COPPE/UFRJ

DECOMPOSIÇÕES PARA COLORAÇÃO DE ARESTAS E COLORAÇÃO TOTAL DE GRAFOS

Raphael Carlos Santos Machado

Tese de Doutorado apresentada ao Programa de Pós-graduação em Engenharia de Sistemas e Computação, COPPE, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Engenharia de Sistemas e Computação.

Orientador: Celina Miraglia Herrera de Figueiredo, D.Sc.

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Não, ninguém faz samba só

porque prefere.

(J. Nogueira e P. C. Pinheiro)

Resumo da Tese apresentada à COPPE/UFRJ como parte dos requisitos necessários para a obtenção do grau de Doutor em Ciências (D.Sc.)

DECOMPOSIÇÕES PARA COLORAÇÃO DE ARESTAS E COLORAÇÃO TOTAL DE GRAFOS

Raphael Carlos Santos Machado

Janeiro/2010

Orientador: Celina Miraglia Herrera de Figueiredo, D.Sc.

Programa: Engenharia de Sistemas e Computação

Esta tese propõe a aplicação a coloração de arestas e coloração total de técnicas já consolidadas no contexto de coloração de vértices. Aplicamos tais técnicas de forma a obter resultados de complexidade de coloração de arestas e coloração total restritos a classes de grafos, tais como grafos join, grafos cobipartidos, partial-grids, grafos outerplanares, grafos chordless, grafos unichord-free, bipartidos unichord-free e {square,unichord}-free. Os resultados obtidos mostram a independência entre os problemas de coloração de arestas e de coloração total e permitem compreender melhor a relação — e as distinções — entre estes problemas clássicos de coloração.

Abstract of Thesis presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Doctor of Science (D.Sc.)

DECOMPOSITIONS FOR EDGE-COLOURING AND TOTAL-COLOURING

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The present thesis considers the application to edge-colouring and total-colouring of decomposition techniques well established in the vertex-colouring scenario. We apply such decomposition techniques to obtain complexity results for edge-colouring and total-colouring in graph classes, such as join graphs, cobipartite graphs, partial-grids, outerplanar graphs, chordless graphs, unichord-free graphs, bipartite unichord-free graphs, and {square,unichord}-free graphs. The obtained results allow a better undertanding on the relations — and distinctions — between these classical graph colouring problems.

Sumário

Capítulo 1

Apresentação

Grafo é uma ferramenta matemática que modela **relações** entre elementos de um conjunto discreto. Um grafo G é um par ordenado (V, E), onde V é um conjunto estudado e E é uma coleção de pares de elementos de V que representa relações entre estes elementos[1]. Se os pares em E são ordenados, dizemos que G é um *grafo direcionado*, caso contrário, dizemos que G é um *grafo não-direcionado.* A presente tese considera problemas em grafos não-direcionados, de forma que, a partir daqui, assumiremos que os grafos estudados são não-direcionados, a menos que explicitamente mencionado.

O grafo $G = (V, E)$ pode ser representado graficamente associando-se "pontos" aos elementos de V e "ligando" dois pontos associados a $v_1 \in V$ e $v_2 \in V$ se, e somente se, $v_1v_2 \in E$, conforme o exemplo da Figura 1.1

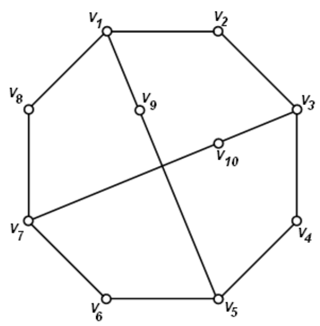


Figura 1.1: Exemplo de grafo $G = (V, E)$ com $V = \{v_1, ..., v_{10}\}$ e $E = \{v_1v_2, v_2v_3, ...,$ $v_7v_8, v_8v_1, v_1v_9, v_5v_9, v_3v_{10}, v_7v_{10}\}$.

Devido à representação gráfica do grafo G, os elementos de V são chamados

[1]Por simplicidade escreveremos apenas uv quando desejarmos nos referir à aresta $\{u, v\}$.

vértices, enquanto os elementos de E são chamados *arestas*. Os vértices e arestas de um grafo são denominados *elementos* deste grafo.

O grande alcance das aplicações da Teoria dos Grafos vem do fato de que as "relações" representadas pelos conjuntos das arestas podem ser, virtualmente, qualquer tipo de relação entre os elementos de V, os quais também podem ser oriundos dos mais diversos campos. Se $V = \{c_1, ..., c_n\}$ é um conjunto de computadores ligados em rede, uma aresta c_ic_j pode indicar a existência de um link direto de comunicação entre os computadores c_i e c_j. Se, por outro lado, $V = \{t_1, ..., t_n\}$ é o conjunto dos componentes de uma placa de um circuito eletrônico, então uma aresta t_it_j pode indicar que terminais dos componentes t_i e t_j devem estar no mesmo potencial, ou seja, deve haver um "curto circuito" entre eles. Em um cenário completamente diferente, poderíamos ter um conjunto $V = \{p_1, ..., p_n\}$ de pessoas que freqüentam determinado ambiente e aresta a p_ip_j representando a existência de algum tipo de relação entre estas pessoas, a qual poderia ser, por exemplo, de amizade ou de inimizade, conforme o problema modelado. O modelo de grafos permite que se concentre esforços nos aspectos combinatórios relevantes de um problema.

Coloração de grafos

Uma importante área da teoria dos grafos é a denominada *Coloração de Grafos*. Tal área está intimamente ligada a modelos de problemas de conflito. De uma forma geral, o problema de coloração de grafos busca associar valores aos elementos de um grafo, de tal forma que elementos[2] relacionados recebam valores diferentes. A origem da expressão *Coloração de Grafos* vem do clássico problema de colorir mapas, no qual territórios vizinhos – ou seja, territórios relacionados através da relação "vizinhança" — devem receber cores diferentes. Referirmo-nos-emos por "cores" aos valores atribuídos a elementos de um grafo em um problema de coloração.

O objetivo geral do problema de coloração é obter uma atribuição de cores que minimize o número de cores utilizadas. No caso da coloração de mapas, por exemplo, um problema que ficou por muito tempo em aberto é o de saber o número mínimo suficiente para colorir mapas de tal forma que territórios vizinhos sempre recebessem

[2]Ao longo desta tese, os elementos a serem coloridos serão vértices e/ou arestas.

cores diferentes. Tal problema foi resolvido na segunda metade do século XX e hoje sabe-se que com 4 cores é possível colorir qualquer mapa [1].

Nesta tese, consideramos três problemas clássicos de coloração. Denominamos *coloração de vértices* uma atribuição de cores aos vértices de um grafo, enquanto *coloração de arestas* é uma atribuição de cores às arestas de um grafo. Finalmente, uma *coloração total* é uma atribuição de cores a todos os elementos de um grafo, ou seja, seus vértices e arestas.

A tese

Esta tese é um trabalho sobre coloração de arestas e coloração total, sobre decomposições de grafos, e sobre complexidade computacional. Mais precisamente, estudamos técnicas de decomposição já consolidadas em coloração de vértices e investigamos as possibilidades de aplicações de tais técnicas aos problemas de coloração de arestas e coloração total. Ao longo deste trabalho, pudemos estabelecer resultados de complexidade computacional dos problemas de coloração de arestas e coloração total restritos a diversas classes.

A principal contribuição desta tese é o uso de informações estruturais e de decomposição — tradicionalmente associados ao problema de coloração de vértices — com o objetivo de resolver problemas de coloração de arestas e coloração total. O desenvolvimento de tais técnicas premitiu obter resultados para uma série de classes de grafos: join, cobipartidos, partial-grids, outerplanares, chordless, unichord-free, bipartidos unichord-free e {square,unichord}-free. Em particular, destacamos o interesse especial nesta última classe, a qual constitui um exemplo inédito e surpreendente de classe de grafos para a qual o problema de coloração de arestas é NP-completo, mas cujos grafos, a menos de ciclos e grafos completos, são todos de Tipo 1, de forma que o problema de coloração de total é resolvido em tempo polinomial.

Outro importante aspecto das técnicas desenvolvidas nesta tese é o fato de que tais técnicas são eminentemente algorítmicas. De uma forma geral, mostramos como combinar colorações de grafos ditos "básicos" de forma a construir colorações de grafos mais sofisticados. Em particular, para problemas dinâmicos, nos quais um grafo pode "crescer" através de operações de composição com outros grafos, nossas técnicas permitem obter colorações para o novo grafo efetuando um mínimo de alterações nos grafos originais — os "operandos" da composição.

Evolução da tese

Este trabalho iniciou-se em janeiro de 2007 com o estudo de um primeiro conjunto de trabalhos clássicos a respeito de coloração de arestas. Aspectos de decomposição mostraram-se imediatamente importantes. Em particular, a operação join presente na construção de cografos foi considerada particularmente interessante, principalmente devido à existência de diversos artigos investigando a operação no contexto de coloração de arestas [44, 45]. Estabelecemos uma técnica de partição do conjunto de arestas de um grafo em dois conjuntos que induzem um grafo desconexo e um grafo bipartido. Tal técnica foi aplicada de forma a obter resultados inéditos em coloração de arestas de grafos join e grafos cobipartidos. Este primeiro trabalho [30] foi apresentado na conferência *IV Latin American Graphs, Algorithms and Optimization Symposium* (LAGOS 2007) na forma de resumo estendido e, posteriormente, submetido, em sua versão completa, para a edição especial da revista *Discrete Applied Mathematics* dedicada à conferência. O artigo completo foi aceito para publicação.

Em meados de 2007, iniciou-se o estudo do problema da coloração total, com vista a adaptar ao problema as técnicas de coloração de vértices e coloração de arestas até então desenvolvidas. Uma decomposição que se mostrou particularmente útil para a coloração total foi a decomposição por clique 2-cutset — cortes-clique de tamanho dois. Tal decomposição permitiu obter resultados de coloração total em partial-grids e coloração total por listas em grafos outerplanares. Este segundo trabalho [31] foi apresentado na conferência *Cologne-Twente Workshop* (CTW 2008) na forma de resumo estendido e, posteriormente, submetido, em sua versão completa, para a edição especial da revista *Networks* dedicada à conferência. O artigo completo foi aceito para publicação. Mais recentemente, em 2009, voltamos a considerar a classe das *partial-grids* no contexto do problema de reconhecimento. Identificamos a complexidade do reconhecimento de partial-grids com diversos conjuntos de restrições nos graus admitidos para seus vértices. Tal trabalho [40] será apresentado no

International Symposium on Combinatorial Optimization (ISCO 2010) e sua versão completa deverá ser submetida para publicação na edição especial da revista Discrete Applied Mathematics dedicada aos melhores trabalhos da conferência.

A partir de meados de 2008 (logo após o CTW 2008), passamos a buscar novas classes de grafos dotadas de estrutura interessante para problemas de coloração. Tivemos contato com a classe dos grafos que não possuem ciclos com corda única — grafos *unichord-free* — recentemente definida e para a qual se havia obtido fortes resultados de decomposição, além de resultados de reconhecimento e de coloração de vértices. Imediatamente, questionamos de que forma e se tais resultados estruturais poderiam ser utilizados para se obter resultados em coloração de arestas e coloração total. Estabelecemos a NP-completude do problema de coloração de arestas na classe, o que motivou a busca por subclasses nas quais o problema fosse polinomial. Um candidato natural era a classe dos grafos {square,unichord}-free, para os quais pudemos mostrar resultados estruturais ainda mais fortes que os dos unichord-free. Para a classe dos grafos {square,unichord}-free, obteve-se uma interessante dicotomia: enquanto o problema de coloração de arestas é NP-completo para grafos com grau máximo 3, o problema é polinomial para todos os outros casos. Foram obtidos resultados adicionais para coloração de arestas no caso de grau máximo 3, os quais apresentam conexões com outras áreas interessantes de coloração de arestas. Os resultados para coloração de arestas de grafos unichord-free e {square,unichord}-free [32] foram apresentados no *ALIO/EURO Workshop on Applied Optimization* (ALIO/EURO 2008) na forma de resumo estendido e, posteriormente, submetido e aceito para publicação em sua versão completa na revista *Theoretical Computer Science* **411** (2010) 1221–1234.

No início de 2009 iniciou-se investigação a respeito do problema de coloração total restrito a grafos unichord-free e {square,unichord}-free. Inicialmente, obteve-se resultado de NP-completude para grafos unichord-free, trabalho que foi apresentado no *Cologne-Twente Workshop* (CTW 2009) na forma de resumo estendido. Em seguida, obteve-se a validade da Conjectura da Coloração Total para a segunda classe, que compôs, juntamente com o resultado de NP-completude, artigo [33] submetido para publicação na edição especial da revista *Discrete Applied Mathematics* dedicada à conferência. Ainda com relação ao problema de coloração total, esta tese contém resultados recentes [34] a respeito da complexidade do problema de coloração total restrito a grafos {square,unichord}-free para o caso grau máximo 3, os quais serão apresentados no International Symposium on Combinatorial Optimization (ISCO 2010). Os resultados deverão ser submetidos na forma publicação na edição especial da revista Discrete Applied Mathematics dedicada aos melhores trabalhos da conferência.

Paralelamente aos grafos {square,unichord}-free, duas outras subclasses de unichord-free foram investigadas no contexto de coloração de arestas e de coloração total. Uma classe é a dos bipartidos unichord-free: embora trivialmente polinomial para coloração de arestas — já que todos são de Classe 1 — o problema de coloração total é NP-completo quando restrito a esta classe. Outra classe é a dos grafos *chordless* (grafos cujos ciclos são todos induzidos) estudada no contexto de grafos que não possuem subdivisão do grafo completo com quatro vértices: mostramos que todo grafo na classe com grau máximo pelo menos 3 é de Classe 1 e de Tipo 1 — e, naturalmente, os problemas de coloração de arestas e de coloração total são polinomiais. Os resultados sobre grafos bipartidos unichord-free e grafos chordless ainda não foram submetidos para publicação.

Organização da tese

A tese está organizada da seguinte forma. No Capítulo 2 apresentamos conceitos, notação, e estado atual das pesquisas em coloração. O Capítulo 3 investiga decomposições por cliques para o problema de coloração total. São apresentadas aplicações da decomposição por cliques de tamanho 2 ao problema de coloração total de partial-grids com ciclo induzido máximo limitado e ao problema de coloração total por listas de grafos outerplanares. Nos Capítulos 4, 5 e 6, estudamos classes associadas a outras decomposições tipicamente relacionadas com o problema de coloração de vértices e investigamos como — e se — essas decomposições podem ser aplicadas aos problemas de coloração de arestas e coloração total. O Capítulo 4 apresenta resultados de complexidade de coloração de arestas e coloração total para a classes de grafos unichord-free, os quais são definidos recursivamente em termos de decomposições "1-cutset", "proper 1-join" e "proper 2-cutset". O Capítulo 5 investiga a classe dos grafos {square,unichord}-free e apresenta resultados de coloração de arestas e coloração total. O Capítulo 6 apresenta duas subclasses adicionais dos grafos unichord-free, os grafos bipartidos unichord-free e os grafos chordless, para os quais obtivemos resultados de complexidade para coloração de arestas e coloração total. No Capítulo 7 apresentamos uma investigação a respeito da decomposição "desconexo-bipartido", definida com o objetivo de melhor compreender as propriedades da operação "join" com relação à coloração de arestas. O estudo da decomposição desconexo-bipartido permitiu alcançar novos resultados com relação à coloração de arestas de "grafos join" e "grafos cobipartidos". Finalmente, o Capítulo 8 contém considerações finais e os próximos trabalhos a serem conduzidos na linha de pesquisa em coloração de grafos via decomposição.

Com o objetivo de facilitar a leitura da tese, optamos por não inserir integralmente todas as demonstrações dos resultados obtidos. De uma forma geral, inserimos "esboços" das demonstrações e remetemos o leitor às referências onde se encontram os detalhes da demontração. As seguintes referências possuem resultados desta tese e, portanto, encontram-se anexadas a ela:

- A decomposition for total-colouring partial-grids and list-total-colouring outerplanar graphs [31].
- Complexity dicothomy on degree constrained VLSI layouts with unit length edges [40].
- Chromatic index of graphs with no cycle with unique chord [32].
- Total chromatic number of unichord-free graphs [33].
- Total chromatic number of {square,unichord}-free graphs [34].
- Decompositions for edge-coloring join graphs and cobipartite graphs [30].

Capítulo 2

Introdução

2.1 Grafos

Um grafo é um par ordenado $G = (V, E)$ onde $V := V(G)$ é denominado o conjunto dos seus *vértices* e $E := E(G)$ é denominado o conjunto de suas *arestas*. Denotamos por $S(G) := V(G) \cup E(G)$ o conjunto dos *elementos* de G, isto é, seus vértices e arestas. Dois vértices u e v de um grafo são ditos *adjacentes* se este grafo possui uma aresta uv. Neste caso, a aresta uv é dita *incidente* aos vértices u e v, os quais, por sua vez, são denominados *extremos* da aresta uv. Duas arestas que possuam um extremo comum são ditas *adjacentes*.

A *vizinhança aberta* de um vértice $v \in V(G)$, denotada por $N_G(v)$, é o conjunto $N_G(v) := \{u \in V(G) | uv \in E(G)\}$. A *vizinhança fechada* de um vértice $v \in V(G)$, denotada por $N_G[v]$, é o conjunto $N_G[v] := N_G(v) \cup \{v\}$. Dado um conjunto de vértices $V' \subseteq V(G)$, denotamos por $N_G(V')$ e $N_G[V']$, respectivamente, os conjuntos $\cup_{v \in V'} N_G(v)$ e $\cup_{v \in V'} N_G[v]$. O subgrafo de G induzido por V', denotado por $G[V']$, possui conjunto de vértices V' e conjunto de arestas $\{uw \in E | u \in V' \wedge w \in V'\}$.

O *grau* de um vértice $v \in V(G)$ é o número de arestas de G incidentes a v, e é denotado por $deg_G(v)$. Quando o grafo G estiver claro no contexto, será usado simplesmente $deg(v)$ para denotar o grau de v. O grau máximo de um vértice em um grafo G é denotado por $\Delta(G)$. Um grafo cujos vértices possuem todos os mesmo grau (d) é dito (d-)regular. Denotamos por $\Lambda(G)$ o conjunto dos vértices de G que possuem grau máximo $\Delta(G)$. O grau de uma aresta uv, denotado por $deg(uv)$, é o maior dos graus de seus extremos, ou seja, $deg(uv) = \max\{deg(u), deg(v)\}$.

2.2 Coloração de Grafos

Uma coloração de um grafo $G = (V(G), E(G))$ é uma função $\pi : S' \rightarrow \texttt{C}$ que atribui cores de $\texttt{C}$ a um conjunto de elementos $S' \subset S(G)$ de tal forma que elementos incidentes ou adjacentes $x \in S(G)$ e $y \in S(G)$ sempre recebem cores diferentes – ou seja, $\pi(x) \neq \pi(y)$. Se $\texttt{C} = \{1, 2, ..., k\}$, então π é dito uma *k-coloração*. Em particular, se $S' = V(G)$, então π é uma *k-coloração de vértices*; se $S' = E(G)$, então π é uma *k-coloração de arestas*; se $S' = S(G)$, então π é uma *k-coloração total*.

O *número cromático* de um grafo G, denotado por $\chi(G)$, é o menor inteiro k para o qual existe uma k-coloração dos vértices de G. O *índice cromático* de um grafo G, denotado por $\chi'(G)$, é o menor inteiro k para o qual existe uma k-coloração das arestas de G. O *número cromático total* de um grafo G, denotado por $\chi_T(G)$, é o menor inteiro k para o qual existe uma k-coloração total de G. São NP-difíceis os problemas de se determinar o número cromático [27], o índice cromático [26] e o número cromático total [42] de um grafo. Dessa forma, muito da pesquisa na área de coloração de grafos é feita no sentido de se determinar classes de grafos para as quais os problemas sejam polinomiais.

2.2.1 Coloração de arestas

O problema de coloração de arestas, aqui denotado por `CHRIND`, é o problema de decisão que tem como entrada um grafo G e um inteiro k e pergunta se existe uma k-coloração das arestas de G. O índice cromático de um grafo está estreitamente relacionado com o seu grau máximo. Observe que todo grafo G precisa de pelo menos $\Delta(G)$ cores para que suas arestas sejam coloridas: uma vez que existe pelo menos um vértice que possui $\Delta(G)$ arestas nele incidentes, então essas $\Delta(G)$ arestas precisarão de $\Delta(G)$ cores diferentes para serem coloridas. Um importante resultado obtido por Vizing [48] é que todo grafo simples G pode ter suas arestas coloridas com $\Delta(G) + 1$ cores. Dessa forma, existem apenas dois valores possíveis para o índice cromático de G: ou $\chi'(G) = \Delta(G)$ ou $\Delta(G) + 1$. Grafos cujo índice cromático é igual ao grau máximo são ditos *Classe 1*, enquanto se o índice cromático excede o grau máximo por uma unidade, são ditos *Classe 2*.

Apesar da enorme restrição com relação aos possíveis valores de $\chi'(G)$, lembramos que é NP-completo decidir se o índice cromático de um grafo é igual ao seu grau máximo [26]. De fato, o problema é NP-completo mesmo quando restrito a grafos r-regulares, para cada grau fixo $r \geq 3$, e permanece NP-completo para as seguintes classes restritas de grafos [9]:

- grafos de comparabilidade (logo grafos perfeitos) regulares de grau $r \geq 3$;
- grafos de linha de bipartidos (logo grafos de linha e grafos clique) regulares de grau $r \geq 3$;
- grafos r-regulares sem ciclo induzido de tamanho k, para cada $r \geq 3$ e $k \geq 3$;
- grafos cúbicos de cintura k, para cada $k \geq 4$.

Classes para as quais o problema de coloração de arestas é polinomial incluem as seguintes:

- grafos bipartidos [27];
- grafos split-indiferença [37];
- grafos série-paralelo [12];
- grafos multipartidos completos [25];
- grafos com vértice universal [38].

A complexidade do problema de coloração de arestas está aberta para diversas classes muito estudadas e de estrutura bastante forte e conhecida, para as quais apenas resultados parciais foram publicados, tais como cografos [2], grafos join [44, 45, 30], grafos cobipartidos [30], grafos planares [43, 49], grafos cordais, e diversas subclasses dos cordais, tais como grafos split [13], grafos de indiferença [16, 18] e grafos de intervalo [6, 18]. O grande número de publicações – mesmo de resultados parciais – mostra o enorme interesse que existe no problema de coloração de arestas restrito a classes de grafos.

Uma importante ferramenta no contexto de coloração de arestas é o conceito de *subgraph-overfullness*. Um grafo $G = (V, E)$ é *overfull* se $|E| > \Delta(G)\lfloor |V|/2 \rfloor$. Um grafo é *subgraph-overfull* se possui um subgrafo de mesmo grau máximo que seja overfull. Todo grafo subgraph-overfull é Classe 2, mas o inverso nem sempre é verdade [17].

2.2.2 Coloração total

O problema de coloração total, aqui denotado por *TOTCHR*, é o problema de decisão que tem como entrada um grafo G e um inteiro k e pergunta se existe uma k-coloração total de G. Assim como o índice cromático, o número cromático total de um grafo também está relacionado com o seu grau máximo. Observe que todo grafo G precisa de pelo menos $\Delta(G) + 1$ cores para que seus elementos (arestas e vértices) sejam coloridos: uma vez que existe pelo menos um vértice que possui $\Delta(G)$ arestas nele incidentes, então essas $\Delta(G)$ arestas precisarão de $\Delta(G)$ cores diferentes para serem coloridas, e uma cor adicional será necessária para colorir o próprio vértice.

Uma importante conjectura, denominada Conjectura da Coloração Total (ou TCC — "Total Colouring Conjecture") afirma que todo grafo simples G possui uma coloração total com $\Delta(G) + 2$ cores [3]. Dessa forma, existiriam apenas dois valores possíveis para o número cromático total de um grafo: $\chi_T(G) = \Delta(G)+1$ ou $\Delta(G)+2$. Um grafo G com $\chi_T(G) = \Delta(G) + 1$ é dito *Tipo 1*, enquanto um grafo G com $\chi_T(G) = \Delta(G) + 2$ é dito *Tipo 2*.

É NP-completo decidir se o número cromático total de um grafo é igual ao seu grau máximo mais a unidade [42]. De fato, o problema é NP-completo mesmo quando restrito a grafos bipartidos r-regulares [41], para cada grau fixo $r \geq 3$. Existem classes restritas conhecidas para as quais o problema de coloração total é polinomial, algumas das quais enumeramos a seguir:

- um ciclo G possui número cromático total $\chi_T(G) = \Delta(G)+1 = 3$ se $|V(G)| = 0$ mod 3, e $\chi_T(G) = \Delta(G) + 2 = 4$ caso contrário [51];

- um grafo completo G possui número cromático total $\chi_T(G) = \Delta(G) + 1$ se $|V(G)|$ é ímpar, e $\chi_T(G) = \Delta(G) + 2$ caso contrário [3, 51];

- um grafo bipartido completo $G = K_{m,n}$ possui número cromático total $\chi_T(G) = \Delta(G) + 1 = \max\{m, n\} + 1$ se $m \neq n$, e $\chi_T(G) = \Delta(G) + 2 = m + 2 = n + 2$ caso contrário [3, 51];

- uma grade $G = P_m \times P_n$ possui número cromático total $\chi_T(G) = \Delta(G) + 2$ se $G = P_2$ ou $G = C_4$, e $\chi_T(G) = \Delta(G) + 1$ caso contrário [10];
- um grafo série-paralelo G possui número cromático total $\chi_T(G) = \Delta(G) + 2$ se $G = P_2$ ou $G = C_n$ com $n \neq 0 \mod 3$, e $\chi_T(G) = \Delta(G) + 1$ caso contrário [22, 52, 54].

Assim como no caso da coloração de arestas, existem diversas classes de grafos para as quais o problema de coloração total está aberto e apenas são conhecidos resultados parciais. Tais classes incluem: grafos planares [51, 53], grafos cordais, e classes de grafos relacionadas com os cordais, tais como split [13], dually chordal [18], intervalo [6] e indiferença [16, 18]. O elevado número de publicações — mesmo de resultados parciais — mostra o interesse que existe no problema de coloração total restrito a classes de grafos.

2.3 Decomposições

Decomposição é uma das palavras chaves da presente tese. Refere-se a dividir um grafo em blocos, com o objetivo de resolver um determinado problema. A idéia é que o problema possua solução mais fácil nos blocos obtidos pela decomposição, para que, em seguida, se tente combinar as soluções dos blocos em uma solução para o grafo original.

Dado um grafo G e um conjunto de vértices $X \subset V(G)$, dizemos que X é um corte (*cutset*) de G se o subgrafo induzido $G \setminus X = G[V(G) \setminus X]$ é desconexo. Se $|X| = q$, dizemos que X é um *q-corte*. Denotando os componentes conexos de $G \setminus X$ por $H_1, ..., H_k$, dizemos que os subgrafos induzidos $G_1 = G[V(H_1) \cup X], ..., G_k = G[V(H_k) \cup X]$ de G são os X-componentes (X-*components*) de G. O conceito de *bloco* é mais geral, construído de acordo com o problema e a classe estudada. Enquanto no Capítulo 3 os blocos de decomposição de G por um corte $X \subset V(G)$ são exatamente os X-components de G, nos Capítulos 4 e 5 os blocos de decomposição podem ser construídos através da inclusão de elementos adicionais não pertencentes a G. O objetivo desta tese é o desenvolvimento e a aplicação de técnicas que permitam obter colorações ótimas de um grafo a partir da combinação de colorações de seus blocos de decomposição.

Capítulo 3

Decomposição por cliques de tamanho 2

3.1 Introdução

O objetivo desta tese é investigar como técnicas de decomposição tradicionalmente úteis ao problema de coloração de vértices podem ser aplicadas à coloração de arestas e à coloração total. Nesta linha, torna-se natural iniciar nossas investigações com um tipo de decomposição extremamente útil à coloração de vértices, a decomposição por cliques. Para os propósitos deste capítulo os *blocos* de decomposição de um grafo G pelo conjunto de vértices $X \subset V(G)$ são os X-componentes de G. O objetivo buscado neste capítulo é obter uma coloração total de um grafo G, a partir de colorações totais de seus blocos de decomposição.

Uma *clique* em um grafo é um conjunto de vértices adjacentes dois-a-dois. Uma decomposição bastante estudada para a coloração de vértices é aquela baseada em cortes do tipo clique (*clique cutsets*), ou seja, cortes que são cliques. Dizemos que X é um n-corte clique (*clique n-cutset*) de G se X é uma clique com n vértices e é um corte de G. Se X é um clique cutset de um grafo G e colorações ótimas de vértices são conhecidas para cada bloco, então é imediato combinar tais colorações em uma coloração ótima dos vértices de G. Mais precisamente, basta renomear as cores de tal forma que as cores dos vértices do corte coincidam em cada X-componente.

Considerando o problema da coloração total, se um clique cutset X possui exatamente um vértice x, então é possível combinar $(\Delta(G) + 1)$-colorações totais dos blocos em uma $(\Delta(G) + 1)$-coloração total do grafo original G: basta determinar uma $(\Delta(G)+1)$-total-coloração cada X-componente de tal forma que a cor de x seja a mesma e as cores das arestas de todos os blocos incidentes a X sejam duas-a-duas distintas. (De fato, ao considerar o problema de coloração total em classes fechadas em relação a decomposições por cortes de 1 vértice, podemos assumir que os grafos são *biconexos*, ou seja, não possuem 1-corte.) Para o caso $|X| \geq 2$, entretanto, não há resultado tão bem-comportado. Observe, na Figura 3.1, um exemplo onde G possui grau máximo 3 e X é um clique 2-cutset. Os dois X-componentes de G são 4-total-coloríveis. No entanto, o grafo G não possui uma 4-coloração total. Exemplos similares podem ser construídos para grafos com graus mais elevados. Tal fato nos motiva a investigar sob quais condições é possível combinar total-colorações modificando apenas elementos "em torno" de um clique 2-cutset. Neste capítulo apresentamos duas aplicações da decomposição por clique 2-cutsets ao problema da coloração total.

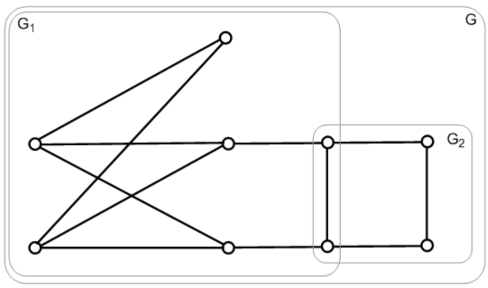


Figura 3.1: Ambos os blocos G_1 e G_2 são 4-total-coloríveis. No entanto, G não é 4-total-colorível.

Conforme já mencionamos, o objetivo de se decompor um grafo é obter uma solução a partir da combinação de soluções para os blocos. Uma forma de entender a idéia é pensar na decomposição recursiva de um grafo até que se obtenha um conjunto de grafos indecomponíveis, os quais são denominados *básicos* (veja a Figure 3.2). Uma vez resolvido o problema para cada grafo básico, busca-se combinar as soluções progressivamente, até que se tenha uma solução para o grafo original.

Uma decomposição é dita *extremal* [47] se pelo menos um dos blocos de decomposição é básico. Possuir uma decomposição extremal é útil porque pelo menos um dos blocos pertence ao conjunto "restrito" dos grafos básicos (veja um exemplo de decomposição extremal na Figura 3.3). O Lema 1 enuncia que todo grafo não-básico

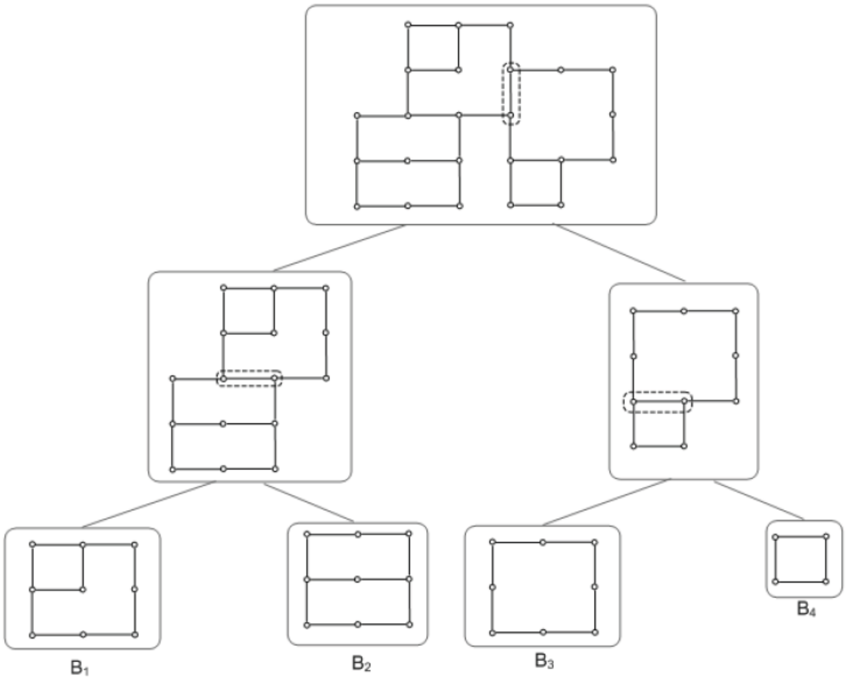


Figura 3.2: Árvore de decomposição com relação a clique 2-cutsets: B_1, B_2, B_3 e B_4 são os blocos ou grafos básicos de decomposição.

possui uma decomposição extremal por clique 2-cutsets.

Lema 1 (Machado and Figueiredo — Lema 1 de [31]) *Seja G um grafo que possui clique 2-cutset. Nestas condições, o grafo G possui clique 2-cutset tal que pelo menos um dos blocos é básico.*

Esboço. Basta escolher, dentre todas as possíveis decomposições, uma que minimize um dos blocos de decomposição. □

Nas Seções 3.2 e 3.3 estudamos duas classes cujos conjuntos de grafos básicos com relação a clique 2-cutsets possuem propriedades úteis, as quais descrevemos a seguir. No caso das partial-grids de grau máximo até 3 e com ciclo induzido máximo limitado, mostramos que o conjunto dos grafos básicos é finito. No caso dos grafos outerplanares, embora os grafos básicos não formem um conjunto finito, eles se resumem a ciclos, uma classe extremamente estruturada.

3.2 Número cromático total de partial-grids

Um grafo $G_{m\times n}$, sendo $m, n \geq 1$, com conjunto de vértices $V(G_{m\times n}) = \{1, ..., m\} \times \{1, ..., n\}$ e conjunto de arestas $E(G_{m\times n}) = \{(i,j)(k,l) : |i-k| + |j-l| = 1,$

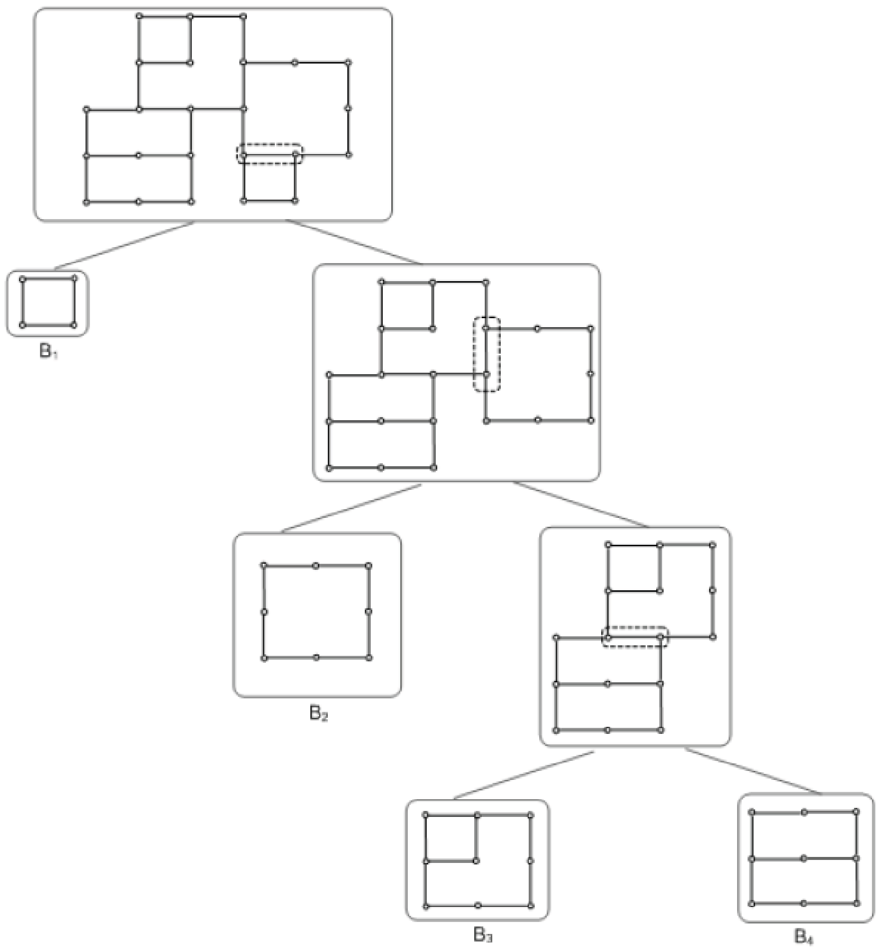


Figura 3.3: Uma árvore de decomposição extremal com relação a clique 2-cutsets para o mesmo grafo da Figura 3.2.

$(i,j),(k,l) \in V(G_{m\times n})\}$, ou um gráfico isomórfico a $G_{m\times n}$, é chamado *grid*. Uma *partial-grid* é um subgrafo arbitrário de uma grid. Partial-grids possuem estrutura mais sofisticada do que grids; por exemplo, o reconhecimento de grids é polinomial [8], enquanto o problema é NP-completo para partial-grids [4, 20, 40], refletindo a maior complexidade desta última classe. A coloração total de partial-grids provou ser um problema desafiador: enquanto as partial-grids de grau máximo 1, 2 ou 4 podem ser coloridos através de aplicação direta de resultados de coloração total para grids e ciclos, o caso de grau máximo 3 continua incompleto [10]. Assim, o último passo em direção a uma completa classificação de partial-grids é considerar os subcasos restantes de grau máximo 3.

Um grafo é *c-cordal* [15] se não possui ciclo induzido maior que c. Apresentamos uma decomposição por clique 2-cutsets que fornece um método para a coloração total de subclasses de partial-grids nas quais existe um limite superior para o tamanho do ciclo induzido máximo, ou seja, partial-grids c-cordais. A aplicabilidade da decomposição proposta vem do fato de que, para cada inteiro positivo fixo c, o conjunto de grafos básicos com relação à decomposição de partial-grids c-cordais por clique 2-cutsets é finito, conforme enunciado mais à frente, na Proposição 1. Como resultado, a tarefa de determinar o número cromático total de partial-grids c-cordais de grau máximo 3 reduz-se a exibir 4-colorações totais apropriadas de um número finito de grafos.

O Lema 2 considera uma decomposição por clique 2-cutset de grafos biconexos com grau máximo 3.

Lema 2 (Machado and Figueiredo — Lema 2 de [31]) *Seja G um grafo biconexo com grau máximo 3. Se $X = \{u, v\}$ é um 2-corte clique de G, então $G \setminus X$ possui exatamente duas componentes conexas.*

Conforme observado na Figura 3.1 da Seção 3.1, o fato de os blocos básicos da decomposição de um grafo possuírem 4-total-colorações **não** é suficiente para que este grafo possua uma 4-total-coloração. Portanto, é necessário determinar uma propriedade de coloração mais forte para os blocos básicos, conforme denominamos a seguir, uma *coloração fronteiriça* ("coloração fronteiriça"). Um *par fronteiriço* ("par fronteiriço") de uma partial-grid com grau máximo 3 ou menos é um conjunto de dois vértices adjacentes de grau 2. Seja G uma partial-grid com grau máximo 3 ou menos e seja $\{u, v\}$ um par fronteiriço de G. Denote u' (resp. v') o vizinho de u (resp. v) em $V(G) \setminus \{u, v\}$. Dado um par fronteiriço $\{u, v\}$, dizemos que uma 4-total-coloração π de G *satisfaz* (u, v) se:

1. $\pi(u) = \pi(vv')$; e

2. $\pi(\{u'u, u, uv, v, vv'\}) = \{1, 2, 3, 4\}$.

Se π satisfaz (u, v) ou π satisfaz (v, u), então dizemos que π *satisfaz* $\{u, v\}$ – observe a Figura 3.4, onde exibimos as duas únicas formas de uma 4-total-coloração satisfazer $\{u, v\}$ (a menos de renomeações de cores).

Dizemos que uma 4-total-coloração de um grafo G é uma *coloração fronteiriça* se esta coloração satisfaz cada par fronteiriço de G. Na Seção 3.3, extendemos o

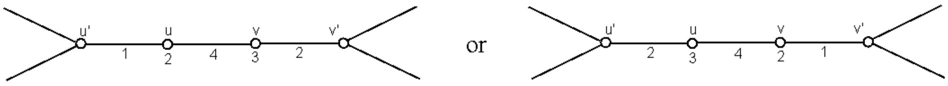


Figura 3.4: As duas 4-total-colorações que satisfazem $\{u, v\}$. A coloração à esquerda satisfaz (u, v), enquanto a coloração à direita satisfaz (v, u).

conceito de coloração fronteiriça, pemitindo o uso de mais de quatro cores. Uma observação importante acerca de colorações fronteiriça é a seguinte propriedade de "inversão":

Observação 1 (Machado and Figueiredo — Observação 1 de [31]) *Seja G um grafo biconexo de grau máximo 3, seja π uma 4-total-coloração de G, e seja $X = \{u, v\}$ um 2-corte clique de G que define as X-componentes G_1 e G_2. A restrição $\pi|_{G_1}$ satisfaz (u, v) se, e somente se, a restrição $\pi|_{G_2}$ satisfaz (v, u). Além disso, se G_1 possui coloração fronteiriça que satisfaz (u, v) e G_2 possui coloração fronteiriça que satisfaz (v, u), então G possui coloração fronteiriça.*

Segue o principal resultado da seção.

Teorema 1 (Machado and Figueiredo — Teorema 1 de [31]) *Seja G uma partial-grid 8-cordal com grau máximo 3 ou menos. O grafo G é 4-total-colorível.*

Esboço. A idéia da demonstração do teorema é simplesmente "descer" na árvore de decomposição escolhendo, para cada nó, uma coloração fronteiriça que não crie conflitos com os elementos já coloridos. Basta, assim, exibir as colorações fronteiriças dos blocos básicos, o que é feito na Figura 3.7 no final da seção. □

Corolário 1 (Machado and Figueiredo — Corolário 1 de [31]) *Toda partial-grid 8-chordal com grau máximo 3 é Tipo 1.*

Apesar de o Teorema 1 considerar partial-grids com ciclo induzido máximo até 8, poderia ser possível estender o resultado para partial-grids c-cordais com maiores limites para os ciclos induzidos, bastando exibir colorações fronteiriças dos grafos básicos para valores maiores de c (da mesma forma como é feito no Teorema 1). Uma vez que, para cada c fixo, existe um número finito de partial-grids indecomponíveis – como enunciado na Proposição 1 – a busca por estas colorações poderia ser automatizada através do uso de programas de computadores.

Proposição 1 (Machado and Figueiredo — Proposição 1 de [31]) *Para cada $c \geq 4$ fixo, existe um número finito de partial-grids c-cordais que não possuem clique 2-cutset.*

Na aplicação de decomposição por clique 2-cutset para a coloração total de partial-grids, a restrição sobre o tamanho do maior ciclo induzido possui a conseqüência de limitar o conjunto de grafos básicos a um conjunto finito. Na Seção 3.3, consideramos uma classe cujos grafos básicos não formam um conjunto finito; entretanto, os elementos deste conjunto são bastante restritos – a saber, são ciclos.

