsilon$ then*

$$\|u_j(x,\cdot) - v_j(x,\cdot)\| \leq \epsilon \exp\big(\frac{4^{-j+1}\pi^2}{2\alpha} x^2\big)$$

where α satisfies $0 < \alpha \leq |K(x)| < +\infty$ as in the definition of the problem (1.1). For j such that $4^{-j} \leq \frac{\alpha}{2\pi^2} \log \epsilon^{-1}$ we have

$$\|u_j(x,\cdot) - v_j(x,\cdot)\| \leq \epsilon^{1-x^2}$$

Proof. $u_j(x,t) = \sum_{l\in\mathbb{Z}} w_l(x)\varphi_{jl}(t)$, $v_j(x,t) = \sum_{l\in\mathbb{Z}} \widetilde{w}_l(x)\varphi_{jl}(t)$ where w and $\widetilde{w}$ are solutions of the Galerkin problem (3.3) with conditions $w(0) = \gamma$ and $\widetilde{w}(0) = \widetilde{\gamma}$, respectively. So, by lemma 3.3 and linearity of (3.3) we have

$$\begin{aligned}
\|u_j(x,\cdot) - v_j(x,\cdot)\| &= \|w(x) - \widetilde{w}(x)\| \\
&\leq \|\gamma - \widetilde{\gamma}\| \exp(4^{-j+1}\pi^2 \int_0^x \int_0^s \frac{1}{|K(\tau)|}\, d\tau ds) \\
&\leq \epsilon \exp(4^{-j+1}\pi^2 \int_0^x \int_0^s \frac{1}{\alpha}\, d\tau ds) \\
&= \epsilon \exp(4^{-j} \frac{2\pi^2}{\alpha} x^2)
\end{aligned}$$

For $j = j(\epsilon)$ such that $4^{-j} \leq \frac{\alpha}{2\pi^2} \log \epsilon^{-1}$, we have

$$\|u_j(x,\cdot) - v_j(x,\cdot)\| \leq \epsilon \exp(x^2 \log \epsilon^{-1}) = \epsilon^{1-x^2}$$

which completes the proof. □

4 Reguralization

In this section we consider (1.1), for the functions $g \in L^2(R)$ such that $\widehat{g}(\xi) \exp(\xi^2/(2\alpha)) \in L^2(R)$, where $\widehat{g}$ is the Fourier Transform of g. The Inverse Fourier Transform of $\exp(-\frac{\xi^2+|\xi|}{2\alpha})$, for instance, satisfies this condition. Define

$$f := \widehat{g}(\xi) \exp\big(\frac{\xi^2}{2\alpha}\big) \in L^2(R) \tag{4.1}$$

Proposition 4.1. *If $u(x,t)$ is a solution of problem (1.1), then*

$$\|u(x,\cdot) - P_j u(x,\cdot)\| \leq \|f\|_{L^2(\mathbb{R})} \exp(-\frac{2}{9}\frac{\pi^2}{\alpha}\ 4^{-j}(1-x^2))$$

where f is given by (4.1).

Proof. From (2.4) and (3.1), we have

$$\begin{aligned}
\|(I-P_j)u(x,\cdot)\| &\leq \|\chi_j \widehat{u}(x,\cdot)\| \\
&= \Big[\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |\widehat{u}(x,\xi)|^2 \, d\xi\Big]^{1/2} \\
&\leq \Big[\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |\widehat{g}(\xi)|^2 \exp[2\xi^2 \int_0^x \int_0^s \frac{1}{|K(\tau)|}\, d\tau\, ds]\, d\xi\Big]^{1/2}
\end{aligned}$$

Then

$$\begin{aligned}
\|(I-P_j)u(x,\cdot)\| &\leq \Big[\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |\widehat{g}(\xi)|^2 \exp(\xi^2 \frac{x^2}{\alpha})\, d\xi\Big]^{1/2} \\
&\leq \Big[\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |f(\xi)|^2 \exp(-\frac{\xi^2}{\alpha}) \exp(\frac{\xi^2}{\alpha} x^2) d\xi\Big]^{1/2} \\
&= \Big[\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |f(\xi)|^2 \exp(-\frac{\xi^2}{\alpha}(1-x^2))\, d\xi\Big]^{1/2}
\end{aligned}$$

For $|x| < 1$,

$$\begin{aligned}\|(I-P_j)u(x,\cdot)\| &\leq \Big[\int_{\mathbb{R}} |f(\xi)|^2\, d\xi\Big]^{1/2} \exp(-\frac{(4/9)\pi^2 4^{-j}}{2\alpha}(1-x^2)) \\ &\leq \|f\|_{L^2(\mathbb{R})} \exp(-\frac{2}{9}\frac{\pi^2}{\alpha}4^{-j}(1-x^2))\end{aligned}$$

which completes the proof. □

Proposition 4.2. *If u is a solution of problem (1.1) and u_{j-1} is a solution of the approximating problem in V_{j-1} then*

$$\widehat{u}(x,\xi) = \widehat{u}_{j-1}(x,\xi) \quad \textit{for } |\xi| \leq \frac{4}{3}\pi 2^{-j} \tag{4.2}$$

