Uniqueness of the Solution of a Hyperbolic or Elliptic Differential Equation Problem

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Abstract

We consider the problem $K(x)u_{xx} = u_{tt}$, $0 < x < 1$, $t \geq 0$, with boundary condition $u(0,t) = g(t) \in L^2$ and $u_x(0,t) = 0$, where $K(x)$ is continuous and does not come close to zero. This problem is ill-posed in the sense that, if the solution exists, it does not depend continuously on g. We prove that it has at most one solution $u(x, \cdot)$ in the Sobolev space $H^1(R)$, by assuming that $1/K(x)$ is Lipschitz.

1 Introduction

In a previous work [2], we studied the following parabolic partial differential equation problem with variable coefficients:

$$K(x)u_{xx}(x,t) = u_t(x,t), \quad t \geq 0, \quad 0 < x < 1$$
$$u(0,\cdot) = g, \quad u_x(0,\cdot) = 0$$
$$0 < \alpha \leq K(x) < +\infty, \quad K \textit{ continuous.}$$

By assuming that $\frac{1}{K(x)}$ is Lipschitz, we proved that the existence of a solution $u(x,.)$ in the Sobolev space[1]

$$H^1(R) = \left\{ f \in L^2(R) \quad / \quad \frac{d}{dx} f \in L^2(R) \right\}$$

for this problem, implies its uniqueness.

Now, we will extend those results to the problem:

$$\begin{aligned} K(x)u_{xx}(x,t) &= u_{tt}(x,t), \quad t \geq 0,\ 0 < x < 1 \\ u(0,\cdot) &= g, \quad u_x(0,\cdot) = 0 \\ 0 < \alpha \leq |K(x)| &< +\infty, \quad \text{K } \textit{continuous.} \end{aligned} \tag{1.1}$$

Note that problem (1.1) will be hyperbolic when $K(x) > 0$ and elliptic when $K(x) < 0$. We assume $g \in L^2(R)$, when it is extended as vanishing for $t < 0$, and the problem to have a solution $u(x,\cdot) \in H^1(R)$, when it is extended as vanishing for $t < 0$.

We would like to point out that our result is weaker than the overall uniqueness of a solution $u(\cdot,\cdot)$ of problem (1.1), which cannot be discussed without further conditions on this problem. Our uniqueness result supposes that $x \in (0,1)$ is fixed and it is the solution $u(x,\cdot) \in H^1(R)$, as function of the second variable, which is proved to be unique. More precisely, a solution $u(x,\cdot)$ can only be modified in a subset of $[0,+\infty)$ of measure zero.

Our approach follows quite closely to the one used in [2].

We will use the results given in a previous technical report titled: *Regularization of an Ill-Posed Partial Differential Equation Problem.* However, to facilitate the reading, we present these results again.

In section 2, we construct the Meyer multiresolution analysis. In section 3, we get the estimates of the numerical stability and the convergence of the wavelet Galerkin method. In section 4 we prove the uniqueness of the solution. In note 1 we show that problem (1.1) is ill-posed in the sense that a small disturbance on the boundary specification g, can produce a big alteration on its solution, if it exists.

For a function $h \in L^1(R) \bigcap L^2(R)$ its Fourier Transform is given by $\widehat{h}(\xi) := \int_{\mathbb{R}} h(x)e^{-ix\xi}dx$. We use the notation e^x and $\exp x$ indistinctly.

[1] where the derivate is in the distribution sense

2 Meyer Multiresolution analysis

Definition A *Multiresolution analysis*, as defined in [1], is a sequence of closed subspaces V_j in $L^2(\mathbb{R})$, called *scaling spaces*, satisfying:

(M1) $V_j \subseteq V_{j-1}$ for all $j \in \mathbb{Z}$

(M2) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$

(M3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

(M4) $f \in V_j$ if and only if $f(2^j \cdot) \in V_0$

(M5) $f \in V_0$ if and only if $f(\cdot - k) \in V_0$ for all $k \in \mathbb{Z}$

(M6) There exists $\phi \in V_0$ such that $\{\phi_{0k} : k \in \mathbb{Z}\}$ is an orthonormal basis in V_0, where $\phi_{jk}(x) = 2^{-j/2}\phi(2^{-j}x - k)$ for all $j, k \in \mathbb{Z}$. The function ϕ is called the *scaling function* of the Multiresolution analysis.

The scaling function of the Meyer Multiresolution Analysis is the function φ defined by its Fourier Transform:

$$\widehat{\varphi}(\xi) := \begin{cases} 1, & |\xi| \leq \frac{2\pi}{3} \\ \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi| - 1\right)\right], & \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3} \\ 0, & |\xi| > \frac{4\pi}{3} \end{cases}$$

where ν is a differentiable function satisfying

$$\nu(x) = \begin{cases} 0 & if \quad x \leq 0 \\ 1 & if \quad x \geq 1 \end{cases}$$

and

$$\nu(x) + \nu(1 - x) = 1$$

The associated mother wavelet ψ, called Meyer's Wavelet, is given by (see [1])

$$\widehat{\psi}(\xi) = \begin{cases} e^{i\xi/2} \sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi| - 1\right)\right], & \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3} \\ e^{i\xi/2} \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\xi| - 1\right)\right], & \frac{4\pi}{3} \leq |\xi| \leq \frac{8\pi}{3} \\ 0, & |\xi| > \frac{8\pi}{3} \end{cases}$$