3.3 Grafos outerplanares: um resultado em coloração total por listas

Um grafo é *outerplanar* se possui uma representação planar na qual todo vértice incide na face externa [50]. As arestas da face externa (ou seja, incidentes na face externa) são denominadas *arestas externas*, enquanto as demais arestas são chamadas *arestas internas*. Se um grafo outerplanar é biconexo, então a fronteira da face externa é um ciclo Hamiltoniano chamado *ciclo externo*. Uma caracterização alternativa pode ser dada usando o conceito de homeomorfismo. Dado um grafo G, uma *subdivisão* de uma aresta $uv \in E(G)$ gera um grafo G' tal que $V(G') = V(G) \cup \{x\}$ e $E'(G) = (E(G) \setminus \{uv\}) \cup \{ux, xv\}$, onde x é um novo vértice. Se grafos G e H podem ser obtidos do mesmo grafo, a partir de uma seqüências de subdivisões, então dizemos que G e H são *homeomórficos*. Um grafo é outerplanar se e somente se [50] não possui subgrafo homeomórfico ao K_4 ou ao $K_{3,2}$.

Uma generalização natural do problema de coloração total é o problema de coloração total por listas. Uma instância do problema de coloração total por listas consiste em um grafo G e uma coleção $\{L_x\}_{x \in S(G)}$ que associa um conjunto de cores — chamada *lista* — a cada elemento de G. Deseja-se saber se existe uma coloração total π de G tal que $\pi(x) \in L_x$ para todo elemento x de G. Se tal coloração existe, dizemos que G é *total-colorível a partir das listas* $\{L_x\}_{x \in S(G)}$. O problema de coloração total por listas é NP-completo, mesmo quando a entrada é restrita a grafos outerplanares biconexos [55] de grau máximo 3. Entretanto, existem algumas condições suficientes conhecidas para que um grafo outerplanar biconexo G com $\Delta(G) \geq 3$ seja total-colorível a partir das listas $\{L_x\}_{x \in S(G)}$. Se todas as listas possuem $\Delta(G) + 1$ cores, então G é total-colorível a partir das listas $\{L_x\}_{x \in S(G)}$ [22, 52, 54] (de fato, o resultado vale para a superclasse de grafos série-paralelo). Um resultado mais forte é obtido por [55]: basta que $|L_{uw}| = \max\{deg(uw) + 1, 5\}$ para cada aresta uw e $|L_v| = \min\{5, \Delta + 1\}$ para cada vértice v.

Nesta seção, mostramos o poder da decomposição por clique 2-cutsets obtendo um resultado em coloração total por listas restrito a grafos outerplanares biconexos. Lembramos que o problema de coloração total por listas é NP-completo mesmo quando restrito a grafos outerplanares biconexos [55]. Ainda assim, alguns resultados "positivos" são conhecidos quando as listas satisfazem determinadas condições [55]: dado um grafo G biconexo outerplanar e uma coleção $\{L_x\}_{x \in S(G)}$ associada aos elementos $S(G) = V(G) \cup E(G)$ de G tal que $|L_{uw}| = \max\{deg(uw) + 1, 5\}$ para cada aresta uw e $|L_v| = \min\{5, \Delta + 1\}$ para cada vértice v, o grafo G é total-colorível a partir das listas $\{L_x\}_{x \in S(G)}$. Provamos um resultado ligeiramente diferente: um grafo G biconexo outerplanar é ainda total-colorível a partir das listas se $|L_{uw}| = deg(uw) + 1$ para cada aresta uw e $|L_v| = 7 - \delta_{deg(v),3} - 2\delta_{deg(v),2}$ para cada vértice v, ou seja, $|L_v| = 5$ se $deg(v) = 2$, $|L_v| = 6$ se $deg(v) = 3$, se $|L_v| = 7$ caso contrário. Comparamos as condições suficientes propostas com aquelas de [55]: nestas novas condições, as listas associadas a arestas possuem, possivelmente, menos cores, enquanto as listas associadas a vértices possuem, possivelmente, mais cores. A técnica utilizada para demonstrar este resultado é similar àquela utilizada na Seção 3.2, uma vez que determinamos uma coloração total de um grafo biconexo outerplanar decompondo este grafo por cliques 2-cutsets e obtendo colorações totais de cada um dos blocos básicos. A Observação 2 estuda os blocos básicos de decomposição de grafos outerplanares por clique 2-cutsets.

Observação 2 (Machado and Figueiredo — Observação 2 de [31]) *Seja G um grafo biconexo outerplanar. Ou G possui um clique 2-cutset ou G é um ciclo.*

No que se segue, definimos o significado dos conceitos de *cor livre*, *par fronteiriço* e *coloração fronteiriça* no contexto de coloração total por listas de grafos biconexos outerplanares.

Cor Livre. Seja G um grafo e (S_L, S_C) uma partição dos elementos $S(G) = V(G) \cup E(G)$ de um grafo G tal que cada elemento y em S_C possui cor $\pi(y)$ e a cada elemento x em S_L está associado um conjunto de cores L_x. O conjunto $F_\pi(z)$ de *free colours* em um elemento $z \in S_L$ é o conjunto de cores L_z que não são usados por π em qualquer dos elementos de S_C incidentes ou adjacentes a z. Este conceito de free colour capta, no contexto de coloração total por listas, a idéia de identificar as cores disponíveis para se colorir um elemento.

Par fronteiriço e **coloração fronteiriço** de um grafo biconexo outerplanar. Seja H um grafo biconexo outerplanar e seja G um subgrafo biconexo de H. Dizemos que um par $\{v_i, v_j\}$ de vértices adjacentes de G é um *par fronteiriço* se a aresta v_iv_j é uma aresta externa de G, mas é uma aresta interna de H (veja a Figura 3.5). Denote por $(v_0, ..., v_k, v_0)$ o ciclo externo de G. Denote por v_i' (resp. v_i'') o vizinho de v_i em $V(H) \setminus V(G)$ que pertence à mesma face $\mathcal{F}_1$ de H que v_{i-1} (resp. mesma face $\mathcal{F}_2$ de H que v_{i+1}), como mostrado na Figura 3.5. Suponha que a cada elemento $x \in S(H) \setminus S(G)$ é associado um conjunto L_x de cores. Dizemos que π é uma *coloração fronteiriça* de G se, a cada vértice $v_i \in \{v_0, ..., v_k\}$ no ciclo externo de G, podemos associar cores $l_G(v_i)$ e $r_G(v_i)$ com as seguintes propriedades (lembre que os índices são tomados módulo $k + 1$).

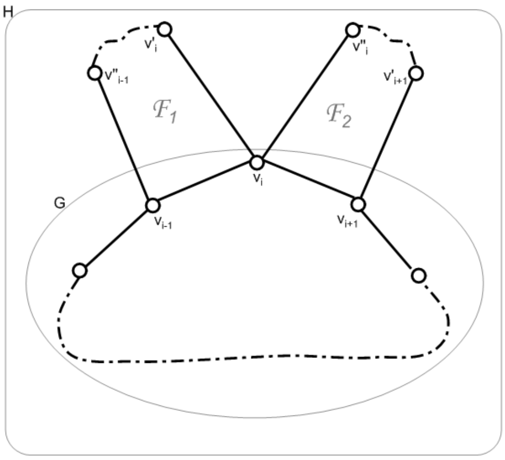


Figura 3.5: No exemplo, $\{v_{i-1}, v_i\}$ e $\{v_i, v_{i+1}\}$ são par fronteiriços de G com relação a H.

1. Se $\{v_i, v_{i-1}\}$ é um par fronteiriço, então $l_G(v_i) \in F_\pi(v_iv_i')$ e dizemos que $l_G(v_i)$ está *definido* (*defined*); caso contrário, $l_G(v_i)$ está *indefinido* (*undefined*).

2. Se $\{v_i, v_{i+1}\}$ é um par fronteiriço, então $r_G(v_i) \in F_\pi(v_iv_i'')$ e dizemos que $r_G(v_i)$ está *definido* (*defined*); caso contrário, $r_G(v_i)$ está *indefinido* (*undefined*).

3. Se ambos $l_G(v_i)$ e $r_G(v_i)$ estão definidos, então $l_G(v_i) \neq r_G(v_i)$. Se ambos $r_G(v_{i-1})$ e $l_G(v_i)$ estão definidos, então $r_G(v_{i-1}) \neq l_G(v_i)$. Se ambos $r_G(v_i)$ e $l_G(v_{i+1})$ estão definidos, então $r_G(v_i) \neq l_G(v_{i+1})$.

A definição de coloração fronteiriça capta a propriedade de ser possível estender uma total-coloração de um grafo através da adição de blocos básicos. As free colours na aresta v_iv_i' (resp. v_iv_i'') são as cores $L_{v_iv_i'}$ (resp. $L_{v_iv_i''}$) que não são usadas em v_i ou qualquer de suas arestas incidentes. Observe um exemplo na Figura 3.6.

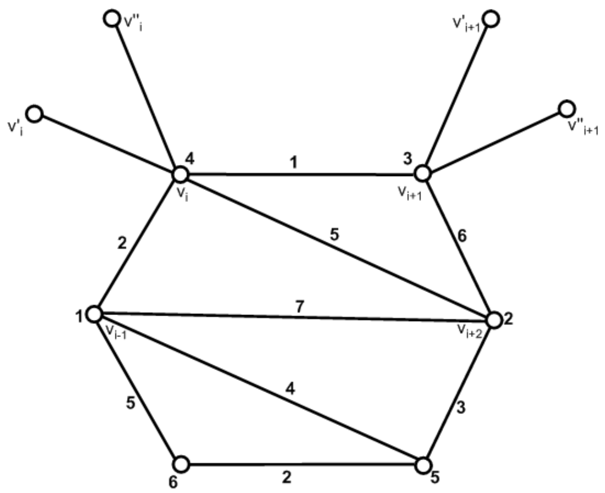


Figura 3.6: No exemplo, suponha que $L_{v_iv_i'} = \{1,2,3,6,7,8\}$, $L_{v_iv_i''} = \{2,3,5,6,7,8\}$, $L_{v_{i+1}v_{i+1}'} = \{1,2,3,4,5,6\}$, and $L_{v_{i+1}v_{i+1}''} = \{1,2,3,6,7,8\}$. Se denotamos por π a total-coloração mostrada na figura, então $F_\pi(v_iv_i') = \{3,6,7,8\}$, $F_\pi(v_iv_i'') = \{3,6,7,8\}$, $F_\pi(v_{i+1}v_{i+1}') = \{2,4,5\}$, e $F_\pi(v_{i+1}v_{i+1}'') = \{2,7,8\}$. Podemos escolher $l_G(v_i) = 7$, $r_G(v_i) = 6$, $l_G(v_{i+1}) = 5$, e $r_G(v_{i+1}) = 7$, de tal forma que π é uma coloração fronteiriça.

Segue o principal resultado da presente seção (usamos o delta de Kronecker: $\delta_{i,j} = 1$ if $i = j$ e $\delta_{i,j} = 0$ if $i \neq j$).

Teorema 2 (Machado and Figueiredo — Teorema 2 de [31]) *Seja G um grafo biconexo outerplanar e seja $\{L_x\}_{x\in S(G)}$ uma coleção de listas tal que $|L_{uw}| = deg(uw)+1$ para cada aresta uw e $|L_v| = 7 - \delta_{deg(v),3} - 2\delta_{deg(v),2}$ para cada vértice v. O grafo G pode ser total-colorido a partir de $\{L_x\}_{x\in S(G)}$.*

3.4 Considerações Finais

Neste capítulo, investigamos aplicações de decomposições determinadar por 2-cortes clique ao problema de coloração total. Nossa abordagem se baseia em decompor um grafo recursivamente até se obter um conjunto de grafos básicos. Solucionando-se apropriadamente o problema para os grafos básicos, esperamos combinar estas soluções em uma solução do grafo original. A técnica foi aplicada para obter resultados de coloração total de partial-grids 8-cordais – observe, na Figura 3.7, colorações dos grafos básicos – e de coloração total por listas de grafos outerplanares biconexos. Fechamos a seção com algumas observações.

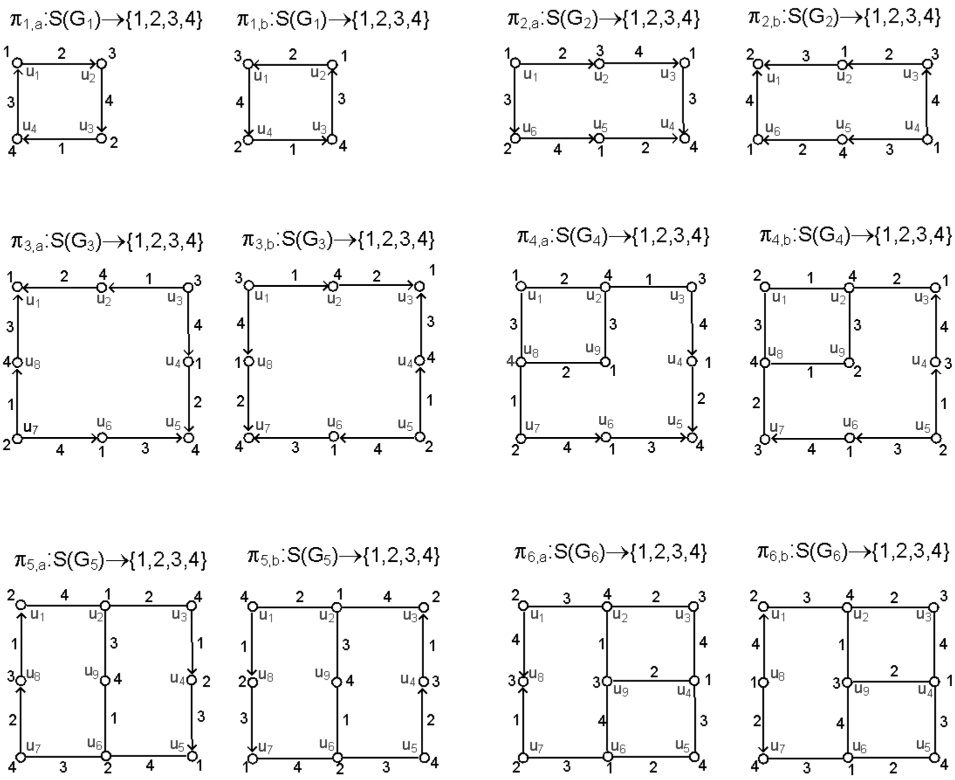


Figura 3.7: Colorações fronteiriças para a prova do Teorema 1. Com o objetivo de indicar se a coloração fronteiriça satisfaz (u, v) ou (v, u), cada aresta uv está representada por uma seta apontando para u se a coloração satisfaz (u, v), e apontando para v se a coloração satisfaz (v, u). Enfatizamos, entretanto, que **não** estamos lidando com grafos direcionados.

A primeira observação relaciona-se com o conceito de **resultado estrutural**, que descrevemos brevemente a seguir. Um *resultado estrutural* caracteriza uma classe de grafos por meio de operações de composição, ou seja, afirma-se que todo grafo obtido de um conjunto de grafos básicos a partir de uma série de operações de composição está em uma determinada classe. Observe que o resultado enunciado na Observação 2 não é um resultado estrutural, mas, sim, um **resultado de decomposição**: todo grafo outerplanar pode ser decomposto por clique 2-cutsets em um conjunto de ciclos. No entanto, seria possível contruir um resultado estrutural correspondente: basta observar que todo grafo G obtido de dois grafos outerplanares através da identificação de uma aresta é outerplanar. Como o foco desta tese não é a caracterização de classes através de resultados estruturais, deixamos os detalhes da demonstração para o leitor interessado.

A segunda observação refere-se à existência de algoritmos eficientes de coloração-total. A idéia de coloração de fronteira é definida de tal maneira que não são necessárias recolorações dos elementos já coloridos. Tal propriedade é apropriada para a construção de algoritmos gulosos de coloração-total para as classes investigadas neste capítulo. Mais uma vez, deixamos ao leitor interessado a tarefa de investigar as potencialidades da técnica para a construção de algoritmos eficientes.

O conceito (e a necessidade) de uma “coloração fronteiriça” ilustra uma importante lição: não é sempre que uma coloração pode ser “extendida” através de um corte. Assim, para mostrar que determinada partial-grid ou grafo outerplanar é Tipo 1, precisamos colorir um dos blocos de decomposição com uma coloração especial, que permite a sua extensão ao outro bloco. A idéia de colorir um dos blocos “preparando o terreno” para a coloração do outro bloco será retomada outras vezes nesta tese, no Capítulo 5 — coloração total de grafos {square,unichord}-free, caso grau máximo 3 — e no Capítulo 6 — coloração total de grafos chordless, caso grau máximo 3.

Capítulo 4

Grafos unichord-free

Seguindo a linha de investigação de aplicações de decomposições à coloração de arestas e à coloração total, tomamos contato com uma classe de grafos definidas por Trotignon e Vušković [47]. Trata-se dos grafos *unichord-free* — grafos que não possuem ciclo com corda única. Trotignon e Vušković [47] obtiveram fortes resultados de decomposição para estes grafos e usam tais resultado para provar que o problema da coloração de vértices é polinomial para a classe. Neste capítulo, mostramos que ambos os problemas de coloração de arestas e coloração total são NP-completos, motivando a busca por subclasses para as quais os problemas sejam polinomiais, o que é feito nos Capítulos 5 e 6. O objetivo do presente capítulo é investigar a complexidade da coloração de arestas e coloração total para a classe.

4.1 Estrutura dos grafos unichord-free

Recentemente, Trotignon e Vušković [47] investigaram a classe $\mathcal{C}$ dos grafos que não possuem ciclo com corda única. A principal motivação para o estudo desta classe é a busca por "teoremas estruturais" em classes definidas por famílias de subgrafos proibidos. Basicamente, tais resultados estruturais enunciam que todo grafo em uma determinada classe pode ser construído partindo-se de um conjunto de grafos ditos "básicos", através da aplicação de uma série de operações de "colagem". Outra propriedade interessante encontrada nesta classe é que trata-se de uma classe χ-*bounded*, conceito introduzido por Gyárfás [21] como uma extensão natural de grafos perfeitos. Uma família de grafos $\mathcal{G}$ é χ-*bounded* com *função* χ-*binding* f se, para todo subgrafo induzido G' de $G \in \mathcal{G}$, $\chi(G') \leq f(\omega(G'))$, onde $\chi(G')$ denota o número cromático de G' e $\omega(G')$ denota o tamanho da maior clique de G'. Também nessa área, a maior parte da pesquisa ocorre no sentido de se compreender para que escolhas de subgrafos proibidos a família resultante é χ-bounded (veja [39] para um survey). Note que grafos perfeitos são uma família χ-bounded com função χ-binding $f(x) = x$, e grafos perfeitos são caracterizados pela exclusão de ciclos induzidos ímpares e seus complementos. Além disso, pelo Teorema de Vizing, a classe dos grafos de linha de grafos simples é uma família χ-bounded com função χ-binding $f(x) = x+1$ (este bound especial é conhecido como *Vizing bound*) e grafos de linha são caracterizados por nove subgrafos (induzidos) proibidos [50]. A classe $\mathcal{C}$ é, também, χ-bounded com o Vizing bound [47]. Também em [47] são obtidos os seguintes resultados para grafos em $\mathcal{C}$: um algoritmo $\mathcal{O}(nm)$ para coloração ótima de vértices, um algoritmo $\mathcal{O}(n+m)$ para o tamanho da clique máxima, um algoritmo de reconhecimento $\mathcal{O}(nm)$, e a NP-completude do problema de conjunto independente máximo.

Nesta seção, apresentamos os resultados de decomposição de [47] para a classe dos grafos unichord-free. Também exibimos novos resultados de decomposição para a classe dos grafos {*square,unichord*}*-free*. Estes resultados são usados ao longo deste capítulo e do próximo para obter resultados sobre a complexidade de coloração de arestas e coloração total na classe dos unichord-free. Os resultados de decomposição de que falamos são da seguinte forma: se é G um grafo de determinada classe, então, ou G pertence a um determinado conjunto de grafos ditos "básicos" ou G pode ser decomposto através de uma decomposição conhecida. Antes de enunciarmos os resultados de decomposição, precisamos definir o conjunto dos grafos básicos e quais são os cortes usados nas decomposições.

O *grafo de Petersen* é um grafo com vértices $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ de tal forma que $a_1a_2a_3a_4a_5a_1$ e $b_1b_2b_3b_4b_5b_1$ são ciclos sem corda, e tal que as únicas arestas entre algum a_i e algum b_i são a_1b_1, a_2b_4, a_3b_2, a_4b_5, a_5b_3. Denotamos por P o grafo de Petersen e por P^* o grafo obtido de P através da remoção de um vértice P. Observe que $P \in \mathcal{C}$.

O *grafo de Heawood* é um grafo bipartido cúbico com vértices $\{a_1, \ldots, a_{14}\}$ de tal forma que $a_1a_2 \ldots a_{14}a_1$ é um ciclo, e tal que as outras arestas são a_1a_{10}, a_2a_7,

a_3a_{12}, a_4a_9, a_5a_{14}, a_6a_{11}, a_8a_{13}. Denotamos por H o grafo de Heawood e por H^* o grafo obtido de H através da remoção de um vértice. Observe que $H \in \mathcal{C}$.

Um grafo é *fortemente 2-bipartido* (*strongly 2-bipartite*) se não possui ciclo incuzido com quatro vértices e é bipartido com bipartição (X, Y) onde todo vértice em X possui grau 2 e todo vértice em Y possui grau pelo menos 3. Um grafo fortemente 2-bipartido está na classe $\mathcal{C}$ porque toda corda em um ciclo é uma aresta entre vértices de grau pelo menos 3, logo, todo ciclo em um grafo fortemente 2-bipartido é sem-corda.

Para os propósitos do presente trabalho, um grafo G é dito *básico*[1] se

1. G é um grafo completo, ou um ciclo induzido com pelo menos cinco vértices, ou um grafo fortemente 2-bipartido, ou um subgrafo induzido (não necessariamente próprio) do grafo de Petersen ou do grafo de Heawood; e
2. G não possui decomposição por 1-corte, 2-corte próprio ou 1-junção própria (definidos a seguir).

Denotamos por $\mathcal{C}_B$ o conjunto dos grafos básicos. Observe que $\mathcal{C}_B \subseteq \mathcal{C}$.

- Um *1-corte* (*1-cutset*) de um grafo conexo $G = (V, E)$ é um vértice v tal que V pode ser particionado em conjuntos X, Y e $\{v\}$, de tal forma que não existe aresta entre X e Y. Dizemos que (X, Y, v) é um *split* deste 1-corte.
- Um *2-corte próprio* (*proper 2-cutset*) de um grafo conexo $G = (V, E)$ é um par de vértices não adjacentes a, b, ambos com grau pelo menos 3, tais que V pode ser particionado em conjuntos X, Y e $\{a, b\}$, de tal forma que: $|X| \geq 2$, $|Y| \geq 2$; não existe aresta entre X e Y, e ambos $G[X \cup \{a, b\}]$ e $G[Y \cup \{a, b\}]$ possuem um caminho entre a e b. Dizemos que (X, Y, a, b) é um *split* deste 2-corte próprio.
- Um *1-junção* (*1-join*) de um grafo $G = (V, E)$ é uma partição de V em conjuntos X e Y tais que existem conjuntos A, B satisfazendo:
  - $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq Y$;

[1] Pela definição de [47], um grafo básico não é, em geral, indecomponível. Entretanto, nossa definição sutilmente diferente reduz o conjunto dos grafos básicos, facilitando a demonstração de determinados resultados.

- $|X| \geq 2$ e $|Y| \geq 2$;
- todas as arestas possíveis entre A e B estão presentes em G;
- não existe aresta entre $X \setminus A$ e Y ou entre $Y \setminus B$ e X.

Dizemos que (X, Y, A, B) é um *split* deste 1-junção.

Uma *1-junção própria* (*proper 1-join*) é uma 1-junção tal que A são B conjuntos estáveis de G com tamanho pelo menos 2.

Podemos, agora, enunciar o principal resultado de decomposição de [47] para grafos unichord-free:

Teorema 3 (Trotignon and Vušković [47]) *Se $G \in \mathcal{C}$ é conexo, então:*

- *$G \in \mathcal{C}_B$, ou*
- *G possui um 1-corte, ou*
- *G possui 2-corte próprio, ou*
- *G possui uma 1-junção própria.*

O *bloco* G_X (resp. G_Y) de um grafo G com relação a um 1-corte com split (X, Y, v) é $G[X \cup \{v\}]$ (resp. $G[Y \cup \{v\}]$).

O *bloco* G_X (resp. G_Y) de um grafo G com relação a uma 1-junção com split (X, Y, A, B) é o grafo obtido de $G[X]$ (resp. $G[Y]$) através da adição de um vértice $y \in Y$ adjacente a todo vértice de A (resp. $x \in X$ adjacente a todo vértice de completo a B). Vértices x, y são chamados *marcadores* (*markers*) de seus blocos.

Os *blocos* G_X e G_Y de um grafo G com relação a um 2-corte próprio com split (X, Y, a, b) são definidos a seguir. Se existe um vértice c de G tal que $N_G(c) = \{a, b\}$, então seja $G_X = G[X \cup \{a, b, c\}]$ e $G_Y = G[Y \cup \{a, b, c\}]$. Caso contrário, o bloco G_X (resp. G_Y) é o grafo obtido de $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) através da adição de um novo vértice c adjacente a a, b. O vértice c é chamado *marker* do bloco G_X (resp. G_Y).

Os blocos com relação a 1-cortes, 2-corte próprios e 1-junção próprias são construídos de tal forma que eles permanecem em $\mathcal{C}$, caso o grafo original esteja em $\mathcal{C}$, conforme o Lema 3.

Lema 3 (Trotignon and Vušković [47]) *Sejam G_X e G_Y blocos de decomposição de G com relação a um 1-corte, uma 1-junção própria ou um 2-corte próprio. Neste caso, $G \in \mathcal{C}$ se, e somente se, $G_X \in \mathcal{C}$ e $G_Y \in \mathcal{C}$.*

Observe que o grafo de Petersen e o grafo de Heawood podem aparecer como bloco de decomposição com relação a uma 1-junção própria, conforme mostrado na Figura 4.1. Entretanto, estes grafos nunca podem aparecer como bloco de decomposição com relação a um 2-corte próprio, porque não possuem vértices de grau 2 que possam fazer o papel de marcador. Sejam P^* e H^* os grafos obtidos, respecti-

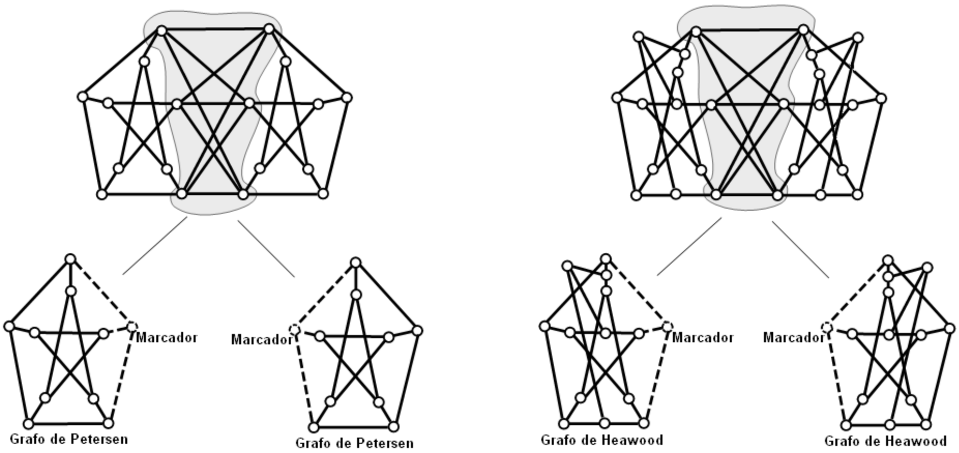


Figura 4.1: Árvore de decomposição com relação a 1-junções próprias. No grafo à esquerda, os blocos básicos de decomposição são duas cópias do grafo de Petersen. No grafo à direita, os blocos básicos de decomposição são duas cópias do grafo de Heawood.

vamente, do grafo de Petersen e do grafo de Heawood, através da remoção de um vértice. Observe que os grafos P^* e H^* podem aparecer como bloco de decomposição com relação a uma decomposição por 2-corte próprio, conforme mostrado na Figura 4.2.

Apresentamos, anteriormente, resultados que permitem decompor um grafo de $\mathcal{C}$ em blocos básicos: o Teorema 3 mostra que todo grafo em $\mathcal{C}$ possui um 1-corte, um 2-corte próprio ou uma 1-junção própria, enquanto Lema 3 mostra que os blocos gerados com relação por estas decomposições são, também, unichord-free. Neste trabalho, obtivemos resultados similares para grafos {square,unichord}-free. Como discutimos na seguinte observação [7], para o propósito de coloração de arestas e coloração total, basta considerar grafos biconexos.

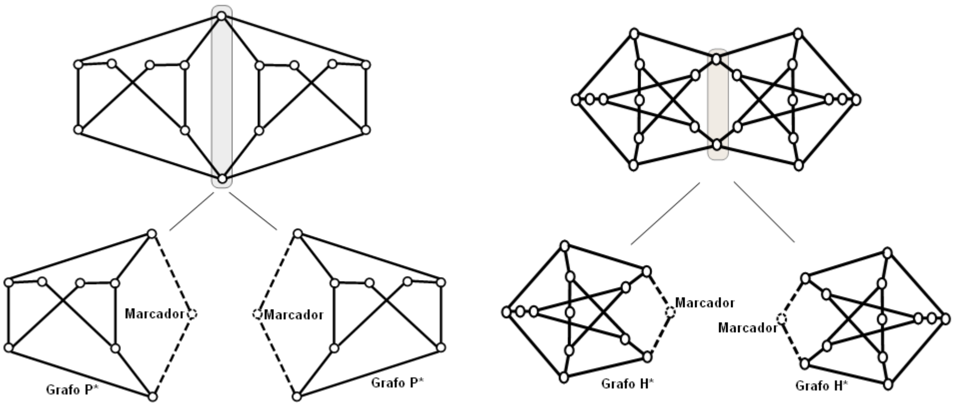


Figura 4.2: Árvore de decomposição com relação a 2-cortes próprios. No grafo à esquerda, os blocos básicos de decomposição são duas cópias de P^*. No grafo à direita, os blocos básicos de decomposição são duas cópias de H^*.

Observação 3 *Seja G um grafo conexo que possui um 1-corte com split (X, Y, v). O índice cromático de G é $\chi'(G) = \max\{\chi'(G_X), \chi'(G_Y), \Delta(G)\}$ e o número cromático total de G é $\chi_T(G) = \max\{\chi_T(G_X), \chi_T(G_Y), \Delta(G) + 1\}$.*

Pela Observação 3, se ambos os blocos G_X e G_Y são $\Delta(G)$-aresta-coloríveis (resp. $\Delta(G) + 1$-total-coloríveis), então G também o é. Ou seja, uma vez determinado o índice cromático (número cromático total) das componentes biconexas de um grafo, é fácil determinar o índice cromático (número cromático total) deste grafo. Logo, podemos focar atenção nos grafos biconexos de $\mathcal{C}'$.

Teorema 4 (Trotignon and Vušković [47]) *Se $G \in \mathcal{C}'$ é biconexo, então, ou $G \in \mathcal{C}_B$, ou G possui um 2-corte próprio.*

O Teorema 4 é conseqüência imediata do Teorema 3: como G não possui quadrado, G não pode possuir uma 1-junção própria, e como G é biconexo, G não pode possuir um 1-corte.

A seguir, no Lema 4, mostramos que os blocos de decomposição de um grafo biconexo de $\mathcal{C}'$ com relação a um 2-corte próprio são também grafos biconexos de $\mathcal{C}'$.

Lema 4 (Machado, Figueiredo and Vušković — Lemma 4 de [32]) *Seja $G \in \mathcal{C}'$ um grafo biconexo e seja (X, Y, a, b) split de 2-corte próprio de G. Então, ambos G_X e G_Y são grafos biconexos de $\mathcal{C}'$.*

Observe que o Lema 3 é, de certa forma, mais forte que o Lema 4. Enquanto o Lema 3 enuncia que um grafo está em $\mathcal{C}$ **se, e somente se,** os blocos de decomposição também estão em $\mathcal{C}$, o Lema 4 estabelece apenas uma direção: **se** um grafo é um grafo biconexo de $\mathcal{C}'$, **então** os blocos de decomposição também o são. Para o propósito de coloração de arestas e coloração total, não existe necessidade de estabelecer a "volta", ou seja, o "somente se". De todo modo, é possível verificar que, se ambos os blocos G_X e G_Y gerados a partir de uma decomposição 2-corte próprio de um grafo G são grafos biconexos de $\mathcal{C}'$, então G é um grafo biconexo de $\mathcal{C}'$.

O lema seguinte mostra que todo grafo biconexo não-básico de $\mathcal{C}'$ possui uma decomposição tal que um dos blocos é básico.

Lema 5 (Machado, Figueiredo and Vušković [32]) *Todo grafo biconexo $G \in \mathcal{C}' \setminus \mathcal{C}_B$ possui um 2-corte próprio tal que um dos blocos de decomposição é básico.*

Esboço. Basta escolher, dentre todos os split de 2-corte próprio, um que minimize um dos blocos de decomposição. □

4.2 NP-completude de coloração de arestas

Na presente seção, apresentamos o resultado de NP-completude do problema de coloração de arestas restrito a grafos unichord-free. De fato, a NP-completude vale para grafos Δ-regulares de $\mathcal{C}$ com grau fixo $\Delta \geq 3$. Observamos que a construção de Cai e Ellis [9] que prova a NP-completude de grafos r-regulares sem ciclo induzido de tamanho k obtém grafos unichord-free. De todo modo, uma construção bem mais simples é apresentada em [9].

Usamos o termo `CHRIND`(P) para denotar o problema de se determinar o índice cromático de um grafo restrito a entradas com propriedade P. Por exemplo, `CHRIND`(grafo unichord-free) denota o seguinte problema:

INSTÂNCIA: um grafo G unichord-free.

QUESTÃO: vale $\chi'(G) = \Delta(G)$?

O seguinte teorema [26, 28] estabelece a NP-completude do problema de se determinar o índice cromático de grafos Δ-regulares com grau pelo menos 3:

Teorema 5 *([26, 28]) Para $\Delta \geq 3$, `CHRIND`(grafo Δ-regular) é NP-completo.*

O grafo Q_n, com $n \geq 3$, da Figura 4.3, é obtido do grafo bipartido completo $K_{n,n}$ a partir da remoção de uma aresta xy e adição de novos vértices pendentes u e v adjacentes a x e y, respectivamente. O grafo Q'_n, da Figura 4.3, é obtido de Q_n através da identificação dos vértices u e v em um novo vértice w. Observe que Q'_n é um grafo de grau máximo n, e possui $2n + 1$ vértices e $n^2 + 1$ arestas. Logo, Q'_n é overfull [17] e, portanto, Classe 2. O Lema 6 investiga as propriedades do grafo Q_n,

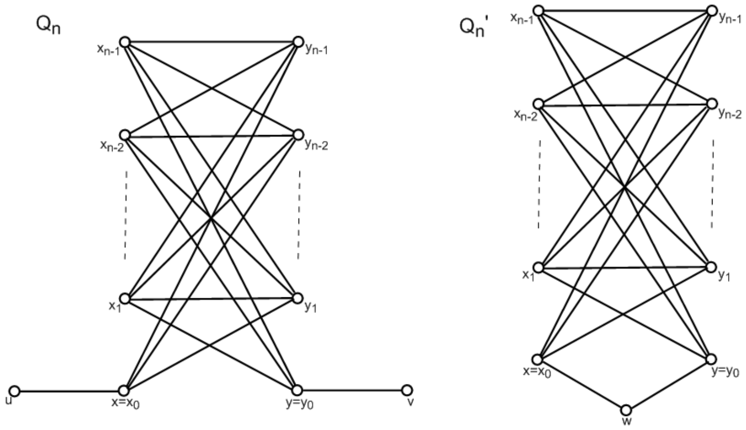


Figura 4.3: "Gadget" de NP-completude Q_n e grafo Q'_n.

o qual é usado como "gadget" na prova de NP-completude do Teorema 6.

Lema 6 (Machado, Figueiredo and Vušković — Lemma 1 de [32]) *Grafo Q_n é n-aresta-colorível, e em toda n-aresta coloração de Q_n, as arestas ux e vy recebem a mesma cor.*

Esboço. O resultado decorre do fato de que o grafo obtido de Q_n através da identificação dos vértices pendentes é Overfull e, portanto, Classe 2. □

O Teorema 6 estabelece a NP-completude de coloração de arestas de grafos unichord-free regulares para cada grau $\Delta \geq 3$.

Teorema 6 (Machado, Figueiredo and Vušković — Teorema 2 de [32]) *Para $\Delta \geq 3$, `CHRIND`(grafo Δ-regular de $\mathcal{C}$) é NP-completo.*

Esboço. Redução de `CHRIND`(grafo Δ-regular) por "local replacement", na qual cada aresta de G é substituída por uma cópia de Q_Δ. □

O grafo construído na demonstração [32] do Teorema 6 é 3-partido, de tal forma que o seguinte resultado segue:

Teorema 7 (Machado, Figueiredo and Vušković — Teorema 3 de [32]) *Para* $k \geq 3, \Delta \geq 3$, *CHRIND(grafo k-partido* Δ*-regular) é NP-completo.*

4.3 NP-completude de coloração total

Na presente seção, enunciamos a NP-completude do problema de coloração total restrito à classe $\mathcal{C}$ de grafos unichord-free. De fato, a NP-completude vale para grafos regulares de $\mathcal{C}$ com grau fixo $\Delta \geq 3$. A prova é inspirada no trabalho de McDiarmid and Sánchez-Arroyo [41, 42], mas guarda diferenças fundamentais de forma a evitar ciclos com corda única.

Usamos `TOTCHR`(P) para denotar o problema de se determinar o número cromático total restrito a grafos com propriedade P. Por exemplo, `TOTCHR`(grafo unichord-free) denota o seguinte problema:

INSTÂNCIA: grafo G unichord-free.

QUESTÃO: é $\chi_T(G) = \Delta(G) + 1$?

O seguinte teorema [41, 42] estabelece a NP-completude do problema de coloração total restrito a grafos bipartidos Δ-regulares, para cada grau fixo $\Delta \geq 3$:

Teorema 8 (McDiarmid and Sanchez-Arroyo [41, 42]) *Para* $\Delta \geq 3$, *TOTCHR(grafo bipartido* Δ*-regular) é NP-completo.*

Observe a Figura 4.4. O grafo S_t, para $t \geq 3$, é obtido do grafo bipartido

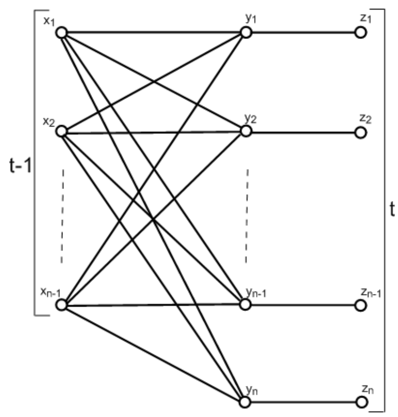


Figura 4.4: Grafo S_t

completo $K_{t-1,t}$, a partir da adição de t arestas pendentes adjacentes aos t vértices de grau $t-1$. O grafo S_t possui a seguinte propriedade:

Lema 7 (McDiarmid e Sánchez-Arroyo [41]) *Considere o grafo S_t, onde $t \geq 3$.*

1. *Existe $(t+1)$-total-coloração de S_t em que cada um dos vértices $y_1, y_2, ..., y_t$ recebe uma cor diferente.*

2. *Em qualquer $(t+1)$-total-coloração de S_t, todas as arestas pendentes recebem a mesma cor.*

O grafo S_t é peça básica na construção dos componentes usados na prova de NP-completude da presente seção. A seguir, construímos o grafo bipartido $H_{n,t}$, com $n \geq 2$ e $t \geq 3$, colocando juntas duas cópias de S_t e identificando $t-n$ arestas pendentes da primeira cópia com arestas pendentes da segunda cópia. Note que, exceto pelas $2n$ arestas pendentes, cada um dos outros vértices de $H_{n,t}$ possui grau t. O grafo $H_{n,t}$ é mostrado na Figura 4.5.

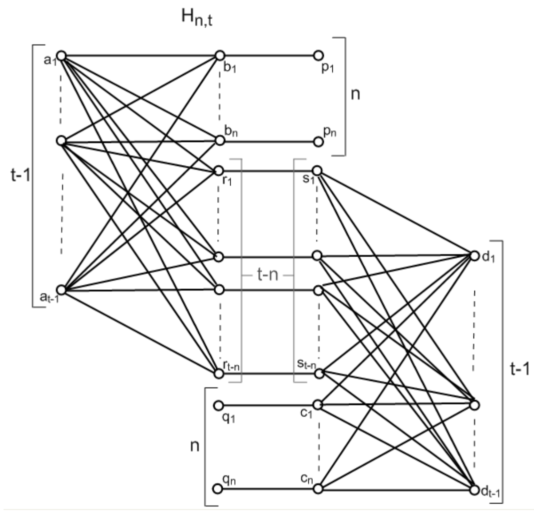


Figura 4.5: Grafo $H_{n,t}$.

Lema 8 (Machado, Figueiredo and Vušković — Lema 2 de [33]) *Considere o grafo $H_{n,t}$ com $t \geq 5$ e $n = \lceil (t+1)/2 \rceil$.*

1. *Considere uma $(t+1)$-total-coloração parcial π' de $H_{n,t}$ em que as arestas pendentes estão coloridas com a mesma cor e os vértices pendentes também estão coloridos (e nada mais está colorido). Esta $(t+1)$-total-coloração parcial se estende para uma $(t+1)$-total-coloração de $H_{n,t}$.*

2. *Em qualquer $(t+1)$-total-coloração de $H_{n,t}$, as arestas pendentes recebem a mesma cor.*

OLema 9 considera os casos de $H_{n,t}$ com $n = \lceil (t+1)/2 \rceil$ não cobertos pelo Lema 8

Lema 9 (Machado, Figueiredo and Vušković — Lema 3 de [33]) *Considere o grafo $H_{n,t}$ com $n = 2$ e $t = 3$ ou $n = 3$ e $t = 4$.*

1. *Considere uma $(t+1)$-total-coloração parcial π' de $H_{n,t}$ em que as arestas pendentes estão coloridas com a mesma cor e os vértices pendentes estão coloridos nem todos com a mesma cor (e nada mais está colorido). Esta $(t+1)$-total-coloração parcial se estende para uma $(t+1)$-total-coloração de $H_{n,t}$.*

2. *Em qualquer $(t+1)$-total-coloração de $H_{n,t}$, as arestas pendentes recebem a mesma cor.*

O grafo de substituição ("replacement graph") original R de [41] possui ciclos com corda única. Modificamos e estendemos R a uma família R_t, $t \geq 3$, de grafos de substituição em $\mathcal{C}$, como segue. Tome $t+1$ cópias de $H_{n,t}$, com $n = \lceil (t+1)/2 \rceil$, e denote estas cópias por $H^{(1)}, H^{(2)}, ..., H^{(t+1)}$. O grafo de substituição R_t é tal que cada cópia de $H_{n,t}$ em R_t possui, ou uma aresta pendente – a qual é chamada *real*, ou duas arestas pendentes – uma das quais é chamada *real*. Para isso, identifique cada um dos t vértices pendentes de $H^{(i)}$, $i = 1, 2, ..., t+1$, com um vértice pendente de $H^{(j)}$ distinto, $j \neq i$. Observe, na Figura 4.6, a construção de R_3 (resp. R_4) a partir da substituição dos vértices de K_4 (resp. K_5) por cinco cópias de $H_{2,3}$ (resp. $H_{3,4}$).

Observe que, se t é par, então existem duas arestas pendentes em cada cópia de $H_{\lceil (\Delta+1)/2 \rceil, t}$, uma das quais é chamada *real* – a outra é chamada *não-real*. Se t é ímpar, então existe uma aresta pendente em cada cópia de $H_{\lceil (\Delta+1)/2 \rceil, t}$, e cada uma delas é chamada *real*.