Consequently,

$$P_j u(x,\cdot) = P_j u_{j-1}(x,\cdot) \tag{4.3}$$

Proof. Let $\Lambda(x,\xi) = \widehat{u}(x,\xi) - \widehat{u}_{j-1}(x,\xi)$. We will show that $\Lambda(x,\xi) = 0$ for $|\xi| \leq \frac{4}{3}\pi 2^{-j}$. Consider the approximating problem in V_{j-1}:

$$\begin{gathered} K(x)(u_{j-1})_{xx} = P_{j-1}(u_{j-1})_{tt} \quad t \in \mathbb{R},\ 0 < x < 1 \\ u_{j-1}(0,\cdot) = P_{j-1}g, \quad (u_{j-1})_x(0,\cdot) = 0 \\ u_{j-1}(x,\cdot) \in V_{j-1} \end{gathered}$$

Applying the Fourier transform with respect to time, we have

$$K(x)(\widehat{u}_{j-1})_{xx}(x,\xi) = \widehat{P}_{j-1}[(u_{j-1})_{tt}]\widehat{\ }(x,\xi) = \widehat{P}_{j-1}(-\xi^2 \widehat{u}_{j-1}(x,\xi))$$

for $0 \leq x < 1$, $\xi \in \mathbb{R}$, with the conditions: $\widehat{u}_{j-1}(0,\xi) = \widehat{P}_{j-1}\widehat{g}(\xi)$ and $(\widehat{u}_{j-1})_x(0,\cdot) = 0$. Now, by (2.3),

$$\widehat{P}_{j-1}(-\xi^2\widehat{u}_{j-1}(x,\xi)) = -\xi^2\widehat{u}_{j-1}(x,\xi) \quad \text{and} \quad \widehat{P}_{j-1}\widehat{u}(0,\xi) = \widehat{u}(0,\xi)$$

for $|\xi| \leq \frac{4}{3}\pi 2^{-j}$. Thus, for $|\xi| \leq \frac{4}{3}\pi 2^{-j}$, we have

$$\begin{aligned} &K(x)\Lambda_{xx}(x,\xi) + \xi^2\Lambda(x,\xi) \\ &= K(x)\widehat{u}_{xx}(x,\xi) - K(x)(\widehat{u}_{j-1})_{xx}(x,\xi) + \xi^2[\widehat{u}(x,\xi) - \widehat{u}_{j-1}(x,\xi)] = 0 \end{aligned}$$

$$\Lambda(0,\xi) = \widehat{u}(0,\xi) - \widehat{u}_{j-1}(0,\xi) = \widehat{u}(0,\xi) - \widehat{P}_{j-1}\widehat{g}(\xi) = \widehat{u}(0,\xi) - \widehat{P}_{j-1}\widehat{u}(0,\xi) = 0$$

$$\Lambda_x(0,\xi) = \widehat{u}_x(0,\xi) - (\widehat{u}_{j-1})_x(0,\xi) = 0$$

Hence, for $|\xi| \leq \frac{4}{3}\pi 2^{-j}$, fixed, $\Lambda(x,\xi)$ is solution on $0 \leq x < 1$ of the problem

$$K(x)\Lambda_{xx}(x,\xi) + \xi^2\Lambda(x,\xi) = 0, \quad 0 < x < 1$$
$$\Lambda(0,\xi) = 0, \quad \Lambda_x(0,\xi) = 0$$

This problem has an unique solution $\Lambda(x,\xi) = 0$, for all $x \in [0,1)$. Thus,

$$\widehat{u}(x,\xi) = \widehat{u}_{j-1}(x,\xi) \quad \text{for } |\xi| \leq \frac{4}{3}\pi 2^{-j}$$

Now, (4.3) is consequence of (4.2) and the definition of $\widehat{P}_j$. □

Theorem 4.3 (**Regularization**). *Let u be a solution of (1.1) with the condition $u(0,\cdot) = g$, and let f be given by (4.1). Let v_{j-1} be a solution of (3.2) in V_{j-1} for the boundary specification $\widetilde{g}$ such that $\|g - \widetilde{g}\| \leq \epsilon$. If $j = j(\epsilon)$ is such that $4^{-j} = \frac{\alpha}{8\pi^2}\log\epsilon^{-1}$, then*

$$\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| \leq \epsilon^{1-x^2} \;+\; \|f\|_{L^2(R)} \cdot \epsilon^{\frac{1}{36}(1-x^2)}$$