We will consider the Meyer Multiresolution Analysis with scaling function φ. We have

$$\begin{aligned} \widehat{\psi_{jk}}(\xi) &= \int_{\mathbb{R}} \psi_{jk}(x) e^{-ix\xi} dx \\ &= \int_{\mathbb{R}} 2^{-\frac{j}{2}} \psi(2^{-j}x - k) e^{-ix\xi} dx \\ &= \int_{\mathbb{R}} 2^{j/2} \psi(y - k) e^{-i2^j y\xi} dy \\ &= 2^{j/2} \int_{\mathbb{R}} \psi(t) e^{-i2^j (t+k)\xi} dt \\ &= 2^{j/2} \int_{\mathbb{R}} \psi(t) e^{-i2^j t\xi - i2^j k\xi} dt = 2^{j/2} e^{-i2^j k\xi} \widehat{\psi}(2^j \xi) \end{aligned}$$

Since $\operatorname{supp}(\widehat{\psi}) = \left\{\xi : \frac{2}{3}\pi \leq |\xi| \leq \frac{8}{3}\pi\right\}$ we have that

$$\operatorname{supp}(\widehat{\psi_{jk}}) = \left\{\xi; \frac{2}{3}\pi 2^{-j} \leq |\xi| \leq \frac{8}{3}\pi 2^{-j}\right\} \quad \forall k \in \mathbb{Z} \tag{2.1}$$

Furthermore,

$$\operatorname{supp}(\widehat{\varphi_{jk}}) = \left\{\xi; |\xi| \leq \frac{4}{3}\pi 2^{-j}\right\} \quad \forall k \in \mathbb{Z} \tag{2.2}$$

Now we consider the orthogonal projection onto V_j, $P_j : L^2(R) \to V_j$,

$$P_j f(t) = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{jk} \rangle \varphi_{jk}(t)$$

The hypothesis M1 and M2 imply that $\lim_{j \to -\infty} P_j f = f$, for all $f \in L^2(R)$. This means that from a representation of f in a given scale, we can get f by adding details which are given at higher frequencies. By (2.2), we see that P_j filters away the frequencies higher than $\frac{4}{3}\pi 2^{-j}$ (low pass filter).

We have, for all $f \in L^2(R)$,

$$\begin{aligned} f &= P_j f - P_j f + f = P_j f + (I - P_j) f \\ &= \sum_{k \in \mathbb{Z}} \langle f, \varphi_{jk} \rangle \varphi_{jk} + \sum_{l \leq j} \sum_{k \in \mathbb{Z}} \langle f, \psi_{lk} \rangle \psi_{lk} \end{aligned}$$

Since, by (2.1), $\widehat{\psi_{lk}}(\xi) = 0$ for all $l \leq j$ and $|\xi| \leq \frac{2}{3}\pi 2^{-j}$, this implies

$$\widehat{P_j f}(\xi) = \widehat{f}(\xi) \quad \text{for } |\xi| \leq \frac{2}{3}\pi 2^{-j} \tag{2.3}$$

Considering the corresponding orthogonal projections in the frequency space, $\widehat{P}_j : L^2(R) \to \widehat{V}_j = \overline{\text{span}\{\widehat{\varphi_{jk}}\}_{k \in \mathbb{Z}}}$,

$$\widehat{P}_j f = \sum_{k \in \mathbb{Z}} \frac{1}{2\pi} \langle f, \widehat{\varphi_{jk}} \rangle \widehat{\varphi_{jk}}$$

we have

$$\widehat{P}_j \widehat{f} = \sum_{k \in \mathbb{Z}} \frac{1}{2\pi} \langle \widehat{f}, \widehat{\varphi_{jk}} \rangle \widehat{\varphi_{jk}} = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{jk} \rangle \widehat{\varphi_{jk}} = \widehat{P_j f}$$

Then (2.3) implies that

$$\begin{aligned} \|(I - P_j) f\| &= \frac{1}{\sqrt{2\pi}} \|[(I - P_j) f]^\wedge\| = \frac{1}{\sqrt{2\pi}} \|(I - \widehat{P}_j) \widehat{f}\| \\ &= \frac{1}{\sqrt{2\pi}} \|(I - \widehat{P}_j) \chi_j \widehat{f}\| \leq \|\chi_j \widehat{f}\| \end{aligned} \tag{2.4}$$

where χ_j is the characteristic function in $(-\infty, -\frac{2}{3}\pi 2^{-j}] \cup [\frac{2}{3}\pi 2^{-j}, +\infty)$.

3 Stability and Regularization

In this section we approach the Ill-posed problem (1.1) by well-posed problems, and we show, with an estimate error, the convergence of the wavelet method used. The next lemma is given in [3].

Lemma 3.1. *Let u and v be positive continuous functions, $x \geq a$ and $c > 0$. If $u(x) \leq c + \int_a^x \int_a^s v(\tau)u(\tau)\, d\tau ds$ then*

$$u(x) \leq c \exp\Big(\int_a^x \int_a^s v(\tau)\, d\tau ds \Big).$$

Applying the Fourier Transform with respect to time in Problem (1.1), we obtain the following problem in the frequency space:

$$\widehat{u}_{xx}(x,\xi) = \frac{-\xi^2}{K(x)} \widehat{u}(x,\xi), \quad 0 < x < 1,\ \xi \in R$$
$$\widehat{u}(0,\xi) = \widehat{g}(\xi), \quad \widehat{u}_x(0,\cdot) = 0$$

whose solution satisfies

$$\widehat{u}(x,\xi) \leq |\widehat{g}(\xi)| + \int_0^x \int_0^s \frac{\xi^2}{|K(\tau)|} \widehat{u}(\tau,\xi)\, d\tau ds$$