Lema 10 (Machado, Figueiredo and Vušković — Lema 4 de [33]) *Considere o grafo R_t, $t \geq 3$.*

1. *Toda $(t+1)$-total-coloração parcial de R_t na qual as $t+1$ arestas pendentes reais possuem cores diferentes (duas a duas distintas) e os vertices pendentes reais estão coloridos (e mais nada está colorido) estende a uma $(t+1)$-total-coloração de R_t,*

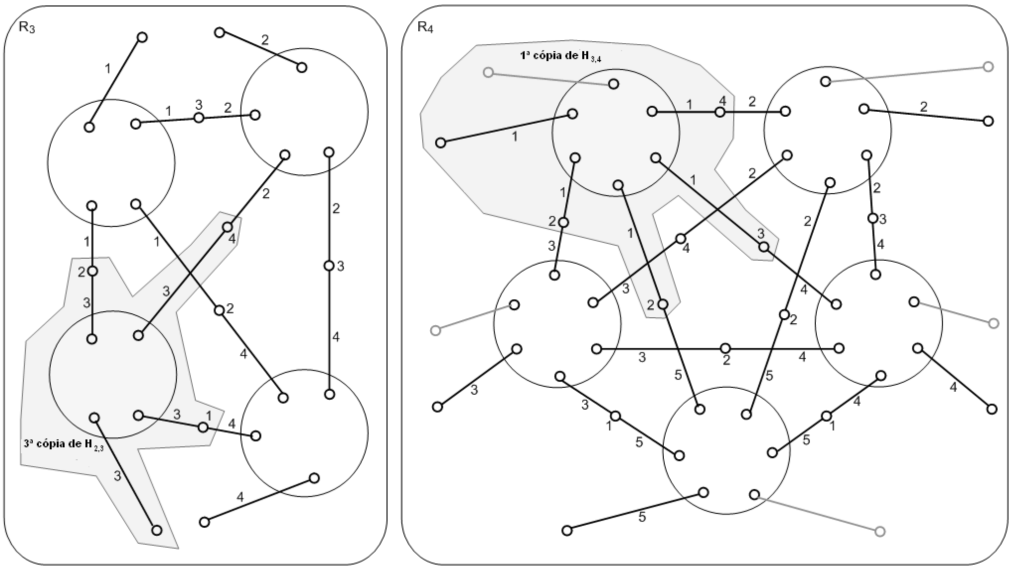


Figura 4.6: “Replacement graphs” R_3 e R_4.

2. *Em toda* $(t+1)$*-total-coloração de* R_t*, as* $t+1$ *arestas pendentes reais possuem cores diferentes (duas a duas distintas).*

O grafo forçador (“forcer graph” [41]) $F_{n,t}$, com $n \geq 2$ e $t \geq 3$, é construído através do encadeamento de n cópias do grafo $H_{2,t}$, conforme mostrado na Figura 4.7. Observe que o grafo $F_{n,t}$ possui $2n$ vertices pendentes (grau 1) e que cada um dos

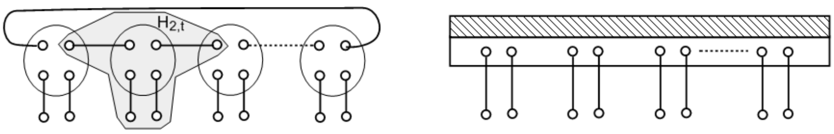


Figura 4.7: O “forcer graph” e sua representação esquemática.

vertices restantes possui grau t.

Lema 11 (McDiarmid e Sánchez-Arroyo [41]) *Considere o grafo* $F = F_{n,t}$*, com* $n \geq 2$ *e* $t \geq 3$.

1. *Considere uma* $(t+1)$*-total-coloração parcial de* F *na qual cada aresta pendente possui a mesma cor e cada vértice pendente está colorido (e nada mais está colorido). Então, esta coloração estende para uma* $(t+1)$*-total-coloração de* F*.*

2. *Em toda* $(t+1)$*-total-coloração de* F *toda aresta pendente possui a mesma cor.*

O Teorema 9 estabelece a NP-completude da coloração total de grafos unichord-free Δ-regulares, para cada grau fixo $\Delta \geq 3$. Antes de enunciar o Teorema 9 para grafos regulares, apresentamos um resultado ligeiramente diferente: Lema 12 prova a NP-completude do problema $P_{\Delta,\delta}$ =TOTCHR(grafo de $\mathcal{C}$ com grau máximo Δ, grau mínimo $\geq \delta$, e tal que toda aresta é incidente a um vértice de grau máximo) para $\delta = 1$. O Teorema 9 obtém um grafo regular a partir de uma estratégia inovadora de indução no grau mínimo.

Lema 12 (Machado, Figueiredo and Vušković — Lema 6 de [33]) *Para $\Delta \geq 3$, o problema $P_{\Delta,1}$ é NP-completo.*

Teorema 9 (Machado, Figueiredo and Vušković — Teorema 3 de [33]) *Para $\Delta \geq 3$, TOTCHR(grafo Δ-regular unichord-free) é NP-completo.*

Ressaltamos que a nossa estratégia indutiva não é usada em [41]. Os gadgets construídos em [41] são regulares, enquanto os nossos gadgets unichord-free não o são. Assim, enquanto [41] usa indução no grau máximo, fazemos uso de indução no grau mínimo.

4.4 Considerações finais

Neste capítulo, mostramos que, apesar da forte estrutura dos grafos unichord-free, tanto o problema de coloração de arestas quanto o problema de coloração total são NP-completos quando restritos à classe. Isto motiva a busca por subclasses nas quais os problemas sejam polinomiais, o que é feito nos próximos dois capítulos.

Chamamos atenção, também, para o resultado de NP-completude do problema de coloração de arestas restrito a grafos 3-partidos. Embora possa parecer um tanto desvinculado dos outros resultados deste trabalho, a coloração de arestas de grafos 3-partidos — e, portanto, 3-vértice-coloríveis — nos dá um pouco de intuição a respeito da dificuldade de colorir as arestas de grafos, mesmo que estes sejam "bem-comportados" em relação à coloração de vértices. Tal resultado é, neste sentido, análogo ao resultado NP-completude de coloração total restrito a bipartidos. E tais resultados vão ao encontro do fato de existirem tantas classes para as quais o problema de coloração de vértices é polinomial e os problemas de coloração de arestas e de coloração total são NP-completos ou estão em aberto.

Capítulo 5

Grafos {square,unichord}-free

Observe que os grafos unichord-free compõem uma classe com fortes resultados estruturais mas, ainda assim, NP-completa para coloração de arestas e coloração total. Neste capítulo e no próximo, buscamos subclasses de $\mathcal{C}$ para as quais o problema de coloração de arestas ou de coloração total seja polinomial. Neste capítulo, investigamos os grafos {square,unichord}-free. No próximo capítulo, investigamos as classes dos grafos bipartidos unichord-free e dos grafos chordless.

5.1 Coloração de arestas

Nesta seção mostramos as propriedades peculiares dos grafos {square,unichord}-free com relação à coloração de arestas: enquanto o problema é NP-completo para o caso de grau máximo 3 — provado na Subseção 5.1.1 — a coloração de arestas é polinomial para todos os outros casos — como mostrado na Subseção 5.1.2. Na Subseção 5.1.3, aprofundamos as investigações sobre o caso de grau máximo 3, obtendo duas subclasses para as quais coloração de arestas é polinomial. A Subseção 5.1.4 resume nossas observações acerca do problema de coloração de arestas restrito a grafos {square,unichord}-free.

5.1.1 NP-completude de coloração de arestas

Considere a classe $\mathcal{C}'$ composta pelos grafos de $\mathcal{C}$ que não possuem quadrados. A estrutura dos grafos de $\mathcal{C}'$ é mais forte que a dos grafos de $\mathcal{C}$, e é descrita em detalhes na Seção 4.1. Ainda assim, o problema de coloração de arestas é NP-completo para entradas de $\mathcal{C}'$, como demonstrado no Teorema 10. Vale observar que a demonstração de Cai e Ellis [9] para a NP-completude de coloração de arestas de grafos cúbicos sem quadrados gera grafos que **possuem** ciclos com corda única. Além disso, observamos que o gadget Q_Δ usado na demonstração da NP-completude de coloração de arestas de grafos unichord-free **possui** quadrados. Dessa forma, uma construção alternativa foi necessária para a NP-completude em $\mathcal{C}'$, construção esta baseada no gadget $\tilde{P}$ mostrado na Figura 5.1. O grafo $\tilde{P}$ é construído de tal forma que a identificação de seus vértices pendentes gera um grafo isomórfico a P^*, que é um grafo Classe 2 não-overfull [24, 11]. As propriedades de $\tilde{P}$ com relação à coloração de arestas são descritas no Lema 13.

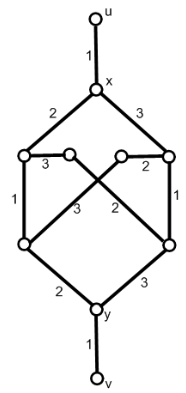


Figura 5.1: Grafo $\tilde{P}$ munido de 3-aresta-coloração.

Lema 13 *(*Machado, Figueiredo e Vušković — Lema 2 de [32]*) O grafo $\tilde{P}$ é 3-aresta-colorível, e em qualquer 3-aresta-coloração de $\tilde{P}$, as arestas ux e vy recebem a mesma cor.*

Esboço. O resultado decorre do fato de que o grafo obtido de $\tilde{P}$ através da identificação dos vértices pendentes é Classe 2. □

Teorema 10 *(*Machado, Figueiredo e Vušković — Teorema 4 de [32]*) `CHRIND`(grafo {square,unichord}-free com grau máximo 3) é NP-completo.*

Esboço. `CHRIND`(grafo cúbico) por “local replacement”, na qual cada aresta de G é substituída por uma cópia de $\tilde{P}$. □

O grafo construído na demonstração do Teorema 10 não é regular, como é comum se observar em provas de NP-completude. De fato, como mostramos na Seção 5.1.3, o problema de coloração de arestas pode ser resolvido em tempo polinomial se a entrada é restrita a grafos {square,unichord}-free cúbicos.

5.1.2 Coloração de arestas de grafos {square,unichord}-free com grau máximo pelo menos 4

Na Seção ?? consideramos o problema de coloração de arestas restrito a $\mathcal{C}$ e obtivemos um resultado de NP-completude. Na Seção 5.1.1, consideramos a subclasse $\mathcal{C}'$ e verificamos que, mesmo para esta classe bastante restrita e estruturada, a NP-completude ainda vale. Nesta seção aplicamos os resultados de decomposição da Seção 4.1 e mostramos que o problema de coloração de arestas é polinomial para grafos de $\mathcal{C}'$ com grau máximo pelo menos 4. De fato, os únicos Classe 2 no conjunto são os grafos completos de ordem ímpar.

Descrevemos, a seguir, a técnica utilizada para colorir as arestas de um grafo $\mathcal{C}'$ através da combinação das colorações de arestas de seus blocos com relação a uma decomposição por 2-cutset. Observe que o fato de um determinado grafo F ser isomórfico a um bloco B obtido a partir de uma decomposição por 2-corte próprio de G **não** implica que G contenha F: possivelmente B é construído através da adição de um vértice marker. Isto é ilustrado no exemplo da Figura 5.2, onde G é P^*-free e, ainda assim, o grafo P^* aparece como bloco de uma decomposição por 2-corte próprio de G.

O leitor irá observar que nem sempre é necessário que os blocos de decomposição de G possuam $\Delta(G)$-aresta-coloração para que G possua $\Delta(G)$-aresta-coloração: o grafo G da Figure 5.2 é 3-aresta-colorível, enquanto P^* não é. Esta é uma importante observação: possivelmente, as arestas incidentes ao marker de um bloco de decomposição não são arestas reais do grafo original, ou então já foram coloridas em uma coloração de arestas do outro bloco, de tal forma que não necessitam ser coloridas.

Observação 4 *Considere um grafo $G \in \mathcal{C}'$ com as seguintes propriedades:*

- (X, Y, a, b) *é split de um 2-corte próprio de G;*
- $\tilde{G_Y}$ *obtido de G_Y a partir da remoção do marker se este marker não é um vértice real de G;*

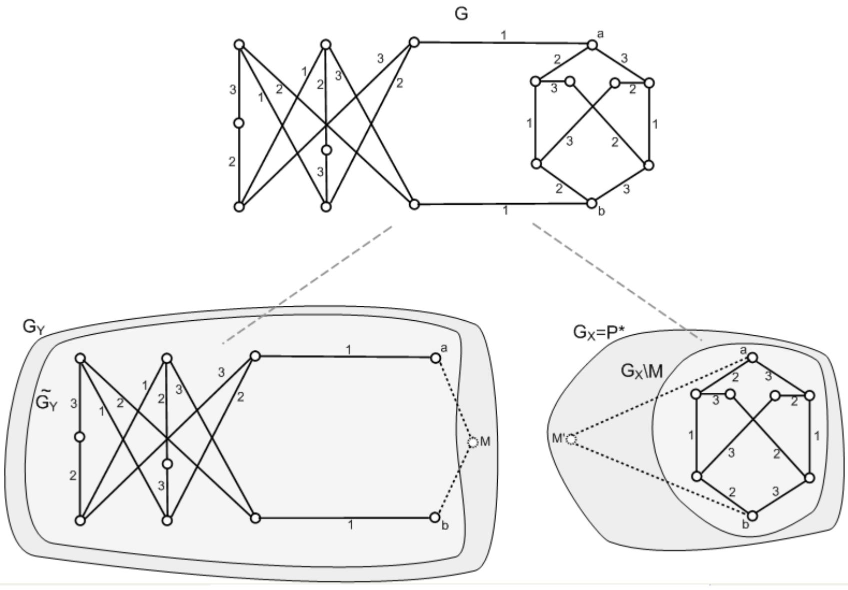


Figura 5.2: Exemplo de decomposição com relação a um 2-corte próprio $\{a, b\}$. Observe que o marker e as arestas incidentes – identificadas por linhas tracejadas – não pertencem ao grafo original.

- *$\tilde{\pi_Y}$ é uma $\Delta(G)$-aresta-coloração de $\tilde{G_Y}$;*
- *F_a (resp. F_b) é o conjunto das cores em $\{1, 2, ..., \Delta\}$ que não são usadas por $\tilde{\pi_Y}$ em qualquer aresta de $\tilde{G_Y}$ incidente a a (resp. b).*

Se existe uma $\Delta(G)$-aresta-coloração π_X de $G_X \setminus M$, onde M é o marker de G_X, tal que cada cor utilizada em uma aresta incidente a a (resp. b) está em F_a (resp. F_b), então G é Δ-aresta-colorível.

A observação acima mostra que, de forma a estender uma $\Delta(G)$-aresta-coloração de $\tilde{G_Y}$ a uma $\Delta(G)$-aresta-coloração de G, basta colorir as arestas de $G_X \setminus M$ de tal forma que as cores das arestas incidentes a a (resp. b) não estejam usadas nas arestas de $\tilde{G_Y}$ incidentes a a (resp. b). Isto garante que nenhum conflito é criado. Além disso, não há necessidade de colorir as arestas incidentes ao marker M de G_X: se este marker é um vértice de G, então suas arestas já estão coloridas por $\tilde{\pi}$, caso contrário, suas arestas não são arestas reais de G. No exemplo da Figura 5.2, exibimos uma 3-aresta-coloração $\tilde{\pi_Y}$ de $\tilde{G_Y}$. Usando a notação da Observação 4, $F_a = \{2, 3\}$ e $F_b = \{2, 3\}$. Exibimos, também, uma 3-aresta-coloração de $G_X \setminus M$ tal que as cores das arestas incidentes a a são $\{2, 3\} \subseteq F_a$ as cores das arestas incidentes a b são $\{2,3\} \subseteq F_b$. Assim, pela Observação 4, podemos combinar as colorações $\tilde{\pi_Y}$ e π_X em uma 3-aresta-coloração de G, o que é feito na Figura 5.2.

Investigamos, agora, como obter uma $\Delta(G)$-aresta-coloração de um grafo $G \in \mathcal{C}'$ através da combinação de $\Delta(G)$-aresta-colorações de seus blocos de decomposição por um 2-corte próprio quando um dos blocos é básico. Mais precisamente, o Lema 14 mostra como isso pode ser feito se um dos blocos é básico. Em seguida, obtemos, no Teorema 11 e seu Corolário 2, uma caracterização para os grafos Classe 2 de $\mathcal{C}'$ com grau máximo $\Delta \geq 4$ que estabelece a polinomialidade do problema de coloração de arestas restrito a grafos {square,unichord}-free com grau máximo pelo menos 4.

Lema 14 *(*Machado, Figueiredo e Vušković — Lema 6 de [32]*) Seja $G \in \mathcal{C}'$ um grafo de grau máximo $\Delta \geq 4$ e seja (X, Y, a, b) split de 2-corte próprio, de tal forma que G_X seja básico. Se G_Y é Δ-aresta-colorível, então G é Δ-aresta-colorível.*

Esboço. Assumimos, como hipótese de indução, que G_Y é λ-aresta-colorível e estendemos a coloração para as arestas de G_X, considerando cada possível grafo básico G_X. □

A partir do Lema 14 é possível determinar em tempo polinomial o índice cromático de grafos {square,unichord}-free com grau máximo pelo menos 4, conforme mostramos no Teorema 11 e seu Corolário 2.

Teorema 11 *(*Machado, Figueiredo e Vušković — Teorema 9 de [32]*) Se λ é um inteiro de valor pelo menos 4 e G é um grafo {square,unichord}-free não-completo e com grau máximo $\Delta(G) \leq \lambda$, então G é λ-aresta-colorível.*

Corolário 2 *(*Machado, Figueiredo e Vušković — Corolário 1 de [32]*) Um grafo conexo $G \in \mathcal{C}'$ de grau máximo $\Delta \geq 4$ é Classe 2 se, e somente se, é um grafo completo de ordem ímpar.*

5.1.3 Coloração das arestas de grafos {square,unichord}-free com grau máximo 3

Os grafos {square,unichord}-free possuem estrutura ainda mais forte que a dos unichord-free; ainda assim, o problema de coloração de arestas é NP-completo mesmo restrito a entradas {square,unichord}-free. Observe que a NP-completude vale para grafos em $\mathcal{C}'$ com grau máximo $\Delta = 3$. Nesta seção, aprofundamos nossas investigações acerca dos grafos {square,unichord}-free com grau máximo $\Delta = 3$, fornecendo duas subclasses para as quais o problema de coloração de arestas pode ser resolvido em tempo polinomial: grafos {square,unichord}-free cúbicos e grafos {square,6-hole,unichord}-free.

Grafos {square,unichord}-free cúbicos

Nesta subseção, provamos a polinomialidade do problema de coloração de arestas restrito a grafos {square,unichord}-free cúbicos. Isto é conseqüência direta do Lema 15, que enuncia que todo grafo cúbico conexo mas não-biconexo é Classe 2, e Lema 16, que enuncia que os únicos grafos biconexos Classe 2 em $\mathcal{C}'$ são o grafo de Petersen, o grafo de Heawood e o grafo completo de ordem 4.

Lema 15 *(*Machado, Figueiredo e Vušković — Lema 7 de [32]*) Seja G um grafo cúbico conexo. Se G possui um 1-corte, então G é Classe 2.*

Esboço. Se G possui um 1-corte v, então v é o único vértice de grau 2 de um subgrafo G' tal que todos os outros vértices possuem grau 3. Logo, G' é overfull [17] e, portanto, G é subgraph-overfull [17] e, assim, Classe 2. □

Lema 16 *(*Machado, Figueiredo e Vušković — Lema 8 de [32]*) Seja $G \in \mathcal{C}'$ um grafo biconexo. Se G é cúbico, então G é isomórfico ao grafo de Petersen, ao grafo de Heawood ou ao grafo completo de ordem 4.*

Esboço. Basta observar que, se G possui 2-corte próprio, então G possui vértice de grau 2. □

O teorema a seguir é conseqüência direta do lema anterior.

Teorema 12 *(*Machado, Figueiredo e Vušković — Teorema 10 de [32]*) Seja $G \in \mathcal{C}'$ um grafo cúbico conexo. O grafo G é Classe 1 se, e somente se, G é biconexo e não é isomórfico ao grafo de Petersen.*

Grafos {square,6-hole,unichord}-free

Na presente seção, mostramos a polinomialidade do problema de coloração de arestas restrito aos grafos {square,6-hole,unichord}-free. Isto é conseqüência do Lema 17, uma variação para 3-aresta-coloração do Lema 14, e que é provado usando-se a mesma técnica.

Lema 17 *(*Machado, Figueiredo e Vušković — Lema 9 de [32]*) Seja $G \in \mathcal{C}'$ um grafo conexo de grau máximo 3 ou menor e seja (X, Y, a, b) split de 2-corte próprio, de tal forma que G_X é básico mas não isomórfico a P^*. Se G_Y é 3-aresta-colorível, então G é 3-aresta-colorível.*

Lembre-se de que o gadget $\tilde{P}$ da Figura 5.1 é construído a partir de P^*. A NP-completude da coloração de arestas em $\mathcal{C}'$ é obtida como conseqüência de que $P^* \in \mathcal{C}'$. Usando o Lema 17, podemos provar que, se o grafo especial P^* não aparece como folha na árvore de decomposição, isto é, como bloco básico após a aplicação recursiva de decomposição 2-corte próprio de um grafo biconexo $G \in \mathcal{C}'$ de grau máximo 3, então G é Classe 1.

Teorema 13 *(*Machado, Figueiredo e Vušković — Teorema 11 de [32]*) Seja $G \in \mathcal{C}'$ um grafo conexo de grau máximo 3. Se G não possui um 6-ciclo induzido por vértices de grau 3, então G é Classe 1.*

Esboço. A hipótese implica a inexistência do grafo de Petersen como bloco de decomposição de G, de forma que o Lema 17 pode ser aplicado. □

Corolário 3 *(*Machado, Figueiredo e Vušković — Corolário 2 de [32]*) Todo grafo conexo 6-hole-free de $\mathcal{C}'$ de grau máximo 3 é Classe 1.*

Uma questão natural é se a proibição de 6-holes tornaria fácil a coloração de arestas em $\mathcal{C}'$, e a resposta é **negativa**: o grafo constuído na demonstração do Teorema 6 é 6-hole-free.

Teorema 14 *(*Machado, Figueiredo e Vušković — Teorema 12 de [32]*) Para cada $\Delta \geq 3$, `CHRIND`(grafo {6-hole,unichord}-free Δ-regular) é NP-completo.*

5.1.4 Observações sobre coloração de arestas de grafos unichord-free e {square,unichord}-free

As tabelas 5.1 e 5.2 resumem os resultados em coloração de arestas alcançados nas seções anteriores.

Classe	$\Delta = 3$	$\Delta \geq 4$	regular
grafos unichord-free	NP-completo	NP-completo	NP-completo
grafos {square,unichord}-free	NP-completo	Polinomial	Polinomial
grafos {6-hole,unichord}-free	NP-completo	NP-completo	NP-completo
grafos {square,6-hole,unichord}-free	Polinomial	Polinomial	Polinomial

Tabela 5.1: Complexidade computacional do problema de coloração de arestas em subclasses de grafos unichord-free.

Os resultados das tabelas 5.1 e 5.2 mostram que, mesmo para classes de grafos com forte estrutura e poderosas decomposições, o problema de coloração de arestas pode ser bastante difícil.

A classe de grafos investigada em um primeiro momento é a classe $\mathcal{C}$ de grafos que não possuem ciclo com corda única. Os grafos não-básicos desta classe podem ser decompostos [47] através de decomposições por 1-cortes, 2-corte próprios ou proper 1-joins. Provamos que o problema de coloração de arestas é NP-completo para grafos em $\mathcal{C}$. Consideramos, então, a subclasse $\mathcal{C}' \subseteq \mathcal{C}$ cujos grafos são os grafos de $\mathcal{C}$ que não possuem 4-ciclo induzido (square). Ao proibir squares, evitamos a presença de operações do tipo junção (join), "tradicionalmente difíceis" de lidar em coloração de arestas [2, 44, 45]. Ou seja, cada grafo não-básico de $\mathcal{C}'$ pode ser decomposto através de 1-cortes e 2-corte próprios. Para a subclasse $\mathcal{C}'$ obtivemos

Classe	$k \leq 2$	$k \geq 3$
grafos k-partidos	Polinomial	NP-completo

Tabela 5.2: Dicotomia de complexidade para coloração de aresta em grafos multipartidos.

uma interessante dicotomia: o problema de coloração de arestas é NP-completo para grafos com grau máximo 3, mas polinomial para grafos com grau máximo **diferente** de 3. Além disso, determinamos uma condição necessária para que um grafo $G \in \mathcal{C}'$ de grau máximo 3 seja Classe 2. Esta condição é a seguinte: possuir o grafo P^* – um subgrafo do grafo de Petersen – como folha da árvore de decomposição. Como conseqüência, se ambos 4-holes e 6-holes são proibidos, então o índice cromático de unichord-free graphs pode ser determinado em tempo polinomial. Os resultados descritos possuem conexões com outras áreas de pesquisa em coloração de arestas, como descrevemos nas observações a seguir.

A primeira observação se refere à dicotomia encontrada em $\mathcal{C}'$. Tal dicotomia apresenta grande interesse, uma vez que, ao que consta, esta é a primeira classe para a qual coloração de arestas é NP-completo para grafos com determinado grau fixo Δ mas é polinomial para grafos com grau máximo $\Delta' > \Delta$, como o leitor pode verificar nos resultados de NP-completude encontrados na literatura e revisados na presente tese. Vale lembrar que, em termos de coloração total, o caso de grau máximo 3 é, notoriamente, "fonte de problemas" — no sentido de que é freqüente a necessidade de técnicas alternativas ou mais sofisticadas para se estabelecer resultados para este caso. Os resultados desta seção mostram que também em coloração de arestas o caso de grau máximo 3 pode ser um "divisor de águas" — em particular no caso dos grafos {square,unichord}-free, em que o caso de grau máximo 3 é o único NP-completo. É também interessante observar que os únicos grafos regulares em $\mathcal{C}'$ são o grafo de Petersen, o grafo de Heawood, os grafos completos e os ciclos. Como conseqüência, o problema de coloração de arestas é NP-completo quando restrito a $\mathcal{C}'$, mas polinomial quando restrito a grafos **regulares** de $\mathcal{C}'$.

Uma segunda observação é relacionada com o estudo dos *snarks* [46]. Um *snark* é um grafo cúbico sem ponte com índice cromático 4. Para evitar casos triviais ("fáceis"), snarks são geralmente definidos como tendo cintura pelo menos 5. O estudo dos snarks está fortemente relacionado com o Teorema das Quatro Cores. Pelo resultado do Lema 16, o único snark **não-trivial** que **não** possui ciclo com corda única é o grafo de Petersen.

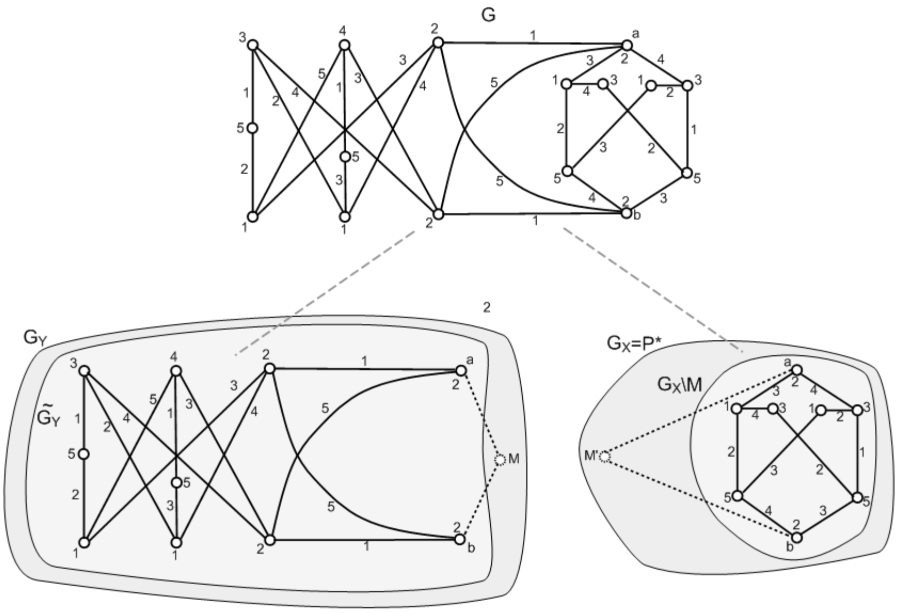
Finalmente, a terceira observação se refere ao problema de se determinar o índice cromático de um grafo k-partido, ou seja, de um grafo cujos vértices podem ser particionados em k conjuntos independentes. O problema é sabidamente polinomial para $k = 2$ e para multipartidos completos [27, 25]. Entretanto, não existe resultado explícito na literatura considerando a complexidade de determinar o índice cromático de grafos k-partidos para $k \geq 3$. Como vimos, a partir da prova do Teorema 6 pode-se concluir a NP-completude da coloração de arestas de grafos k-partidos r-regulares, para cada $k \geq 3$, $r \geq 3$.

5.2 Coloração total

5.2.1 Conjectura de Coloração Total para grafos {square,unichord}-free

Na presente seção mostramos a validade da TCC para grafos {square,unichord}-free através da demonstração de que grafos não-completos {square,unichord}-free com grau máximo pelo menos 4 são Tipo 1.

A técnica utilizada para obter uma coloração total de um grafo {square,unichord}-free através da combinação de colorações totais de seus blocos de decomposição por 2-corte próprio é similar à utilizada na Seção 5.1.2 para colorir arestas (veja a Figura 5.2.1).



Observação 5 *Considere um grafo $G \in \mathcal{C}'$ com as seguintes propriedades:*

- *(X, Y, a, b) é split de 2-corte próprio de G;*

- *$\tilde{G_Y}$ é obtido de G_Y através da remoção do marker se este marker não for vértice real de G;*

- *$\tilde{\pi_Y}$ é uma $\Delta(G) + 1$-total-coloração de $\tilde{G_Y}$;*

- *F_a (resp. F_b) é o conjunto das cores em $\{1, 2, ..., \Delta + 1\}$ não utilizadas por π_Y em a (resp. b) nem em outra aresta de $\tilde{G_Y}$ incidente a a (resp. b).*

Se existe uma $\Delta(G) + 1$-total-coloração π_X de $G_X \setminus M$, onde M é o marker de G_X, tal que $\pi_X(a) = \pi_Y(a)$ (rep. $\pi_X(b) = \pi_Y(b)$) e cada cor usada em uma aresta incidente a a (resp. b) está em F_a (resp. F_b), então G é $\Delta + 1$-total-colorível.

A observação acima mostra que, para estender uma $\Delta(G) + 1$-total-coloração de $\tilde{G_Y}$ a uma $\Delta(G) + 1$-total-coloração de G, é necessário colorir os elementos de $G_X \setminus M$ de tal forma que as cores de a, b, e das arestas incidentes a estes não criem conflitos com as cores dos elementos de $\tilde{G_Y}$.

No exemplo da Figura 5.2.1, exibimos uma 5-total-coloração $\tilde{\pi_Y}$ de $\tilde{G_Y}$. Na notação da Observação 5, $F_a = \{3, 4\}$ e $F_b = \{3, 4\}$. Exibimos, também, uma 5-total-coloração de $G_X \setminus M$ tal que as cores de a e b são as mesmas que em $\tilde{G_Y}$, as cores das arestas incidentes a a são $\{3, 4\} \subseteq F_a$ e as cores das arestas incidentes a b são $\{3, 4\} \subseteq F_b$. Logo, pela Oservação 5, podemos combinar as 5-total-colorações $\tilde{\pi_Y}$ e π_X em uma 5-total-coloração de G, como mostrado na Figura 5.2.1.

Investigamos, agora, como $(\Delta(G)+1)$-total-colorir um grafo $G \in \mathcal{C}'$ combinando $(\Delta(G) + 1)$-total-colorações de seus blocos com relação a um 2-corte próprio. Mais precisamente, o Lema 18 mostra como isso pode ser feito se um dos blocos é básico. Em seguida, obtemos, no Teorema 15 e seu Corolário 4, resultados que permite determinar o número cromático total de grafos de $\mathcal{C}'$ com grau máximo pelo menos 4.

Lema 18 *(*Machado e Figueiredo — Lema 11 de [33]*) Seja $G \in \mathcal{C}'$ um grafo de grau máximo $\Delta \geq 4$ e seja (X, Y, a, b) split de 2-corte próprio, de tal forma que G_X seja básico. Se G_Y é $(\Delta + 1)$-total-colorível, então G é $(\Delta + 1)$-total-colorível.*

Esboço. Supomos, por indução, que G_Y é $\Delta + 1$-total-colorível e estendemos a coloração para os elementos de G_X, considerando cada possível bloco básico G_X.

Teorema 15 *(*Machado e Figueiredo — Teorema 8 de [33]*) Se λ é inteiro de valor pelo menos 4 e G é um grafo conexo não-completo de $\mathcal{C}'$ com grau máximo $\Delta(G) \leq \lambda$, então G é $\lambda + 1$-total-colorível.*

Corolário 4 *(*Machado e Figueiredo — Corolário 1 de [33]*) Um grafo conexo $G \in \mathcal{C}'$ com grau máximo $\Delta \geq 4$ é Tipo 2 se, e somente se, é um grafo completo de ordem par.*

5.2.2 O caso grau máximo 3

Nesta seção, apresentamos o resultado de que, à exceção do grafo completo com quatro vértices, todos os grafos {square,unichord}-free com grau máximo 3 são Tipo 1. Observe que a técnica utilizada na Seção 5.2.1 não pode ser aplicada para o caso de grau máximo 3 usando apenas 4 cores. Isto porque existem certas condições para as quais a 4-total-coloração de G_Y não poderia ser estendida para G_X sem o uso de cores adicionais, como é o caso do exemplo da Figura 5.3. Para que seja possível

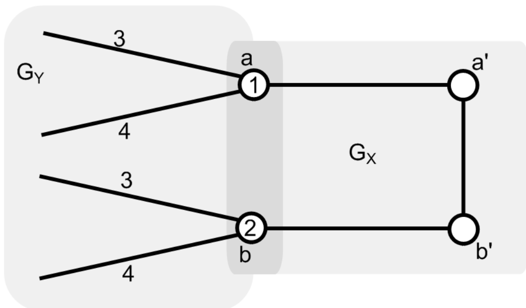


Figura 5.3: 4-coloração-total de G_Y não estensível a G_X.

4-total-colorir grafos {square,unichord}-free com grau máximo 3, é necessário que se adapte a técnica da Seção 5.2.1 de forma a evitar o acontecimento de situações como a da Figura 5.3. Para isso, resgataremos a idéia de "coloração fronteiriça" apresentada no Capítulo 3. Basicamente, nosso objetivo, ao definir uma nova "coloração fronteiriça", é garantir a existência de uma coloração dos elementos a, b, aa' e bb' que permita que os elementos a', b' e $a'b'$ sejam coloridos adequadamente (o que não acontece no exemplo da Figura 5.3. Uma condição que permite isso é garantindo que as cores de a e b sejam iguais (condição de vértice) ou que as cores de aa' e bb' sejam iguais (condição de aresta), mas que ambos não ocorram.

Teorema 16 (Machado e Figueiredo — Corolário 1 de [34]) *Todo grafo {square,unichord}-free não-completo de grau máximo 3 é Tipo 1.*

A Tabela 5.3 resume a complexidade computacional dos problemas de coloração restritos a $\mathcal{C}$ e a $\mathcal{C}'$.

Problema \ Classe	$\mathcal{C}$	$\mathcal{C}'$, $\Delta \geq 4$	$\mathcal{C}'$, $\Delta = 3$
Coloração de Vértices	Polinomial [47]	Polinomial [47]	Polinomial [47]
Coloração de Arestas	NP-Completo [32]	Polinomial [32]	NP-Completo [32]
Coloração Total	NP-Completo [33]	Polinomial [33]	Polinomial*

Tabela 5.3: Complexidade computacional dos problemas de coloração restritos a $\mathcal{C}$ e a $\mathcal{C}'$.

Devemos fazer algumas observações a respeito dos resultados acima. Primeiro, observe a interessante dicotomia com relação à coloração de arestas de grafos {square,unichord}-free. Uma vez que a técnica utilizada em [33] para a coloração total de grafos {square,unichord}-free funciona apenas para o caso de grafos com grau máximo pelo menos 4, era de se esperar que uma dicotomia semelhante valesse para o problema de coloração total. O que verificamos, no entanto, é que tal dicotomia não existe, e o problema de coloração total é polinomial quando restrito a grafos {square,unichord}-free, inclusive para o caso de grau máximo 3. De todo modo, é interessante notar como abordagens distintas tiveram de ser usadas para resolver os casos $\Delta \geq 4$ e $\Delta = 3$.

5.3 Considerações finais

Intuitivamente, tende-se a pensar que o problema de coloração-total é "mais difícil" que o problema de coloração de arestas — no sentido de que classes com estrutura suficiente para que o problema de coloração de arestas seja fácil fatalmente levariam, também, à solução do problema de coloração de arestas. Os motivos para isso são diversos:

- a NP-completude de coloração total é obtida através de redução de coloração de arestas;
- o problema de coloração total é NP-completo para grafos bipartidos — a classe mais “bem comportada possível”, em termos de coloração de arestas;
- mesmo a Conjectura da Coloração Total parece um desafio imenso, estando aberta para grafos planares, por exemplo.

Ao final deste capítulo, no entando, dispomos de uma classe surpreendente composta apenas de grafos Tipo 1 (a menos do K_4) e para a qual o problema de coloração de arestas é NP-completo. A determinação dos resultados obtidos neste capítulo para grafos {square,unichord}-free certamente contribui muito para o entendimento da relação (e da independência!) dos problemas de coloração de arestas e coloração total.

Capítulo 6

Outras Classes - resultados recentes

Neste capítulo estudamos mais duas subclasses dos unichord-free. Embora, no decorrer da tese, o estudo destas duas classes tenha sido iniciado em paralelo com os grafos {square,unichord}-free, apenas muito recentemente obtivemos os resultados de complexidade para elas, de forma que optamos por incluí-las em um capítulo à parte. Assim como no restante da tese, os resultados aqui apresentados possuem apenas esboço de demonstração. No entanto, devido ao fato de os resultados serem muito recentes, ainda não dispomos de artigo (ou minuta) contendo demonstrações completas. Ainda assim, registramos que tais demonstrações são variações sobre técnicas já apresentadas na tese, de forma que o leitor interessado não terá maiores dificuldades em verificá-las. Na Seção 6.1, investigamos os grafos bipartidos unichord-free. Naturalmente, o problema de coloração de arestas é polinomial quando restrito a estes grafos, já que são todos Classe 1. Mostramos que o problema de coloração total, no entanto, permanece NP-completo, na classe. Na Seção 6.2 consideramos a classe dos grafos tais que todo ciclo é sem corda, denominados *chordless*. Para esta classe, mostramos que, se o grau máximo é pelo menos 3, então todos os grafos são, ao mesmo tempo, Classe 1 e Tipo 1.

6.1 Grafos bipartidos unichord-free

Uma vez demonstrada a NP-completude do problema de coloração total restrito a grafos unichord-free, e considerando o resultado clássico [42] de NP-completude de coloração total para bipartidos, é uma questão natural investigar a complexidade de coloração total na interseção das duas classes. Nesta seção, mostramos que o problema de coloração total é NP-completo para grafos bipartidos unichord-free. A demonstração do resultado é feita a partir de redução do problema de coloração de arestas. A técnica é similar àquela já utilizada no Capítulo 4, com algumas variações efetuadas com o objetivo de obter um grafo bipartido.

Considere, inicialmente, o grafo H'_t, construído a partir de três cópias do grafo S_t mostrado na Figura 6.1, no Capítulo 4. Observe que o grafo H'_t possui $3t-4$ vértices pendentes, $2t-2$ dos quais são coloridos de vermelho, enquanto os $t-2$ restantes são coloridos de verde. Os vértices pendentes vermelhos são utilizados através da identificação com vértices reais de outras cópias de H'_t, de modo a construir o grafo R'_t.

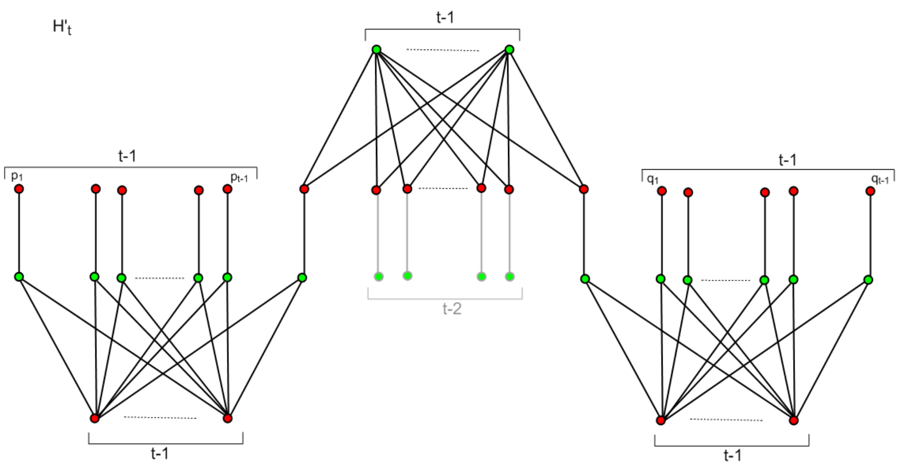


Figura 6.1: Grafo H'_t.

O grafo R'_t é construído a partir de $t+1$ cópias de H'_t, de maneira similar ao grafo H'_t da Figura 4.6 do Capítulo 4. A Figura 6.2 mostra o grafo R'_t munido de uma bicoloração. Observe que, de cada cópia de H'_t em R'_t, restam $t-2$ vértices pendentes vermelhos e $t-2$ vértices pendentes verdes.

Finalmente, apresentamos o grafo $F'_{n,t}$ que, de maneira similar ao grafo $F'_{n,t}$ na Figura 4.7 do Capítulo 4, é construído a partir de n cópias de H'_t. Na Figura 6.3,

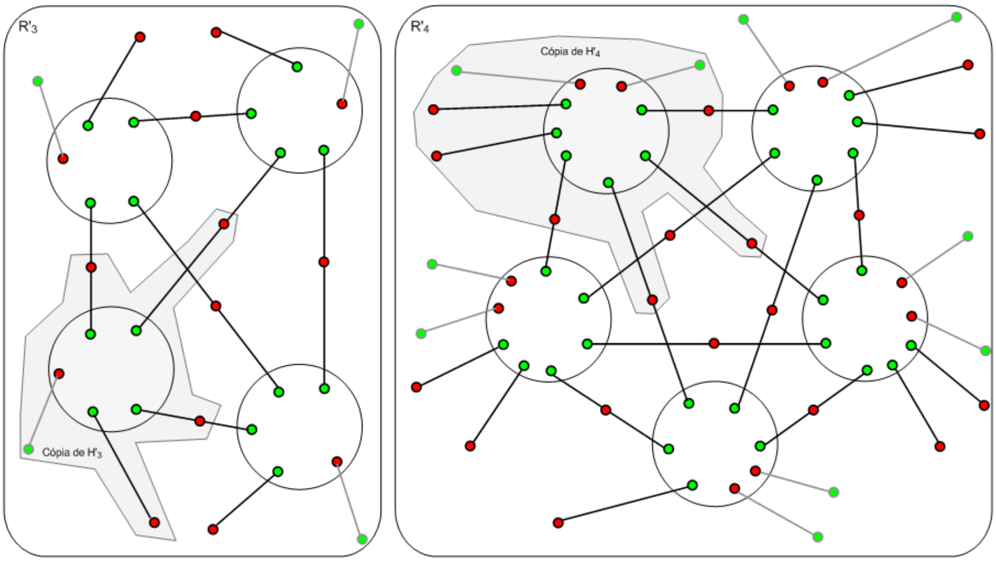


Figura 6.2: "Replacement graphs" R'_3 e R'_4.

representamos, apenas quatro dos vértices reais de cada cópia de H'_t (caso $t > 3$, haverá mais vértices reais). Observe que também exibimos uma bicoloração dos vértices de $F_{n,t}$, e que metade dos vértices pendentes reais recebe a cor verde, e a outra metade, recebe a cor vermelha.