Proof.

$$\begin{aligned}\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| &\leq \|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot) + P_j u(x,\cdot) - u(x,\cdot)\| \\ &\leq \|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot)\| + \|P_j u(x,\cdot) - u(x,\cdot)\|.\end{aligned}$$

Let u_{j-1} be a solution of (3.2) in V_{j-1} for the boundary specification g. By (4.3), $P_j u(x,\cdot) = P_j u_{j-1}(x,\cdot)$. Thus, by theorem 3.4, we have

$$\begin{aligned}\|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot)\| &= \|P_j v_{j-1}(x,\cdot) - P_j u_{j-1}(x,\cdot)\| \\ &\leq \|v_{j-1}(x,\cdot) - u_{j-1}(x,\cdot)\| \leq \epsilon^{1-x^2}\end{aligned}$$

Now, by proposition 4.1,

$$\|P_j u(x,\cdot) - u(x,\cdot)\| \leq \|f\|_{L^2(\mathbb{R})} \exp(-\frac{2}{9}\frac{\pi^2}{\alpha}\; 4^{-j}(1-x^2)) \leq \|f\|_{L^2(\mathbb{R})} \cdot \epsilon^{\frac{1}{36}(1-x^2)}$$

Then $\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| \leq \epsilon^{1-x^2} + \|f\|_{L^2(R)}\epsilon^{\frac{1}{36}(1-x^2)}$ □

Conclusion

We have considered solutions $u(x,\cdot) \in H^1(R)$ for the problem $K(x)u_{xx} = u_{tt}$, $0 < x < 1$, $t \geq 0$, with boundary specification g and $u_x(0,\cdot) = 0$, where $K(x)$ is continuous, $0 < \alpha \leq |K(x)| < +\infty$, and $\widehat{g}(\xi)\exp(\xi^2/(2\alpha)) \in L^2(R)$. Utilizing a wavelet Galerkin method with the Meyer multiresolution analysis, we regularize the ill-posedness of the problem, approaching it by well-posed problems in the scaling spaces and we shown the convergence of the wavelet Galerkin method applied to our problem, with an estimate error. The results obtained apply to the hyperbolic ($K(x) > 0$) and to the elliptic ($K(x) < 0$) case.

Notes: 1) Consider the problem

$$u_{xx}(x,t) = u_{tt}(x,t), \quad t \geq 0,\ 0 < x < 1$$
$$u(0,\cdot) = g_n, \quad u_x(0,\cdot) = 0\,,$$

where

$$g_n(t) = \begin{cases} n^{-2}\cos\sqrt{2}nt, & \text{if } 0 \leq t \leq t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

The solution of this problem is

$$u_n(x,t) = \begin{cases} \sum_{j=0}^{\infty} n^{-2}\cos(\sqrt{2}nt + j\pi)\frac{(\sqrt{2}nx)^{2j}}{(2j)!}, & \text{if } 0 \leq t \leq t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

Note that $g_n(t)$ converges uniformly to zero as n tends to infinity, while for $x > 0$, the solution $u_n(x,t)$ does not tend to zero.

Now consider the Laplace equation with Cauchy conditions on x:

$$u_{xx}(x,t) + u_{tt}(x,t) = 0, \quad t \geq 0,\ 0 < x < 1$$
$$u(0,\cdot) = g_n, \quad u_x(0,\cdot) = 0\,,$$

where

$$g_n(t) = \begin{cases} n^{-2}\cos\sqrt{2}nt, & \text{if } 0 \leq t \leq t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

The solution of this problem is

$$u_n(x,t) = \begin{cases} \sum_{j=0}^{\infty} n^{-2}\cos(\sqrt{2}nt)\frac{(\sqrt{2}nx)^{2j}}{(2j)!}, & \text{if } 0 \le t \le t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

We have that $g_n(t)$ converges uniformly to zero as n tends to infinity, while for $x > 0$, the solution $u_n(x,t)$ does not tend to zero.

2) Note that $(\varphi_{jl})'' \notin V_j$. In fact, if $(\varphi_{jl})'' \in V_j$ then $(\varphi_{jl})'' = \sum_{k\in Z} \alpha_k \varphi_{jk}$. Hence

$$\widehat{(\varphi_{jl})''} = \sum_{k\in Z} \alpha_k \widehat{\varphi_{jk}}$$

So, we would have

$$-2^{j/2}e^{-i2^j l\xi}\xi^2\widehat{\varphi}(2^j\xi) = \sum_{k\in Z} \alpha_k 2^{j/2}e^{-i2^{j/2}\xi}\widehat{\varphi}(2^j\xi)$$

This equality implies $\xi^2 = \sum_{k\in Z} -\alpha_k e^{-i[2^j(k-l)\xi]}$.

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