Then, by lemma 3.1, for $\widehat{g}(\xi) \neq 0$, we have

$$|\widehat{u}(x,\xi)| \leq |\widehat{g}(\xi)| \exp\Big[\xi^2 \int_0^x \int_0^s \frac{1}{|K(\tau)|}\, d\tau ds\Big] \tag{3.1}$$

Lemma 3.2. *The operator $D_j(x)$ defined by*

$$[(D_j)_{lk}(x)]_{l\in\mathbb{Z},\ k\in\mathbb{Z}} = \Big[\frac{1}{K(x)} \langle \varphi''_{jl}, \varphi_{jk} \rangle\Big]_{l\in\mathbb{Z}, k\in\mathbb{Z}}$$

satisfies the following three conditions: 1) $(D_j)_{lk}(x) = (D_j)_{kl}(x)$
2) $(D_j)_{lk}(x) = (D_j)_{(l-k)0}(x)$. Hence, $(D_j)_{lk}(x)$ is a Töplitz matrix.
3) $\|D_j(x)\| \leq \frac{\pi^2 4^{-j+1}}{|K(x)|}$

Proof. 1) Since φ and φ' are reals and $\varphi_{jk}(x) \to 0$, $\varphi'_{jk}(x) \to 0$, when $x \to \pm\infty$, two integrations by parts give the result.

2) Since $\phi_{jm}(t) = 2^{-j/2}\phi(2^{-j}t - m)$, the substitution $2^{-j}s = 2^{-j}t - k$ in $(D_j)_{lk}(x)$ gives:

$$\begin{aligned}(D_j)_{lk}(x) &= \frac{1}{K(x)} \int_{\mathbb{R}} \varphi''_{jl}(t)\varphi_{jk}(t) dt = \frac{1}{K(x)} \int_{\mathbb{R}} \varphi''_{j(l-k)}(s)\varphi_{j0}(s) ds \\ &= (D_j)_{(l-k)0}(x)\end{aligned}$$

3) We have

$$\|D_j(x)\| = \Big\| \frac{1}{K(x)} B_j \Big\| = \frac{1}{|K(x)|} \|B_j\|$$

where $(B_j)_{lk} = \langle \varphi''_{jl}, \varphi_{jk} \rangle$. From results 1) and 2), we have $(B_j)_{lk} = (B_j)_{kl}$, $(B_j)_{lk} = -\frac{1}{2\pi} \int_{\mathbb{R}} \xi^2 e^{-i(l-k)\xi 2^j} |\widehat{\varphi_{j0}}(\xi)|^2 d\xi = (B_j)_{(l-k)0}$ and $(B_j)_{lk}$ is a Töplitz matrix. We will show that $\|B_j\| \le \pi^2 4^{-j+1}$. Thus, we will have

$$\|D_j(x)\| \le \frac{\pi^2 4^{-j+1}}{|K(x)|}$$

For $|t| \le \pi 2^{-j}$,

$$\begin{aligned}\Gamma_j(t) = &- 2^{-j} \big[(t - 2^{-j+1}\pi)^2 |\widehat{\varphi_{j0}}(t - 2^{-j+1}\pi)|^2 + t^2 |\widehat{\varphi_{j0}}(t)|^2 \\ &+ (t + 2^{-j+1}\pi)^2 |\widehat{\varphi_{j0}}(t + 2^{-j+1}\pi)|^2 \big]\end{aligned}$$

Extend Γ_j periodically to $\mathbb{R}$ and expand it in Fourier series as

$$\Gamma_j(t) = \sum_{k \in \mathbb{Z}} \gamma_k e^{ikt2^j}$$

We have $\gamma_k = b_k$ for all k, where b_k is the element in diagonal k of B_j. In fact, since $\widehat{\varphi_{j0}}(t) = 0$ for $|t| \ge \frac{4}{3}\pi 2^{-j}$, it follows that

$$\begin{aligned}\gamma_k &= \frac{1}{2^{-j+1}\pi} \int_{-\pi 2^{-j}}^{\pi 2^{-j}} \Gamma_j(t) e^{-ikt2^j} dt \\ &= -\frac{1}{2\pi} \int_{-\pi 2^{-j}}^{\pi 2^{-j}} (t - 2^{-j+1}\pi) |\widehat{\varphi_{j0}}(t - 2^{-j+1}\pi)|^2 e^{-ikt2^j} dt \\ &\quad -\frac{1}{2\pi} \int_{-\pi 2^{-j}}^{\pi 2^{-j}} t |\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j} dt \\ &\quad -\frac{1}{2\pi} \int_{-\pi 2^{-j}}^{\pi 2^{-j}} (t + 2^{-j+1}\pi) |\widehat{\varphi_{j0}}(t + 2^{-j+1}\pi)|^2 e^{-ikt2^j} dt\end{aligned}$$

Making a change of variable, we obtain:

$$
\begin{aligned}
\gamma_k &= -\frac{1}{2\pi}\int_{-3\pi 2^{-j}}^{-\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}dt - \frac{1}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}dt \\
&\quad -\frac{1}{2\pi}\int_{\pi 2^{-j}}^{3\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}dt \\
&= -\frac{1}{2\pi}\int_{-3\pi 2^{-j}}^{3\pi 2^{-j}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}dt \\
&= -\frac{1}{2\pi}\int_{\mathbb{R}} t|\widehat{\varphi_{j0}}(t)|^2 e^{-ikt2^j}dt = b_k
\end{aligned}
$$