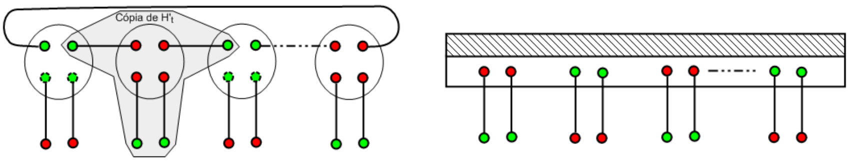


Figura 6.3: O "forcer graph" e sua representação esquemática.

Uma vez definidos nossos novos "gadgets", podemos enunciar os lemas análogos aos do Capítulo 4.

Lema 19 *Considere o grafo H'_t.*

1. *Considere uma $(t+1)$-total-coloração parcial π' de H'_t em que as arestas pendentes estão coloridas com a mesma cor e os vértices pendentes também estão coloridos (e nada mais está colorido). Esta $(t+1)$-total-coloração parcial estende para uma $(t+1)$-total-coloração de $H_{n,t}$.*

2. *Em qualquer $(t+1)$-total-coloração de H'_t, as arestas pendentes recebem a mesma cor.*

Lema 20 *Considere o grafo R'_t, $t \geq 3$.*

1. *Toda $(t+1)$-total-coloração parcial de R'_t na qual as $t+1$ arestas pendentes reais possuem cores diferentes (duas a duas distintas) e os vértices pendentes reais estão coloridos (e mais nada está colorido) estende a uma $(t+1)$-total-coloração de R'_t,*

2. *Em toda $(t+1)$-total-coloração de R'_t as $t+1$ arestas pendentes reais possuem cores diferentes (duas a duas distintas).*

Lema 21 (McDiarmid e Sánchez-Arroyo [41]) *Considere o grafo $F = F'_{n,t}$, com $n \geq 2$ e $t \geq 3$.*

1. *Considere uma $(t+1)$-total-coloração parcial de F na qual cada aresta pendente possui a mesma cor e cada vértice pendente está colorido (e nada mais está colorido). Então, isto estende para uma $(t+1)$-total-coloração de F.*

2. *Em toda $(t+1)$-total-coloração de F toda aresta pendente possui a mesma cor.*

As demonstrações dos três lemas acima são bastante similares às demonstrações dos lemas análogos do Capítulo 4, de modo deixamos ao leitor a tarefa de verificá-las em detalhes. Uma vez dotados dos lemas para estes novos gadgets, somos capazes de demonstrar a NP-completude de coloração total para grafos bipartidos unichord-free.

Teorema 17 *Para $\Delta \geq 3$, `TOTCHR`(grafo bipartido unichord-free com grau máximo Δ) é NP-completo.*

Prova (esboço):

Seja G uma instância do problema NP-completo `CHRIND`(grafo Δ-regular). Lembre-se de que coloração de arestas está em NP. Construiremos uma instância G' de `TOTCHR`(grafo bipartido unichord-free com grau máximo Δ) tal que G' é $\Delta+1$-total-colorível se e somente se G é Δ-aresta-colorível. O grafo G' é construído da seguinte forma:

1. Construa G'' substituindo cada vértice v de G por uma cópia de R'_Δ, identificando Δ de suas arestas pendentes (uma proveniente de cada cópia de H'_t) com

as Δ arestas de G incidentes a v. As arestas pendentes devem ser escolhidas de forma a manter G' bipartido. Como cada cópia de H'_t possui pelo menos uma aresta incidente a vértices vermelho e uma aresta incidente a vértice verde, é sempre possível fazer essa escolha apropriadamente. Observe que uma das $\Delta + 1$ cópias de H'_t em R'_Δ não tem suas arestas pendentes utilizadas, denominadas *arestas forçáveis*.

2. Construa G' identificando identificando as arestas forçáveis (uma de cada cópia de R'_Δ associado a um vértice) com as arestas de um forcer graph. Novamente, esta identificação deve ser feita de maneira a manter o grafo G' bipartido. Como cada cópia de H'_t em R'_Δ possui arestas forçáveis incidentes a vértices verdes e vermelhos, assim como as arestas pendentes do próprio forcer graph, é possível fazer esta identificação apropriadamente.

A Figura 6.4 mostra um exemplo da construção de G'.

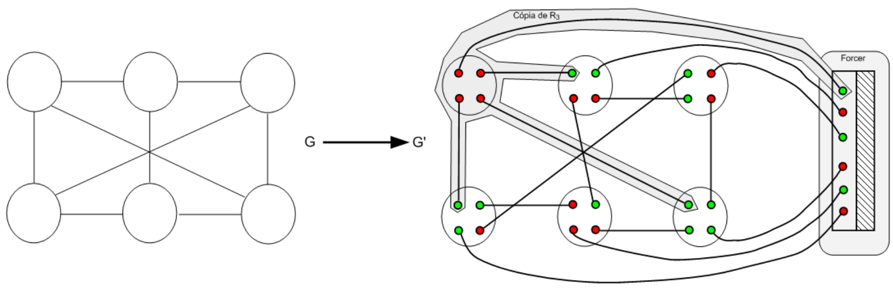


Figura 6.4: .

Por questão de simplicidade, na demonstração do Teorema 17, construímos um grafo G' não-regular. Ressaltamos, entretanto, que a técnica de indução no grau mínimo desenvolvida no Capítulo 4 pode, igualmente, ser aplicada à presente classe. Dessa forma, o problema de coloração total é NP-completo mesmo restrito a grafos bipartidos unichord-free regulares.

6.2 Grafos chordless

Um grafo é denominado *chordless* se todo ciclo é sem corda [29]. Dessa forma, a classe dos grafos chordless é, claramente, uma subclasse da classe dos unichord-free. Além disso, tais grafos aparecem no contexto de grafos que não possuem subdivisão do K_4 — grafos *ISK$_4$-free* — como um dos blocos básicos com relação à decomposição por 2-corte próprios. Nesta seção, mostramos que os problemas de coloração de arestas e de coloração total são polinomiais quando restritos a grafos chordless. De fato, a coloração de vértices dos grafos ISK_4-free depende da coloração de arestas dos grafos chordless de grau máximo 3. Nesta seção, mostramos que todos os grafos chordless de grau máximo pelo menos 3 são Classe 1 e Tipo 1.

6.2.1 Estrutura dos grafos chordless

O seguinte resultado de decomposição [29] mostra que os grafos básicos da decomposição de grafos chordless por 1-cortes e 2-cortes próprios são os grafos *esparsos*, ou seja, grafos tais que toda aresta é incidente a um vértice de grau no máximo 2.

Teorema 18 (Lévêque, Maffray e Trotignon [29]) *Se G é um grafo chordless, então G é esparso, ou G admite 1-corte ou G admite 2-corte próprio.*

Mostramos um teorema de decomposição para grafos esparsos.

Teorema 19 *Se G é um grafo biconexo esparso, então G é fortemente 2-bipartido, ou $G = K_{2,n}$ com $n \geq 2$, ou $G = C_n$, ou $G = P_2$, ou G admite 1-corte ou G admite 2-corte próprio.*

Prova (esboço):

Seja G um grafo esparso e suponha que G não é fortemente 2-bipartido nem um ciclo nem um caminho. Como G é esparso, então o conjunto $D = \{v \in V | deg_G(v) \geq 3\}$ é independente. Como G não é fortemente 2-bipartido, vale uma das alternativas a seguir:

1. $V \setminus D$ não é conjunto independente; ou

2. G possui quadrado; ou

3. G possui vértice de grau 1.

Suponha que $V \setminus D$ não é conjunto independente e sejam u e v dois vértices adjacentes de $V \setminus D$. Existe um caminho $(a, a'..., u, v, ...b', b)$ entre a e b tal que a e b estão em D e todos os outros vértices do caminho estão em $V \setminus D$. Logo, $(\{a', ..., u, v, ..., b'\}, V \setminus \{a, a'..., u, v, ...b', b\}, a, b)$ é split de um 2-corte próprio de G.

Agora, suponha que G possui quadrado $abcd$. Se exatamente um vértice, digamos a, possui grau maior que 2 então $(\{b, c, d\}, V \setminus \{a, b, c, d\}, a)$ é split de um 1-corte. Se exatamente dois vértices, digamos a e c, possuem grau maior que 2, então estes vértices são não-adjacentes e $(\{b, d\}, V \setminus \{a, b, c, d\}, a, c)$ é split de um 2-corte próprio de G. Observe que não é possível que haja mais de dois vértices com grau maior que 2, pois isso implica que dois deles são adjacentes.

Finalmente, se G possui um vértice v de grau 1, então $G = P_2$ ou o único vizinho de v é um 1-corte de G. □

A partir dos Teoremas 18 e 19, podemos enunciar o seguinte resultado:

Corolário 5 *Se G é grafo chordless, então G é fortemente 2-bipartido, ou $G = K_{2,n}$ com $n \geq 2$, ou $G = C_n$, ou $G = P_2$, ou G admite 1-corte ou G admite 2-corte próprio.*

Como já discutimos em outros capítulos desta tese, para o objetivo de resolver problemas de coloração de arestas e coloração total, podemos trabalhar apenas com grafos biconexos. Dado um grafo chordless G decomponível por um 2-corte próprio com split (X, Y, a, b), definimos o bloco G_X como $G[X \cup \{a, b\}] \cup M$ adicionando-se um vértice marcador ("marker") M somente se não existe vértice em X adjacente a ambos a e b. O bloco G_Y é definido analogamente.

Como vimos usando ao longo desta tese, um grafo é básico se é indecomponível. De acordo com o Teorema 5, os grafos básicos de decomposição de grafos chordless com relação a 1-cortes e 2-corte próprios são grafos fortemente 2-bipartido, $K_{2,n}$ com $n \geq 2$, $G = C_n$, e $G = P_2$ — em particular, P_2 aparece apenas como bloco de decomposição com relação a 1-cortes. O Teorema 20 mostra a existência de decomposição extremal em grafos chordless.

Teorema 20 *Se G é grafo chordless biconexo não-básico, então G admite 2-corte próprio com split (X, Y, a, b) tal que G_X é indecomponível.*

Demonstração (esboço). Análoga à do Lema 5. □

6.2.2 Índice cromático de grafos chordless

Nesta seção aplicamos os resultados da Seção 6.2.1 de forma a mostrar que todo grafo chordless não-completo de grau máximo pelo menos 3 é Classe 1, mostrando que o índice cromático de grafos chordless é um problema polinomial. A demonstração utiliza um resultado clássico de Borodin, Kostochka, and Woodall [7].

Teorema 21 (Borodin, Kostochka, and Woodall [7]) *Seja $G = (V, E)$ um grafo bipartido e seja $\mathcal{L} = \{L_e\}_{e \in E}$ uma coleção de listas de cores que associa, a cada aresta $uv \in E$, uma lista L_{uv} de cores. Se, para cada aresta $uv \in E$, vale $|L_{uv}| \geq \max\{deg_G(u), deg_G(v)\}$, então existe uma coloração de arestas π de G tal que, para cada aresta $uv \in E$, vale $\pi(uv) \in L_{uv}$.*

Teorema 22 *Todo grafo chordless G com $\Delta(G) \geq 3$ é $\Delta(G)$-aresta-colorível.*

Prova (esboço):

Se G é básico, então G é fortemente 2-bipartido ou $G = K_{2,3}$, em ambos os casos $\Delta(G)$-aresta-colorível. Logo, podemos assumir que G não é básico e possui 1-corte ou 2-corte próprio.

Suponha que G possui um 1-corte com split (X, Y, v) e assuma, como hipótese de indução, que ambos G_X e G_Y são $\Delta(G)$-aresta-coloríveis. Sejam π_X e π_Y $\Delta(G)$-aresta-colorações de G_X e G_Y tais que as cores das arestas de G_X incidentes a v são diferentes das cores das arestas de G_Y incidentes a v. Construímos uma $\Delta(G)$-aresta-coloração π de G definindo $\pi_X(w)$, se $w \in X$, e $\pi_Y(w)$, caso contrário.

Suponha, agora, que G possui 2-corte próprio com split (X, Y, a, b) tal que G_X é básico e suponha, pela hipótese de indução, que G_Y possui uma $\Delta(G)$-aresta-coloração π_X. Denote por M o marker de G_X.

Caso 1. $G_X = K_{2,n}$, $n \geq 2$. Denote por F_a (resp. F_b) o conjunto das cores disponíveis em a (resp. b) com relação às cores usadas nas arestas de G_Y incidentes a a (resp. b). Associe o conjunto F_a às arestas de G_X incidentes em a e o conjunto F_b às arestas de G_X incidentes em b. Pelo Teorema 21, existe uma coloração das arestas de G_X a partir destas listas.

Caso 2. G_X é fortemente 2-bipartido. Similar ao Caso 1.

Caso 3. G_X é um ciclo — $G_X \setminus \{M\} = P_n$ é um caminho. Podemos assumir $n \geq 4$. Primeiramente, pinte as arestas de G_X incidentes em a ou em b. Então, pinte as arestas remanescentes de G_X em qualquer ordem. $\square$

6.2.3 Número cromático total de grafos chordless

Nesta seção aplicamos os resultados da Seção 6.2.1 de forma a mostrar que todo grafo chordless não-completo de grau máximo pelo menos 3 é Tipo 1, de forma que determinar o número cromático total de grafos chordless é um problema polinomial. Primeiramente consideramos o caso de grau máximo pelo menos 4, em seguida consideramos o já tradicionalmente patológico caso de grau máximo 3.

Obsevamos que a técnica utilizada para colorir totalmente grafos chordless é similar àquela utilizada no Capítulo 5 para grafos {square,unichord}-free. Em particular, o mesmo tipo de problema acontece no caso de grafos com grau máximo 3, de forma que é necessário recorrer à técnica de frontier colouring.

Grau máximo pelo menos 4

Consideramos, primeiramente, a coloração total de grafos com grau máximo pelo menos 4.

Teorema 23 *Todo grafo chordless G não-completo com $\Delta(G) \geq 4$ é $(\Delta(G) + 1)$-total-colorível.*

Prova (esboço):

Similar ao Lema 11 de [33] □

Grau máximo 3

Nesta seção, enunciamos que todo grafo chordless não-completo com grau máximo 3 é Tipo 1. Observamos que as técnicas utilizadas nesta seção não diretamente são aplicáveis se apenas 4 cores estão disponíveis. Neste caso, uma condição mais forte sobre os elementos já coloridos é necessária — é preciso resgatar o conceito de "frontier-colouring", já definido no Capítulo 5.

Teorema 24 *Todo grafo chordless G não-completo com $\Delta(G) = 3$ é $(\Delta(G) + 1)$-total-colorível.*

Prova (esboço):

Similar ao Lema 11 de [34] □

6.3 Considerações finais

Ao final deste capítulo, dispomos de mais duas subclasses dos unichord-free para as quais as complexidades de coloração de arestas e coloração total é conhecida. A classe dos grafos bipartidos unichord-free é, naturalmente, polinomial para coloração de arestas — todos os grafos são Classe 1 — por outro lado, com relação à coloração total, o problema é NP-completo. Já a classe dos grafos chordless é polinomial tanto para coloração de arestas quanto para coloração total. Os resultados adicionais deste capítulo reiteram a riqueza dos grafos unichord-free com relação aos problemas de coloração de arestas e coloração total. A Figura 6.5 resume os resultados obtidos para os grafos unichord-free nos Capítulos 4, 5 e 6.

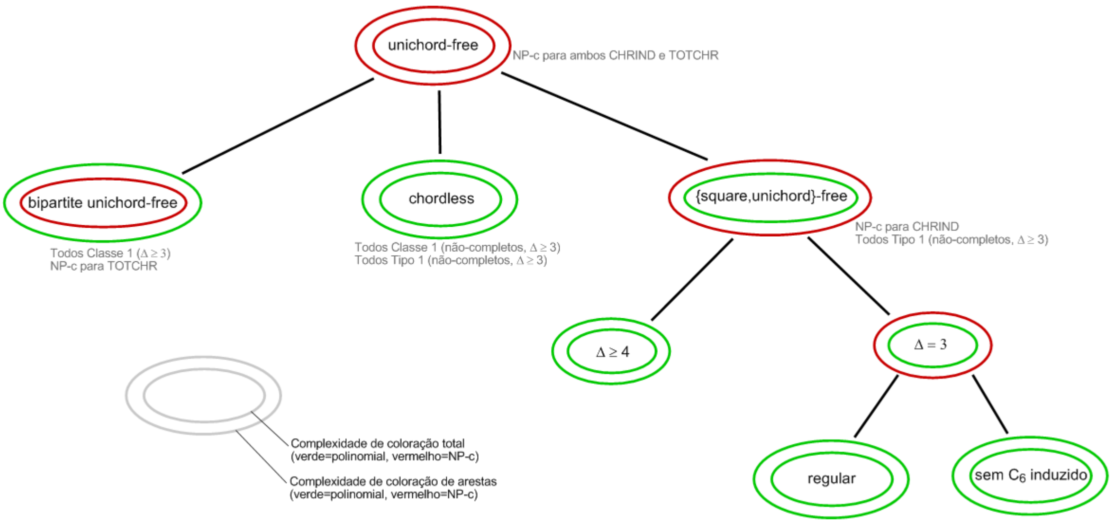


Figura 6.5: Complexidade dos problemas de coloração de arestas e coloração total na hierarquia dos unichord-free.

Capítulo 7

Decomposição Desconexo-Bipartida

7.1 Motivação

No presente capítulo apresentamos uma técnica [30] de partição do conjunto de arestas denominada Decomposição Desconexo-Bipartida. A motivação para o desenvolvimento da técnica é a necessidade de uma melhor compreensão da operação *join* e suas implicações à variação do índice cromático, já que a operação é básica para a construção dos cografos. A determinação do índice cromático de cografos ainda é um problema em aberto na Teoria dos Grafos.

7.2 As decomposições

Dizemos que um grafo $G = (V, E)$ é uma *união* (union) de grafos $G_1 = (V_1, E_1)$ e $G_2 = (V_2, E_2)$ se $V = V_1 \cup V_2$ e $E = E_1 \cup E_2$. Escrevemos $G = G_1 \cup G_2$, dizemos que G é o *grafo união* (union graph), e que G_1 e G_2 são os *operandos* da operação união. Observe que, se $E_1 \cap E_2 = \emptyset$, então $\max\{\Delta(G_1), \Delta(G_2)\} \leq \Delta(G) \leq \Delta(G_1) + \Delta(G_2)$. Usamos o conceito de união de grafos com o objetivo de representar decomposições do conjunto de arestas de um grafo. No que segue, assumiremos que o grafo $G = (V, E)$ é representado como a união de dois grafos $G_1 = (V_1, E_1)$ e $G_2 = (V_2, E_2)$ tais que $V = V_1 = V_2$ e $E_1 \cap E_2 = \emptyset$. Nesta seção, consideramos um dos casos limites para $\Delta(G)$, a saber, $\Delta(G) = \Delta(G_1) + \Delta(G_2)$. Na Seção 7.4, consideramos o outro caso limite $\Delta(G) = \max\{\Delta(G_1), \Delta(G_2)\}$. A Proposição 2 enuncia um resultado simples sobre a união de dois grafos Classe 1.

Proposição 2 (Machado e Figueiredo — Proposição 1 de [30]) *Seja $G = G_1 \cup G_2$ a união de dois grafos tais que $\Delta(G) = \Delta(G_1) + \Delta(G_2)$. Se ambos G_1 e G_2 são Classe 1, então G é Classe 1.*

Esboço. Basta colorir as arestas de G_1 com $Delta(G_1)$ cores e as arestas de G_2 com $Delta(G_2)$ cores adicionais para obter uma $\Delta(G)$-aresta-coloração de G. □

O uso da decomposição descrita na Proposição 2 é um argumento freqüente na identificação de grafos Classe 1. Neste capítulo, consideramos decomposições diferentes. Na decomposição proposta nesta seção, um grafo G ainda é representado como a união de dois grafos G_1 e G_2 tais que $\Delta(G) = \Delta(G_1) + \Delta(G_2)$. Entretanto, diferentemente da decomposição descrita na Proposição 2, somente um dos grafo é requerido ser Classe 1 – assumamos, sem perda de generalidade, que este grafo Classe 1 é G_1. Denotando por $\Lambda(G)$ o conjunto dos vértices de G que possuem grau máximo, uma condição suficiente que garante o grafo união G ser Classe 1 é dada pela Proposição 3, cuja demonstração depende de outros resultados enunciados ao longo desta seção.

Proposição 3 (Machado e Figueiredo — Proposição 2 de [30]) *Seja $G = G_1 \cup G_2$ a união de um grafo Classe 1, $G_1 = (V, E_1)m$ e um grafo $G_2 = (V, E_2)$ tais que $\Delta(G) = \Delta(G_1) + \Delta(G_2)$. Se nenhuma aresta de E_1 possui ambos os extremos em $N_{G_2}(\Lambda_{G_2})$, então G é Classe 1.*

A Figura 7.1 mostra um exemplo de união de um grafo Classe 1 e um grafo Classe 2 satisfazendo às condições da Proposição 3.

A prova da Proposição 3, é conseqüência dos Lemas 22, 23 e 24. O Lema 22 é uma condição clássica a respeito do núcleo (core) de um grafo que é suficiente para que este grafo seja Classe 1.

Lema 22 (Fournier [19]) *Se o núcleo $G[\Lambda_G]$ de G é acíclico, então G é Classe 1.*

O Lema 23 a seguir é obtido através de uma aplicação do procedimento de recoloração de Vizing [48, 35]. O lema apresenta condição sob as quais podemos

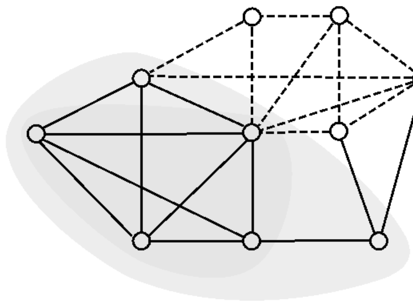


Figura 7.1: Exemplo de união de um grafo G_1 que é Classe 1 e um grafo G_2 que é Classe 2 satisfazendo as condições da Proposição 3. As arestas de G_1 são tracejadas e $\Delta(G_1) = 4$. As arestas de G_2 são contínuas e $\Delta(G_2) = 4$. O grafo união possui grau 8. Identificamos o conjunto Λ_{G_2}, envolvendo seus vértices na área cinza-escuro, e o conjunto $N_{G_2}(\Lambda_{G_2})$ envolvendo seus vértices na área cinza-claro. Observe que toda aresta tracejada possui extremo fora da área cinza-claro.

adicionar uma aresta a um grafo Classe 1 de tal maneira que o grafo resultante é, ainda, Classe 1.

Lema 23 (Machado e Figueiredo — Lema 4 de [30]) *Seja $G = (V, E)$ um grafo Classe 1 e seja uv uma não-aresta de G que possui extremo, digamos u, não-adjacente a um delta-vértice de G, ou seja, $Adj_G(u) \cap \Lambda_G = \emptyset$. Nestas condições, $G' = (V, E \cup \{uv\})$ é Classe 1.*

O Lema 24 é um caso especial da Proposição 3, quando $\Delta(G_1) = 1$.

Lema 24 (Machado e Figueiredo — Lema 5 de [30]) *Considere os grafos $G_1 = (V, E_1)$ e $G_2 = (V, E_2)$, onde E_1 é um emparelhamento tal que nenhuma aresta possui ambos os extremos em $N_{G_2}(\Lambda_{G_2})$. Se o grafo união $G = G_1 \cup G_2$ possui grau máximo $\Delta(G) = \Delta(G_2) + 1$, então G é Classe 1.*

Esboço. O Lema 24 é provado duas etapas. Na primeira etapa, adiciona-se uma única aresta a G_2 de forma a incrementar o grau de G_2 e obter um grafo que, pelo Lema 22, é Classe 1. Em seguida, adicionam-se as arestas restantes de tal forma que sempre se pode aplicar o Lema 23, obtendo-se, assim, em todos os passos, um grafo Classe 1. □

A partir do Lema 24, é possível demonstrar a validade da Proposição 3. De fato, G possui um emparelhamento que "toca" todos os delta-vértices de G. Logo, pelo

Lemma 24, o grafo $G' := (V, E_2 \cup M)$ é Classe 1. Observe que $G'' := (V, E_1 \setminus M)$ é Classe 1. Pela Proposição 2, o grafo $G = G' \cup G''$ é Classe 1. □

7.3 Aplicações

7.3.1 Decomposição Desconexo-Bipartida

Nesta seção utilizamos os resultados da Seção 7.2 de forma a desenvolver uma ferramenta de decomposição para a exibição de grafos Classe 1. Em uma *DB-Decomposição*, um grafo é decomposto na união entre um grafo desconexo e um grafo bipartido, de acordo com a seguinte notação. Considere um grafo $G = (V, E)$ e uma partição (V_L, V_R) de V, com $V_L, V_R \neq \emptyset$. Denotamos por $B_G(V_L, V_R)$ o grafo que possui mesmo conjunto de vértices que G, mas cujas arestas são aquelas arestas de G que possuem um extremo em V_L e outro extremos em V_R, ou seja, $B_G(V_L, V_R) = (V, \{uv \in E | (u \in V_L) \wedge (v \in V_R)\})$. Denotamos por $D_G(V_L, V_R)$ o grafo que possui mesmo conjunto de vértices que G, mas cujas arestas são aquelas arestas de que possuem ambos os extremos em V_L ou ambos os extremos em V_R, ou seja $D_G(V_L, V_R) = (V, \{uv \in E | (u, v \in V_L) \vee (u, v \in V_R)\})$. Quando a partição estiver clara no contexto, escreveremos apenas B_G e D_G. Note que B_G é bipartido, D_G é desconexo, e $B_G \cup D_G = G$.

Duas classes de grafos aparecem naturalmente no contexto de DB-Decomposições. A primeira classe são os grafos join, que são grafos que possuem partição (V_L, V_R) tal que o grafo bipartido $B_G(V_L, V_R)$ associado é completo. A segunda classe é a dos grafos cobipartidos, nos quais existe partição (V_L, V_R) tal que o grafo desconexo $D_G(V_L, V_R)$ associado é a união de duas cliques disjuntas. Nesta seção, usamos DB-Decomposições para obter novos resultados na coloração de arestas de grafos join e grafos cobipartidos.

7.3.2 Join graphs

O join é uma operação entre dois grafos que aparece no contexto de cografos. Um grafo $G = (V, E)$ é o *join* entre dois grafos $G_L = (V_L, E_L)$ e $G_R = (V_R, E_R)$ disjuntos por vértices se $V = V_L \cup V_R$ e $E = E_L \cup E_R \cup \{uv : (u \in V_L) \wedge (v \in V_R)\}$. Neste caso, escrevemos $G = G_L + G_R$ e dizemos que G é um *grafo join*. Observe que $B_G(V_L, V_R)$ é um grafo bipartido completo. Os grafos nas Figuras 7.2 e 7.3 são exemplos de grafos join.

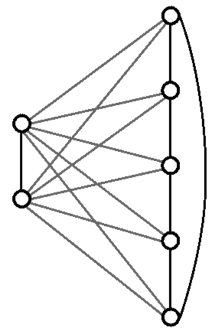


Figura 7.2: Join entre $G_L = P_2$ e $G_R = C_5$.

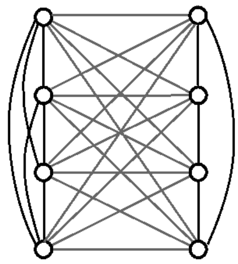


Figura 7.3: Join entre $G_L = K_4$ e $G_R = C_4$.

O índice cromático de grafos join é um tópico freqüentemente estudado em coloração de arestas [13, 38, 25, 44]. Sabe-se [44] que um um grafo join $G = G_L + G_R$ com $|V(G_L)| \leq |V(G_R)|$ e $\Delta(G_L) > \Delta(G_R)$ é Classe 1. O Teorema 25 é conseqüência direta da Proposição 3 e apresenta uma condição suficiente distinta para que um grafo join seja Classe 1.

Teorema 25 (Machado e Figueiredo — Teorema 6 de [30]) *Seja $G = G_L + G_R$ um grafo join entre $G_L = (V_L, E_L)$ e $G_R = (V_R, E_R)$. Se $\Delta(G_R) < |V_R| - |V_L|$, então G é Classe 1.*

Vale registrar que existem grafos Classe 1 satisfazendo às condições do Teorema 25 mas que não atendem às condições do resultado de [44]. Como um exempo, considere o grafo da Figura 7.2, que é o join entre $G_L = P_2$ e $G_R = C_5$. Existem, também, grafos join Classe 1 cobertos pelo resultado de [44] mas não pelo Teorema 25, como é o caso do join entre $G_L = K_4$ e $G_R = C_4$, mostrado na Figura 7.3. Ou seja, nenhum destes dois resultados generaliza o outro.

Observe, ainda, que a desigualdade $\Delta(G_R) < |V_R| - |V_L|$ é justa ("tight"). Para um exemplo de grafo join Classe 2 com $\Delta(G_R) = |V_R| - |V_L|$, considere um grafo completo de ordem ímpar $G = K_{2t+1}$, que pode ser visto como o join entre $G_L = K_1$ e $G_R = K_{2k}$.

A seguinte tabela resume os resultados sobre grafos join obtidos em [44] e no presente trabalho, destacando os casos em aberto.

	$\|V_L\| < \|V_R\|$	$\|V_L\| = \|V_R\|$
$\Delta(G[V_L]) < \Delta(G[V_R])$	Resultados parciais (este trabalho).	Todos os grafos são Classe 1 [44].
$\Delta(G[V_L]) = \Delta(G[V_R])$	Resultados parciais ([44] e este trabalho).	Resultados parciais [44].
$\Delta(G[V_L]) > \Delta(G[V_R])$	Todos os grafos são Classe 1 [44].	Todos os grafos são Classe 1 [44].

7.3.3 Grafos Cobipartidos

Um grafo $G = (V, E)$ é cobipartido se seu complemento é bipartido. Então existe uma partição (V_L, V_R) de V tal que ambos $G[V_L]$ e $G[V_R]$ são grafos completos; chamamos tal partição de *cobipartição*. Denotamos $d_L(V_L, V_R) = \max\{d_{BG(V_L,V_R)}(v) | v \in V_L\}$ e $d_R(V_L, V_R) = \max\{d_{BG(V_L,V_R)}(v) | v \in V_R\}$ e assumimos, sem perda de generalidade, que $|V_L| \leq |V_R|$. Quando a partição está clara, no contexto, escrevemos apenas d_L e d_R. Além disso, consideramos apenas grafos cobipartidos conexos com $V_L, V_R \neq \emptyset$; de outra forma, o problema reduz-se a determinar o índice cromático de grafos completos. Provamos, como conseqüência imediata dos Teoremas 26 e 27, que toda partição (V_L, V_R) de um grafo cobipartido Classe 2 é tal que $V_L < V_R$ e $d_L > d_R$. As definições de $d_L(V_L, V_R)$ e $d_R(V_L, V_R)$ e o seguinte resultado do Teorema 26 são análogos a um resultado de Chen, Fu e Ko (Lema 3.4 de [13]) para grafos split.

Teorema 26 (Machado e Figueiredo — Teorema 7 de [30]) *Seja $G = (V, E)$ um grafo cobipartido conexo com cobipartição (V_L, V_R), $0 < |V_L| < |V_R|$. Se $d_L(V_L, V_R) \leq d_R(V_L, V_R)$, então G é Classe 1.*

A prova do Teorema 26 é conseqüência direta da Proposição 3. O Teorema 27 enuncia que, se $|V_L| = |V_R|$, o grafo cobipartido é Classe 1.

Teorema 27 (Machado e Figueiredo — Teorema 8 de [30]) *Seja* $G = (V, E)$ *um grafo conexo cobipartido com cobipartição* (V_L, V_R). *Se* $|V_L| = |V_R|$, *então* G *é Classe 1.*

Esboço. A prova do teorema é feita em duas etapas. Primeiro, colore-se as arestas $E(G[V_L]) \cup E(G[V_R]) \cup M$, onde $M \subset E(B_G)$ é um matching, usando $|V_L| = |V_R|$ cores. Em seguida, adiciona-se as arestas de $B_G \setminus M$ obtendo-se, pela Proposição 2, um grafo Classe 1. □

Corolário 6 (Machado e Figueiredo — Corolário 9 de [30]) *Seja* $G = (V, E)$ *um grafo cobipartido conexo Classe 2. Para toda cobipartição* (V_L, V_R) *com* $|V_L| \leq |V_R|$, *vale* $|V_L| < |V_R|$ *e* $d_L > d_R$.

O caso restante para a classificação completa dos grafos cobipartidos é $d_L > d_R$ – observe que existem grafos Classe 2 satisfazendo a essa relação, como é o caso dos grafos completos de ordem ímpar.

7.4 núcleo e semi-núcleo de um grafo

O *núcleo* (*core*) de um grafo G é o subgrafo induzido por seus delta-vértices, ou seja, $G[\Lambda(G)]$. O *semi-núcleo* (*semi-core*) de G é o subgrafo induzido pelos delta-vértices e seus vizinhos, ou seja, $G[N_G(\Lambda_G)]$. Diversos trabalhos [23, 24] investigam a relação entre a estrutura do núcleo de um grafo e o seu índice cromático. Usamos os resultados da Seção 7.2 para mostrar como o núcleo e o semi-núcleo de um grafo podem fornecer informação sobre o índice cromático deste grafo.

O Teorema 28 enuncia que o índice cromático de um grafo é igual ao índice cromático de seu semi-core. A demonstração do Teorema 28 faz uso de um outro resultado a respeito da união de grafos na qual o grafo resultante possui grau máximo igual ao maior dos graus máximos dos operandos. A Proposição 4 dá uma condição suficiente para que o grafo união, neste caso, seja Classe 1.

Proposição 4 (Machado e Figueiredo — Proposição 10 de [30]) *Seja* $G = G_1 \cup G_2$ *o grafo união de um grafo* $G_1 = (V, E_1)$ *de Classe 1 e um grafo* $G_2 = (V, E_2)$ *tais que* $\Delta(G) = \Delta(G_1)$. *Se* $\Lambda_G = \Lambda_{G_1}$ *nenhuma aresta de* G_2 *possui ambos os extremos em* $N_{G_1}(\Lambda_{G_1})$, *então* G *é Classe 1.*

Esboço. Adiciona-se a G_1 as arestas de G_2 em uma seqüência tal que é sempre possível aplicar o Lemma 23, obtendo-se, a cada adição de aresta, um grafo Classe 1. □

A partir da Proposição 4, é possível mostrar que o índice cromático de um grafo depende apenas de seu semi-core.

Teorema 28 (Machado e Figueiredo — Teorema 11 de [30]) *Para todo grafo G, vale $\chi'(G) = \chi'(G[N_G(\Lambda_G)])$.*

Esboço. Aplique a Proposição 4 a $G[N_G(\Lambda_G)]$ e $G' = (V(G), E(G) \setminus E(G[N_G(\Lambda_G)]))$. □

Os resultados da presente seção encontram conexões com resultados clássicos de coloração de arestas. Considere o seguinte teorema de Hilton e Cheng.

Teorema 29 *(Hilton e Cheng [23]) Seja $G = (V, E)$ um grafo conexo Classe 2 com $\Delta(G[\Lambda_G]) \leq 2$. Então:*

1. *G é crítico (ou seja, a exclusão de qualquer aresta torna G um grafo Classe 1);*
2. *$\delta(G[\Lambda_G]) = 2$;*
3. *$\delta(G) = \Delta(G) - 1$ exceto se G for ciclo ímpar; e*
4. *$Adj_G(\Lambda_G) = V$.*

Mostramos que a última afirmativa do Teorema 29 é, na verdade, conseqüência das duas primeiras. Considere a seguinte proposição e seu corolário imediato.

Proposição 5 (Machado e Figueiredo — Proposição 13 de [30]) *Se $G = (V, E)$ é grafo crítico, então $G[N_G(\Lambda_G)] = G$.*

Corolário 7 (Machado e Figueiredo — Corolário 14 de [30]) *Seja $G = (V, E)$ um grafo crítico. Se $G[\Lambda_G]$ não possui vértice isolado, então $Adj_G(\Lambda_G) = V$.*

Corolário 7 mostra que a última afirmativa do Teorema 29 não depende do fato de $G[\Lambda_G]$ possuir grau máximo 2 ou menos, mas sim, segue da criticalidade de G e do fato de que $\delta(G[\Lambda_G]) = 2 \geq 1$ (o que implica que $G[\Lambda_G]$ não possui vértice isolado).

7.5 Considerações finais

Decomposição Desconexo-Bipartida. Sabemos que é sempre possível decompor um grafo G como a união entre um grafo desconexo $D_G(V_L, V_R)$ e um grafo bipartido $B_G(V_L, V_R)$. Para mostrar que determinado grafo $G = (V, E)$ é Classe 1, pode-se aplicar a estratégia de encontrar uma partição (V_L, V_R) de V com as seguintes propriedades:

1. $\Delta(G[V_L]) > \Delta(G[V_R])$, e
2. $\Lambda_{B_G(V_L,V_R)} \cap \Lambda_{G_L} \neq \emptyset$.

Se a primeira condição é satisfeita, todo delta-vértice deD_G está em V_L. Além disso,, todo vizinho de delta-vértice de D_G está, também, em V_L, ou seja, $N_{D_G}(\Lambda_{D_G}) \subset V_L$. Assim, toda aresta de B_G – um grafo Classe 1 – possui ao menos um extremo fora de $N_{D_G}(\Lambda_{D_G})$, a saber, o extremos em V_R. Além disso, a segunda condição garante $\Delta(G) = B_G(V_L, V_R) + D_G(V_L, V_R)$. Dessa forma, as condições da Proposição 3 são satisfeitas. Denominamos tal partição especial como *C1-DB-Decomposição*. Se um grafo G possui uma C1-DB-Decomposição, então G é Classe 1. Esta estratégia foi aplicada para encontrar grafos join e grafos cobipartidos Classe 1. Logo, uma questão natural é: pode esta estratégia ser aplicada a grafos Classe 1 mais gerais? A Figura 7.4 mostra um exemplo de C1-DB-decomposição de um grafo.

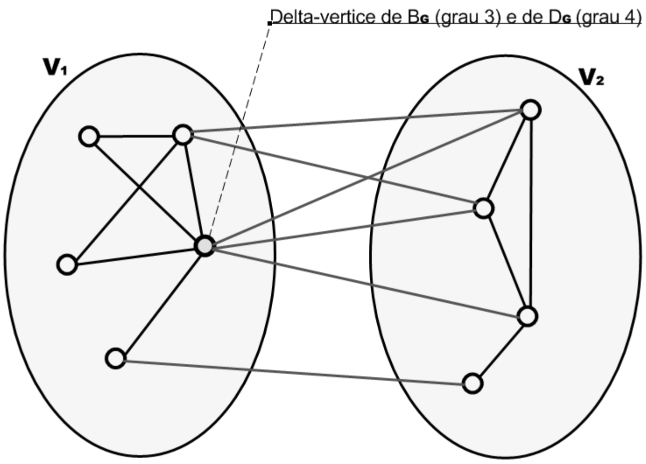


Figura 7.4: Examplo de C1-DB-decomposição.

Apesar de possuir uma C1-DB-Decomposição ser suficiente para ser Classe 1, existem grafos Classe 1 que não possuem C1-DB-Decomposição – considere, por exemplo o 4-ciclo $G = C_4$. Seria interessante, então, saber para que classes de grafos possuir uma C1-DB-Decomposição é equivalente a ser Classe 1, e se existe algoritmo polinomial para determinar a existência de tal partição – pelo menos para aquelas classes de grafos. O problema pode ser visto como um problema de decisão cuja entrada é um grafo G e cuja resposta é SIM se, e somente se, G possui uma C1-DB-Decomposição.

Capítulo 8

Conclusão

8.1 Considerações Finais

A principal contribuição desta tese é o desenvolvimento de métodos e técnicas para coloração de arestas e coloração total utilizando resultados de decomposição tradicionalmente aplicados à coloração de vértices. Nesta linha, utilizamos decomposições baseadas em cortes clique, cortes estáveis e operações join de modo a obter resultados inéditos de coloração de arestas e coloração total em classes como partial-grids, outerplanares, unichord-free (e subclasses), grafos join e cobipartidos. Também são de grande valor resultados que mostram condições em que tais decomposições **não** são úteis aos problemas de coloração de arestas e coloração total, como é o caso dos resultados de NP-completude obtidos nesta tese. Neste sentido, este trabalho é, também, um trabalho sobre a complexidade de problemas de coloração. De fato, apresentamos resultados bastante interessantes, especialmente aqueles ligados à classe dos grafos {square,unichord}-free. Tal classe, além de diferenciar a complexidade de coloração de arestas de acordo com o grau máximo — NP-completo para grau máximo 3, e polinomial para os outros casos — funciona como uma classe separadora para os problemas de coloração total e coloração de arestas — uma classe supreendente cujos grafos são todos Tipo 1 mas o problema de coloração de arestas é NP-completo. Também foi obtido interessante resultado que relaciona os problemas de coloração de vértices e coloração de arestas — a NP-completude de coloração de arestas restrita a grafos 3-coloríveis. Tais resultados certamente possibilitam uma melhor percepção a respeito da relação — e da independência — entre estes três problemas clássicos de coloração. De fato, ao final desta tese, podemos construir uma pequena tabela que deixa bem clara a independência entre os problemas de coloração — ou seja, como a resolução de um problema não necessariamente fornece informação para a solução de outro.

Classe	col. de vértices	col. de arestas	col. total
grafos	NP-completo	NP-completo	NP-completo
grafos tripartidos	Polinomial	NP-completo	NP-completo
grafos bipartidos	Polinomial	Polinomial	NP-completo
grafos {square,unichord}-free	Polinomial	NP-completo	Polinomial

Tabela 8.1: Complexidade dos problemas de coloração restritos a classes de grafos.

8.2 Futuros Trabalhos

Este é um trabalho em andamento. A literatura de coloração de vértices proporciona inúmeros resultados estruturais e de decomposição cuja aplicação à coloração de arestas e à coloração total deve ser experimentada. E, à medida em que mais ferramentas e técnicas vão sendo desenvolvidas, maiores serão as perpectivas de solução dos problemas clássicos de coloração de arestas e coloração total. Elencamos, a seguir, alguns dos problemas que têm sido considerados, mais recentemente, em nossas pesquisas, e para os quais esperamos obter resultados nos próximos anos.

- Coloração de arestas e total de grafos sem subdivisão de K_4 — grafos ISK_4-free. Investigações preliminares mostram a NP-completude de coloração de arestas na classe. Entretanto, a proibição adicional de wheels (rodas) parece tornar a classe bem mais estruturada — de fato, todos os grafos básicos são Classe 1. Um trabalho que já foi iniciado é o de determinar a complexidade dos problemas de coloração de arestas e coloração total na classe.

- Coloração por listas. Associados aos problemas de coloração de arestas e coloração total, existem as versões por listas (ver Capítulo 3, Seção 3.3). A utilização das técnicas desenvolvidas nesta tese para a obtenção dos "choice numbers" (e verificação da LECC[1] e LTCC[2]) das diversas classes investigadas é uma meta para os próximos anos.

[1]List-Edge-Colouring Conjecture

[2]List-Total-Colouring Conjecture

Capítulo 9

Anexo: Manuscrito "Decompositions for edge-coloring join graphs and cobipartite graphs"

Decompositions for edge-coloring join graphs and cobipartite graphs ⋆

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Abstract

An edge-coloring is an association of colors to the edges of a graph, in such a way that no pair of adjacent edges receive the same color. A graph G is Class 1 if it is edge-colorable with a number of colors equal to its maximum degree $\Delta(G)$. To determine whether a graph G is Class 1 is NP-complete [9]. First, we propose edge-decompositions of a graph G with the goal of edge-coloring G with $\Delta(G)$ colors. Second, we apply these decompositions for identifying new subsets of Class 1 join graphs and cobipartite graphs. Third, the proposed technique is applied for proving that the chromatic index of a graph is equal to the chromatic index of its semi-core, the subgraph induced by the maximum degree vertices and their neighbors. Finally, we apply these decomposition tools to a classical result [6] that relates the chromatic index of a graph to its core, the subgraph induced by the maximum degree vertices.

Key words: edge-coloring, chromatic index, join graph, cobipartite graph, core
PACS: 02.10.Ox

1 Introduction

Let $G = (V, E)$ be a simple connected graph (i.e., without loops or multiple edges). The maximum degree of a vertex in G is denoted by $\Delta(G)$, and the minimum degree is denoted by $\delta(G)$. A vertex of maximum degree is called a *delta-vertex* and we denote by Λ_G the set of delta-vertices of G. We denote by $Adj_G(v)$ the set of vertices of G adjacent to vertex v, and by

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$N_G(v)$ the set $Adj_G(v) \cup \{v\}$. Similarly, for a set of vertices $Z \subset V$, we denote $Adj_G(Z) = \cup_{v \in Z} Adj_G(v)$ and $N_G(Z) = Adj_G(Z) \cup Z$. The graph denoted $G[Z]$ is the subgraph of G induced by Z. In particular, $G[\Lambda_G]$ is the *core* of G, and $G[N_G(\Lambda_G)]$ is its *semi-core*.

A k-*edge-coloring* of a graph $G = (V, E)$ is a function $\pi : E \rightarrow \mathcal{C}$, with $|\mathcal{C}| = k$, such that no two edges incident to the same vertex receive the same color. A *partial edge-coloring* is a function $\pi : E' \rightarrow \mathcal{C}$, $E' \subsetneq E$, such that no two edges incident to the same vertex receive the same color. Given an edge-coloring or a partial edge-coloring, we say that a color f is a *free color* at vertex u if f is not used in any of the edges incident to u. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-coloring. Vizing's theorem [12] states that $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$, defining the *classification problem*: graphs with $\chi'(G) = \Delta(G)$ are said to be *Class 1*; graphs with $\chi'(G) = \Delta(G) + 1$ are said to be *Class 2*. It is NP-complete to determine [9] whether a graph is Class 1, even when the input is restricted to cubic graphs [9], comparability graphs [1] and line graphs [1]. On the other hand, we can determine in polynomial time whether a graph is Class 1 when the input is restricted to bipartite graphs [4], graphs with a universal vertex [11], complete multipartite graphs [8] and complete split graphs [2]. Besides, the complexity of the problem is unknown for important graph classes, such as cographs, chordal graphs, and indifference graphs.

In this paper we propose techniques for decomposing the edge-set of a graph G and searching for a $\Delta(G)$-edge-coloring of this graph. The technique is applied for optimally edge-coloring some join graphs and cobipartite graphs. We also prove that the chromatic index of a graph G and of its semi-core $G[N_G(\Lambda_G)]$ are the same. Finally, we show that, for a critical graph $G = (V, E)$ whose core $G[\Lambda_G]$ has no isolated vertices, if holds $Adj_G(\Lambda_G) = V$, an application to a classical result [6].

2 The decompositions

We say that a graph $G = (V, E)$ is the *union* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ if $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. We write $G = G_1 \cup G_2$, we say that G is the *union graph*, and that G_1 and G_2 are the *operands* of the union operation. Observe that, if $E_1 \cap E_2 = \emptyset$, then $\max\{\Delta(G_1), \Delta(G_2)\} \leq \Delta(G) \leq \Delta(G_1) + \Delta(G_2)$. We use the concept of union of graphs with the aim of representing decompositions of the edge-set of a graph. In the sequel, we assume a graph $G = (V, E)$ is represented as the union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $V = V_1 = V_2$ and $E_1 \cap E_2 = \emptyset$. In this section, we consider the limit case $\Delta(G) = \Delta(G_1) + \Delta(G_2)$. In Section 4, we consider the other limit case $\Delta(G) = \max\{\Delta(G_1), \Delta(G_2)\}$. Observe the following proposition, which considers the union of two Class 1 graphs.

Proposition 1 *Let $G = G_1 \cup G_2$ be a union of two graphs such that $\Delta(G) = \Delta(G_1) + \Delta(G_2)$. If both G_1 and G_2 are Class 1, then G is Class 1.*

Proof:

Since G_1 is Class 1, we can color its edges using $\Delta(G_1)$ colors; and since G_2 is also Class 1, we can color its edges using $\Delta(G_2)$ additional colors. So, we can color all edges of G using $\Delta(G_1) + \Delta(G_2) = \Delta(G)$ colors. □

The use of the decomposition described in Proposition 1 is a frequent argument in the identification of Class 1 graphs. In the present work, we consider a different decomposition. In the proposed decomposition, a graph G is also expressed as the union of two graphs G_1 and G_2 such that $\Delta(G) = \Delta(G_1) + \Delta(G_2)$. But, differently from the decomposition described in Proposition 1, only one of the graphs is required to be Class 1 — let us assume, without loss of generality, that this Class 1 graph is G_1. A sufficient condition that guarantees that the union graph G is Class 1 is given by Proposition 2.

Proposition 2 *Let $G = G_1 \cup G_2$ be the union of a Class 1 graph $G_1 = (V, E_1)$ and a graph $G_2 = (V, E_2)$ such that $\Delta(G) = \Delta(G_1) + \Delta(G_2)$. If no edge of E_1 has both endvertices in $N_{G_2}(\Lambda_{G_2})$, then G is Class 1.*

Figure 1 shows an example of union of a Class 1 graph and a Class 2 graph satisfying the conditions of Proposition 2, in such a way that the union graph is Class 1.

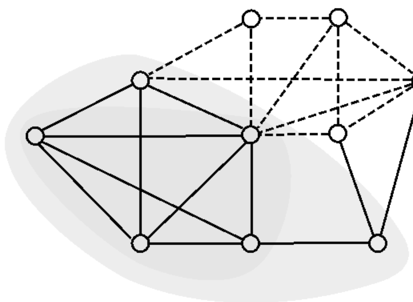

Fig. 1. Example of union of a Class 1 graph G_1 and a Class 2 graph G_2 satisfying the conditions of Proposition 2. The edges of G_1 are dashed and $\Delta(G_1) = 4$. The edges of G_2 are the black edges and $\Delta(G_2) = 4$. The union graph has degree 8. We identify the set Λ_{G_2}, by enclosing these vertices in a grey area, and the set $N_{G_2}(\Lambda_{G_2})$ by enclosing these vertices in a lighter grey area. Observe that every dashed edge has an endvertex outside the lighter grey area.

For the proof of Proposition 2, we use Lemmas 3, 4 and 5. Lemma 3 is a classical sufficient condition on the core due to Fournier that ensures G to be Class 1.

Lemma 3 *(Fournier [5]) If the core $G[\Lambda_G]$ of G has no cycle, then G is Class 1.*

Lemma 4 is an application of Vizing's recoloring procedure [10,12]. It shows conditions under which we can add an edge to a Class 1 graph in such a way that the resulting graph is also Class 1.

Lemma 4 *Let $G = (V, E)$ be a Class 1 graph and let uv be a non-edge of G which has an endvertex, say u, that is not adjacent to a delta-vertex of G, that is, $Adj_G(u) \cap \Lambda_G = \emptyset$. Then $G' = (V, E \cup \{uv\})$ is Class 1.*

Proof:

If $\Delta(G') = \Delta(G) + 1$, then G' has at most two delta-vertices, which are u and/or v. So, the core of G' has at most two vertices and cannot have cycles. By Lemma 3, graph G' is Class 1.

Now, suppose $\Delta(G') = \Delta(G)$ and let $\pi : E \to \mathcal{C}$ be a partial $\Delta(G')$-edge-coloring of G'. By assumption, every neighbor of u in G has degree smaller than $\Delta(G)$ and, therefore, has at least one free color in $\mathcal{C}$. In these conditions, we can apply Vizing's recoloring procedure [10,12] to color uv and recolor the other edges of G in order to get a $\Delta(G')$-edge-coloring of G'. $\square$

Lemma 5 is a special case of Proposition 2, when $\Delta(G_1) = 1$. The proof of this lemma follows two steps. In the first step we add to a graph an edge that increases the maximum degree, obtaining a resulting graph that is Class 1. In the second step, we add the other edges of the matching, one at a time, showing that we always meet the conditions of Lemma 4, so that the resulting graph is also Class 1.

Lemma 5 *Consider the graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, where E_1 is a matching such that no edge has both endvertices in $N_{G_2}(\Lambda_{G_2})$. If the union graph $G = G_1 \cup G_2$ has maximum degree $\Delta(G) = \Delta(G_2) + 1$, then G is Class 1.*

Proof:

Since $\Delta(G) = \Delta(G_2)+1$, there is an edge in E_1 incident to a delta-vertex of G_2. Call this edge u_1v_1, where $v_1 \in \Lambda_{G_2}$. Now, denote the edges in $E_1 \setminus \{u_1v_1\}$ by $u_2v_2, ..., u_kv_k$, in such a way that u_i is an endvertex of edge u_iv_i which does not belong to $N_{G_2}(\Lambda_{G_2})$. Now, let $G^0 = G_2$ and $G^i = (V, E_2 \cup \{u_1v_1, ..., u_iv_i\})$. Observe that G^i is obtained from G^{i-1} by adding edge u_iv_i. We prove the lemma by induction on i.

Graph G^1 is obtained from G^0 by adding edge u_1v_1. Since u_1v_1 is incident to a delta-vertex of G^0, we have that $\Delta(G^1) = \Delta(G^0) + 1$. Observe that G^1 has at most two delta-vertices, which are v_1 and, possibly, u_1. So, the core of G^1 has no cycle and, by Lemma 3, G^1 is Class 1.

Now, suppose $G^i = (V, E^i)$ is Class 1 and consider the graph $G^{i+1} = (V, E^i \cup u_{i+1}v_{i+1})$, obtained from G^i by adding the non-edge $u_{i+1}v_{i+1}$. Observe that, since E_1 is a matching, each vertex in G^i can have its degree increased by at most one, with respect to G_2. So, only a delta-vertex of G_2 can be a delta-vertex of G^i, which implies $\Lambda_{G^i} \subset \Lambda_{G_2}$ and $N_{G^i}(\Lambda_{G^i}) \subset N_{G^i}(\Lambda_{G_2})$. Observe, also, that $N_{G^i}(\Lambda_{G_2}) \subset N_{G_2}(\Lambda_{G_2}) \cup \{u_1, ..., u_i, v_1, ..., v_i\}$. So, the following relation holds:

$$N_{G^i}(\Lambda_{G^i}) \subset N_{G^i}(\Lambda_{G_2}) \subset N_{G_2}(\Lambda_{G_2}) \cup \{u_1, ..., u_i, v_1, ..., v_i\}.$$

By assumption, $u_{i+1} \notin N_{G_2}(\Lambda_{G_2})$. Since E_1 is a matching, $u_{i+1} \notin \{u_1, ..., u_i, v_1, ..., v_i\}$. So, $u_{i+1} \notin N_{G_2}(\Lambda_{G_2}) \cup \{u_1, ..., u_i, v_1, ..., v_i\}$ and consequently $u_{i+1} \notin N_{G^i}(\Lambda_{G^i})$. By Lemma 4, $G^{i+1} = (V, E^i \cup u_{i+1}v_{i+1})$ is Class 1. $\square$

Now we give the proof of Proposition 2 in two steps. In the first step, we add to a graph G_2 a matching obtained from a coloring class of a Class 1 graph G_1 and show, using Lemma 5, that the resulting graph G_2' is Class 1. In the second step, we add the remaining edges of G_1 to G_2' and show that we have a union of two Class 1 graphs satisfying the conditions of Proposition 1, in such a way that the union graph is also Class 1.

Proof (of Proposition 2):

Let $\pi : E_1 \to \{1, ..., \Delta(G_1)\}$ be an edge-coloring of G_1 and consider the matching $M_1 = \{e \in E_1 | \pi(e) = 1\}$, determined by the edges with color 1 in π. Now, consider the graph $G_1' = (V, E_1 \setminus M_1)$, obtained from G_1 by removing the edges in M_1. Observe that M_1 covers all delta-vertices of G_1, so that G_1' is $(\Delta(G_1) - 1)$-edge-colorable and has degree $\Delta(G_1') = \Delta(G_1) - 1$. So, G_1' is Class 1.

Now, consider the graph $G_2' = (V, E_2 \cup M_1)$, obtained from G_2 by adding the edges of M_1. Since $\Delta(G) = \Delta(G_1) + \Delta(G_2)$, there is a vertex x which is delta-vertex of both G_1 and G_2. Since x is a delta-vertex of G_1, this vertex has no free color in π, so that there is an edge $e \in M_1$ incident to x. So,

$\Delta(G_2') = \Delta(G_2) + 1$. By hypothesis, every edge of G_1, and, in particular, in M_1, has one endvertex not in $N_{G_2}(\Lambda_{G_2})$. So, by Lemma 5, G_2' is Class 1.

Now, observe that $G_1' \cup G_2' = G_1 \cup G_2$ and $\Delta(G_1' \cup G_2') = \Delta(G_1 \cup G_2) = \Delta(G_1) + \Delta(G_2) = (\Delta(G_1') - 1) + (\Delta(G_2') + 1) = \Delta(G_1') + \Delta(G_2')$. Since both G_1' and G_2' are Class 1, we can use Proposition 1 and conclude that $G = G_1' \cup G_2'$ is Class 1. □

3 Applications

3.1 Disconnected-Bipartite Decomposition

In this section we use the results of Section 2 to develop a decomposition tool for exhibiting Class 1 graphs. In a *DB-Decomposition*, a graph is decomposed into a disconnected graph and a bipartite graph, according to the following notation. Consider a graph $G = (V, E)$ and a partition (V_L, V_R) of V, with $V_L, V_R \neq \emptyset$. We denote by $B_G(V_L, V_R)$ the graph with same vertex set as G, but whose edges are the edges of G with one endvertex in V_L and the other in V_R, that is, $B_G(V_L, V_R) = (V, \{uv \in E | (u \in V_L) \wedge (v \in V_R)\})$. We denote by $D_G(V_L, V_R)$ the graph with same vertex set as G, but whose edges are the edges of G with both endvertices in V_L or both in V_R, that is, $D_G(V_L, V_R) = (V, \{uv \in E | (u, v \in V_L) \vee (u, v \in V_R)\})$. When the partition is clear in the context, we write only B_G and D_G. Note that B_G is bipartite, D_G is disconnected, and $B_G \cup D_G = G$.

Two graph classes arise naturally in the context of DB-Decompositions. The first class is that of join graphs, which are graphs that have a partition (V_L, V_R) such that the associated bipartite graph $B_G(V_L, V_R)$ is complete bipartite — that is, it has all "all possible edges". The second class is that of cobipartite graphs, which have a partition (V_L, V_R) such that the associated disconnected graph $D_G(V_L, V_R)$ is a union of two disjoint cliques — analogously, it also has "all possible edges". In this section, we use DB-Decompositions for obtaining new results on the edge coloring of join graphs and cobipartite graphs.

3.2 Join graphs

The join is an operation between two graphs and appears in the context of cographs, which are defined recursively by means of this operation. A graph $G = (V, E)$ is the *join* of two vertex disjoint graphs $G_L = (V_L, E_L)$ and $G_R = (V_R, E_R)$ if $V = V_L \cup V_R$ and $E = E_L \cup E_R \cup \{uv : (u \in V_L) \wedge (v \in V_R)\}$.

In this case, we write $G = G_L + G_R$ and we say that G is a *join graph*. Observe that $B_G(V_L, V_R)$ is a complete bipartite graph. The graphs in Figures 2 and 3 are examples of join graphs.

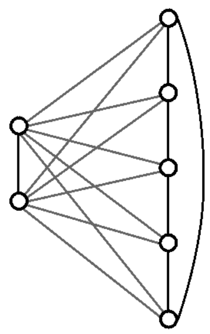

Fig. 2. Join of $G_L = P_2$ and $G_R = C_5$.

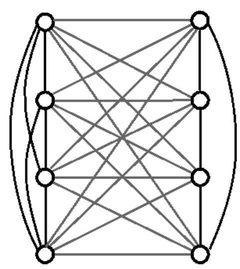

Fig. 3. Join of $G_L = K_4$ and $G_R = C_4$.

The chromatic index of join graphs was already studied [2,3,8,11]. We remark that some classes of join graphs have been classified [2,8,11]. It is known [3] that a join graph $G = G_L + G_R$ with $|V(G_L)| \leq |V(G_R)|$ and $\Delta(G_L) > \Delta(G_R)$ is Class 1. We show in Theorem 6 a different sufficient condition for a join graph to be Class 1:

Theorem 6 *Let $G = G_L + G_R$ be the join graph of $G_L = (V_L, E_L)$ and $G_R = (V_R, E_R)$. If $\Delta(G_R) < |V_R| - |V_L|$, then G is Class 1.*

Proof:

Let $G' = G_R \cup B_G(V_L, V_R)$. The degree in G' of any vertex $v \in V_L$ is $|V_R|$. So, since $|V_R| > \Delta(G_R) + |V_L|$, the delta-vertices of G' are the vertices of V_L. Observe that these vertices are an independent set of G' and, in particular, they induce an acyclic graph. So, by Lemma 3, graph G' is Class 1.

Now, observe that $G = G' \cup G_L$ and that $\Delta(G) = |V_R| + \Delta(G_L) = \Delta(G') + \Delta(G_L)$. Besides, G' is Class 1 and all of its edges have at least one endvertex not in $N_{G_L}(\Lambda_{G_L})$, which is the endvertex in V_R. So, by Proposition 2, G is Class 1. $\square$

We observe that there are Class 1 graphs satisfying the conditions of Theorem 6 but not of the result of [3]. For an example, consider the graph of Figure 2, which is the join of $G_L = P_2$ and $G_R = C_5$. There are, also, Class 1 graphs which are covered by the result of [3] but do not satisfy the assumptions of Theorem 6, such as the join of $G_L = K_4$ and $G_R = C_4$, shown in Figure 3. That is, none of these two results generalizes the other.

We observe, also, that the inequality $\Delta(G_R) < |V_R| - |V_L|$ is tight. For an example of a Class 2 join graph with $\Delta(G_R) = |V_R| - |V_L|$, consider a complete graph of odd order $G = K_{2t+1}$, which can be viewed as the join of $G_L = K_1$ and $G_R = K_{2k}$.

The following table merges the results on join graphs of the present work and [3], and highlight the open cases.

	$\|V_L\| < \|V_R\|$	$\|V_L\| = \|V_R\|$
$\Delta(G[V_L]) < \Delta(G[V_R])$	Partial results (this work).	All graphs are Class 1 [3].
$\Delta(G[V_L]) = \Delta(G[V_R])$	Partial results ([3] and this work).	Partial results [3].
$\Delta(G[V_L]) > \Delta(G[V_R])$	All graphs are Class 1 [3].	All graphs are Class 1 [3].

3.3 Cobipartite graphs

A graph $G = (V, E)$ is cobipartite if its complement is a bipartite graph. If this is the case, there exists a partition (V_L, V_R) of V such that both $G[V_L]$ and $G[V_R]$ are complete graphs; we call such partition a *cobipartition*. We denote $d_L(V_L, V_R) = \max\{d_{B_G(V_L,V_R)}(v) | v \in V_L\}$ and $d_R(V_L, V_R) = \max\{d_{B_G(V_L,V_R)}(v) | v \in V_R\}$, and we assume, without loss of generality, that $|V_L| \leq |V_R|$. When the partition is clear in the context, we write only d_L and d_R. We consider only connected cobipartite graphs with $V_L, V_R \neq \emptyset$, otherwise the problem is reduced to determining the chromatic index of complete graphs. We prove, as immediate consequence of Theorems 7 and 8, that every cobipartition (V_L, V_R) of a Class 2 graph is such that $V_L < V_R$ and $d_L > d_R$. The definitions of $d_L(V_L, V_R)$ and $d_R(V_L, V_R)$ and the following result of Theorem 7 are analogous to a result of Chen et al. (Lemma 3.4 of [2]) for split graphs.

Theorem 7 *Let $G = (V, E)$ be a connected cobipartite graph with cobipartition (V_L, V_R), $0 < |V_L| < |V_R|$. If $d_L(V_L, V_R) \leq d_R(V_L, V_R)$, then G is Class 1.*

Proof:

Let $D_G = D_G(V_L, V_R)$ and $B_G = B_G(V_L, V_R)$. Since the delta-vertices of D_G are in V_R, all edges of B_G have one endvertex in $V_{D_G} \setminus N_{D_G}(\Lambda_{D_G})$, which are those in V_L. Remember that B_G is bipartite and, so, is Class 1. Since $d_R(V_L, V_R) \geq d_L(V_L, V_R)$, we have $\Delta(B_G \cup D_G) = \Delta(B_G) + \Delta(D_G)$ and we can apply Proposition 2 to show that $G = D_G \cup B_G$ is Class 1. □

We prove in Theorem 8 that, if $|V_L| = |V_R|$, the cobipartite graph is Class 1.

Theorem 8 *Let $G = (V, E)$ be a connected cobipartite graph with cobipartition (V_L, V_R). If $|V_L| = |V_R|$, then G is Class 1.*

Proof:

Denote $D_G = D_G(V_L, V_R)$ and $B_G = B_G(V_L, V_R)$. Let π_B be a $\Delta(B_G)$-edge-coloring of B_G. Let $G' = (V, E')$, where E' is a matching defined by edges colored with an arbitrary color c in π_B. We show that $G_M = D_G \cup G'$ is a Class 1 graph. If $|V_L|$ is even, then D_G is Class 1. Since $\Delta(G_M) = \Delta(D_G) + 1 = \Delta(D_G) + \Delta(G')$, by Proposition 1, G_M is Class 1. Now, suppose $|V_L|$ is odd and color the edges of the complete graphs $G[V_L]$ and $G[V_R]$ with colors $\{1, 2, ..., |V_L|\}$ in such a way that, if uv is an edge of G', then the free color of u and v is the same. Now, we can color each edge $u_iv_i \in E(G')$ with the color in $\{1, 2, ..., |V_L|\}$ which does not appear in u_i and v_i, obtaining a $|V_L|$-edge-coloring of G_M. So, G_M is Class 1.

Now consider graph $B'_G = B_G \setminus E'$, obtained from B_G after removing the edges in E'. Observe that E' covers all maximum degree vertices of B_G, so that $\Delta(B_{G'}) = \Delta(B_G) - 1$ and $B_{G'}$ is Class 1. Besides, G_M has degree $\Delta(G_M) = \Delta(D_G) + 1$ and is also Class 1. Since $G = G_M \cup B_{G'}$ has degree $\Delta(G) = (\Delta(D_G) + 1) + (\Delta(B_G) - 1) = \Delta(G_M) + \Delta(B_{G'})$, we can apply Proposition 1 and conclude that G is Class 1. □

Corollary 9 *Let $G = (V, E)$ be a Class 2 connected cobipartite graph. Then, for every cobipartition (V_L, V_R) with $|V_L| \leq |V_R|$, it holds $|V_L| < |V_R|$ and $d_L > d_R$.*

Proof:

Immediate, from Theorems 7 and 8. □

We observe that, for the remaining case $d_L > d_R$, there exist Class 2 graphs, for instance, the complete graphs of odd order.

4 Core and semi-core of a graph

The core of a graph G is the subgraph induced by its delta-vertices, that is, $G[\Lambda_G]$. The semi-core of G is the subgraph induced by the delta-vertices and their neighbors, that is, $G[N_G(\Lambda_G)]$. Several works [6,7] investigate the relation between the structure of the core and the chromatic index of a graph. We use the results of Section 2 for showing how the core and the semi-core of a graph can provide information on its chromatic index.

We show in Theorem 11 that the chromatic index of a graph is equal to the chromatic index of its semi-core. For the proof, we use a result on union of graphs where the union graph has maximum degree equal to the largest maximum degree of the operands. Proposition 10 gives a sufficient condition for the union graph to be Class 1, and its proof is similar to the proof of Proposition 2.

Proposition 10 *Let $G = G_1 \cup G_2$ be the union graph of a Class 1 graph $G_1 = (V, E_1)$ and a graph $G_2 = (V, E_2)$ such that $\Delta(G) = \Delta(G_1)$. If $\Lambda_G = \Lambda_{G_1}$ and no edge of G_2 has both endvertices in $N_{G_1}(\Lambda_{G_1})$, then G is Class 1.*

Proof:

Denote the edges in E_2 by $u_1v_1, ..., u_kv_k$, in such a way that u_i is an endvertex of edge u_iv_i which does not belong to $N_{G_1}(\Lambda_{G_1})$. Now, let $G^0 = G_1$ and $G^i = (V, E_1 \cup \{u_1v_1, ..., u_iv_i\})$, for $i = 1, ..., k$. Observe that G^i is obtained from G^{i-1} by adding edge u_iv_i. We prove the lemma by induction on i.

Graph $G^0 = G_1$ is Class 1 by assumption. Now, suppose G^i is Class 1. Remember that no edge u_iv_i, for $i = 1, ..., k$, is incident to a delta-vertex of G_1, so that $\Lambda_{G^0} = \Lambda_{G^1} = ... = \Lambda_{G^k}$ and $N_{G^0}(\Lambda_{G^0}) = N_{G^1}(\Lambda_{G^1}) = ... = N_{G^k}(\Lambda_{G^k})$. So, u_{i+1} does not belong to $N_{G^i}(\Lambda_{G^i})$ and, by Lemma 4, G^{i+1} is Class 1. □

We show, using Proposition 10, that the chromatic index of a graph depends only on its semi-core.

Theorem 11 *Let $G = (V, E)$ be a graph. Then $\chi'(G) = \chi'(G[N_G(\Lambda_G)])$.*

Proof:

Observe that $G[N_G(\Lambda_G)]$ is a subgraph of G with the same maximum degree. So, if $G[N_G(\Lambda_G)]$ is Class 2, then G is also Class 2 and both graphs have the same chromatic index.

Now suppose $G[N_G(\Lambda_G)]$ is Class 1. Let G' be the graph obtained from G after the exclusion of the edges of the semi-core. Remember that G and $G[N_G(\Lambda_G)]$ have the same maximum degree and the same set of delta-vertices. Moreover, no edge of G' can have both endvertices in $N_{G[\Lambda_G]}(\Lambda_{G[\Lambda_G]})$, because $G[N_G(\Lambda_G)]$ is induced. So, by Proposition 10, $G = G[N_G(\Lambda_G)] \cup G'$ is Class 1, that is, G and $G[N_G(\Lambda_G)]$ have the same chromatic index. □

We use the previous result in connection with a classical theorem [6] on the core of a graph. The mentioned theorem is the following:

Theorem 12 *(Hilton and Cheng [6]) Let $G = (V, E)$ be a connected Class 2 graph with $\Delta(G[\Lambda_G]) \leq 2$. Then:*

(1) G is critical (that is, the exclusion of any edge turns G into a Class 1 graph);
(2) $\delta(G[\Lambda_G]) = 2$;
(3) $\delta(G) = \Delta(G) - 1$ unless G is an odd cycle; and
(4) $Adj_G(\Lambda_G) = V$.

We show that the last claim in Theorem 12 is, actually, a consequence of first two. Consider the following proposition and its immediate corollary:

Proposition 13 *If $G = (V, E)$ is a critical graph, then $G[N_G(\Lambda_G)] = G$.*

Proof:

Suppose that $G[N_G(\Lambda_G)] \neq G$. Since $G[N_G(\Lambda_G)]$ is an induced subgraph of G, there must exist a vertex $v \in V$ which does not belong to the semi-core $G[N_G(\Lambda_G)]$ of G. Let $H = G \setminus \{v\}$ be the graph obtained from G after the exclusion of vertex v. Clearly $H[N_H(\Lambda_H)] = G[N_G(\Lambda_G)]$ and, by Theorem 11, the chromatic indices of H and G are the same. So, G cannot be critical. □

Corollary 14 *Let $G = (V, E)$ be a critical graph. If $G[\Lambda_G]$ has no isolated vertices, then $Adj_G(\Lambda_G) = V$.*

Proof:

Since $G[\Lambda_G]$ has no isolated vertices, $Adj_{G[\Lambda_G]}(\Lambda_G) = \Lambda_G$ and $Adj_G(\Lambda_G) = N_G(\Lambda_G)$. Since G is critical, by Proposition 13, $N_G(\Lambda_G) = V$. □

Corollary 14 shows that the last claim of Theorem 12 does not depend on the fact that $G[\Lambda_G]$ has maximum degree at most two, but follows from the criticality of G and from the fact that $\delta(G[\Lambda_G]) = 2 \geq 1$ (which implies that $G[\Lambda_G]$ has no isolated vertices).

5 Concluding remarks

The disconnected-bipartite decomposition. We have seen that we can always decompose a connected graph G by representing it as the union of a disconnected graph $D_G(V_L, V_R)$ and a bipartite graph $B_G(V_L, V_R)$. In order to show that a given graph $G = (V, E)$ is Class 1, we have applied a number of times the strategy of finding a partition (V_L, V_R) of V with the following two properties:

(1) $\Delta(G[V_L]) > \Delta(G[V_R])$,
(2) $\Lambda_{B_G(V_L,V_R)} \cap \Lambda_{G_L} \neq \emptyset$.

If the first condition is satisfied, every delta-vertex of D_G is in V_L. Moreover, every neighbor of a delta-vertex of D_G is also in V_L, that is, $N_{D_G}(\Lambda_{D_G}) \subset V_L$. So, every edge of B_G — a Class 1 graph — has at least one endvertex not in $N_{D_G}(\Lambda_{D_G})$, which is the endvertex at V_R. Besides, the second condition guarantees that $\Delta(G) = B_G(V_L, V_R) + D_G(V_L, V_R)$. So the conditions of Proposition 2 are met. We call such partition a *C1-DB-Decomposition*. If G has a C1-DB-Decomposition, then it is Class 1. This strategy has been successfully applied to some join graphs and cobipartite graphs. A natural question is: can this strategy be applied to more general Class 1 graphs? Figure 4 shows an example of a C1-DB-decomposition of a graph.

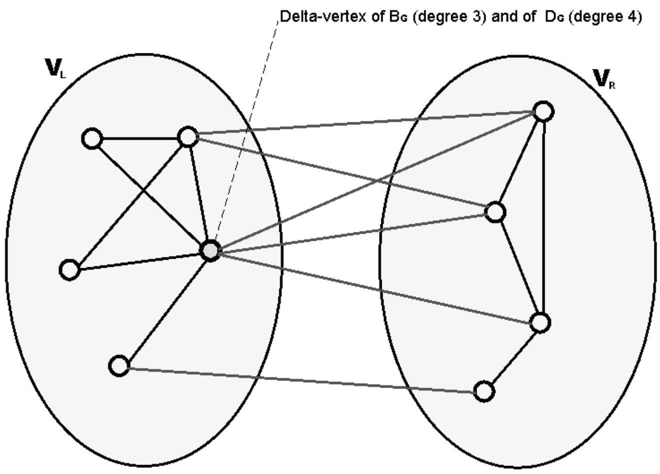

Fig. 4. Example of a C1-DB-decomposition.

Observe that, although having a C1-DB-Decomposition is sufficient for being Class 1, there are Class 1 graphs that do not have a C1-DB-Decomposition — we mention, for example, the 4-cycle $G = C_4$. The reader will verify that, in the two vertex-partitions of the 4-cycle where $|V_L| = |V_R| = 2$, the degrees of $G[V_L]$ and $G[V_R]$ are **equal**, and in the partition where $|V_L| = 3$ and $|V_R| = 1$, the maximum degrees of $B_G(V_L, V_R)$ and $D_G(V_L, V_R)$ are **not** added in the union operation. It would be interesting to know for which restricted graph classes having a C1-DB-Decomposition is equivalent to being Class 1, and whether there exists a polynomial time algorithm for determining if a graph has a C1-DB-Decomposition — at least for those restricted classes. This problem can be viewed as a decision problem whose input is a graph $G = (V, E)$ and whose answer is YES if G has a C1-DB-Decomposition.

Final Considerations. In this work we investigated decomposition tools for the edge-coloring classification problem and applied those tools to subclasses of join graphs and cobipartite graphs. We also considered the role of the core and the semi-core of a graph with respect to edge-coloring.

The decomposition tools investigated in this work generalize techniques which were applied to solve the edge-coloring classification problem in various graph classes. Besides, the ideas here investigated give origin to an interesting combinatorial problem related to the existence of a vertex partition with special properties, and which will hopefully help to edge-color optimally new graph classes.

References

[1] L. Cai and J. A. Ellis. *NP-Completeness of edge-coloring some restricted graphs.* Discrete Appl. Math. **30** (1991) 15–27.

[2] B. L. Chen, H.-L. Fu, and M. T. Ko. *Total chromatic number and chromatic index of split graphs.* J. Combin. Math. Combin. Comput. **17** (1995) 137–146.

[3] C. De Simone and C. P. de Mello. *Edge-colouring of join graphs.* Theoret. Comput. Sci. **355** (2006) 364–370.

[4] S. Fiorini and R. J. Wilson. *Edge-colouring of graphs.* Pitman, London (1977).

[5] J.-C. Fournier. *Méthode et théorème générale de coloration des arêtes.* J. Math. Pures Appl. **56** (1977) 437–453.

[6] A. J. W. Hilton and Z. Cheng. *The chromatic index of a graph whose core has maximum degree 2.* Discrete Appl. Math. **101** (1992) 135–147.

[7] A. J. W. Hilton and Z. Cheng. *On the edge-colouring of a graph whose core has maximum degree two.* J. Combin. Math. Combin. Comput. **21** (1996) 97–108.

[8] D. G. Hoffman and C. A. Rodger. *The chromatic index of complete multipartite graphs.* J. Graph Theory **16** (1992) 159–163.

[9] I. Holyer. *The NP-completeness of edge-coloring.* SIAM J. Comput. **10** (1981) 718–720.

[10] J. Misra and D. Gries. *A constructive proof of Vizing's theorem.* Inf. Proc. Lett. **41** (1992) 131–133.

[11] M. Plantholt. *The chromatic index of graphs with a spanning star.* J. Graph Theory **5** (1981) 45–53.

[12] V. G. Vizing. *On an estimate of the chromatic class of a p-graph.* Metody Diskret. Analiz. **3** (1964) 25–30. In Russian.

Capítulo 10

Anexo: Manuscrito "A decomposition for total-colouring partial-grids and list-total-colouring outerplanar graphs"

A decomposition for total-colouring partial-grids and list-total-colouring outerplanar graphs*

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Abstract

The total chromatic number $\chi_T(G)$ is the least number of colours sufficient to colour the elements (vertices and edges) of a graph G in such a way that no incident or adjacent elements receive the same colour. In the present work we obtain two results on total-colouring. First, we extend the set of partial-grids classified with respect to the total-chromatic number, by proving that every 8-chordal partial-grid of maximum degree 3 has total chromatic number 4. Second, we prove a result on list-total-colouring biconnected outerplanar graphs. If for each element x of a biconnected outerplanar graph G there exists a set L_x of colors such that $|L_{uw}| = \max\{deg(u) + 1, deg(w) + 1\}$ for each edge uw and $|L_v| = 7 - \delta_{deg(v),3} - 2\delta_{deg(v),2}$ (where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$) for each vertex v, then there is a total-colouring π of graph G such that $\pi(x) \in L_x$ for each element x of G. The technique used in these two results is a decomposition by a cutset of two adjacent vertices, whose properties are discussed in the paper.

Keywords: total-colouring, list-total-colouring, graph decompositions, partial-grids, outerplanar graphs.

1 Introduction

In the present paper we deal with simple connected graphs. A graph G has vertex set $V(G)$ and edge set $E(G)$. An *element* of G is one of its vertices or edges and the set of elements of G is denoted $S(G) = V(G) \cup E(G)$. Two vertices $u, v \in V(G)$ are *adjacent* if $uv \in E(G)$; two edges $e_1, e_2 \in E(G)$ are *adjacent* if they share a common endvertex; a vertex u and an edge e

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are *incident* if u is an endvertex of e. The *degree of a vertex* v in G, denoted $deg_G(v)$, is the number of edges of G incident to v. The *degree of an edge* uw in G, denoted $deg_G(uw)$, is the value $\max\{deg_G(u), deg_G(w)\}$. When graph G is clear from the context, we use $deg(v)$ and $deg(uw)$. We use the standard notation of C_n and P_n for cycle-graphs and path-graphs, respectively.

A *total-colouring* is an association of colours to the elements of a graph in such a way that no adjacent or incident elements receive the same colour. The *total chromatic number* of a graph G, denoted $\chi_T(G)$, is the least number of colours sufficient to total-colour this graph. Clearly, $\chi_T(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. The Total Colouring Conjecture (TCC) states that every graph G can be total-coloured with $\Delta(G)+2$ colours. By the TCC only two values would be possible for the total chromatic number of a graph: $\chi_T(G) = \Delta(G)+1$ or $\Delta(G)+2$. If a graph G has total chromatic number $\Delta(G)+1$, then G is said to be *Type 1*; if G has total chromatic number $\Delta(G)+2$, then G is said to be *Type 2*. The TCC is open since 1964, exposing how challenging the problem of total-colouring is.

It is NP-complete [17] to determine whether the total chromatic number of a graph G is $\Delta(G)+1$. In fact, the problem remains NP-complete when restricted to r-regular bipartite inputs [16], for each fixed $r \geq 3$. The problem is polynomial for a few very restricted graph classes, some of which we enumerate next:

- a cycle-graph G has $\chi_T(G) = \Delta(G)+1 = 3$ if $|V(G)| = 0 \mod 3$, and $\chi_T(G) = \Delta(G)+2 = 4$ otherwise [25];
- a complete graph G has $\chi_T(G) = \Delta(G)+1$ if $|V(G)|$ is odd, and $\chi_T(G) = \Delta(G)+2$ otherwise [25];
- a complete bipartite graph $G = K_{m,n}$ has $\chi_T(G) = \Delta(G)+1 = \max\{m,n\}+1$ if $m \neq n$, and $\chi_T(G) = \Delta(G)+2 = m+2 = n+2$ otherwise [25];
- a grid $G = P_m \times P_n$ has $\chi_T(G) = \Delta(G)+2$ if $G = P_2$ or $G = C_4$, and $\chi_T(G) = \Delta(G)+1$ otherwise [8];
- a series-parallel graph G has $\chi_T(G) = \Delta(G)+2$ if $G = P_2$ or $G = C_n$ with $n = 0 \mod 3$, and $\chi_T(G) = \Delta(G)+1$ otherwise [13, 21, 24] (the same result holds for the subclass of outerplanar graphs).

The complexity of the total-colouring problem is unknown for several important and well studied graph classes. A first example is the class of partial-grids, which are arbitrary subgraphs of grids. Remark that, when the maximum degree is 1, 2 or 4, a partial-grid can be total-coloured as a path-graph, a cycle-graph or a grid. The remaining case of maximum degree 3 is considered in [8]: partial-grids with at most three vertices of maximum degree and partial-grids such that every induced cycle has size 4 are Type 1. The complexity of total-colouring planar graphs is unknown; in fact, even the TCC is not yet settled for the class [22]. The complexity of total-colouring is open for the class of chordal graphs, and the partial results for the related classes of interval graphs [3], split graphs [5] and dually chordal graphs [6] expose the interest in the total-colouring problem restricted to chordal graphs. Another class for which the complexity of total-colouring is unknown is the class of join graphs: there are known results only for very restricted subclasses, such as the join between a complete inequibipartite graph and a path [14] and the join between a complete bipartite graph and a cycle [15], all of which are Type 1.

A natural generalization of total-colouring is the list-total-colouring problem. An instance of the list-total-colouring problem consists of a graph G and a collection $\{L_x\}_{x \in S(G)}$ which associates a set of colours – called *list* – to each element of G. It is asked whether there is a total-colouring π of G such that $\pi(x) \in L_x$ for each element x of G. If such a total-colouring exists, then we say that G is *total-colourable from the lists* $\{L_x\}_{x \in S(G)}$. The list-total-colouring problem is NP-complete when the input is restricted to biconnected outerplanar graphs of maximum degree at least 3 [27]. However, there are some known sufficient conditions for a biconnected outerplanar graph G with $\Delta(G) \geq 3$ to be total-coloured from the lists $\{L_x\}_{x \in S(G)}$. If every list has exactly $\Delta(G) + 1$ colours, then G is total-colourable from the lists $\{L_x\}_{x \in S(G)}$ [13, 21, 24] (in fact, the result holds for the superclass of series-parallel graphs). A stronger result is given by [27]: it suffices $|L_{uw}| = \max\{deg(uw) + 1, 5\}$ for each edge uw and $|L_v| = \min\{5, \Delta + 1\}$ for each vertex v.

In the present work we obtain two results on total-colouring. First, we extend the set of partial-grids classified with respect to the total-chromatic number, by proving that every 8-chordal partial-grid of maximum degree 3 is Type 1. This result provides further evidence to the conjecture [9] that every partial-grid of maximum degree 3 is Type 1. Second, we obtain a new result on list-total-colouring biconnected outerplanar graphs: if there is a list of colours L_x associated to each element of a graph G such that $|L_{uw}| = deg(uw) + 1$ for each edge uw and $|L_v| = 7 - \delta_{deg(v),3} - 2\delta_{deg(v),2}$ (where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$) for each vertex v, then G is colourable from the lists $\{L_x\}_{x \in S(G)}$.

In Section 2 we describe the proposed technique of decomposition by clique 2-cutsets. In Section 3, we total-colour every 8-chordal partial-grid of maximum degree 3 with four colours. In Section 4, we prove the result on list-total-colouring biconnected outerplanar graphs. Section 5 contains final considerations.

2 The decomposition technique

We propose a decomposition technique to total-colour structured graph classes. Next, we present the idea of decomposing a graph. Given a graph G and a set of vertices $X \subset V(G)$, we say that X is a *cutset* of G if the induced subgraph $G \setminus X = G[V(G) \setminus X]$ is not connected. If $|X| = n$, we say that X is an *n-cutset.* If we denote the connected components of $G \setminus X$ by $H_1, ..., H_k$, we say that the induced subgraphs $G_1 = G[V(H_1) \cup X], ..., G_k = G[V(H_k) \cup X]$ of G are the *X-components* of G. The concept of *block* is more general [18] but, for the purposes of the present work, the *blocks* of decomposition of a graph G by a set of vertices $X \subset V(G)$ are the X-components of G. The main goal of decomposing a graph is trying to solve a problem for this graph by combining the solutions of its blocks. The goal of the present work is to obtain a $(\Delta(G) + 1)$-total-colouring of G from $(\Delta(G) + 1)$-total-colourings of its blocks.

A *clique* in a graph is a set of pairwise adjacent vertices. A well studied decomposition for the vertex-colouring problem is the one based on *clique cutsets*, that is, cutsets that are cliques. We say that X is a *clique n-cutset* of G if X is a clique on n vertices and it is a cutset of G. If X is a clique cutset of a graph G and optimum vertex-colourings are known for each block, it is immediate to combine those colorings into an optimum vertex-colouring of G. More precisely, we just exchange the colours of the vertices of each X-component in such a way that the colours of the vertices in cutset X agree.

For the total-colouring problem, if a clique cutset X has exactly one vertex x, then it is possible to combine $(\Delta(G) + 1)$-total-colourings of the blocks of decomposition into a $(\Delta(G) + 1)$-total-colouring of the original graph G: just $(\Delta(G) + 1)$-total-colour each X-component in such a way that the colour of x is the same and the colours of its incident edges are all different. In fact, when total-colouring graph classes which are closed under decompositions by 1-cutsets, we may assume the graphs are biconnected. If $|X| \geq 2$, however, there is no such a well-behaved result. Observe in Figure 1 an example where G has maximum degree 3 and X is a clique 2-

cutset. Both X-components of G are 4-total-colourable. However, graph G does not have a 4-total-colouring. Similar examples can be constructed for graphs of larger degrees. This motivates us to investigate under which conditions we can combine total-colourings around a clique cutset. In the present work, we give two applications of the decomposition by clique 2-cutsets to the total-colouring problem.