Now, $\|B_j\| = \sup_{\|f\|=1}\|B_j f\|$ where $\|f\|^2 = \sum_{k\in\mathbb{Z}}|f_k|^2$. Let $F(t) = \sum_{k\in\mathbb{Z}} f_k e^{ikt2^j}$ and define $W(t) = \Gamma_j(t)F(t)$. We have

$$W(t) = \sum_{k\in\mathbb{Z}} \omega_k e^{ikt2^j} \quad \text{and} \quad \omega_k = \sum_{l\in\mathbb{Z}} b_{k-l} f_l = (B_j f)_k$$

Hence

$$
\begin{aligned}
\|\omega\|^2 = \sum_{k\in\mathbb{Z}}|\omega_k|^2 &= \frac{1}{2\pi 2^{-j}}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} |W(t)|^2 dt \\
&= \frac{1}{2\pi 2^{-j}}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} |\Gamma_j(t)F(t)|^2 dt \\
&\leq \sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|^2 \frac{1}{2\pi 2^{-j}}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} |F(t)|^2 dt \\
&= \sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|^2 \|f\|^2
\end{aligned}
$$

Then

$$\|B_j\| \leq \sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)|$$

On the other hand, Γ_j is an odd function. Hence

$$\sup_{|t|\leq \pi 2^{-j}} |\Gamma_j(t)| = \sup_{0\leq t\leq \pi 2^{-j}} |\Gamma_j(t)|$$

But, for $0 \leq t \leq \pi 2^{-j}$, we have $t + \pi 2^{-j+1} \geq \pi 2^{-j+1}$ and $t - \pi 2^{-j+1} \leq -\pi 2^{-j}$. Hence

$$\widehat{\varphi_{j0}}(t + \pi 2^{-j+1}) = 0 \quad \text{and} \quad |\widehat{\varphi_{j0}}(t - \pi 2^{-j+1})|^2 \leq |\widehat{\varphi_{j0}}(t)|^2$$

for $t \in [0, \pi 2^{-j}]$. Thus

$$\begin{aligned}
\sup_{0\le t\le \pi 2^{-j}} |\Gamma_j(t)| &\le 2^{-j} \sup_{0\le t\le \pi 2^{-j}} [t^2 + (t-\pi 2^{-j+1})^2] |\widehat{\varphi_{j0}}(t)|^2 \\
&\le 2^{-j}\pi^2 4^{-j+1} \sup_{0\le t\le \pi 2^{-j}} |\widehat{\varphi_{j0}}(t)|^2 \\
&= \pi 4^{-j+1} \sup_{0\le t\le \pi 2^{-j}} |\widehat{\varphi}(2^j t)|^2 \\
&= \pi^2 4^{-j+1} \sup_{0\le s\le \pi} |\widehat{\varphi}(s)|^2
\end{aligned}$$

By definition of $\widehat{\varphi}$ we have $|\widehat{\varphi}(s)|^2 \le 1$ for $0 \le s \le \pi$. Then

$$\sup_{0\le t\le \pi 2^{-j}} |\Gamma_j(t)| \le \pi^2 4^{-j+1}$$

Thus

$$\|D_j(x)\| = \frac{1}{|K(x)|}\|B_j\| \le \frac{1}{|K(x)|} \sup_{|t|\le \pi 2^{-j}} |\Gamma_j(t)| \le \frac{\pi^2 4^{-j+1}}{|K(x)|}$$

which completes the proof of lemma 3.2. □

Let us now consider the following approximating problem[2] in V_j:

$$\begin{cases} K(x)\, u_{xx}(x,t) = P_j u_{tt}(x,t), \qquad t \ge 0, \qquad 0 < x < 1 \\ u(0,\cdot) = P_j g \\ u_x(0,\cdot) = 0 \\ u(x,t) \in V_j \end{cases} \tag{3.2}$$

Its variational formulation is

$$\begin{cases} \langle K\,(x)\; u_{xx} - u_{tt}\;,\; \varphi_{jk}\rangle = 0 \\ \langle u(0,\cdot)\;,\; \varphi_{jk}\rangle = \langle P_j g\,,\, \varphi_{jk}\rangle\,, \qquad \langle u_x(0,\cdot)\;,\; \varphi_{jk}\rangle = \langle 0\,,\, \varphi_{jk}\rangle\,, \qquad k \in Z \end{cases}$$

where φ_{jk} is the orthonormal basis of V_j given by the scaling function φ.

[2]The projection in the first equation of (3.2) is needed because we can have $\varphi \in V_j$ with $\varphi'' \notin V_j$ (see note 2 below).

Consider u_j a solution of the approximating problem (3.2), given by $u_j(x,t) = \sum_{l\in\mathbb{Z}} w_l(x)\varphi_{jl}(t)$. Then, we have $(u_j)_{tt}(x,t) = \sum_{l\in\mathbb{Z}} w_l(x)\varphi''_{jl}(t)$ and $(u_j)_{xx}(x,t) = \sum_{l\in\mathbb{Z}} w''_l(x)\varphi_{jl}(t)$. Therefore,

$$K(x)(u_j)_{xx}(x,t) - (u_j)_{tt}(x,t) = K(x)\sum_{l\in\mathbb{Z}} w''_l(x)\varphi_{jl}(t) - \sum_{l\in\mathbb{Z}} w_l(x)\varphi''_{jl}(t)$$