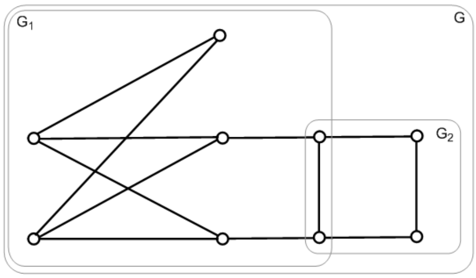

Figure 1: In any 4-total-colouring of G_1, the two edges incident to the clique 2-cutset have the same colour [17]. So, if G is 4-total-colourable, then the 4-cycle-graph G_2 has a 4-total-colouring such that the free colours at two consecutive vertices of the cycle are the same, and this is not possible.

As we have already mentioned, the goal of decomposing a graph is to obtain a solution for a problem by combining the solutions for the blocks of decomposition. Another way of viewing this idea is recursively decomposing a graph until obtaining a set of indecomposable graphs, which are called *basic* (see Figure 2). Once we can solve a problem for each basic graph, we try to combine the solutions until we have a solution for the original graph.

A decomposition is *extremal* if at least one of the blocks of decomposition is basic. Having an extremal decomposition is useful because one of the blocks is in the “restricted” set of basic graphs (see an example of extremal decomposition in Figure 3). Lemma 1 proves that every non-basic graph has an extremal decomposition by clique 2-cutsets.

Lemma 1. *Let G be a graph that has a clique 2-cutset. Graph G has a clique 2-cutset such that at least one of the blocks is basic.*

Proof:

Let X be a clique 2-cutset chosen over all clique 2-cutsets of G in such a way that one of the blocks of decomposition, denoted H, has minimum size. Suppose H has a clique 2-cutset Y and let $H_1, H_2, ..., H_p$ be the blocks of

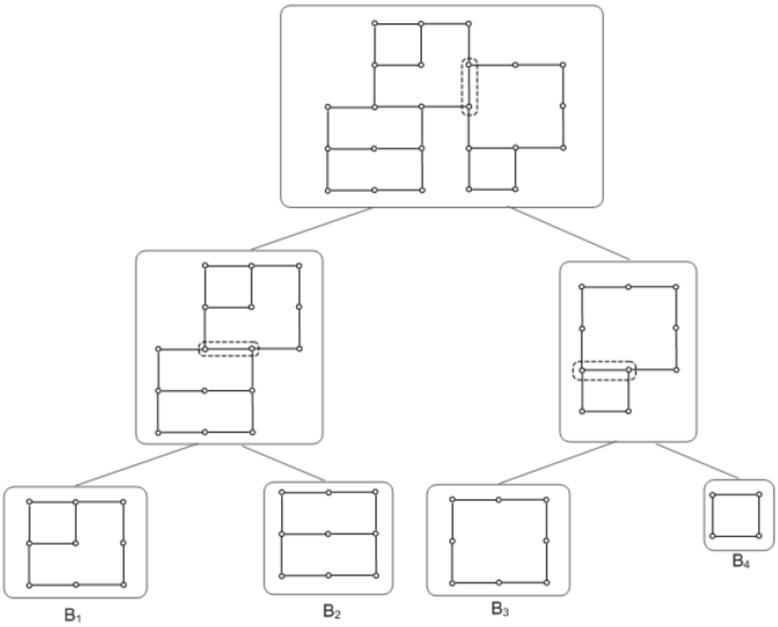

Figure 2: A decomposition tree with respect to clique 2-cutsets: B_1, B_2, B_3 and B_4 are the basic blocks of decomposition.

decomposition of graph H by cutset Y. Then Y is also a clique 2-cutset of G and at least one of $H_1, H_2, ..., H_p$ is a block of decomposition of graph G by cutset Y, contradicting the minimality of H. □

In Sections 3 and 4 we study two classes whose sets of basic graphs with respect to clique 2-cutset decompositions have useful properties, which we describe next. In the case of partial-grids of maximum degree at most 3 with bounded size of maximum induced cycle, we prove that the set of basic graphs is finite. This allows to prove that a specific colouring property – which we define in Section 3 – holds for each basic graph by simply exhibiting this colouring property in each of the basic graphs individually. In the case of outerplanar graphs, the basic graphs with respect to clique 2-cutset decompositions are cycle-graphs. Although the set of basic graphs is not finite, they compose a very structured class, in such a way that we can prove a specific colouring property – which we define in Section 4.

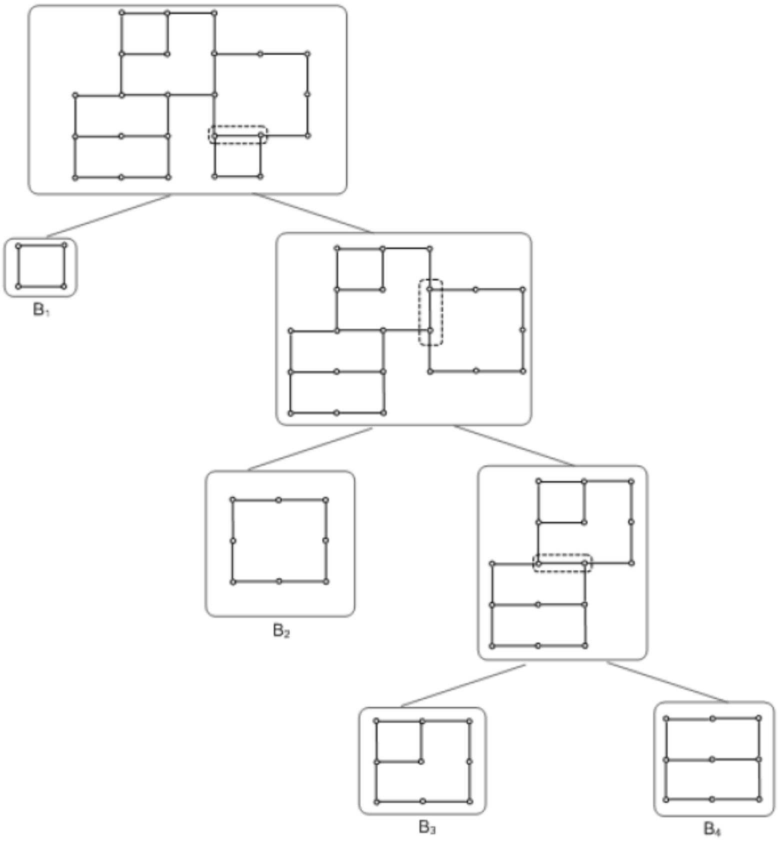

Figure 3: An extremal decomposition tree with respect to clique 2-cutsets constructed for the graph of Figure 2.

3 Partial-grids

A graph $G_{m\times n}$, where $m, n \geq 1$, with vertex set $V(G_{m\times n}) = \{1, ..., m\} \times \{1, ..., n\}$ and edge set $E(G_{m\times n}) = \{(i, j)(k, l) : |i - k| + |j - l| = 1, (i, j), (k, l) \in V(G_{m\times n})\}$, or a graph isomorphic to $G_{m\times n}$, is called a *grid*. A *partial-grid* is an arbitrary subgraph of a grid. Partial-grids are harder to work with than grids; for instance, recognition of grids is polynomial [7], whereas the problem is NP-complete for partial-grids [2, 12]. The total-colouring of partial-grids proved to be a challenging problem. While the partial-grids of maximum degree 1, 2 or 4 can be coloured by direct application of the total-colouring results for grids and cycles, the case of maximum degree 3 remains unfinished [8]. So, the last step towards a complete classification of partial-grids is to consider the remaining subcases of maximum degree 3.

A graph is *c-chordal* [10] if it does not have an induced cycle of size larger than c. The decomposition by clique 2-cutsets provides a method to total-colour subclasses of partial-grids where there exists a bound on the size of the maximum induced cycle. The applicability of our proposed decomposition comes from the fact that, for fixed c, the set of basic graphs with respect to the decomposition of c-chordal partial-grids by clique 2-cutsets is finite, as we show in Proposition 1. As a consequence, we obtain that the task of determining the total chromatic number of c-chordal partial-grids of maximum degree 3 is reduced to that of exhibiting suitable 4-total-colourings of a finite number of graphs.

Lemma 2 considers the clique 2-cutset decomposition of biconnected graphs with maximum degree 3.

Lemma 2. *Let G be a biconnected graph of maximum degree 3. If $X = \{u, v\}$ is a clique 2-cutset of G, then $G \setminus X$ contains exactly two connected components.*

Proof:

Suppose $G \setminus X$ has more than two connected components and denote by H_1, H_2 and H_3 three of these components. Since G is biconnected, each H_i, $i = 1, 2, 3$, has at least one vertex w_i adjacent to vertex u and one vertex w'_i adjacent to vertex v. So, each of the vertices u and v has at least four neighbors in G, which is a contradiction to the fact that G has maximum degree 3. □

As observed in Figure 1 of Section 2, the fact that the basic blocks of decomposition have a 4-total-colouring is **not** sufficient for a graph to be 4-total-colourable. So, we need a stronger colouring property for the basic blocks – a frontier-colouring, which we define next. A *frontier-pair* of a partial-grid of maximum degree 3 is a set of two adjacent vertices of degree 2. Let G be a partial-grid of maximum degree 3 and let $\{u, v\}$ be a frontier-pair of G. Denote u' (resp. v') the neighbor of u (resp. v) in $V(G) \setminus \{u, v\}$. Given a frontier pair $\{u, v\}$, we say that a 4-total-colouring π of G *satisfies* (u, v) if:

1. $\pi(u) = \pi(vv')$; and

2. $\pi(\{u'u, u, uv, v, vv'\}) = \{1, 2, 3, 4\}$.

If π satisfies (u, v) or π satisfies (v, u), then we say that π *satisfies* $\{u, v\}$ – please refer to Figure 4, where we exhibit the only two possible ways a 4-total-colouring may satisfy $\{u, v\}$.

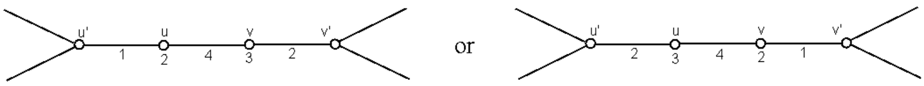

Figure 4: The two 4-total-colourings that satisfy $\{u, v\}$. The colouring on the left satisfies (u, v), while the colouring on the right satisfies (v, u).

We say that a 4-total-colouring of a graph G is a *frontier-colouring* if this colouring satisfies each frontier-pair of G. Remark that, at least for the purposes of the present section, a frontier-colouring is always a 4-total-colouring. So, when we say that a graph has a frontier-colouring, this implicitly means that this graph has a 4-total-colouring. In Section 4, we extend the concept of frontier-colouring by allowing the use of more than four colours. An important observation about frontier-colourings is the following "inversion" property:

Observation 1. *Let G be a biconnected graph of maximum degree 3, let π be a 4-total-colouring of G, and let $X = \{u, v\}$ be a clique 2-cutset of G which defines the X-components G_1 and G_2. The restriction $\pi|_{G_1}$ satisfies (u, v) if and only if the restriction $\pi|_{G_2}$ satisfies (v, u). Moreover, if G_1 has a frontier-colouring that satisfies (u, v) and G_2 has a frontier-colouring that satisfies (v, u), then G has a frontier-colouring.*

Proof:

Denote by u' (resp u'') the vertex in $V(G_1)\backslash X$ (resp. $V(G_2)\backslash X$) adjacent to u, and denote by v' (resp v'') the vertex in $V(G_1) \setminus X$ (resp. $V(G_2) \setminus X$) adjacent to v. Suppose that $\pi|_{G_1}$ satisfies (u, v) and assume, w.l.o.g., that the colours of elements $u'u$, u, uv, v and vv' are, respectively, 1, 2, 4, 3 and 2, as shown in Figure 5.

Clearly, in any such 4-total-colouring of G, the colours of edges uu'' and vv'' are 3 and 1, respectively, so that $\pi|_{G_2}$ satisfies (v, u). By symmetry, if $\pi|_{G_2}$ satisfies (v, u), then $\pi|_{G_1}$ satisfies (u, v).

Finally, if G_1 has a frontier-colouring that satisfies (u, v) and G_2 has a frontier-colouring that satisfies (v, u), then we can construct a frontier-colouring of G by choosing the frontier-colourings of G_1 and G_2 in such a way that the colours of u, v and uv agree. $\square$

Next, we prove the main result of the present section.

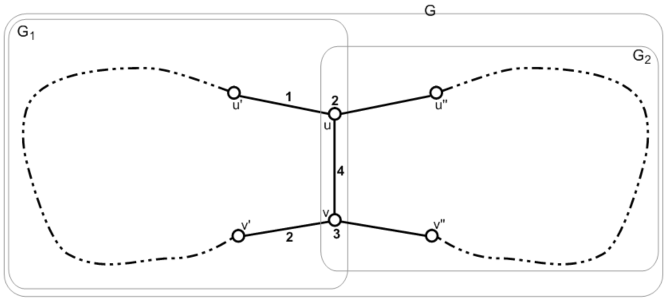

Figure 5: Inversion: the elements already coloured uniquely determine the colours of edges uu'' and vv''.

Theorem 1. *Let G be an 8-chordal partial-grid of maximum degree at most 3. Graph G is 4-total-colourable.*

Proof:

Actually we prove the stronger result that G has a 4-total-colouring which is a frontier-colouring. As we discuss in Section 2, we may assume that G is biconnected. Please refer to Figure 6 where we have depicted the six possible basic graphs $G_1, ..., G_6$ obtained from decomposing an 8-chordal partial-grid by clique 2-cutsets. In Figure 6 each of the six basic graphs G_i has two frontier-colourings $\pi_{i,a}$ and $\pi_{i,b}$ such that, for each frontier-pair $\{u, v\}$ of G_i, frontier-colouring $\pi_{i,a}$ satisfies (u, v) if and only if frontier-colouring $\pi_{i,b}$ satisfies (v, u).

If G is basic, then G has a frontier-colouring; in fact, as shown in Figure 6, each basic graph $G_1, ..., G_6$ has at least two frontier-colourings.

If G is not basic, then, by Lemmas 1 and 2, graph G has a clique 2-cutset $X = \{u, v\}$ with two blocks G' and B such that B is basic. Suppose, by induction hypothesis, that G' has a frontier-colouring. Let $\pi_{G'}$ be a frontier-colouring of G' and assume, w.l.o.g, that $\pi_{G'}$ satisfies (u, v). As exhibited in Figure 6, basic block B has a frontier-colouring π_B which satisfies (v, u). Moreover, we can choose π_B in such a way that $\pi_B(u) = \pi_{G'}(u)$, $\pi_B(uv) = \pi_{G'}(uv)$ and $\pi_B(v) = \pi_{G'}(v)$. A frontier-colouring π of G can be constructed as follows:

$$\pi(x) = \begin{cases} \pi_{G'}(x), \text{ if } x \in S(G'); \\ \pi_B(x), \text{ if } x \in S(B) \setminus S(G'). \square \end{cases}$$

Corollary 1. *Every 8-chordal partial-grid of maximum degree 3 is Type 1.*

Although Theorem 1 considers partial-grids with maximum induced cy-

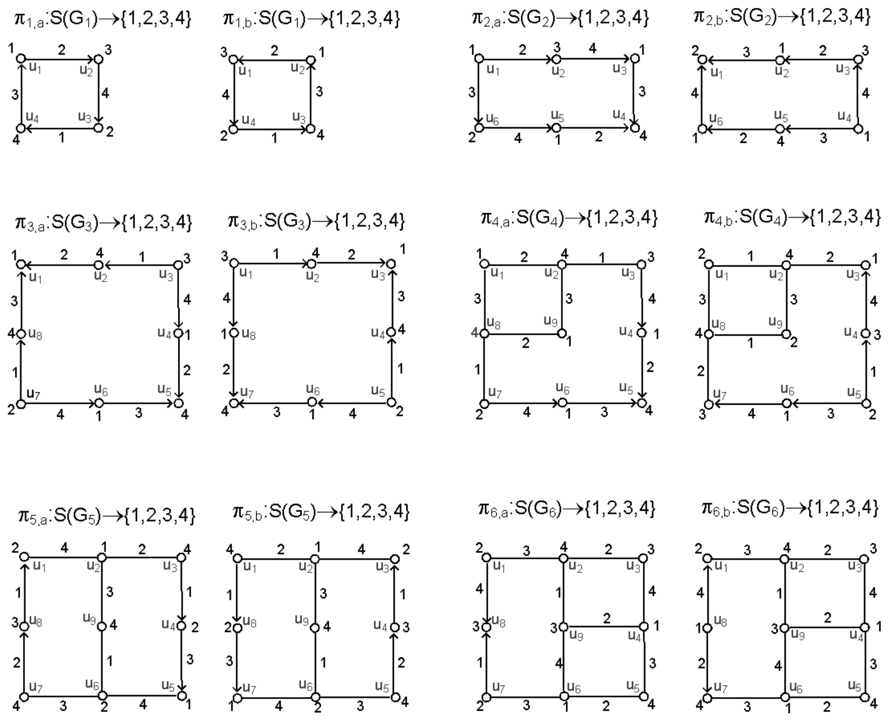

Figure 6: Total-colourings for the proof of Theorem 1. With the aim of indicating whether a frontier-colouring satisfies (u, v) or (v, u), each edge uv is represented by an arrow pointing to u if the colouring satisfies (u, v), and pointing to v if the colouring satisfies (v, u). Nevertheless, we emphasize that we are **not** dealing with directed graphs.

cle of size at most 8, it may be possible to extend this result to c-chordal partial-grids with larger bounds on the maximum induced cycles if one is capable of exhibiting, for larger values of c, frontier-colourings of the basic blocks, as done in Theorem 1. Since, for each fixed c, there is a finite number of these indecomposable partial-grids – as proved in Proposition 1 – the search for these colourings could be automatized by computer software.

Proposition 1. *For each fixed $c \geq 4$, there is a finite number of c-chordal partial-grids that do not have a clique 2-cutset.*

Proof:

Observe that a partial-grid G that does not have a clique 2-cutset is a biconnected planar graph. So, in any planar representation of G, there is a cycle C such that each vertex of $G \setminus C$ is in the "interior" – in the geometric sense – of C. We claim that C is induced, as if it has a chord, then this chord is a clique 2-cutset of G. Since G is c-chordal, cycle C has at most c vertices. Since each vertex of $G \setminus C$ is in the interior of C and the vertices of G have integer cartesian coordinates, G has a finite number $x \leq (1 + c/4)^2$ of vertices. The result follows from the fact that there is a finite number of partial-grids with a fixed number x of vertices. □

In the application of the clique 2-cutset decomposition to total-colour partial-grids, the restriction on the size of the maximum induced cycle has the consequence of limiting the set of basic graphs to a finite set. In Section 4 we consider a class whose basic blocks do not form a finite set. Nevertheless, the elements of this set are very restricted – namely, cycle-graphs.

4 Outerplanar graphs

A graph is *outerplanar* if it has a planar representation in which every vertex is in the external face [23]. Such representation, called *outerplanar representation*, is unique. The edges in the external face are called *external edges*, while the remaining edges are called *internal edges*. If an outerplanar graph is biconnected, then the boundary of the external face is a hamiltonian cycle called *external cycle*. An alternative characterization of outerplanar graphs can be given using the concept of homeomorphism. Given a graph G, a *subdivision* of edge $uv \in E(G)$ originates a graph G' such that $V(G') = V(G) \cup \{x\}$ and $E'(G) = (E(G) \setminus \{uv\}) \cup \{ux, xv\}$, where x is a new vertex. If graphs H and G can be obtained from the same graph by a series of subdivisions, then we say that G and H are *homeomorphic*.

In the present section, we prove a result on the list-total-colouring problem restricted to biconnected outerplanar graphs. We recall that the list-total-colouring problem is NP-complete for biconnected outerplanar graphs [27]. However, positive results are known when the lists satisfy the following conditions [27]: given a biconnected outerplanar graph G and a collection $\{L_x\}_{x \in S(G)}$ associated to the elements $S(G) = V(G) \cup E(G)$ of graph G such that $|L_{uw}| = \max\{deg(uw) + 1, 5\}$ for each edge uw and $|L_v| = \min\{5, \Delta + 1\}$ for each vertex v, graph G is total-colourable from the lists $\{L_x\}_{x \in S(G)}$. We prove a different result: a biconnected outerplanar graph G is still list-total-colourable if $|L_{uw}| = deg(uw) + 1$ for each edge uw and $|L_v| = 7 - \delta_{deg(v),3} - 2\delta_{deg(v),2}$ for each vertex v, that is, $|L_v| = 5$ if $deg(v) = 2$, $|L_v| = 6$ if $deg(v) = 3$, and $|L_v| = 7$ otherwise. We compare the proposed sufficient conditions with those of [27]: in these new conditions, the lists associated to the edges have, possibly, fewer colours, while the lists associated to the vertices have, possibly, more colours. The technique to prove this result is similar to that used in Section 3, as we total-colour a biconnected outerplanar graph by decomposing this graph by clique 2-cutsets and total-colouring each of the basic blocks. Observation 2 shows that the basic blocks of the decomposition of outerplanar graphs are cycle-graphs.

Observation 2. *Let G be a biconnected outerplanar graph. Either G has a clique 2-cutset or G is a cycle-graph.*

Proof:

Consider an outerplanar representation of G and denote by $(v_0, ..., v_k, v_0)$ the external cycle. Suppose that G is not a cycle-graph. Then there is an edge v_iv_j in G with $j > i$ and $|j - i| \neq 1 \mod k + 1$. Observe that there is no edge uw between a vertex $u \in \{v_0, v_1, ..., v_{i-1}, v_{j+1}, v_{j+2}, ..., v_k\}$ and a vertex $w \in \{v_{i+1}, v_{i+2}, ..., v_{j-2}, v_{j-1}\}$, because such an edge uw could not be an internal edge – because uw and v_iv_j would cross – nor an external edge – because u and w are not consecutive vertices of the external cycle. So, $\{v_i, v_j\}$ is a clique 2-cutset of G. □

In what follows, we redefine the concepts of free colours, frontier-pair and frontier-colouring in the context of list-total-colouring biconnected outerplanar graphs.

Free colours. Let G be a graph and (S_L, S_C) a partition of the elements $S(G) = V(G) \cup E(G)$ of graph G such that each element y in S_C is coloured $\pi(y)$ and to each element x in S_L it is associated a set of colours L_x. The set $F_\pi(z)$ of the *free colours* at an element $z \in S_L$ is the set of the colours in L_z which are not used by π in any of the elements in S_C incident or adjacent to z. This concept of free colour captures, in the context of list-total-colouring, the idea of describing the colours available to colour an element.

Frontier-pair and **frontier-colouring** of a biconnected outerplanar graph. Let H be a biconnected outerplanar graph and let G be a biconnected subgraph of H. We say that a pair $\{v_i, v_j\}$ of adjacent vertices of G is a *frontier-pair* if edge v_iv_j is an external edge of G, but it is an internal edge of H (see Figure 7). Denote by $(v_0, ..., v_k, v_0)$ the external cycle of G, where indices are taken modulo $k + 1$. Denote by v_i' (resp. v_i'') the neighbor of v_i in $V(H) \setminus V(G)$ which belongs to the same face $\mathcal{F}_1$ of H as v_{i-1} (resp. same face $\mathcal{F}_2$ of H as v_{i+1}), as shown in Figure 7. Suppose that to each element $x \in S(H) \setminus S(G)$ it is associated a set L_x of colours. We say that π is a *frontier-colouring* of G if, to each vertex $v_i \in \{v_0, ..., v_k\}$ at the external cycle of G, we can associate the colours $l_G(v_i)$ and $r_G(v_i)$ with the following properties (recall that indices are taken modulo $k + 1$).

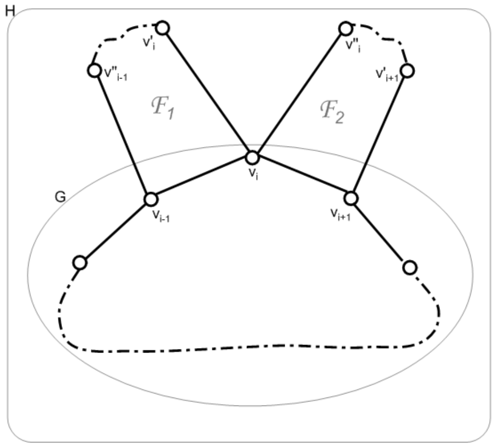

Figure 7: In the example, $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$ are frontier-pairs of G with respect to H.

1. If $\{v_i, v_{i-1}\}$ is a frontier-pair, then $l_G(v_i) \in F_\pi(v_i v_i')$ and we say that $l_G(v_i)$ is *defined*; otherwise, $l_G(v_i)$ is *undefined*.

2. If $\{v_i, v_{i+1}\}$ is a frontier-pair, then $r_G(v_i) \in F_\pi(v_i v_i'')$ and we say that $r_G(v_i)$ is *defined*; otherwise, $r_G(v_i)$ is *undefined*.

3. If both $l_G(v_i)$ and $r_G(v_i)$ are defined, then $l_G(v_i) \neq r_G(v_i)$. If both $r_G(v_{i-1})$ and $l_G(v_i)$ are defined, then $r_G(v_{i-1}) \neq l_G(v_i)$. If both $r_G(v_i)$ and $l_G(v_{i+1})$ are defined, then $r_G(v_i) \neq l_G(v_{i+1})$.

The definition of frontier-colouring captures the property of being possible to extend the total-colouring of a subgraph G by the addition of basic blocks. The free colours at edge $v_i v_i'$ (resp. $v_i v_i''$) are the colours in $L_{v_i v_i'}$ (resp. $L_{v_i v_i''}$) which are not used in v_i or any of its incident edges. Observe an example in Figure 8.

We prove the main result of the present section (we use the Kronecker delta: $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$).

Theorem 2. *Let G be a biconnected outerplanar graph and let $\{L_x\}_{x \in S(G)}$ be a collection of lists such that $|L_{uw}| = deg(uw) + 1$ for each edge uw and $|L_v| = 7 - \delta_{deg(v),3} - 2\delta_{deg(v),2}$ for each vertex v. Graph G can be total-coloured from $\{L_x\}_{x \in S(G)}$.*

Proof:

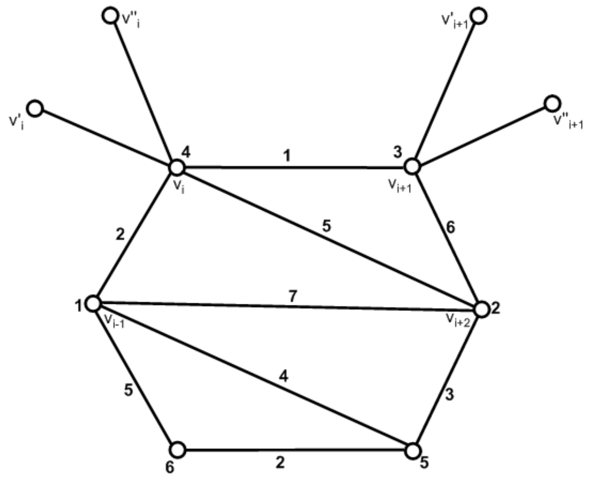

Figure 8: In the example, suppose that $L_{v_iv'_i} = \{1,2,3,6,7,8\}$, $L_{v_iv''_i} = \{2,3,5,6,7,8\}$, $L_{v_{i+1}v'_{i+1}} = \{1,2,3,4,5,6\}$, and $L_{v_{i+1}v''_{i+1}} = \{1,2,3,6,7,8\}$. If we denote by π the total-colouring shown in the figure, then $F_\pi(v_iv'_i) = \{3,6,7,8\}$, $F_\pi(v_iv''_i) = \{3,6,7,8\}$, $F_\pi(v_{i+1}v'_{i+1}) = \{2,4,5\}$, and $F_\pi(v_{i+1}v''_{i+1}) = \{2,7,8\}$. We can choose $l_G(v_i) = 7$, $r_G(v_i) = 6$, $l_G(v_{i+1}) = 5$, and $r_G(v_{i+1}) = 7$, so that π is a frontier-colouring.

We prove that every biconnected subgraph G' of G has a total-colouring from $\{L_x\}_{x\in S(G')}$ which is a frontier-colouring. The proof is by induction on the number of faces.

Let G_1 be a biconnected subgraph of G which has only one face, that is, G_1 is a chordless cycle $(w_0, ..., w_p, w_0)$. We show that G_1 has a total-colouring from $\{L_x\}_{x\in S(G_1)}$ which is a frontier-colouring, where the indices are taken modulo $p + 1$.

1. Colour the edges $w_0w_1, w_1w_2, ..., w_pw_0$ in any order using, at each step, a colour in $L_{w_iw_{i+1}}$ not used in $w_{i-1}w_i$ nor in $w_{i+1}w_{i+2}$.

2. Define, in any order and where it can be defined, the values $l_{G_1}(w_0)$, $r_{G_1}(w_0)$, $l_{G_1}(w_1)$, $r_{G_1}(w_1)$,..., $l_{G_1}(w_p)$, $r_{G_1}(w_p)$. At each step, let the colour of $l_{G_1}(w_i)$ be a colour in $L_{w_iw'_i}$ not used in $w_{i-1}w_i$ nor w_iw_{i+1} nor $r_{G_1}(w_{i-1})$ nor $r_{G_1}(w_i)$ (resp. let the colour of $r_{G_1}(w_i)$ be a colour in $L_{w_iw''_i}$ not used in $w_{i-1}w_i$ nor w_iw_{i+1} nor $l_{G_1}(w_i)$ nor $r_{G_1}(w_{i+1})$).

3. Colour the vertices $w_0, ..., w_p$ in any order. At each step, let the colour

of w_i be a colour in L_{w_i} not used in w_{i-1} nor w_{i+1} nor $w_{i-1}w_i$ nor w_iw_{i+1} nor $l_{G_1}(w_i)$ nor $r_{G_1}(w_i)$

Now, let G_k be a biconnected outerplanar subgraph of G with k faces and denote by $(v_0, ..., v_q, v_0)$ the external cycle of G_k. From now on, the indices are taken modulo $q + 1$. Let $(v_i, y_1, y_2, ..., y_t, v_{i+1}, v_i)$, for some $i \in \{0, 1, 2, ..., q\}$, be a cycle of G which determines a face not contained in G_k, and let $G_{k+1} = (V(G_k) \cup \{y_1, y_2, ..., y_t\}, E(G_k) \cup \{v_iy_1, y_1y_2, ..., y_tv_{i+1}\})$ (please refer to Figure 9). Assume, by induction hypothesis, that G_k has a total-colouring from $\{L_x\}_{x\in S(G_k)}$ which is a frontier-colouring. We prove that G_{k+1} has a total-colouring from $\{L_x\}_{x\in S(G_{k+1})}$ which is a frontier-colouring.

1. Let the colours of the original elements of G_k be the same.
2. Let the colour of v_iy_1 (resp. y_tv_{i+1}) be $r_{G_k}(v_i)$ (resp. $l_{G_k}(v_{i+1})$).
3. Define $l_{G_{k+1}}(v_i) = l_{G_k}(v_i)$, if $l_{G_k}(v_i)$ is defined.
4. Define $r_{G_{k+1}}(v_{i+1}) = r_{G_k}(v_{i+1})$, if $r_{G_k}(v_{i+1})$ is defined.
5. Define $r_{G_{k+1}}(v_i)$, if $r_{G_{k+1}}(v_i)$ can be defined, as some colour in $L_{v_iv_i''}$ not used in v_i nor $v_{i-1}v_i$ nor v_iv_{i+1} nor $l_{G_{k+1}}(v_i)$ nor v_iy_1.
6. Define $l_{G_{k+1}}(v_{i+1})$, if $l_{G_{k+1}}(v_{i+1})$ can be defined, as some colour in $L_{v_{i+1}v'_{i+1}}$ not used in v_{i+1} nor v_iv_{i+1} nor $v_{i+1}v_{i+2}$ nor y_tv_{i+1} nor $r_{G_{k+1}}(v_{i+1})$.
7. Colour the edges $y_1y_2, y_2y_3, ..., y_{t-1}y_t$ in any order.
8. Define, in any order and when they can be defined, the values $l_{G_{k+1}}(y_1)$, $r_{G_{k+1}}(y_1)$, $l_{G_{k+1}}(y_2)$, $r_{G_{k+1}}(y_2)$,..., $l_{G_{k+1}}(y_t)$, $r_{G_{k+1}}(y_t)$.
9. Colour the vertices $y_1, y_2, ..., y_t$ in any order.
10. Each of the other values $l_{G_{k+1}}$ and $r_{G_{k+1}}$ is the same as in G_k.

Then, we have a frontier-colouring of G_{k+1} and the result follows by induction. □

5 Final considerations

In the present work we investigate applications of graph decompositions to the total-colouring problem. Our basic approach is to recursively decompose a graph until getting a set of basic graphs. By solving the total-colouring

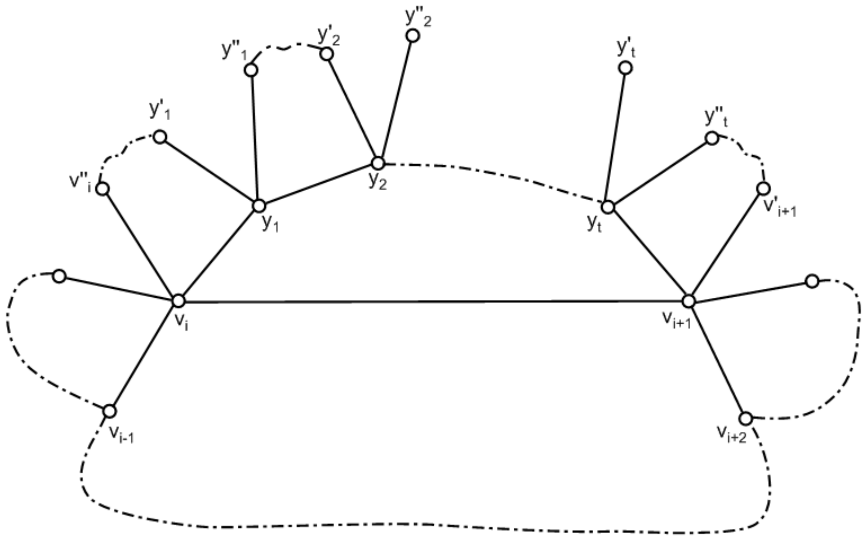

Figure 9: Construction of graph G_{k+1} by "adding one face".

on this set of basic graphs, we expect to be able to combine these solutions into a solution for the original graph. This approach is used to obtain total-colouring results for two classes of graphs, the 8-chordal partial-grids and the biconnected outerplanar graphs. In the present section, we give two final observations and describe our current work.

The first observation relates to the concept of a "structure result", which we briefly describe. A structure result characterizes a class of graphs by means of decomposition/composition results. The decomposition result states that every graph in a given class can be built starting from basic graphs and "gluing" them together by prescribed composition operations [18]. The composition result states that every graph constructed in this way is in the given class. Observe that Observation 2 is an example of a decomposition result, since it determines that the basic blocks of construction of outerplanar graphs are cycles. However, it is also possible to obtain a composition result for outerplanar graphs: just observe that the graph G obtained from two outerplanar graphs H_1 and H_2 by identifying an edge of H_1 with an edge of H_2 is also outerplanar. Since the focus of this paper is not characterization of classes by structure results, we leave the details of the proof to the interested reader.

The second observation refers to the existence of efficient algorithms to total-colour graphs in the classes investigated in the present work. The idea of frontier-colouring is defined in such a way that no recolouring of the already coloured elements is necessary. Such property is appropriate to the construction of efficient greedy algorithms to total-colour the classes investigated in the present work.

Current work. Currently, we are investigating whether the decomposition technique proposed in the present work can help to obtain new results for list-vertex-colouring and list-edge-colouring problems restricted to outerplanar graphs. With respect to total-colouring, we currently investigate the class of square-free graphs that do not contain a cycle with a unique chord. This class is studied in [18], where a strong structure result is proved and applied to vertex-colouring. The edge-colouring of this class is considered in [11] while the total-colouring is under research.

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References

[1] M. Behzad. *Graphs and their chromatic numbers.* Doctoral Thesis Michigan State University (1965).

[2] S. N. Bhatt and S. S. Cosmadakis. *The complexity of minimizing wire lengths in VLSI layouts.* Inform. Process. Lett. **25** (1987) 263–267.

[3] V. A. Bojarshinov. *Edge and total coloring of interval graphs.* Discrete Appl. Math. **114** (2001) 23–28.

[4] O. V. Borodin. *On the total colouring of planar graphs.* J. Reine Angew. Math. **394** (1989) 180-185.

[5] B.-L. Chen,H.-L. Fu, and M. T. Ko. *Total chromatic number and chromatic index of split graphs.* J. Combin. Math. Combin. Comp. **17** (1995) 137–146.

[6] C. H. de Figueiredo, J. Meidanis, C. P. de Mello. *Total-chromatic number and chromatic index of dually chordal graphs.* Inform. Process. Lett. **70** (1999) 147–152.

[7] G. Burosch and J.-M. Laborde. *Characterization of grid graphs.* Discrete Math. **87** (1991) 85-88.

[8] C. N. Campos, C. P. Mello. *The total chromatic number of some bipartite graphs.* Ars Combin. **88** (2008) 335–347.

[9] C. N. Campos. *O problema da coloração total em classes de grafos.* PhD Thesis – Unicamp (In Portuguese).

[10] D. G. Corneil, F. F. Dragan, and E. Köhler. *On the power of BFS to determine a graph's diameter.* Networks **42** (2003) 209-222.

[11] C. de Figueiredo, R. Machado, and K. Vušković. *Edge-colouring graphs with no cycle with unique chord.* Proceedings of ALIO/EURO Conference 2008.

[12] A. Gregori. *Unit length embedding of binary trees on a square grid.* Inform. Process. Lett. **31** (1989) 167–173.

[13] T. J. Hetherington, D. R. Woodall. *Edge and total choosability of near-outerplanar graphs.* Electron. J. Combin. **13** (2006) #R98.

[14] G. Li, L. Zhang. *Total chromatic number of one kind of join graphs.* Discrete Math. **306** (2006) 1895–1905.

[15] G. Li, L. Zhang. *Total chromatic number of the join of $K_{m,n}$ and C_n.* Chinese Quart. J. Math. **21** (2006) 264–270.

[16] C. J. H. McDiarmid, A. Sánchez-Arroyo. *Total colouring regular bipartite graphs is NP-hard.* Discrete Math. **124** (1994) 155-162.

[17] A. Sánchez-Arroyo. *Determining the total colouring number is NP-hard.* Discrete Math. **78** (1989) 315-319.

[18] N. Trotignon, K. Vušković. *A structure theorem for graphs with no cycle with a unique chord and its consequences.* To appear in J. Graph Theory (available online at http://www.comp.leeds.ac.uk/vuskovi/chord.ps).

[19] V. G. Vizing. *On an estimate of the chromatic class of a p-graph.* Diskret. Analiz **3** (1964) 25-30. In Russian.

[20] W. Weifan and K.-W. Lih. *Choosability, Edge Choosability, and Total Choosability of Outerplane Graphs.* Europ. J. Combinatorics **22** (2001), 71–78.

[21] S.-D. Wang and S.-C. Pang. *The determination of the total-chromatic number of series-parallel graphs.* Graphs Combin. **21** (2005) 531-540.

[22] W. Wang. *Total chromatic number of planar graphs with maximum degree ten.* J. Graph Theory **54** (2007) 91–102.

[23] D. B. West. *Introduction to graph theory, Second Edition.* Prentice Hall (2001).

[24] D. R. Woodall. *Total 4-choosability of series-parallel graphs.* Electron. J. Combin. **13** (2006) #R97.

[25] H. P. Yap. *Total colourings of graphs.* Lecture Notes in Mathematics. Springer-Verlag 1996. Germany.

[26] Z. Zhongfu, Z. Jianxue, and W. Jianfang. *The total chromatic number of some graphs.* Sci. Sinica Ser. A **31** (1988) 1434–1441.

[27] X. Zhou, Y. Matsuo, and T. Nishizeki. *List total colorings of series-parallel graphs.* J. Discrete Algorithms **3** (2005) 47–60

[28] B. Zmazek and J. Žerovnik. *Behzad-Vizing conjecture and Cartesian-product graphs.* Appl. Math. Lett. **15** (2002) 781–784.

Capítulo 11

Anexo: Manuscrito “Complexity dichotomy on degree-constrained VLSI layouts with unit-length edges”

Complexity dichotomy on degree-constrained VLSI layouts with unit-length edges

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Abstract. Deciding whether an arbitrary graph admits a VLSI layout with unit-length edges is NP-complete [1], even when restricted to binary trees [7]. However, for certain graphs, the problem is polynomial or even trivial. A natural step, outstanding thus far, was to provide a broader classification of graphs that make for polynomial or NP-complete instances. We provide such a classification based on the set of vertex degrees in the input graphs, yielding a comprehensive dichotomy on the complexity of the problem, with and without the restriction to trees.

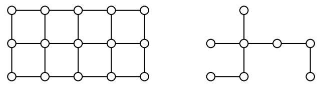
1 Introduction

A *grid* $G_{M\times N}$ has vertex set $V(G_{M\times N}) = \{(i,j) : 1 \leq i \leq M, 1 \leq j \leq N\}$, and edge set $E(G_{M\times N}) = \{(i,j)(k,l) : |i-k| + |j-l| = 1, (i,j), (k,l) \in V(G_{M\times N})\}$ (as illustrated in Figure 1(a)). A *partial grid* is any subgraph (not necessarily induced) of a grid. Grids and partial grids are planar bipartite graphs.

A VLSI layout is defined as a mapping from a graph's vertices to a subset of the points of a grid, along with an incidence-preserving assignment of edges to non-crossing paths in the grid. A partial grid can therefore be characterized as a graph that admits a VLSI layout with only unit-length edges. Two VLSI layouts (equivalently, *embeddings*) are equal if they correspond to the same drawing, short of rotation, translation and reflection. The same VLSI layout, though, might correspond to different assignments—or *grid mappings*—of vertices (respectively, edges) to points (resp. segments) in the grid. Grid embeddings are widely studied not only due to applications in VLSI design [9], but also because they describe efficient simulations of a given parallel architecture by another [8].

Deciding whether a graph admits a unit-length embedding is NP-complete [1], even when restricted to binary trees [7]. The so-called *logic engine paradigm* for proving the NP-hardness of problems in Graph Drawing is described in [4], where the seminal references [1, 7] and further applications [5, 6] are discussed. On the other hand, in the context of Graph Theory, the recognition of partial grids is often stated as an open problem [2, 3].

We consider the complexity dichotomy into polynomial and NP-complete for degree-constrained VLSI layouts with unit-length edges. Let D be a set of non-negative integers. We say a graph is a *D-graph* if the degrees of all its vertices



(a) (b)

Fig. 1. (a) The grid $G_{3,5}$. (b) Unit-length embedding for a {1,2,4}-tree.

are elements of D, e.g. a path is a {1,2}-graph, a cycle is a {2}-graph, a complete graph on n vertices is a $\{n-1\}$-graph etc.[4] (See Figure 1(b) for an example.)

This paper covers the UNIT-LENGTH VLSI problem (alternatively, PARTIAL-GRID RECOGNITION) when the input is restricted to D-graphs, for every possible set D the degrees of the input vertices may belong to. Since the only connected graph containing vertices of degree 0 is a singleton, and because graphs containing vertices of degree 5 or greater cannot possibly be embedded in a 2-dimensional, degree-4 grid, we are interested in the subsets of {1,2,3,4}.

Throughout the text, the term *immersibility* refers to a graph's ability, or the lack thereof, to be embedded in a grid with only unit-length edges. All graphs considered in this paper are connected.

2 Previous NP-completeness results

In [1], Bhatt and Cosmadakis proved that deciding the existence of unit-length embeddings for arbitrary trees is NP-complete. Their proof was based on the reduction of the well-known NP-complete problem NOT-ALL-EQUAL 3CNFSAT (not-all-equal conjunctive-normal-form satisfiability with 3 literals per clause) to the problem of deciding the existence of a unit-length embedding for a special tree called the *extended skeleton* (see Figure 2). This problem is referred to as the BHATT-COSMADAKIS problem.

$D = \{1, 2, 3\}$, $D = \{1, 2, 4\}$, $D = \{1, 2, 3, 4\}$

The seminal proof of Bhatt and Cosmadakis suffices to show that UNIT-LENGTH VLSI is NP-complete for {1,2,4}-trees, since the extended skeleton is itself a {1,2,4}-tree. As a consequence, it is also NP-complete for {1,2,3,4}-trees.[5]

The NP-completeness for {1,2,3}-trees (and consequently for binary trees) was demonstrated by Gregori [7], who conceived an ingenious {1,2,3}-tree called the *U-tree* as a suitable replacement structure with only one special type of unit-length embedding.

[4] Notice that a D-graph G is also a D'-graph, for $D' \supset D$, since it is not required that all elements of D actually appear as the degree of some vertex in G.

[5] We remark that, if the problem is NP-complete for D-trees, given a set D, then it is NP-complete for D-graphs (allowing cycles) and for D'-graphs, $D' \supset D$, as well.

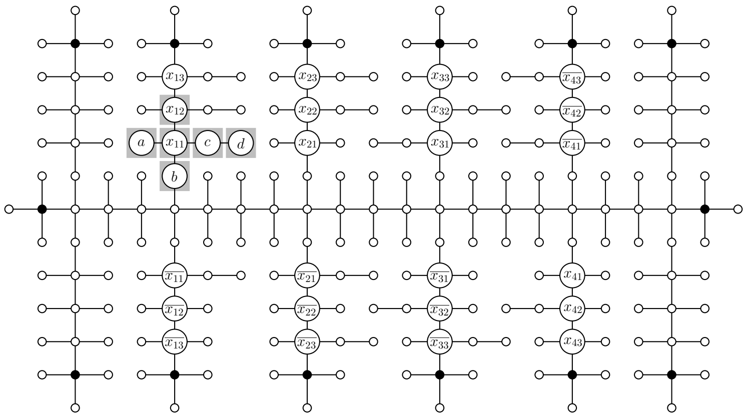


Fig. 2. Unit-length embedding for Bhatt and Cosmadakis's extended skeleton S_φ associated to the 3CNF formula $\varphi = (\overline{x_2} \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee \overline{x_3} \vee \overline{x_4})$.

3 New NP-completeness results

In the forthcoming proofs, we take for granted that Unit-Length VLSI belongs to NP, regardless of the restrictions imposed to its input. One can always check the validity of a given unit-length embedding in polynomial time.

$D = \{1, 3, 4\}$

We introduce a special {1,3,4}-tree called the *windmill tree*, which is shown in Figure 3(a) in a unit-length embedding. Vertices x, y, z, w, painted black in the figure, are called *interconnectors*. It is easy to see that such embedding is unique[6], although it corresponds to different grid mappings (in which appear all circular permutations of the interconnectors).[7]

Let G be a graph. We define the *windmill substitution* as the operation that obtains a graph $W(G)$ such that: (i) there is a bijection between vertices v in G and windmill trees $w(v)$ in $W(G)$; (ii) for each edge uv in G, there is an edge between an interconnector of $w(u)$ and an interconnector of $w(v)$ in $W(G)$, and we say $w(u)$ and $w(v)$ are adjacent. Such interconnectors, which may be chosen arbitrarily, become *active*. As an example, Figure 3(b) illustrates the result of a windmill substitution applied to the subgraph induced by vertex set $\{x_{11}, x_{12}, a, b, c, d\}$ in the extended skeleton (highlighted in Figure 2).

Lemma 1 *Windmill substitution preserves immersibility.*

[6] In reality, it is possible that degree-1 neighbors of interconnectors bend and fall outside the 5×5 grid square that envelops the interconnectors, but this is irrelevant.

[7] Each of the windmill "arms" (one of which is highlighted in Figure 3(a)) are independently tied to the windmill "axis"—its *center*—by an edge, making it possible that they interchange their positions.

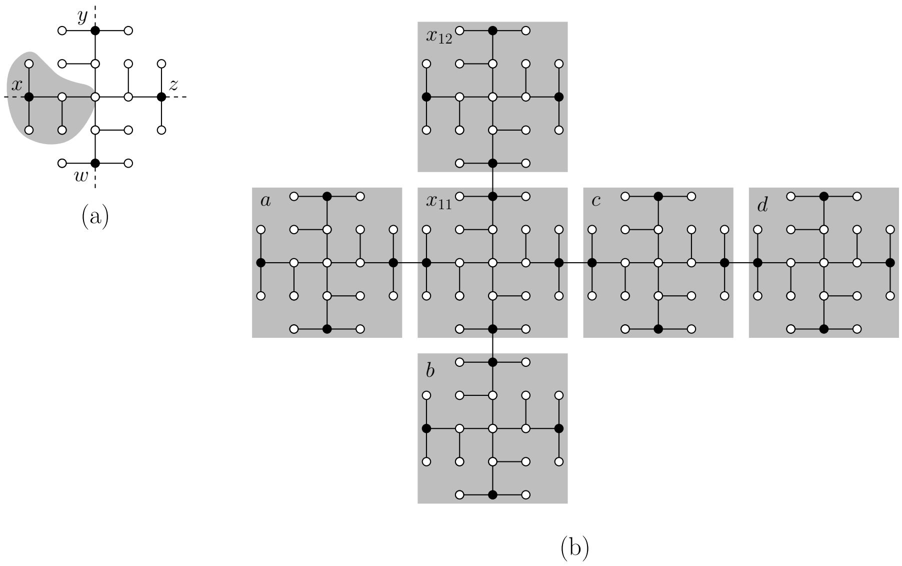


Fig. 3. (a) The {1,3,4} gadget (windmill tree). (b) Windmill substitution.

Proof. Let G be a connected graph. We want to show that G is a partial grid if and only if so is $W(G)$. Suppose $W(G)$ is a partial grid, and let Γ' be a unit-length embedding of $W(G)$. No matter how each windmill tree is embedded, the distance between the centers of two adjacent windmill trees will always be 5. Since G is connected, the distance between the centers of *any* two windmill trees in Γ' will always be, in both directions (vertical/horizontal), a multiple of 5. Thus, by substituting a single vertex v (placed at its center) for each windmill tree $w(v)$, and then depriving Γ' of all lines and columns other than those containing the centers, we get a new grid that is 5 times smaller (on each dimension). Now, by adding to the new grid an edge uv for every pair of adjacent windmill trees $w(u), w(v)$, we obtain a unit-length embedding Γ of G.

For the converse, suppose G can be embedded in an $M \times N$ grid using unit-length edges, and let Γ be such an embedding. Clearly, there will always be a unit-length embedding Γ' for $W(G)$ in a $5M \times 5N$ grid, where each vertex v at coordinate (i, j) in Γ corresponds to a windmill tree $w(v)$ spreading over a 5×5 square, in Γ', whose center has coordinates $(5i, 5j)$. As for the embedding of each windmill tree $w(v)$, it must be such that the positions of its active interconnectors match those of the windmill trees adjacent to $w(v)$—and this is never a problem, since all permutations of interconnectors are possible. □

Theorem 2 Unit-Length VLSI *is NP-complete for* {*1,3,4*}*-trees.*

Proof. Since Unit-Length VLSI is NP-complete for arbitrary trees (see Section 2) and, by Lemma 1, any tree can be polynomially transformed into a

{1,3,4}-tree with the same immersibility, Unit-Length VLSI is NP-complete when restricted to {1,3,4}-trees as well. □

$D = \{2, 3\}$, $D = \{2, 3, 4\}$

We start with a new definition. Let G be a graph. Say vertex $v \in G$ is adjacent to vertices s and t. If, in all unit-length embeddings of G, edges sv and vt can only appear as two consecutive segments of the same grid line (or column), we say we have a pair of *necessarily collinear* edges. Analogously, if sv and vt can only be embedded with a 90° angle between them, we say they are *necessarily orthogonal*. If there is at least one unit-length embedding for G in which sv and vt appear one way, and at least one unit-length embedding for G in which they appear the other way, we say they constitute a pair of *free-angle* edges.

In the graph of Figure 2, it is easy to see that edges ax_{11} and cx_{11} are necessarily collinear, whereas edges ax_{11} and bx_{11} are necessarily orthogonal, and all pairs of edges incident to a vertex painted black are free-angle.

In this section, we introduce a special {2,3}-graph called the *double ladder*. Figure 4(a) presents its only existing unit-length embedding. Vertices x, y, z, w are again regarded as interconnectors. Unlike the windmill tree, the double ladder only admits one circular ordering of the interconnectors in all its feasible embeddings.[8] Consequently, the pairs of *opposed* interconnectors (namely x, z and w, y) and of *consecutive* interconnectors (all other pairs) will always remain the same.

Let G be a graph. We define the *double-ladder substitution* as the operation that obtains the graph $L(G)$ such that: (i) there is a bijection between each vertex v in G and a double ladder $l(v)$ in $L(G)$; and (ii) there is a bijection between each edge uv in G and an edge linking an interconnector of $l(u)$ to an interconnector of $l(v)$ in $L(G)$. Figure 4(b) illustrates the result of a double-ladder substitution applied to that same highlighted subgraph in Figure 2.

The double-ladder substitution, though, does not necessarily preserve the immersibility of the original graph when the active interconnectors are chosen freely. The problem with structures like the double ladder, which present a fixed permutation of the interconnectors, is that they might not mimic the exact behavior of the original vertex they are meant to emulate. In other words, if a pair of opposed (respectively, consecutive) interconnectors of $l(v)$ are chosen to link it to $l(s)$ and $l(t)$ during the double-ladder substitution that originated $L(G)$, those graphs will *always* appear collinearly (resp. orthogonally) on every feasible unit-length embedding of $L(G)$, destroying the equivalence between the immersibility of G and that of $L(G)$ in case $sv, vt \in G$ are necessarily orthogonal (resp. collinear).

In order to preserve the immersibility of the original graph, it would be mandatory that the choice of active interconnectors matched the relative positions of all pairs of edges that are not free-angle. The only problem is, to locate

[8] The outer cycle—please see again Figure 4(a)—ties everything together, fixing the relative positions of the interconnectors.

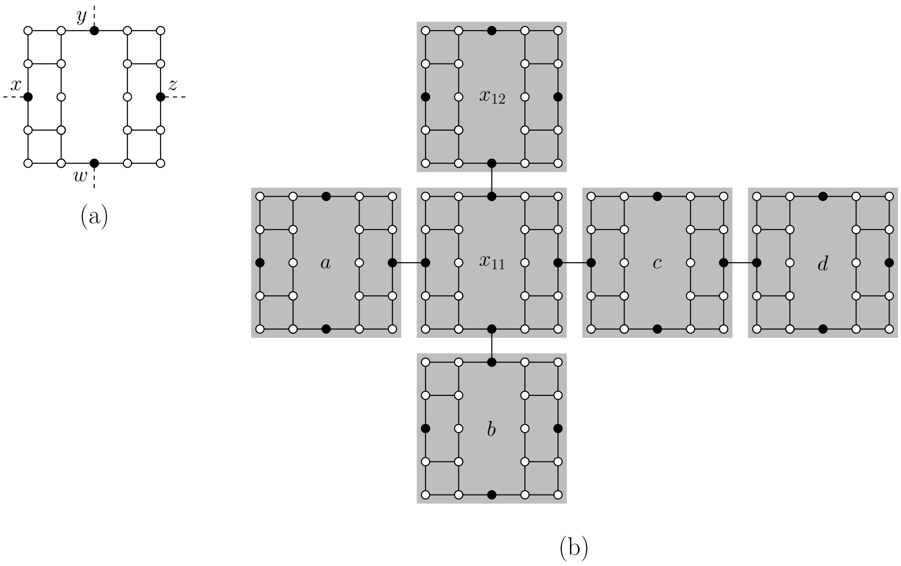


Fig. 4. (a) The {2,3} gadget (double ladder). (b) Double-ladder substitution.

all necessarily collinear and necessarily orthogonal pairs of edges is seldom easy. Fortunately, for extended skeletons, that *is* easy.

Lemma 3 *Double-ladder substitution—with appropriately chosen interconnectors—preserves the immersibility of extended skeletons.*

Proof. The point is, an extended skeleton is a rigid enough structure, so that all pairs of adjacent edges are either necessarily collinear or necessarily orthogonal. As a matter of fact, the absolute orientation (vertical/horizontal) of all edges in an extended skeleton S_φ is always the same in every feasible unit-length embedding of S_φ, and is known beforehand. It is exactly that in the embedding shown in Figure 2.[9] Thus, a double-ladder substitution on graph G, when connecting $l(v)$ to $l(s)$ and $l(t)$ (for $sv, vt \in G$), must select a pair of opposed interconnectors of $l(v)$ if sv, vt are necessarily collinear, a pair of consecutive interconnectors if they are necessarily orthogonal, and an arbitrary pair of interconnectors of $l(v)$ if sv, vt are free-angle.

The remainder of the proof is analogous to that for {1,3,4}-graphs. Since a double ladder occupies a perfect 5×5 grid square in any unit-length embedding, the placement of the double ladder graphs in a unit-length embedding for $L(S_\varphi)$ shall always be met by a feasible placement of S_φ's vertices on a grid that

[9] The sole exception is the orientation of those edges between a black vertex (still in Figure 2) and a vertex of degree 1, which are free to switch positions with one another. However, since they constitute free-angle pairs, they do not impose any restrictions on the choice of interconnectors.

is 5 times smaller. Edges linking one double ladder to another always occur between two adjacent 5×5 squares in the grid, therefore only edges of unit-length will be required, in the reduced grid. And since the interconnectors choice never disagrees with the orientation of the edges in a unit-length embedding of the extended skeleton, a unit-length embedding for an extended skeleton S_φ will always lead to a unit-length embedding (in a grid 5 times larger) for the associated graph $L(S_\varphi)$. □

Theorem 4 UNIT-LENGTH VLSI *is NP-complete for* {*2,3*}- *and* {*2,3,4*}*-graphs.*

Proof. Identical to the {1,3,4} case. By Lemma 3, BHATT-COSMADAKIS reduces to UNIT-LENGTH VLSI for {2,3}-graphs, so the latter problem is NP-complete. As $\{2,3,4\} \supset \{2,3\}$, the NP-completeness for {2,3,4}-graphs follows. □

The acyclic case does not apply, for there are no trees without leaves.

$D = \{1, 3\}$

To prove the NP-completeness of the problem for {1,3}-graphs, our strategy will be identical to that just seen for {2,3}-graphs. We introduce an appropriate gadget (one that is a {1,3}-graph, in this case) and an associated substitution procedure that preserves the immersibility of the extended skeleton.

The gadget we employ is the one shown in Figure 5(a). We call it the *brick wall* graph. Notice that its embedding, which occupies a 9×9 grid square, is unique, except for the possible rotations of the four edges at the corners. As usual, interconnectors are the labeled vertices in the figure.

We define the *brick-wall substitution* analogously to the double-ladder substitution, only replacing the double ladder with the brick wall. Because the interconnectors in the brick wall are degree-1 vertices, and we do not want the result of the brick-wall substitution to contain vertex degrees other than 1 and 3, we need a special way to bind adjacent brick walls, rather then simply linking them to one another by an edge. Figures 5(b) and 5(c) illustrate such bindings, which are also necessary to make the structure formed by two adjacent brick walls rigid, that is, one in which the centers are placed at the same row (or column), exactly 9 segments apart from each other, allowing for no vertical or horizontal play. The result of a brick-wall substitution on the usual subgraph of the extended skeleton is presented in Figure 5(d).

Lemma 5 *Brick-wall substitution—with appropriately chosen interconnectors—preserves the immersibility of extended skeletons.*

Proof. Just like in the double ladder, the interconnectors in the brick wall will always appear in the same circular permutation. This might have posed a problem to the desired immersibility preservation, were it not for the fact that the orientation of all necessarily collinear (or necessarily orthogonal) edges in an extended skeleton is unique and known. Indeed, with interconnectors chosen in the appropriate manner (see proof of Lemma 3), and because two adjacent brick

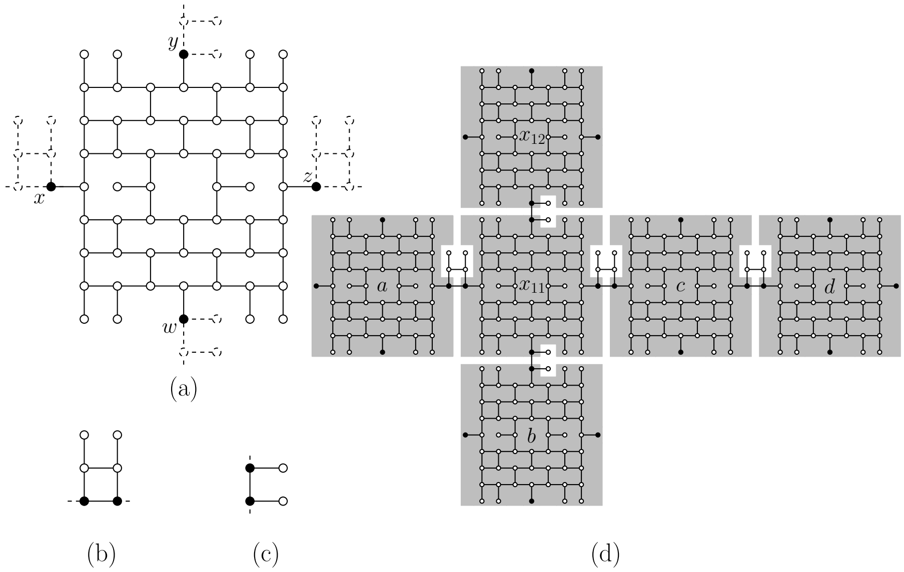


Fig. 5. (a) The {1,3} gadget (brick wall). (b) Horizontal binding. (c) Vertical binding. (d) Brick-wall substitution.

wall graphs shall occupy a rigid 9×18 rectangle, the graph $B(S_\varphi)$ resulting from a brick-wall substitution on an extended skeleton S_φ will admit a unit-length embedding if and only if S_φ does. □

Theorem 6 UNIT-LENGTH VLSI *is NP-complete for {1,3}-graphs.*

Proof. Same strategy, here. By Lemma 5, BHATT-COSMADAKIS reduces to UNIT-LENGTH VLSI for {1,3}-graphs, hence the latter problem is NP-complete. □

Although the NP-completeness of the {1,3,4} case follows (by the superset property), the proof based on the windmill substitution gives a stronger result, namely that UNIT-LENGTH VLSI is NP-complete for {1,3,4}-trees as well. As for the {1,3} case, the current proof cannot be leveraged to trees, once the brick wall graph is cyclic. It is therefore not known whether UNIT-LENGTH VLSI is still NP-complete when restricted to {1,3}-trees.

$D = \{2, 4\}$

The idea is still the same. We define a transformation that, given an extended skeleton S_φ, produces a {2,4}-graph $Q(S_\varphi)$ with the same immersibility.

The replacement structure we use is a simple C_4—shown in Figure 6(a)—whose vertices are regarded as interconnectors. Surprisingly, the C_4 shall replace

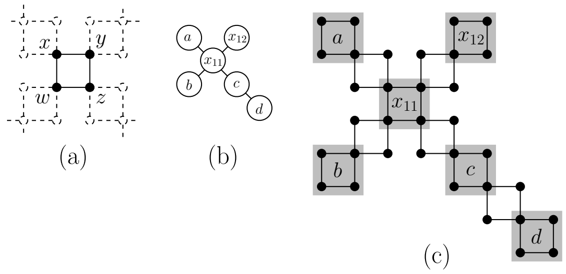


Fig. 6. (a) The {2,4} gadget (C_4). (b) Rotation of 45°. (c) Square substitution.

both vertices and edges of the original graph, in what we call the *square substitution.* Each vertex v of the original graph G gives rise to a C_4 $q(v)$ in the graph $Q(G)$ resulting from the square transformation. Each edge $uv \in G$ corresponds to another C_4, call it $q(uv)$, linking $q(v)$ to $q(u)$ using (diagonally) opposed interconnectors of $q(uv)$. Figure 6(c) shows the result of the square substitution applied to the highlighted subgraph in Figure 2. Notice that it looks as though the original graph has been rotated 45°, as depicted in Figure 6(b).

Lemma 7 *Square substitution—with appropriately chosen interconnectors—preserves the immersibility of extended skeletons.*

Proof. Despite the fixed circular permutation of the interconnectors of a C_4, the foreknowledge of the unique feasible orientation of the edges in all non-free-angle pairs of edges incident to a vertex v allows active interconnectors to be suitably chosen in $q(v)$ (pick two opposed interconnectors of $q(v)$ for necessarily collinear edges incident to v, consecutive interconnectors for necessarily orthogonal edges, an arbitrary pair otherwise). Let S_φ be an extended skeleton and let $Q(S_\varphi)$ be the result of some such "orientation-aware" square substitution. We want to prove that S_φ admits a unit-length embedding if and only if $Q(S_\varphi)$ does.

Suppose S_φ is a partial grid graph. Then, there is a unit-length embedding Γ for S_φ such that the relative position of every pair of edges $sv, vt \in S_\varphi$ matches the only relative position of $q(sv), q(vt)$ allowed by that particular choice of interconnectors of $q(v)$.[10] Now, it is always possible to obtain a unit-length embedding Γ' for $Q(S_\varphi)$ as follows. For each vertex v located at a grid point with coordinates (i, j) in Γ, place the topmost, leftmost vertex of $q(v)$ at $f(i, j) = (2i+2j, -2i+2j)$. Now place $q(uv)$, for every edge $uv \in S_\varphi$, at the only unit-area square that intersects both $q(u)$ and $q(v)$.

For the converse, suppose Γ' is a unit-length embedding for $Q(S_\varphi)$. We will show this implies the existence of a unit-length embedding Γ for S_φ. Without loss of generality, let the topmost vertex in the leftmost column of Γ be located at the grid's origin. The function $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ just defined is clearly bijective. Then,

[10] The existence of such Γ is guaranteed by the way interconnectors were chosen, that is, coping with all restrictions concerning necessarily collinear and orthogonal edges.

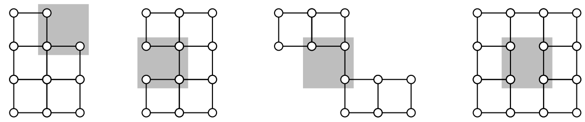


Fig. 7. Proof of Theorem 10: examples of incomplete unit-area squares σ present in connected, non-grid, partial grids.

for each C_4 $q(v)$ located at a unit-area square whose topmost, leftmost corner has coordinates (i, j), i, j even (for these are, by construction, the C_4 associated to vertices, not edges, of S_φ), place vertex v at coordinates $f^{-1}(i, j) = (\frac{i-j}{4}, \frac{i+j}{4})$ of an initially empty embedding Γ. Now link vertices u, v by a unitary segment, in Γ, if there is a C_4 in Γ' intersecting both $q(u)$ and $q(v)$, and Γ is clearly a unit-length embedding for S_φ. □

Theorem 8 UNIT-LENGTH VLSI *is NP-complete for {2,4}-graphs.*

Proof. By Lemma 7, BHATT-COSMADAKIS reduces to UNIT-LENGTH VLSI for {2,4}-graphs, hence the latter is NP-complete. □

Here again, since there are no trees without vertices of degree 1, the problem on {2,4}-graphs cannot be restricted to trees.

4 Polynomially decidable cases

$D = \{1\}$, $D = \{2\}$, $D = \{1, 2\}$

Trivial. The only connected {1}-graph—the P_2—admits a unit-length embedding. A path on n vertices can always be laid out on a straight line of a $1 \times n$ grid, and any even cycle on $2k$ vertices can be embedded on a $2 \times k$ grid. Odd cycles are not bipartite and therefore cannot be partial grids.

$D = \{3\}$, $D = \{4\}$, $D = \{3, 4\}$

Theorem 9 *No unit-length embedding exists for {3}-, {4}- or {3,4}-graphs.*

Proof. Suppose there is a unit-length embedding Γ for a graph with no vertices of degree 1 or 2. Let v be the topmost vertex in the leftmost column of Γ. Since all other vertices are placed below or to the right of v, v can have at most 2 neighbors, a contradiction. □

$D = \{1, 4\}$

Theorem 10 *A {1,4}-graph is a partial grid if and only if its degree-4 vertices induce a grid.* UNIT-LENGTH VLSI *is therefore polynomial for {1,4}-graphs.*

Proof. Let G be a {1,4}-graph. If the subgraph of G induced by all its vertices of degree 4 is a grid, then there is always a unit-length embedding for G, in which the degree-4 vertices occupy all points of an $M \times N$ rectangle, surrounded by the $2(M+N)$ degree-1 vertices, which are necessarily adjacent to the vertices in the boundaries of such rectangle.

Now, let Γ be a unit-length VLSI layout for a {1,4}-graph G, and let G' be the graph induced by all degree-4 vertices of G. Since G is a partial grid, G' is a partial grid as well. Moreover, G' must be connected, since G is itself connected and the vertices in $G \setminus G'$ have degree 1. Suppose, by contradiction, that G' is a connected partial grid that is not a grid graph (i.e. the image of its grid mapping does not correspond to all the points and segments of an $M \times N$ rectangle in the grid). This hypothesis implies the existence of some "incomplete" unit-area square σ (see Figure 7), in Γ, containing at least 2 but no more than 3 edges of G'. Without loss of generality, let $u, v \in G'$ be incident to two such edges and placed at the extremes of a diagonal of σ. Since u and v have degree 4 in G, the two other diagonally opposed corners of σ must correspond to vertices $s, t \in G$ which are necessarily adjacent to both u and v. Thus, the degree of s and t, in G, is at least 2, hence exactly 4, therefore s and t must belong to G' as well. As a result, σ contains 4 edges us, sv, vt, tu of G', a contradiction.

A polynomial-time recognition of grids can be achieved as follows. First, locate the 4 vertices of degree 2 and the $2(M+N)-4$ vertices of degree 3 present in the graph. They define the boundaries of an $M \times N$ rectangle in the grid. Now, recursively place each degree-4 vertex at the fourth point of a unit-area square already containing two of its neighbors (diagonally opposed in the grid) and one of its non-neighbors. Repeat this procedure inwardly, starting from the rectangle corners, until the vertices of degree 4 have matched the inner points of the rectangle (in which case the graph is a grid) or until such matching does not exist (in which case it is not). □

5 Conclusion and open problems

Please refer to Figure 8 for a summary of the dichotomy into polynomial and NP-complete for degree-constrained VLSI layouts with unit-length edges obtained in the present paper. Existing results are duly referenced.

Although NP-complete for {1,3}-graphs, it is still not known whether Unit-Length VLSI can be decided in polynomial time for acyclic inputs. That is to say that the {1,3} case is still a possible "separating entity", i.e. it may happen that, for $D = \{1,3\}$, the problem is NP-complete for D-graphs (in general) yet polynomial for D-trees.

Another question of theoretical interest concerns the existence of replacement D-graphs that always preserve immersibility. With the one exception of the windmill tree, the gadgets introduced herein, while sufficient for the intended proofs, do not guarantee that the immersibility of the original graph is preserved upon substitution when the relative positions of its edges are not known beforehand.

D	D-graphs	D-trees
{1}	P	P
{2}	P	—
{3}	P	—
{4}	P	—
{1,2}	P	P
{1,3}	NP-C	?
{1,4}	P	P
{2,3}	NP-C	—

D	D-graphs	D-trees
{2,4}	NP-C	—
{3,4}	P	—
{1,2,3}	NP-C [7]	NP-C [7]
{1,2,4}	NP-C [1]	NP-C [1]
{1,3,4}	NP-C	NP-C
{2,3,4}	NP-C	—
{1,2,3,4}	NP-C [1]	NP-C [1]

Fig. 8. Complexity dichotomy ("NP-C": NP-complete; "P": polynomial; "—": the corresponding input does not exist; "?": the remaining open case).

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References

1. S. N. Bhatt, S. S. Cosmadakis, The complexity of minimizing wire lengths in VLSI layouts. Inform. Process. Lett. 25 (1987), 263–267.
2. A. Brandstädt, V.B. Le, T. Szymczak, F. Siegemund, H.N. de Ridder, S. Knorr, M. Rzehak, M. Mowitz, N. Ryabova, U. Nagel. Information system on graph class inclusions. WWW document at `http://wwwteo.informatik.uni-rostock.de/isgci/about.html`, 2002.
3. C. N. Campos, C. P. Mello, The total chromatic number of some bipartite graphs. Proc. 7th International Colloquium on Graph Theory, 557–561, Electron. Notes Discrete Math., 22, 2005. Full paper in Ars Combin. 88 (2008), 335–347.
4. G. Di Battista, P. Eades, R. Tamassia, I. G. Tollis, Graph Drawing – Algorithms for the Visualization of Graphs (Prentice Hall, 1999).
5. P. Eades, S. Whitesides, The realization problem for Euclidean minimum spanning trees is NP-hard. Algorithmica 16 (1996), 60–82.
6. P. Eades, S. Whitesides, The logic engine and the realization problem for nearest neighbor graphs. Theoret. Comput. Sci. 169 (1996), 23–37.
7. A. Gregori, Unit-length embedding of binary trees on a square grid. Inform. Process. Lett. 31 (1989), 167–173.
8. Y.-B. Lin, Z. Miller, M. Perkel, D. Pritikin, I. H. Sudborough, Expansion of layouts of complete binary trees into grids. Discrete Appl. Math. 131 (2003), 611–642
9. J. D. Ullman, Computational Aspects of VLSI (Addison-Wesley, 1984).

Capítulo 12

Anexo: Manuscrito “Chromatic index of graphs with no cycle with a unique chord”

Chromatic index of graphs with no cycle with a unique chord

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Abstract

The class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord was recently studied by Trotignon and Vušković [23], who proved for these graphs strong structure results which led to solving the recognition and vertex-colouring problems in polynomial time. In the present paper we investigate how these structure results can be applied to solve the edge-colouring problem in the class. We give computational complexity results for the edge-colouring problem restricted to $\mathcal{C}$ and to the subclass $\mathcal{C}'$ composed of the graphs of $\mathcal{C}$ that do not have a 4-hole. We show that it is NP-complete to determine whether the chromatic index of a graph is equal to its maximum degree when the input is restricted to regular graphs of $\mathcal{C}$ with fixed degree $\Delta \geq 3$. For the subclass $\mathcal{C}'$, we establish a dichotomy: if the maximum degree is $\Delta = 3$, the edge-colouring problem is NP-complete, whereas, if $\Delta \neq 3$, the only graphs for which the chromatic index exceeds the maximum degree are the odd holes and the odd order complete graphs, a characterization that solves edge-colouring problem in polynomial time. We determine two subclasses of graphs in $\mathcal{C}'$ of maximum degree 3 for which edge-colouring is polynomial. Finally, we remark that a consequence of one of our proofs is that edge-colouring in NP-complete for r-regular tripartite graphs of degree $\Delta \geq 3$, for $r \geq 3$.

Keywords: cycle with a unique chord, decomposition, recognition, Petersen graph, Heawood graph, edge-colouring.

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1 Motivation

Let $G = (V, E)$ be a simple graph. The degree of a vertex v in G is denoted $deg_G(v)$, and the maximum degree of a vertex in G is denoted $\Delta(G)$. An *edge-colouring* of G is a function $\pi : E \rightarrow \mathbf{C}$ such that no two adjacent edges receive the same colour $c \in \mathbf{C}$. If $\mathbf{C} = \{1, 2, ..., k\}$, we say that π is a *k-edge-colouring*. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-colouring.

Vizing's theorem [24] states that $\chi'(G) = \Delta(G)$ or $\Delta(G)+1$, defining the *classification problem*: graphs with $\chi'(G) = \Delta(G)$ are said to be *Class 1*, while graphs with $\chi'(G) = \Delta(G) + 1$ are said to be *Class 2*. The *edge-colouring problem* or *chromatic index problem* is the problem of determining the chromatic index of a graph. Edge-colouring is a challenging topic in graph theory and the complexity of the problem is unknown for several important well studied classes. Edge-colouring is NP-complete for regular graphs [13, 17] of degree $\Delta \geq 3$. The problem is NP-complete also for the following classes [5]:

- r-regular comparability (hence perfect) graphs, for $r \geq 3$;
- r-regular line graphs of bipartite graphs (hence line graphs and clique graphs), for $r \geq 3$;
- r-regular k-hole-free graphs, for $r \geq 3$, $k \geq 3$;
- cubic graphs of girth k, for $k \geq 4$.

Graph classes for which edge-colouring is polynomially solvable include the following:

- bipartite graphs [14];
- split-indifference graphs [19];
- series-parallel graphs (hence outerplanar) [14];
- k-outerplanar graphs [2], for $k \geq 1$.

The complexity of edge-colouring is unknown for several well-studied strong structured graph classes, for which only partial results have been reported, such as cographs [1], join graphs [10, 11, 18], cobipartite graphs [18], planar graphs [21, 25], chordal graphs, and several subclasses of chordal graphs such as indifference graphs [8], split graphs [7] and interval graphs [3].

Given a graph F, we say that a graph G *contains* F if graph F is isomorphic to an induced subgraph of G. A graph G is *F-free* if G does not contain F. A *cycle* C in a graph G is a sequence of vertices $v_1 v_2 \ldots v_n v_1$, that are distinct except for the first and the last vertex, such that for $i = 1, \ldots, n-1$, $v_i v_{i+1}$ is an edge and $v_n v_1$ is an edge – we call these edges *the edges of C*. An edge of G with both endvertices in a cycle C is called a *chord* of C if it is not an edge of C. One can similarly define a path and a chord of a path.

A *hole* is a chordless cycle of length at least four and an ℓ*-hole* is a hole of length ℓ. A *triangle* is a cycle of length 3 and a *square* is a 4-hole.

Trotignon and Vušković [23] studied the class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord. The main motivation to investigate this class was to find a structure theorem for it, a kind of result which is not very frequent in the literature. Basically, this structure result states that every graph in $\mathcal{C}$ can be built starting from a restricted set of basic graphs and applying a series of known "gluing" operations. Another interesting property of this class is that it belongs to the family of the χ-bounded graphs, introduced by Gyárfás [12] as a natural extension of perfect graphs. A family of graphs $\mathcal{G}$ is χ*-bounded* with χ*-binding function* f if, for every induced subgraph G' of $G \in \mathcal{G}$, $\chi(G') \leq f(\omega(G'))$, where $\chi(G')$ denotes the chromatic number of G' and $\omega(G')$ denotes the size of a maximum clique in G'. The research in this area is mainly devoted to understanding for what choices of forbidden induced subgraphs, the resulting family of graphs is χ-bounded, see [20] for a survey. Note that perfect graphs are a χ-bounded family of graphs with χ-binding function $f(x) = x$, and perfect graphs are characterized by excluding odd holes and their complements. Also, by Vizing's Theorem, the class of line graphs of simple graphs is a χ-bounded family with χ-binding function $f(x) = x + 1$ (this special upper bound is known as the *Vizing bound*) and line graphs are characterized by nine forbidden induced subgraphs [26]. The class $\mathcal{C}$ is also χ-bounded with the Vizing bound [23]. Also in [23] the following results are obtained for graphs in $\mathcal{C}$: an $\mathcal{O}(nm)$ algorithm for optimal vertex-colouring, an $\mathcal{O}(n + m)$ algorithm for maximum clique, an $\mathcal{O}(nm)$ recognition algorithm, and the NP-completeness of the maximum stable set problem.

In the present paper we consider the complexity of determining the chromatic index of graphs in $\mathcal{C}$. In particular, we investigate how structure results can be used to solve the edge-colouring problem. We also investigate the subclasses obtained from $\mathcal{C}$ by forbidding 4-holes and/or 6-holes. Tables 1 and 2 summarize the main results achieved in the present work.

The results of Tables 1 and 2 show that, even for graph classes with strong structure and powerful decompositions, the edge-colouring problem may be difficult.

The class initially investigated in this work is the class $\mathcal{C}$ of graphs with no cycle with a unique chord. Each non-basic graph in this class can be decomposed [23] by special cutsets: 1-cutsets, proper 2-cutsets or proper

Class	$\Delta = 3$	$\Delta \geq 4$	regular
graphs of $\mathcal{C}$	NP-complete	NP-complete	NP-complete
4-hole-free graphs of $\mathcal{C}$	NP-complete	Polynomial	Polynomial
6-hole-free graphs of $\mathcal{C}$	NP-complete	NP-complete	NP-complete
{4-hole,6-hole}-free graphs of $\mathcal{C}$	Polynomial	Polynomial	Polynomial

Table 1: Complexity dichotomy for edge-colouring in the class of graphs with no cycle with a unique chord.

Class	$k \leq 2$	$k \geq 3$
k-partite graphs	Polynomial	NP-complete

Table 2: Complexity dichotomy for edge-colouring in the class of multipartite graphs.

1-joins. We prove that edge-colouring is NP-complete for graphs in $\mathcal{C}$. We consider, then, a subclass $\mathcal{C}' \subset \mathcal{C}$ whose graphs are the graphs in $\mathcal{C}$ that do not have a 4-hole. By forbidding 4-holes we avoid decompositions by joins, which are difficult to deal with in edge-colouring [1, 10, 11]. That is, each non-basic graph in $\mathcal{C}'$ can be decomposed of 1-cutsets and proper 2-cutsets. For this class $\mathcal{C}'$ we establish a dichotomy: edge-colouring is NP-complete for graphs in $\mathcal{C}'$ with maximum degree 3 and polynomial for graphs in $\mathcal{C}'$ with maximum degree **not** 3. We determine also a necessary condition for a graph $G \in \mathcal{C}'$ of maximum degree 3 to be Class 2. This condition is having graph P^* – a subgraph of the Petersen graph – as a basic block in the decomposition tree. As a consequence, if both 4-holes and 6-holes are forbidden, the chromatic index of graphs with no cycle with a unique chord can be determined in polynomial time. The results achieved in this work have connections with other areas of research in edge-colouring, as we describe in the following three observations.

The first observation refers to the complexity dichotomy result found for class $\mathcal{C}'$. This dichotomy presents great interest since, to the best of our knowledge, this is the first class for which edge-colouring is NP-complete for graphs with a given fixed maximum degree Δ and is polynomial for graphs with maximum degree $\Delta' > \Delta$, as the reader may verify in the NP-completeness results reviewed in the beginning of the present section. It is interesting to observe that the only regular graphs in $\mathcal{C}'$ are the Petersen graph, the Heawood graph, the complete graphs, and the holes. As a consequence, edge-colouring is NP-complete when restricted to $\mathcal{C}'$, but polynomial when restricted to **regular** graphs in $\mathcal{C}'$.

The second observation is related to the study of *snarks* [22]. A *snark* is a cubic bridgeless graph with chromatic index 4. In order to avoid trivial (easy) cases, snarks are commonly restricted to have girth 5 or more and not to contain three edges whose deletion results in a disconnected graph, each of whose components is non-trivial. The study of snarks is closely related to the Four Colour Theorem. By the result of Lemma 8, the only **non-trivial** snark which has **no** cycle with a unique chord is the Petersen graph.

Finally, the third observation refers to the problem of determining the chromatic index of a k-partite graph, that is, a graph whose vertices can be partitioned into k stable sets. The problem is known to be polynomial [14, 16] for $k = 2$ and for complete multipartite graphs. However, there is no explicit result in the literature regarding the complexity of determining the chromatic index of a k-partite graph for $k \geq 3$. From the proof of Theorem 2 we can observe that edge-colouring is NP-complete for k-partite r-regular graphs, for each $k \geq 3$, $r \geq 3$.

The remainder of the paper is organized as follows. In Section 2, we prove NP-completeness results regarding edge-colouring in the classes $\mathcal{C}$ and $\mathcal{C}'$. In Section 3, we review known results on the structure of graphs in $\mathcal{C}$ and obtain stronger structure results for graphs in $\mathcal{C}'$. In Section 4 we show how to determine in polynomial time the chromatic index of a graph in $\mathcal{C}'$ with maximum degree $\Delta \geq 4$. In Section 5 we further investigate graphs in $\mathcal{C}'$ with maximum degree 3: we show that edge-colouring can be solved in polynomial time if the inputs are restricted to regular graphs of $\mathcal{C}'$ and to 6-hole-free graphs of $\mathcal{C}'$.

2 NP-completeness results

In this section, we state NP-completeness results on the edge-colouring problem restricted to the class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord and to the class $\mathcal{C}'$ composed of the graphs in $\mathcal{C}$ that do not contain a 4-hole. First, we prove that edge-colouring is NP-complete for regular graphs of $\mathcal{C}$ with fixed degree $\Delta \geq 3$. We observe that it can be shown that the construction of Cai and Ellis [5] which proves the NP-completeness of r-regular k-hole-free graphs, for $r \geq 3$ and $k \neq 4$, creates a graph with no cycle with a unique chord. Nevertheless, in the present section, we give a simpler construction. Second, we prove that edge-colouring is NP-complete for graphs in $\mathcal{C}'$ with maximum degree $\Delta = 3$. For the proof of this second result, we construct a replacement graph which is not present in any edge-colouring NP-completeness proof we could find in the literature.

We use the term CHRIND(P) to denote the problem of determining the chromatic index restricted to graph inputs with property P. For example, CHRIND(graph of $\mathcal{C}$) denotes the following problem:

INSTANCE: a graph G of $\mathcal{C}$.

QUESTION: is $\chi'(G) = \Delta(G)$?

The following theorem [13, 17] establishes the NP-completeness of determining the chromatic index of Δ-regular graphs of fixed degree Δ at least 3:

Theorem 1. *([13, 17]) For each $\Delta \geq 3$, CHRIND(Δ-regular graph) is NP-complete.*

Please refer to Figure 1. Graph Q_n, for $n \geq 3$, is obtained from the complete bipartite graph $K_{n,n}$ by removing an edge xy, by adding new pendant vertices u and v, and by adding pendant edges ux and vy. Graph Q'_n is obtained from Q_n by identifying vertices u and v into a vertex w. Observe that Q'_n is a graph of maximum degree n, and has $2n+1$ vertices and n^2+1 edges. So, Q'_n is overfull and, hence [9], Class 2. Lemma 1

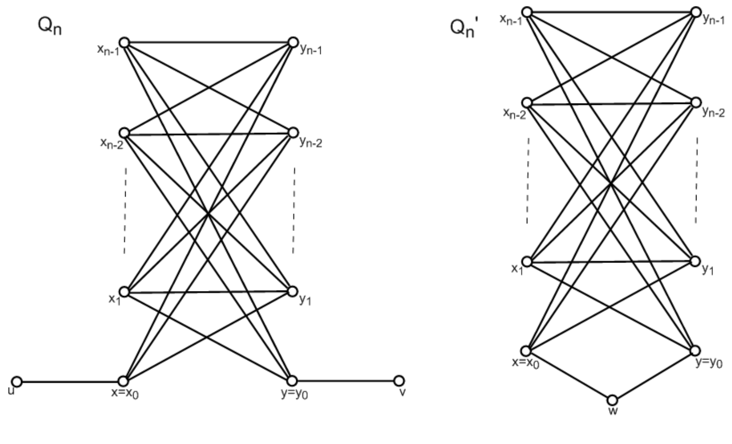

Figure 1: NP-complete gadget Q_n and graph Q'_n.

investigates the properties of graph Q_n, which is used as "gadget" in the NP-completeness proof of Theorem 2.

Lemma 1. *Graph Q_n is n-edge-colourable, and in any n-edge-colouring of Q_n, edges ux and vy receive the same colour.*

Proof:

We use the notation from Figure 1. First, we exhibit an n-edge-colouring of Q_n. Denote by $x_0, ..., x_{n-1}$ (resp. $y_0, ..., y_{n-1}$) the vertices of Q_n which belong to the same partition as x (resp. y), where $x = x_0$ (resp. $y = y_0$). An n-edge-colouring of Q_n is constructed as follows: just let the colour of edge x_iy_j be $(i+j \mod n)+1$ and let the colour of edges x_0u and y_0v be 1.

Now we prove that, in any n-edge-colouring of Q_n, edges ux and vy have the same colour. Suppose there is an n-edge-colouring π of Q_n where ux and vy have different colours. Consider the graph $Q'_n = (V', E')$ obtained from Q_n, by contracting vertices u and v into vertex w. Then we can construct an n-edge-colouring π' of Q'_n by setting $\pi'(e) = \pi(e)$ if $e \in E' \setminus \{ux, vy\}$, $\pi'(wx) = \pi(ux)$ and $\pi'(wy) = \pi(vy)$, which is a contradiction to the fact that Q'_n is Class 2. □

We prove in Theorem 2 the NP-completeness of edge-colouring regular graphs that do not contain a cycle with a unique chord for each fixed degree $\Delta \geq 3$.