Hence

$$\begin{aligned}
&\langle K(x)(u_j)_{xx} - (u_j)_{tt}, \varphi_{jk}\rangle = 0 \Longleftrightarrow \langle\sum_{l\in\mathbb{Z}} K(x)w''_l\varphi_{jl} - \sum_{l\in\mathbb{Z}} w_l\varphi''_{jl}, \varphi_{jk}\rangle = 0\\
&\Longleftrightarrow \sum_{l\in\mathbb{Z}} K(x)w''_l\langle\varphi_{jl},\varphi_{jk}\rangle = \sum_{l\in\mathbb{Z}} w_l\langle\varphi''_{jl},\varphi_{jk}\rangle\\
&\Longleftrightarrow K(x)w''_k = \sum_{l\in\mathbb{Z}} w_l\langle\varphi''_{jl},\varphi_{jk}\rangle \quad k\in\mathbb{Z}\\
&\Longleftrightarrow \frac{d^2}{dx^2}w_k = \sum_{l\in\mathbb{Z}} w_l\frac{1}{K(x)}\langle\varphi''_{jl},\varphi_{jk}\rangle \quad \Longleftrightarrow \quad \frac{d^2}{dx^2}w_k = \sum_{l\in\mathbb{Z}} w_l(D_j)_{lk}(x).
\end{aligned}$$

where, as defined before, $(D_j)_{lk}(x) = \frac{1}{K(x)}\langle\varphi''_{jl},\varphi_{jk}\rangle$. Thus, we get an infinite-dimensional system of ordinary differential equations:

$$\begin{cases} \frac{d^2}{dx^2}w = D_j(x)w \\ w(0) = \gamma, \\ w'(0) = 0 \end{cases} \tag{3.3}$$

where γ is given by

$$P_jg = \sum_{z\in\mathbb{Z}} \gamma_z\varphi_{jz} = \sum_{z\in\mathbb{Z}} \langle g,\varphi_{jz}\rangle\varphi_{jz}$$

Lemma 3.3. *If w is a solution of the evolution problem of second order (3.3), then*

$$\|w(x)\| \le \|\gamma\| \exp\left(4^{-j+1}\pi^2\int_0^x\int_0^s \frac{1}{|K(\tau)|}\,d\tau\,ds\right)$$

Proof. Since $w(x) = \gamma + \int_0^x \int_0^s (D_j)(\tau) w(\tau)\, d\tau ds$,

$$\|w(x)\| \leq \|\gamma\| + \int_0^x \int_0^s \|D_j(\tau)\| \|w(\tau)\|\, d\tau\, ds$$

By lemma 3.2 this implies

$$\|w(x)\| \leq \|\gamma\| + \int_0^x \int_0^s \frac{4^{-j+1}\pi^2}{|K(x)|} \|w(\tau)\|\, d\tau\, ds.$$

Then by lemma 3.1 we have

$$\|w(x)\| \leq \|\gamma\| \exp\Big(4^{-j+1}\pi^2 \int_0^x \int_0^s \frac{1}{|K(\tau)|}\, d\tau ds\Big)$$

which completes the proof. □

Theorem 3.4 (Stability of the wavelet Galerkin method). *Let u_j and v_j be solutions in V_j of the approximating problems (3.2) for the boundary specifications g and $\widetilde{g}$, respectively. If $\|g - \widetilde{g}\| \leq \epsilon$ then*

$$\|u_j(x, \cdot) - v_j(x, \cdot)\| \leq \epsilon \exp\big(\frac{4^{-j+1}\pi^2}{2\alpha} x^2\big)$$

where α satisfies $0 < \alpha \leq |K(x)| < +\infty$ as in the definition of the problem (1.1). For j such that $4^{-j} \leq \frac{\alpha}{2\pi^2} \log \epsilon^{-1}$ we have

$$\|u_j(x, \cdot) - v_j(x, \cdot)\| \leq \epsilon^{1-x^2}$$

Proof. $u_j(x,t) = \sum_{l\in\mathbb{Z}} w_l(x)\varphi_{jl}(t)$, $v_j(x,t) = \sum_{l\in\mathbb{Z}} \widetilde{w}_l(x)\varphi_{jl}(t)$ where w and $\widetilde{w}$ are solutions of the Galerkin problem (3.3) with conditions $w(0) = \gamma$ and $\widetilde{w}(0) = \widetilde{\gamma}$, respectively. So, by lemma 3.3 and linearity of (3.3) we have

$$\begin{aligned}
\|u_j(x, \cdot) - v_j(x, \cdot)\| &= \|w(x) - \widetilde{w}(x)\| \\
&\leq \|\gamma - \widetilde{\gamma}\| \exp(4^{-j+1}\pi^2 \int_0^x \int_0^s \frac{1}{|K(\tau)|}\, d\tau ds) \\
&\leq \epsilon \exp(4^{-j+1}\pi^2 \int_0^x \int_0^s \frac{1}{\alpha}\, d\tau ds) \\
&= \epsilon \exp(4^{-j} \frac{2\pi^2}{\alpha} x^2)
\end{aligned}$$

For $j = j(\epsilon)$ such that $4^{-j} \leq \frac{\alpha}{2\pi^2} \log \epsilon^{-1}$, we have

$$\|u_j(x,\cdot) - v_j(x,\cdot)\| \leq \epsilon \exp(x^2 \log \epsilon^{-1}) = \epsilon^{1-x^2}$$

which completes the proof. □

We will consider problem (1.1), for the functions $g \in L^2(R)$ such that $\widehat{g}(\xi)\exp(\xi^2/(2\alpha)) \in L^2(R)$, where $\widehat{g}$ is the Fourier Transform of g. The Inverse Fourier Transform of $\exp(-\frac{\xi^2+|\xi|}{2\alpha})$, for instance, satisfies this condition. Define

$$f := \widehat{g}(\xi) \exp\big(\frac{\xi^2}{2\alpha}\big) \in L^2(R) \tag{3.4}$$

Proposition 3.5. *If $u(x,t)$ is a solution of problem (1.1), then*

$$\|u(x,\cdot) - P_j u(x,\cdot)\| \leq \|f\|_{L^2(\mathbb{R})} \exp(-\frac{2}{9}\frac{\pi^2}{\alpha}\ 4^{-j}(1-x^2))$$

where f is given by (3.4).

Proof. From (2.4) and (3.1), we have

$$\begin{aligned}
\|(I-P_j)u(x,\cdot)\| &\leq \|\chi_j \widehat{u}(x,\cdot)\| \\
&= \big[\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |\widehat{u}(x,\xi)|^2 \, d\xi\big]^{1/2} \\
&\leq \big[\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |\widehat{g}(\xi)|^2 \exp[2\xi^2 \int_0^x \int_0^s \frac{1}{|K(\tau)|}\, d\tau\, ds]\, d\xi\big]^{1/2}
\end{aligned}$$

Then

$$\begin{aligned}
\|(I-P_j)u(x,\cdot)\| &\leq \big[\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |\widehat{g}(\xi)|^2 \exp(\xi^2 \frac{x^2}{\alpha})\, d\xi\big]^{1/2} \\
&\leq \big[\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |f(\xi)|^2 \exp(-\frac{\xi^2}{\alpha}) \exp(\frac{\xi^2}{\alpha} x^2) d\xi\big]^{1/2} \\
&= \big[\int_{|\xi|>\frac{2}{3}\pi 2^{-j}} |f(\xi)|^2 \exp(-\frac{\xi^2}{\alpha}(1-x^2))\, d\xi\big]^{1/2}
\end{aligned}$$

For $|x| < 1$,

$$\begin{aligned}\|(I-P_j)u(x,\cdot)\| &\leq \Big[\int_{\mathbb{R}} |f(\xi)|^2\, d\xi\Big]^{1/2} \exp\Big(-\frac{(4/9)\pi^2 4^{-j}}{2\alpha}(1-x^2)\Big)\\ &\leq \|f\|_{L^2(\mathbb{R})} \exp\Big(-\frac{2}{9}\frac{\pi^2}{\alpha} 4^{-j}(1-x^2)\Big)\end{aligned}$$

which completes the proof. □

Proposition 3.6. *If u is a solution of problem (1.1) and u_{j-1} is a solution of the approximating problem in V_{j-1} then*

$$\widehat{u}(x,\xi) = \widehat{u}_{j-1}(x,\xi) \quad \textit{for } |\xi| \leq \frac{4}{3}\pi 2^{-j} \tag{3.5}$$

Consequently,

$$P_j u(x,\cdot) = P_j u_{j-1}(x,\cdot) \tag{3.6}$$

Proof. Let $\Lambda(x,\xi) = \widehat{u}(x,\xi) - \widehat{u}_{j-1}(x,\xi)$. We will show that $\Lambda(x,\xi) = 0$ for $|\xi| \leq \frac{4}{3}\pi 2^{-j}$. Consider the approximating problem in V_{j-1}:

$$\begin{gathered} K(x)(u_{j-1})_{xx} = P_{j-1}(u_{j-1})_{tt} \quad t \in \mathbb{R},\ 0 < x < 1\\ u_{j-1}(0,\cdot) = P_{j-1}g, \quad (u_{j-1})_x(0,\cdot) = 0\\ u_{j-1}(x,\cdot) \in V_{j-1}\end{gathered}$$

Applying the Fourier transform with respect to time, we have

$$K(x)(\widehat{u}_{j-1})_{xx}(x,\xi) = \widehat{P}_{j-1}[(u_{j-1})_{tt}]\widehat{\ }(x,\xi) = \widehat{P}_{j-1}(-\xi^2 \widehat{u}_{j-1}(x,\xi))$$

for $0 \leq x < 1$, $\xi \in \mathbb{R}$, with the conditions: $\widehat{u}_{j-1}(0,\xi) = \widehat{P}_{j-1}\widehat{g}(\xi)$ and $(\widehat{u}_{j-1})_x(0,\cdot) = 0$. Now, by (2.3),

$$\widehat{P}_{j-1}(-\xi^2\widehat{u}_{j-1}(x,\xi)) = -\xi^2\widehat{u}_{j-1}(x,\xi) \quad \text{and} \quad \widehat{P}_{j-1}\widehat{u}(0,\xi) = \widehat{u}(0,\xi)$$

for $|\xi| \leq \frac{4}{3}\pi 2^{-j}$. Thus, for $|\xi| \leq \frac{4}{3}\pi 2^{-j}$, we have

$$\begin{aligned}&K(x)\Lambda_{xx}(x,\xi) + \xi^2\Lambda(x,\xi)\\ &= K(x)\widehat{u}_{xx}(x,\xi) - K(x)(\widehat{u}_{j-1})_{xx}(x,\xi) + \xi^2[\widehat{u}(x,\xi) - \widehat{u}_{j-1}(x,\xi)] = 0\end{aligned}$$

$$\Lambda(0,\xi) = \widehat{u}(0,\xi) - \widehat{u}_{j-1}(0,\xi) = \widehat{u}(0,\xi) - \widehat{P}_{j-1}\widehat{g}(\xi) = \widehat{u}(0,\xi) - \widehat{P}_{j-1}\widehat{u}(0,\xi) = 0$$

$$\Lambda_x(0,\xi) = \widehat{u}_x(0,\xi) - (\widehat{u}_{j-1})_x(0,\xi) = 0$$

Hence, for $|\xi| \le \frac{4}{3}\pi 2^{-j}$, fixed, $\Lambda(x,\xi)$ is solution on $0 \le x < 1$ of the problem

$$K(x)\Lambda_{xx}(x,\xi) + \xi^2\Lambda(x,\xi) = 0, \quad 0 < x < 1$$
$$\Lambda(0,\xi) = 0, \quad \Lambda_x(0,\xi) = 0$$

This problem has an unique solution $\Lambda(x,\xi) = 0$, for all $x \in [0,1)$. Thus,

$$\widehat{u}(x,\xi) = \widehat{u}_{j-1}(x,\xi) \quad \text{for } |\xi| \le \frac{4}{3}\pi 2^{-j}$$

Now, (3.6) is consequence of (3.5) and the definition of $\widehat{P}_j$. □

Theorem 3.7 (**Regularization**). *Let u be a solution of (1.1) with the condition $u(0,\cdot) = g$, and let f be given by (3.4). Let v_{j-1} be a solution of (3.2) in V_{j-1} for the boundary specification $\widetilde{g}$ such that $\|g - \widetilde{g}\| \le \epsilon$. If $j = j(\epsilon)$ is such that $4^{-j} = \frac{\alpha}{8\pi^2}\log\epsilon^{-1}$, then*

$$\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| \le \epsilon^{1-x^2} \ + \ \|f\|_{L^2(R)} \cdot \epsilon^{\frac{1}{36}(1-x^2)}$$

Proof.

$$\begin{aligned}\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| &\le \|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot) + P_j u(x,\cdot) - u(x,\cdot)\| \\ &\le \|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot)\| + \|P_j u(x,\cdot) - u(x,\cdot)\|.\end{aligned}$$

Let u_{j-1} be a solution of (3.2) in V_{j-1} for the boundary specification g. By (3.6), $P_j u(x,\cdot) = P_j u_{j-1}(x,\cdot)$. Thus, by theorem 3.4, we have

$$\begin{aligned}\|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot)\| &= \|P_j v_{j-1}(x,\cdot) - P_j u_{j-1}(x,\cdot)\| \\ &\le \|v_{j-1}(x,\cdot) - u_{j-1}(x,\cdot)\| \le \epsilon^{1-x^2}\end{aligned}$$

Now, by proposition 3.5,

$$\|P_j u(x,\cdot) - u(x,\cdot)\| \le \|f\|_{L^2(\mathbb{R})}\exp(-\frac{2}{9}\frac{\pi^2}{\alpha}\ 4^{-j}(1-x^2)) \le \|f\|_{L^2(\mathbb{R})} \cdot \epsilon^{\frac{1}{36}(1-x^2)}$$

Then $\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| \le \epsilon^{1-x^2} + \|f\|_{L^2(R)}\epsilon^{\frac{1}{36}(1-x^2)}$ □

4 Uniqueness of the Solution

The infinite-dimensional system of ordinary differential equations (3.3) can be written in the following way:

$$\begin{cases} \frac{dv}{dx} = D_j(x)w + 0v \\ \frac{dw}{dx} = 0w + v \\ w(0) = \gamma \quad \text{and} \quad v(0) = 0 \end{cases} \qquad \begin{cases} \frac{dV}{dx} = A_j(x)V \\ V(0) = (0, \gamma)^T \end{cases}$$

where $V = (v, w) \in X := l^2(R) \times l^2(R)$, $x \in [0, 1)$ and

$$A_j(x) = \begin{bmatrix} 0 & D_j(x) \\ 1 & 0 \end{bmatrix}$$

with $\|A_j(x)V\|_X = \|(D_j(x)w, v)\|_X = \sqrt{\|D_j(x)w\|_{l^2}^2 + \|v\|_{l^2}^2}$

Lemma 4.1. *For all* $j \in Z$, $A_j(x) : X \longrightarrow X$ *is a uniformly bounded linear operator on* $x \in [0, 1)$.

Proof. By lemma 3.2 and the hypothesis $0 < \alpha \leq |K(x)| < +\infty$, we have

$$\|D_j(x)\| \leq \frac{\pi^2 4^{-j+1}}{|K(x)|} \leq \frac{\pi^2 4^{-j+1}}{\alpha} := K_j$$

If $\|V\|_X = 1$ then $\|w\|_{l^2} \leq 1$ and $\|v\|_{l^2} \leq 1$. So,

$$\|A_j(x)V\|_X = \sqrt{\|D_j(x)w\|_{l^2}^2 + \|v\|_{l^2}^2} \leq \sqrt{K_j^2 + 1}$$

Thus, the operator $A_j(x)$ is uniformly bounded on $x \in [0, 1)$. □

Lemma 4.2. *If* $\frac{1}{K(x)}$ *is Lipschitz on* $[0, 1)$ *then* $x \longmapsto D_j(x)$ *is Lipschitz on* $[0, 1)$, $\forall\, j \in Z$. *Consequently* $x \longmapsto A_j(x)$ *is Lipschitz on* $[0, 1)$.

Proof. $D_j(x) = \frac{1}{K(x)} B_j$, where $(B_j)_{lk} = \langle \varphi''_{jl}\,, \varphi_{jk} \rangle$. We have $\|B_j\| \leq \pi^2 4^{-j+1}$. Then

$$\|D_j(x) - D_j(y)\| \leq \left| \frac{1}{K(x)} - \frac{1}{K(y)} \right| \pi^2 4^{-j+1} \leq L_j \, |x-y|$$

with $L_j = L \cdot \pi^2 4^{-j+1}$, where L is the Lipschitz constant of $\frac{1}{K(x)}$.

Now,

$$\begin{aligned} \|A_j(x) - A_j(y)\| &= \sup_{V \in X, \, \|V\|=1} \|(A_j(x) - A_j(y))\, V\|_X \\ &= \sup_{w \in l^2, \, \|w\|=1} \|(D_j(x) - D_j(y))\, w\|_{l^2} \\ &= \|D_j(x) - D_j(y)\| \\ &\leq L_j \, |x-y| \end{aligned}$$

□

Lemma 4.3. *For each* $j \in Z$*, the operator* $[0,1) \ni x \longmapsto A_j(x)$ *is continuous in the uniform operator topology.*

Proof. Let $x \in [0,1)$ and $\epsilon > 0$. By lemma 4.2, $A_j(x)$ is Lipschitz with Lipschitz constant L_j. Let $\delta_\epsilon := \epsilon / L_j$. We have, for $y \in [0,1)$:

$$|x-y| < \delta_\epsilon \Longrightarrow \|A_j(x) - A_j(y)\| \leq L_j \, |x-y| < L_j \cdot \delta_\epsilon = \epsilon.$$

□

By the previous lemmas, we have:

Theorem 4.4. *The infinite-dimensional system of ordinary differential equations (3.3) has a unique solution.*

Proof. The result follows by lemma 4.1, lemma 4.2, lemma 4.3 above and theorem 5.1 in [4, page 127]. □

Theorem 4.5. *Let u be a solution of problem (1.1) with condition $u(0,\cdot) = g$ where g satisfies (3.4). Then, for any sequence j_n, such that $j_n \longrightarrow -\infty$ as $n \longrightarrow +\infty$, there exists a unique sequence u_{j_n} of solutions of the approximating problems (3.2) in V_{j_n} with conditions $u_{j_n}(0,\cdot) = P_{j_n}g$ and $\forall\, x \in [0,1)$ such that*

$$P_{j_n+1}u_{j_n}(x,\cdot) \longrightarrow u(x,\cdot) \text{ in } L^2.$$

Proof. From Theorem 4.4 each approximating problem has a unique solution. Then the result follows from theorem 3.7, with $\widetilde{g} = g$, since that j and ϵ are functionally related by $4^{-j} = \frac{\alpha}{8\pi^2} \log \epsilon^{-1}$ independently of u. $\square$

Corollary 4.6. *Problem (1.1)* has at most one solution, for each $x \in [0,1)$, where g satisfies *(3.4)* $\square$

Conclusion

We have considered solutions $u(x,\cdot) \in H^1(R)$ for the problem $K(x)u_{xx} = u_{tt}$, $0 < x < 1$, $t \geq 0$, with boundary specification g and $u_x(0,\cdot) = 0$, where $K(x)$ is continuous, $0 < \alpha \leq |K(x)| < +\infty$, $\frac{1}{K(x)}$ is Lipschitz and $\widehat{g}(\xi)\exp(\xi^2/(2\alpha)) \in L^2(R)$. Utilizing a wavelet Galerkin method with the Meyer multiresolution analysis, we regularize the ill-posedness of the problem, approaching it by well-posed problems in the scaling spaces and we shown the convergence of the wavelet Galerkin method applied to our problem, with an estimate error. We have shown that if a solution exists, it is unique. The results obtained apply to the hyperbolic ($K(x) > 0$) and to the elliptic ($K(x) < 0$) case.

Notes: 1) Consider the problem

$$u_{xx}(x,t) = u_{tt}(x,t), \quad t \geq 0,\ 0 < x < 1$$
$$u(0,\cdot) = g_n, \quad u_x(0,\cdot) = 0\,,$$

where

$$g_n(t) = \begin{cases} n^{-2}\cos\sqrt{2}nt, & \text{if } 0 \leq t \leq t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

The solution of this problem is

$$u_n(x,t) = \begin{cases} \sum_{j=0}^{\infty} n^{-2}\cos(\sqrt{2}nt + j\pi)\frac{(\sqrt{2}nx)^{2j}}{(2j)!}, & \text{if } 0 \le t \le t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

Note that $g_n(t)$ converges uniformly to zero as n tends to infinity, while for $x > 0$, the solution $u_n(x,t)$ does not tend to zero.

Now consider the Laplace equation with Cauchy conditions on x:

$$u_{xx}(x,t) + u_{tt}(x,t) = 0, \quad t \ge 0,\ 0 < x < 1$$
$$u(0,\cdot) = g_n, \quad u_x(0,\cdot) = 0\,,$$

where

$$g_n(t) = \begin{cases} n^{-2}\cos\sqrt{2}nt, & \text{if } 0 \le t \le t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

The solution of this problem is

$$u_n(x,t) = \begin{cases} \sum_{j=0}^{\infty} n^{-2}\cos(\sqrt{2}nt)\frac{(\sqrt{2}nx)^{2j}}{(2j)!}, & \text{if } 0 \le t \le t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

We have that $g_n(t)$ converges uniformly to zero as n tends to infinity, while for $x > 0$, the solution $u_n(x,t)$ does not tend to zero.

2) Note that $(\varphi_{jl})'' \notin V_j$. In fact, if $(\varphi_{jl})'' \in V_j$ then $(\varphi_{jl})'' = \sum_{k\in Z}\alpha_k\varphi_{jk}$. Hence

$$\widehat{(\varphi_{jl})''} = \sum_{k\in Z}\alpha_k\widehat{\varphi_{jk}}$$

So, we would have

$$-2^{j/2}e^{-i2^jl\xi}\xi^2\widehat{\varphi}(2^j\xi) = \sum_{k\in Z}\alpha_k 2^{j/2}e^{-i2^{j/2}\xi}\widehat{\varphi}(2^j\xi)$$

This equality implies $\xi^2 = \sum_{k\in Z} -\alpha_k e^{-i[2^j(k-l)\xi]}$.

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