Theorem 2. *For each $\Delta \geq 3$, `CHRIND`(Δ-regular graph in $\mathcal{C}$) is NP-complete.*

Proof:

Let $G = (V, E)$ be an input of the NP-complete problem `CHRIND`(Δ-regular graph). Now, let G' be the graph obtained from G by removing each edge $pq \in E$ and adding a copy of Q_Δ, identifying vertices u and v of Q_Δ with vertices p and q of G. For each edge pq of G, denote H_{pq} the subgraph of G' isomorphic to Q_Δ whose pendant vertices are p and q. Observe that G' is also Δ-regular.

Claim 1: G' can be constructed in polynomial time from G. In fact, we make one substitution – by a copy of Q_Δ – for each edge of G, so that the construction time is linear on the number of edges of G.

Claim 2: if G is Δ-edge-colourable, then so is G'. Let π be a Δ-edge-colouring of G. We construct a Δ-edge-colouring π' of G' in the following way: for each edge pq of G, let the edges of H_{pq} in G' be coloured in such a way that the pendant edges have the colour $\pi(pq)$ – this colouring exists and is described by Lemma 1.

Claim 3: if G' is Δ-edge-colourable, then so is G. Let π' be a Δ-edge-colouring of G'. We construct a Δ-edge-colouring π of G as follows: let the colour in π of each edge pq of G be equal to the colour in π' of the pendant edges of H_{pq} (by Lemma 1, these two pendant edges must receive the same colour).

Claim 4: $G' \in \mathcal{C}$. Suppose G' has a cycle C with a unique chord $\alpha\beta$. Observe that, by construction, every edge of G' – and, in particular, chord $\alpha\beta$ – has both endvertices in the same copy of Q_Δ. Denote by $H_{p'q'}$ this copy and observe that cycle C, when restricted to $H_{p'q'}$, is a path between p' and q', and that $\alpha\beta$ is a unique chord of this path. But there is no path with a unique chord between the pendant vertices of Q_Δ, so that we have a contradiction. □

Observe that graph G' in the proof of Theorem 2 is tripartite with vertex tripartition (P_1, P_2, P_3) determined as follows:

- P_1 is the set whose elements are the original vertices of G and the vertices denoted $y_1, ..., y_\Delta$ in each copy of Q_Δ;
- P_2 is the set whose elements are the vertices denoted x_0 and y_0 in each copy of Q_Δ;
- P_3 is the set whose elements are the vertices denoted $x_1, ..., x_\Delta$ in each copy of Q_Δ.

So, the following result holds:

Theorem 3. *For each $k \geq 3, \Delta \geq 3$, `CHRIND`(Δ-regular k-partite graph) is NP-complete.*

We emphasize that $\mathcal{C}$ is a class with strong structure [23], yet, it is NP-complete for edge-colouring. We manage in Section 4 to define a subclass of $\mathcal{C}$ where edge-colouring is solvable in polynomial time. Consider the class $\mathcal{C}'$ as the subset of the graphs of $\mathcal{C}$ that do not contain a square. The structure of graphs in $\mathcal{C}'$ is stronger than that of graphs in $\mathcal{C}$, and is described in detail in Section 3. Yet, the edge-colouring problem is still NP-complete for inputs in $\mathcal{C}'$, as we prove next in Theorem 4. We recall that the proof of Cai and Ellis [5] for the NP-completeness of edge-colouring cubic square-free graphs generates a graph which **has** a cycle with a unique chord. In addition, remark that the gadget Q_Δ used in the proof of the NP-completeness of edge-colouring graphs with no cycle with a unique chord **has** a square. So, we need an alternative construction, which is based on the gadget $\tilde{P}$ shown in Figure 2. Graph $\tilde{P}$ is constructed in such a way that the identification of its pendant vertices generates a graph isomorphic to P^*, the graph obtained from the Petersen graph by removing one vertex. Graph P^* is a non-overfull Class 2 graph [15, 6]. The properties of $\tilde{P}$ with respect to edge-colouring are described in Lemma 2.

Lemma 2. *Graph $\tilde{P}$ is 3-edge-colourable, and in any 3-edge-colouring of $\tilde{P}$, the edges ux and vy receive the same colour.*

Proof:

Figure 2 shows a 3-edge-colouring of $\tilde{P}$ – observe that edges ux and vy receive the same colour.

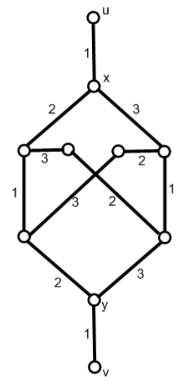

Figure 2: 3-edge-colouring of gadget graph $\tilde{P}$.

The fact that edges ux and vy always receive the same colour is a consequence of P^* being Class 2. The proof is similar to that of Lemma 1, except that gadget $\tilde{P}$ is used instead of Q_Δ. □

Theorem 4. *CHRIND(graph in $\mathcal{C}'$ with maximum degree 3) is NP-complete.*

Proof:

The proof is similar to that of Theorem 2, except that $\Delta = 3$ and gadget $\tilde{P}$ is used instead of Q_Δ. □

Observe that the graph G' constructed in the proof of Theorem 4 is not regular. In fact, as we prove in Section 5.1, the edge-colouring problem can be solved in polynomial time if the input is restricted to cubic graphs of $\mathcal{C}'$.

3 Structure of graphs in $\mathcal{C}$ and $\mathcal{C}'$

The goal of the present section is to review structure results for the graphs in $\mathcal{C}$ and obtain stronger results for the subclass $\mathcal{C}'$. These results are used in Section 4 to edge-colour the graphs in $\mathcal{C}'$ with maximum degree at least 4. In the present section we review the results of Trotignon and Vušković [23] on the structure of graphs in $\mathcal{C}$ and obtain stronger results for graphs in $\mathcal{C}'$.

Let $\mathcal{C}$ be the class of the graphs that do not contain a cycle with a unique chord and let $\mathcal{C}'$ be the class of the graphs of $\mathcal{C}$ that do not contain a square. Trotignon and Vušković give a decomposition result [23] for graphs in $\mathcal{C}$ and graphs in $\mathcal{C}'$ in the following form: every graph in $\mathcal{C}$ or in $\mathcal{C}'$ either belongs to a basic class or has a cutset. Before we can state these decomposition theorems, we define the basic graphs and the cutsets used in the decomposition.

The *Petersen graph* is the graph on vertices $\{a_1, \dots, a_5, b_1, \dots, b_5\}$ so that both $a_1a_2a_3a_4a_5a_1$ and $b_1b_2b_3b_4b_5b_1$ are chordless cycles, and such that the only edges between some a_i and some b_i are a_1b_1, a_2b_4, a_3b_2, a_4b_5, a_5b_3. We denote by P the Petersen graph and by P^* the graph obtained from P by removal of one vertex. Observe that $P \in \mathcal{C}$.

The *Heawood graph* is a cubic bipartite graph on vertices $\{a_1, \dots, a_{14}\}$ so that $a_1a_2 \dots a_{14}a_1$ is a cycle, and such that the only other edges are a_1a_{10}, a_2a_7, a_3a_{12}, a_4a_9, a_5a_{14}, a_6a_{11}, a_8a_{13}. We denote by H the Heawood graph and by H^* the graph obtained from H by removal of one vertex. Observe that $H \in \mathcal{C}$.

A graph is *strongly 2-bipartite* if it is square-free and bipartite with bipartition (X, Y) where every vertex in X has degree 2 and every vertex in Y has degree at least 3. A strongly 2-bipartite graph is in $\mathcal{C}$ because any chord of a cycle is an edge between two vertices of degree at least three, so every cycle in a strongly 2-bipartite graph is chordless.

For the purposes of this work, a graph G is called *basic*[1] if

1. G is a complete graph, a hole with at least five vertices, a strongly 2-bipartite graph, or an induced subgraph (not necessarily proper) of the Petersen graph or of the Heawood graph; and
2. G has no 1-cutset, proper 2-cutset or proper 1-join (all defined next).

We denote by $\mathcal{C}_B$ the set of the basic graphs. Observe that $\mathcal{C}_B \subseteq \mathcal{C}$.

A *cutset* S of a connected graph G is a set of elements, vertices and/or edges, whose removal disconnects G. A decomposition of a graph is the removal of a cutset to obtain smaller graphs, called the *blocks* of the decompositions, by possibly adding some nodes and edges to connected components of $G \setminus S$. The goal of decomposing a graph is trying to solve a problem on the whole graph by combining the solutions on the blocks. For a graph $G = (V, E)$ and $V' \subseteq V$, $G[V']$ denotes the subgraph of G induced by V'. The following cutsets are used in the known decomposition theorems of the class $\mathcal{C}$ [23]:

- A *1-cutset* of a connected graph $G = (V, E)$ is a node v such that V can be partitioned into sets X, Y and $\{v\}$, so that there is no edge between X and Y. We say that (X, Y, v) is a *split* of this 1-cutset.

[1] By the definition of [23], a basic graph is not, in general, indecomposable. However, our slightly different definition helps simplifying some of our proofs.

- A *proper 2-cutset* of a connected graph $G = (V, E)$ is a pair of non-adjacent nodes a, b, both of degree at least three, such that V can be partitioned into sets X, Y and $\{a, b\}$ so that: $|X| \geq 2$, $|Y| \geq 2$; there is no edge between X and Y, and both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain an ab-path. We say that (X, Y, a, b) is a *split* of this proper 2-cutset.

- A *1-join* of a graph $G = (V, E)$ is a partition of V into sets X and Y such that there exist sets A, B satisfying:

  - $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq Y$;
  - $|X| \geq 2$ and $|Y| \geq 2$;
  - there are all possible edges between A and B;
  - there is no other edge between X and Y.

  We say that (X, Y, A, B) is a *split* of this 1-join.

  A *proper 1-join* is a 1-join such that A and B are stable sets of G of size at least two.

We can now state a decomposition result for graphs in $\mathcal{C}$:

Theorem 5. (Trotignon and Vušković [23]) *If $G \in \mathcal{C}$ is connected then either $G \in \mathcal{C}_B$ or G has a 1-cutset, or a proper 2-cutset, or a proper 1-join.*

The *block* G_X (resp. G_Y) of a graph G with respect to a 1-cutset with split (X, Y, v) is $G[X \cup \{v\}]$ (resp. $G[Y \cup \{v\}]$).

The *block* G_X (resp. G_Y) of a graph G with respect to a 1-join with split (X, Y, A, B) is the graph obtained by taking $G[X]$ (resp. $G[Y]$) and adding a node y complete to A (resp. x complete to B). Nodes x, y are called *markers* of their respective blocks.

The *blocks* G_X and G_Y of a graph G with respect to a proper 2-cutset with split (X, Y, a, b) are defined as follows. If there exists a node c of G such that $N_G(c) = \{a, b\}$, then let $G_X = G[X \cup \{a, b, c\}]$ and $G_Y = G[Y \cup \{a, b, c\}]$. Otherwise, block G_X (resp. G_Y) is the graph obtained by taking $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) and adding a new node c adjacent to a, b. Node c is called the *marker* of the block G_X (resp. G_Y).

The blocks with respect to 1-cutsets, proper 2-cutsets and proper 1-joins are constructed in such a way that they remain in $\mathcal{C}$, as shown by Lemma 3.

Lemma 3. (Trotignon and Vušković [23]) *Let G_X and G_Y be the blocks of decomposition of G with respect to a 1-cutset, a proper 1-join or a proper 2-cutset. Then $G \in \mathcal{C}$ if and only if $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$.*

Observe that the Petersen graph and the Heawood graph may appear as a block of decomposition with respect to a proper 1-join, as shown in Figure 3. However, these graphs cannot appear as a block of decomposition with respect to a proper 2-cutset, because they have no vertex with degree 2 to play the role of a marker.

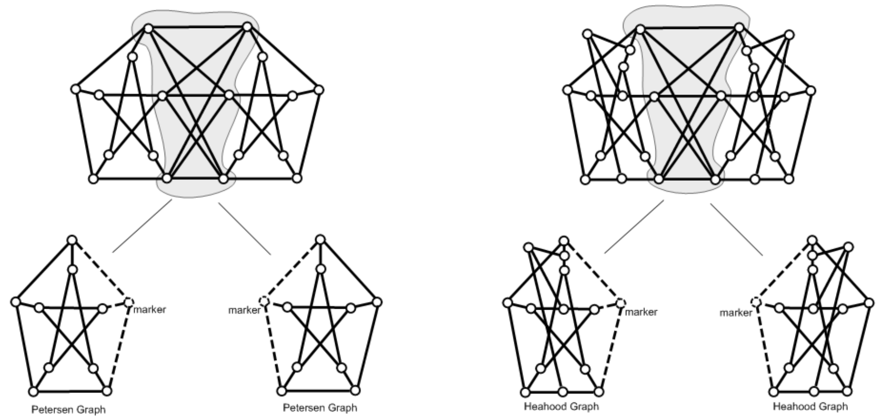

Figure 3: Decomposition trees with respect to proper 1-joins. In the graph on the left, the basic blocks of decomposition are two copies of the Petersen graph. In the graph on the right, the basic blocks of decomposition are two copies of the Heawood graph.

Despite the fact that the Petersen graph and the Heawood graph do not appear as a block of decomposition with respect to a proper 2-cutset, they must be listed as basic blocks, because these graphs, themselves, are in $\mathcal{C}'$. So, the Petersen graph (resp. the Heawood graph) appears as a leaf of exactly one decomposition tree, namely, the decomposition tree of the Petersen graph (resp. the Heawood graph), itself – which is, actually, a trivial decomposition tree. Observe that graphs P^* (Petersen graph minus one vertex) and H^* (Heawood graph minus one vertex) may appear as a block with respect to a proper 2-cutset decomposition, as shown in Figure 4.

We reviewed results that show how to decompose a graph of $\mathcal{C}$ into basic blocks: Theorem 5 states that each graph in $\mathcal{C}$ has a 1-cutset, a proper 2-cutset or a proper 1-join, while Lemma 3 states that the blocks generated with respect to any of these cutsets are still in $\mathcal{C}$. We now obtain similar results for $\mathcal{C}'$. These results are not explicit in [23], but they can be obtained as consequences of results in [23] and by making minor modifications in its proofs. As we discuss in the following observation [4], for the goal of edge-

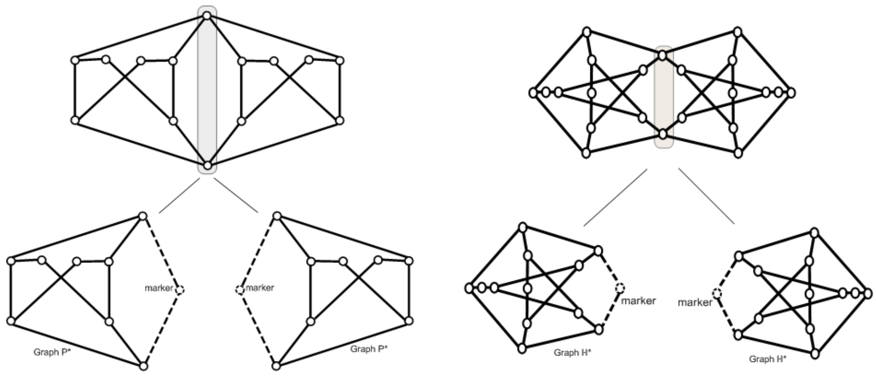

Figure 4: Decomposition trees with respect to proper 2-cutsets. In the graph on the left, the basic blocks of decomposition are two copies of P^*. In the graph on the right, the basic blocks of decomposition are two copies of H^*.

colouring, we only need to consider the **biconnected** graphs of $\mathcal{C}'$.

Observation 1. *Let G be a connected graph with a 1-cutset with split (X, Y, v). The chromatic index of G is $\chi'(G) = \max\{\chi'(G_X), \chi'(G_Y), \Delta(G)\}$.*

By Observation 1, if both blocks G_X and G_Y are $\Delta(G)$-edge-colourable, then so is G. That is, once we know the chromatic index of the biconnected components of a graph, it is easy to determine the chromatic index of the whole graph. So, we may focus our investigation on the biconnected graphs of $\mathcal{C}'$.

Theorem 6. (Trotignon and Vušković [23]) *If $G \in \mathcal{C}'$ is biconnected, then either $G \in \mathcal{C}_B$ or G has a proper 2-cutset.*

Theorem 6 is an immediate consequence of Theorem 5: since G has no 4-hole, G cannot have a proper 1-join, and since G is biconnected, G cannot have a 1-cutset.

Next, in Lemma 4, we show that the blocks of decomposition of a biconnected graph of $\mathcal{C}'$ with respect to a proper 2-cutset, are also biconnected graphs of $\mathcal{C}'$. The proof of Lemma 4 is similar to that of Lemma 5.2 of [23]. For the sake of completeness, the proof, which uses the result of Theorem 7 below, is included here.

Theorem 7. (Trotignon and Vušković [23]) *Let $G \in \mathcal{C}$ be a connected graph. If G contains a triangle then either G is a complete graph, or some vertex of the maximal clique that contains this triangle is a 1-cutset of G.*

Lemma 4. *Let $G \in \mathcal{C}'$ be a biconnected graph and let (X, Y, a, b) be a split of a proper 2-cutset of G. Then both G_X and G_Y are biconnected graphs of $\mathcal{C}'$.*

Proof:

We first prove that G is triangle-free. Suppose G contains a triangle. Then, by Theorem 7, either G is a complete graph, which contradicts the assumption that G has a proper 2-cutset, or G has a 1-cutset, which contradicts the assumption that G is biconnected. So G is triangle-free, and hence by construction, both of the blocks G_X and G_Y are triangle-free.

Now we show that G_X and G_Y are square-free. Suppose w.l.o.g. that G_X contains a square C. Since G is square-free, C contains the marker node M, which is not a real node of G, and $C = MazbM$, for some node $z \in X$. Since M is not a real node of G, we have $deg_G(z) > 2$, otherwise, z would be a marker of G_X. Let z' be a neighbor of z distinct of a and b. Since G is triangle-free, z' is not adjacent to a nor b. Since z is not a 1-cutset, there exists a path P in $G[X \cup \{a, b\}]$ from z' to $\{a, b\}$. We choose z' and P subject to the minimality of P. So, w.l.o.g., $z'Pa$ is a chordless path. Note that b is not adjacent to the neighbor of a along P because G is triangle-free and square-free, so that z is the unique common neighbor of a and b in G. So, by the minimality of P, vertex b does not have a neighbor in P. Now let Q be a chordless path from a to b whose interior is in Y. So, $bzz'PaQb$ is a cycle of G with a unique chord (namely az), contradicting the assumption that $G \in \mathcal{C}$.

By Lemma 3, G_X and G_Y both belong to $\mathcal{C}$, and since G_X and G_Y are both square-free, it follows that G_X and G_Y both belong to $\mathcal{C}'$.

Finally we show that G_X and G_Y are biconnected. Suppose w.l.o.g. that G_X has a 1-cutset with split (A, B, v). Since G is biconnected and $G[X \cup \{a, b\}]$ contains an ab-path, we have that $v \neq M$, where M is the marker of G_X. Suppose $v = a$. Then, w.l.o.g., $b \in B$, and $(A, B \cup Y, a)$ is a split of a 1-cutset of G, with possibly M removed from $B \cup Y$, if M is not a real node of G, contradicting the assumption that G is biconnected. So $v \neq a$ and by symmetry $v \neq b$. So $v \in X \setminus \{M\}$. W.l.o.g. $\{a, b, M\} \subset B$. Then $(A, B \cup Y, v)$ is a split of a 1-cutset of G, with possibly M removed from $B \cup Y$ if M is not a real node of G, contradicting the assumption that G is biconnected. $\square$

Observe that Lemma 3 is somehow stronger than Lemma 4. While Lemma 3 states that a graph is in $\mathcal{C}$ **if and only if** the blocks with respect to any cutset are also in $\mathcal{C}$, Lemma 4 establishes only one direction: **if** a graph is a biconnected graph of $\mathcal{C}'$, **then** the blocks with respect to any cutset are also biconnected graph of $\mathcal{C}'$. For our goal of edge-colouring, there is no need of establishing the "only if" part. Anyway, it is possible to verify that, if both blocks G_X and G_Y generated with respect to a proper 2-cutset of a graph G are biconnected graphs of $\mathcal{C}'$, then G itself is a biconnected graph of $\mathcal{C}'$.

Next lemma shows that every non-basic biconnected graph in $\mathcal{C}'$ has a decomposition such that one of the blocks is basic.

Lemma 5. *Every biconnected graph $G \in \mathcal{C}' \setminus \mathcal{C}_B$ has a proper 2-cutset such that one of the blocks of decomposition is basic.*

Proof:

By Theorem 6 G has a proper 2-cutset. Consider all possible 2-cutset decompositions of G and pick a proper 2-cutset S that has a block of decomposition B whose size is smallest possible. By Lemma 4, $B \in \mathcal{C}'$ and is biconnected. So by Theorem 6, either B has a proper 2-cutset or it is basic. We now show that in fact B must be basic.

Let (X, Y, a, b) be a split with respect to S. Let M be the marker node of G_X, and assume w.l.o.g. that $B = G_X$. Suppose G_X has a proper 2-cutset with split (X_1, X_2, u, v). By minimality of $B = G_X$, $\{a, b\} \neq \{u, v\}$. Assume w.l.o.g. $b \notin \{u, v\}$. Note that since $deg_{G_X}(u) \geq 3$ and $deg_{G_X}(v) \geq 3$, it follows that $M \notin \{u, v\}$. Suppose $a \notin \{u, v\}$. Then w.l.o.g. $\{a, b, M\} \subseteq X_1$, and hence $(X_1 \cup Y, X_2, u, v)$, with M removed if M is not a real node of G, is a proper 2-cutset of G whose block of decomposition G_{X_2} is smaller than G_X, contradicting the minimality of $G_X = B$. Therefore $a \in \{u, v\}$. Then w.l.o.g. $\{b, M\} \subseteq X_1$, and hence $(X_1 \cup Y, X_2, u, v)$, with M removed if M is not a real node of G, is a proper 2-cutset of G whose block of decomposition G_{X_2} is smaller than G_X, contradicting the minimality of $G_X = B$. Therefore G_X does not have a proper 2-cutset, and hence it is basic. □

4 Chromatic index of graphs in $\mathcal{C}'$ with maximum degree at least 4

The first NP-completeness result of Section 2 proves that edge-colouring is difficult for the graphs in $\mathcal{C}$. We consider, further, the subclass $\mathcal{C}'$ and verify that the edge-colouring problem is still NP-complete when restricted to $\mathcal{C}'$.

In the present section we apply the structure results of Section 3 to show that edge-colouring graphs in $\mathcal{C}'$ of maximum degree $\Delta \geq 4$ is polynomial by establishing that the only Class 2 graphs in $\mathcal{C}'$ are the odd order complete graphs. Remark that the NP-completeness holds only for 3-edge-colouring restricted to graphs in $\mathcal{C}'$ with maximum degree 3.

We describe, next, the technique applied to edge-colour a graph in $\mathcal{C}'$ by combining edge-colourings of its blocks with respect to a proper 2-cutset. Observe that the fact that a graph F is isomorphic to a block B obtained from a proper 2-cutset decomposition of G does **not** imply that G contains F: possibly B is constructed by the addition of a marker vertex. This is illustrated in the example of Figure 5, where G is P^*-free, yet, graph P^* appears as a block with respect to a proper 2-cutset of G.

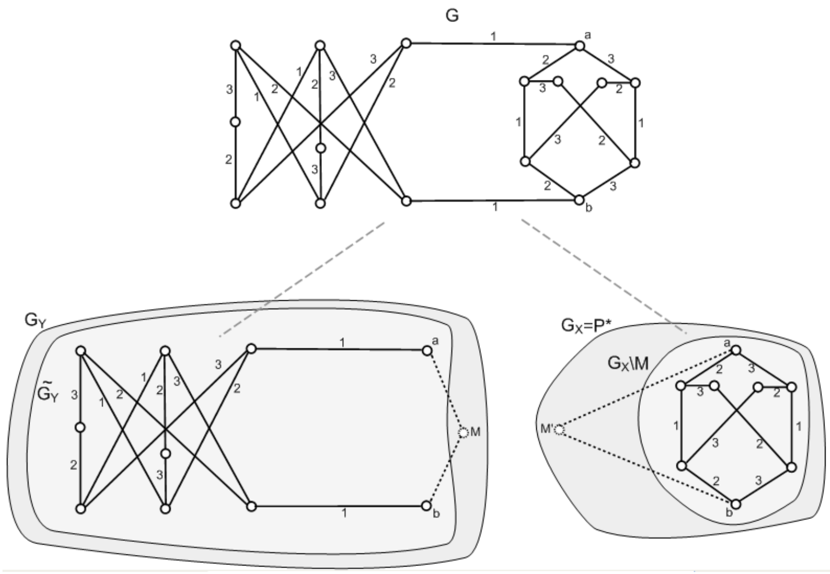

Figure 5: Example of decomposition with respect to a proper 2-cutset $\{a, b\}$. Observe that the marker vertices and their incident edges – identified by dashed lines – do not belong to the original graph.

The reader will also observe that it is not necessary that a block of decomposition of G is $\Delta(G)$-edge-colourable in order that G itself is $\Delta(G)$-edge-colourable: graph G in Figure 5 is 3-edge-colourable, while block P^* is not. This is an important observation: possibly, the edges adjacent to a marker vertex of a block of decomposition are not real edges of the original graph, or are already coloured by an edge-colouring of another block, so that these edges do not need to be coloured.

Observation 2. *Consider a graph $G \in \mathcal{C}'$ with the following properties:*

- *(X, Y, a, b) is a split of a proper 2-cutset of G;*
- *$\tilde{G_Y}$ is obtained from G_Y by removing its marker if this marker is not a real vertex of G;*
- *$\tilde{\pi_Y}$ is a $\Delta(G)$-edge-colouring of $\tilde{G_Y}$;*
- *F_a (resp. F_b) is the set of the colours in $\{1, 2, ..., \Delta\}$ not used by $\tilde{\pi_Y}$ in any edge of $\tilde{G_Y}$ incident to a (resp. b).*

If there exists a $\Delta(G)$-edge-colouring π_X of $G_X \setminus M$, where M is the marker vertex of G_X, such that each colour used in an edge incident to a (resp. b) is in F_a (resp. F_b), then G is Δ-edge-colourable.

The above observation shows that, in order to extend a $\Delta(G)$-edge-colouring of $\tilde{G_Y}$ to a $\Delta(G)$-edge-colouring of G, one must colour the edges of $G_X \setminus M$ in such a way that the colours of the edges incident to a (resp. b) are not used at the edges of $\tilde{G_Y}$ incident to a (resp. b). This guarantees that we create no conflicts. Moreover, there is no need to colour the edges incident to the marker M of G_X: if this marker is a vertex of G, its incident edges are already coloured by $\tilde{\pi}$, otherwise, these edges are not real edges of G. In the example of Figure 5, we exhibit a 3-edge-colouring $\tilde{\pi_Y}$ of $\tilde{G_Y}$. In the notation of Observation 2, $F_a = \{2, 3\}$ and $F_b = \{2, 3\}$. We exhibit, also, a 3-edge-colouring of $G_X \setminus M$ such that the colours of the edges incident to a are $\{2, 3\} \subset F_a$ and the colours of the edges incident to b are $\{2, 3\} \subset F_b$. So, by Observation 2, we can combine colourings $\tilde{\pi_Y}$ and π_X in a 3-edge-colouring of G, as it is done in Figure 5.

Before we proceed and show how to edge-colour graphs in $\mathcal{C}'$ with maximum degree $\Delta \geq 4$, we need to introduce some additional tools and concepts. A *partial k-edge-colouring* of a graph $G = (V, E)$ is a colouring of a subset E' of E, that is, a function $\pi : E' \to \{1, 2, ..., k\}$ such that no two adjacent edges of E' receive the same colour.

The *set of free-colours* at vertex u with respect to a partial-edge-colouring $\pi : E' \to \mathbf{C}$ is the set $\mathbf{C} \setminus \pi(\{uv | uv \in E'\})$. The *list-edge-colouring* problem is described next. Let $G = (V, E)$ be a graph and let $\mathcal{L} = \{L_e\}_{e \in E}$ be a collection which associates to each edge $e \in E$ a set of colours L_e called the *list* relative to e. It is asked whether there is an edge-colouring π of G such that $\pi(e) \in L_e$ for each edge $e \in E$. Theorem 8 is a result on list-edge-colouring which is applied, in this work, to edge-colour some of our basic graphs: strongly 2-bipartite graphs, Heawood graph and its subgraphs, and holes.

Theorem 8. (Borodin, Kostochka, and Woodall [4]) *Let $G = (V, E)$ be a bipartite graph and $\mathcal{L} = \{L_e\}_{e \in E}$ be a collection of lists of colours which associates to each edge $uv \in E$ a list L_{uv} of colours. If, for each edge $uv \in E$, $|L_{uv}| \geq \max\{deg_G(u), deg_G(v)\}$, then there is an edge-colouring π of G such that, for each edge $uv \in E$, $\pi(uv) \in L_{uv}$.*

We investigate, now, how to $\Delta(G)$-edge-colour a graph $G \in \mathcal{C}'$ by combining $\Delta(G)$-edge-colourings of its blocks with respect to a proper 2-cutset. More precisely, Lemma 6 shows how this can be done if one of the blocks is basic. Subsequently, we obtain, in Theorem 9 and Corollary 1, a characterization for graphs in $\mathcal{C}'$ of maximum degree at least 4 of its Class 2 graphs which establishes that edge-colouring is polynomial for these graphs.

Lemma 6. *Let $G \in \mathcal{C}'$ be a graph of maximum degree $\Delta \geq 4$ and let (X, Y, a, b) be a split of proper 2-cutset, in such a way that G_X is basic. If G_Y is Δ-edge-colourable, then G is Δ-edge-colourable.*

Proof:

Denote by M the marker vertex of G_X and let $\tilde{G_Y}$ be obtained from G_Y by removing its marker if this marker is not a real vertex of G. Since $\tilde{G_Y}$ is a subgraph of G_Y, graph $\tilde{G_Y}$ is Δ-edge-colourable. Let π_Y be a Δ-edge-colouring of $\tilde{G_Y}$, i.e. a partial-edge-colouring of G, and let F_a and F_b be the sets of the free colours of a and b, respectively, with respect to the partial edge-colouring π_Y. We show how to extend the partial edge-colouring π_Y to G, as described in Observation 2, that is, by colouring the edges of $G_X \setminus M$. Since a and b are not adjacent, G_X is not a complete graph. Moreover, the block G_X cannot be isomorphic to the Petersen graph or to the Heawood graph, because these graphs are cubic and G_X has a marker vertex M of degree 2. So, G_X is isomorphic to an induced subgraph of P^*, or to an induced subgraph of H^*, or to a strongly 2-bipartite graph, or is a hole.

Case 1. G_X is a strongly 2-bipartite graph.

Since $deg_{G_X}(M) = 2$, vertex M belongs to the bipartition of G_X whose vertices have degree 2. So, vertices a and b belong to the bipartition of G_X whose vertices have degree larger than 2, and $|F_a| \geq deg_{G_X \setminus M}(a) \geq 2$ and $|F_b| \geq deg_{G_X \setminus M}(b) \geq 2$. Associate to each edge of $G_X \setminus M$ incident to a (resp. b) a list of colours equal to F_a (resp. F_b). To each of the other edges of $G_X \setminus M$, associate list $\{1, ..., \Delta\}$. Now, to each edge uv of $G_X \setminus M$, a list of colours is associated whose size is not smaller than $\max\{deg_{G_X \setminus M}(u), deg_{G_X \setminus M}(v)\}$

and, by Theorem 8, there is an edge-colouring π_1 of $G_X \setminus M$ from these lists. Finally, set $\pi := \pi_1$ for the edges of $G_X \setminus M$.

Case 2. G_X is a hole.

In this case, $G_X \setminus M$ is a path. Denote the vertices of $G_X \setminus M$ by $a = x_1, x_2, ..., x_k = b$, in such a way that $x_1x_2...x_k$ is a path. We now show that $k \geq 4$. Since a and b are not adjacent, $k \geq 3$. Suppose that $k = 3$. If M is a real node of G, then G_X is a square and it is an induced subgraph of G, contradicting the assumption that G is square-free. So M is not a real node of G, and hence $G_X \setminus M = X$. But, then, $|X| = 1$, contradicting the definition of a proper 2-cutset. Therefore, $k \geq 4$.

Observe that there is at least one colour c_α in F_a and one colour c_β in F_b. We construct a 3-edge-colouring π of $C_X \setminus M$ by setting $\pi(x_1x_2) := c_\alpha$ and $\pi(x_{k-1}x_k) := c_\beta$, and by colouring the other edges of $G_X \setminus M$ as follows. If $k = 4$, let $\pi(x_2x_3)$ be some colour in $\{1, 2, 3\} \setminus \{c_\alpha, c_\beta\}$, which is clearly a non-empty set. If $k \geq 5$, let $\mathcal{L}_2 = \{L_2, L_3, ..., L_{k-2}\}$ be a collection which associates to each edge x_ix_{i+1} a list of colours L_i such that:

- $L_i = \{1, 2, 3\} \setminus \{c_\alpha\}$, for $i = 2, 3, ..., k-3$, and
- $L_{k-2} = \{1, 2, 3\} \setminus \{c_\beta\}$.

Observe that $G_X \setminus \{M, a, b\}$ is a path, hence bipartite of maximum degree 2, and that $|L_i| \geq 2$ for each $i = 2, ..., k-2$, so that by, Theorem 8, there is an edge-colouring π_2 of $G_X \setminus \{M, a, b\}$ from the lists $\mathcal{L}_2$. Moreover, this colouring creates no conflicts with the colours c_α of x_1x_2 and c_β of $x_{k-1}x_k$, so that we can set $\pi := \pi_2$ for edges $x_2x_3, x_3x_4, ..., x_{k-2}x_{k-1}$.

Case 3. G_X is an induced subgraph of the Heawood graph.

Observe that a and b have only M as common neighbor in G_X, otherwise G_X has a square (recall that Heawood graph is square-free). We construct a 4-edge-colouring of $G_X \setminus M$. Denote the neighbors of a (resp. b) in $G_X \setminus M$ by $a_1, ..., a_x$ (resp. $b_1, ..., b_y$), where $x = deg_{G_X \setminus M}(a)$ (resp. $y = deg_{G_X \setminus M}(b)$). Note that $x, y \in \{1, 2\}$. Observe that F_a (resp. F_b) contains at least x (resp. y) colours, which we denote by $c_{a_1}, ..., c_{a_x}$ (resp. $c_{b_1}, c_{b_2}, ..., c_{b_y}$). Set the colour π of edge aa_i (resp. bb_j), for $i = 1, ..., x$ (resp. for $j = 1, ..., y$), to c_{a_i} (resp. c_{b_j}). Now, associate to each edge incident to a_i and different from aa_i a list of colours $\{1, 2, 3, 4\} \setminus \{c_{a_i}\}$. Similarly, associate to each edge incident to b_j and different of bb_j a list of colours $\{1, 2, 3, 4\} \setminus \{c_{b_j}\}$. Finally, associate to each of the other edges of $G_X \setminus \{M, a, b\}$ the list of colours $\{1, 2, 3, 4\}$. Observe that $G_X \setminus \{M, a, b\}$ is bipartite of maximum degree at most 3 and that each of the lists has 3 or 4 colours, so that, by Theorem 8, there is an edge-colouring π_3 of $G_X \setminus \{M, a, b\}$ from these lists, and we set $\pi := \pi_3$ for the edges of $G_Z \setminus M$.

Case 4.a: $G_X = P^*$.

Observe that there are at least two colours c_{a_1}, c_{a_2} in F_a and two colours c_{b_1}, c_{b_2} in F_b, and that exactly one of the following three possibilities holds:

- $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 0$;
- $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 1$; or
- $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 2$.

In the three cases, it is possible to extend the Δ-edge-colouring π_Y to G by colouring the edges of $G_X \setminus M$, as it is shown on Figure 6.

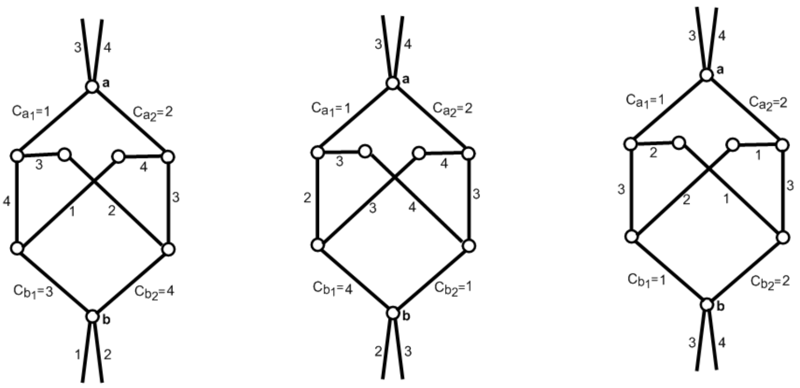

Figure 6: Extending the colouring to the edges of G_X.

Case 4.b: G_X is a proper induced subgraph of P^*.

We need to investigate which are the proper induced subgraphs of P^*. We invite the reader to verify that, except for graph P^{**} shown on the left of Figure 7, each proper induced subgraph of P^* either has a 1-cutset or a proper 2-cutset, and we do not consider it because G_X is assumed basic, or is a hole, which is already considered in Case 2.

There is only one possible choice for the marker M of $G_X = P^{**}$, in the sense that, for any other choice of marker M', we have $G_X \setminus M' = G_X \setminus M$. As in Case 4.a, there are at least two colours c_{a_1}, c_{a_2} in F_a and two colours c_{b_1}, c_{b_2} in F_b, and $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 0$, 1 or 2. In Figure 7 we exhibit three edge-colourings for $P^{**} \setminus M$, one for each possibility. $\square$

Using Lemma 6 we can determine in polynomial time the chromatic index of the graphs of $\mathcal{C}'$, as we show in Theorem 9 and its Corollary 1.

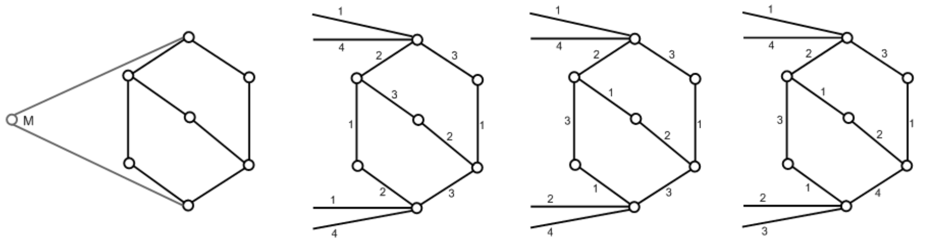

Figure 7: Graph P^{**} and three edge-colourings of $P^{**} \setminus M$ subject to each possible free colour restriction.

Theorem 9. *If λ is an integer at least 4 and G is a connected non-complete graph of $\mathcal{C}'$ with maximum degree $\Delta(G) \leq \lambda$, then G is λ-edge-colourable.*

Proof: We prove the theorem by induction. Let $G \in \mathcal{C}'$ be a connected graph with k vertices such that $\Delta(G) \leq \lambda$ and G is not a complete graph. By Theorem 6 either G is basic, or G has a 1-cutset, or G is biconnected and has a proper 2-cutset.

Suppose G is basic. If G is strongly 2-bipartite, then G is λ-edge-colourable because bipartite graphs are Class 1 and $\Delta(G) \leq \lambda$. If G is not strongly 2-bipartite, then G is a hole or a subgraph of the Petersen graph or of the Heawood graph, so that $\Delta(G) \leq 3 \leq \lambda - 1$ and G is λ-edge-colourable by Vizing's theorem. Assume, as induction hypothesis, that every connected non-complete graph $G' \in \mathcal{C}'$ with $k' < k$ vertices such that $\Delta(G') \leq \lambda$ is λ-edge-colourable.

Suppose G has a 1-cutset with split (X, Y, v). Note that blocks of decomposition G_X and G_Y are induced subgraphs of G and hence both belong to $\mathcal{C}'$. If G_X (resp. G_Y) is complete, then its maximum degree is at most $\lambda - 1$, so that G_X (resp. G_Y) is λ-edge-colourable by Vizing's theorem. If G_X (resp. G_Y) is not complete, G_X (resp. G_Y) is λ-edge-colourable by the induction hypothesis. In any case, both G_X and G_Y are λ-edge-colourable, and hence by Observation 1, graph G is λ-edge-colourable.

Finally, suppose G is biconnected and has a proper 2-cutset. Let (X, Y, a, b) be a split of a proper 2-cutset such that block G_X is basic (note that such a cutset exists by Lemma 5). By Theorem 7, block G_X is not a complete graph. By Lemma 4, block G_Y is in $\mathcal{C}'$. By the induction hypothesis, block G_Y is λ-edge-colourable. By Lemma 6, graph G is λ-edge-colourable. $\square$

Corollary 1. *A connected graph $G \in \mathcal{C}'$ of maximum degree $\Delta \geq 4$ is Class 2 if and only if it is an odd order complete graph.*

Proof:

If G is complete, then the result clearly holds. So, we may assume G is not complete. Just choose $\lambda = \Delta$ in Theorem 9 to prove that every connected non-complete graph of $\mathcal{C}'$ with maximum degree $\Delta(G) \geq 4$ is λ-edge-colourable, hence Class 1. □

5 Graphs of $\mathcal{C}'$ with maximum degree 3

Class $\mathcal{C}'$ has a stronger structure than $\mathcal{C}$, yet, edge-colouring problem is NP-complete for inputs in $\mathcal{C}'$. In fact, the problem is NP-complete for graphs in $\mathcal{C}'$ with maximum degree $\Delta = 3$. In this section, we further investigate graphs in $\mathcal{C}'$ with maximum degree $\Delta = 3$, providing two subclasses for which edge-colouring can be solved in polynomial time: cubic graphs of $\mathcal{C}'$ and 6-hole-free graphs of $\mathcal{C}'$.

5.1 Cubic graphs of $\mathcal{C}'$

In the present section, we prove the polynomiality of the edge-colouring problem restricted to cubic graphs of $\mathcal{C}'$. This is a direct consequence of Lemma 7, which states that every non-biconnected cubic graph is Class 2, and Lemma 8, which states that the only biconnected cubic Class 2 graph in $\mathcal{C}'$ are the Petersen graph, the Heawood graph and the complete graph on four vertices.

Lemma 7. *Let G be a connected cubic graph. If G has a 1-cutset, then G is Class 2.*

Proof:

Denote by (X, Y, v) a split of a 1-cutset of G. Observe that v has degree 1 in exactly one of the blocks G_X and G_Y; assume, w.l.o.g. that this block is G_X. Let G'_X be the graph obtained from G_X by removing vertex v. Observe that G'_X has exactly one vertex of degree 2 and each of the other vertices has degree 3. Since the sum of the degrees of the vertices is even, G'_X has an even number of vertices of degree 3, say n. So, the number of edges in G'_X is $(3n+2)/2$. Since $3\lfloor(n+1)/2\rfloor = 3n/2 < (3n+2)/2$, graph G'_X is overfull, so that G is subgraph-overfull, hence Class 2. □

Lemma 8. *Let $G \in \mathcal{C}'$ be biconnected graph. If G is cubic, then G is isomorphic to the Petersen graph or to the Heawood graph or is a complete graph on four vertices.*

Proof:

Suppose G is not basic. By Lemma 5, G has a proper 2-cutset such that one of the blocks is basic. Let (X, Y, a, b) be a split of such cutset, in such a way that G_X is basic, and denote by M the marker vertex of G_X. If $deg_{G_X}(a) = 1$, vertex M is the only neighbor of a and, clearly, is a 1-cutset of G_X. By Lemma 4, G_X is a biconnected graph of $\mathcal{C}'$. Since G_X is biconnected $deg_{G_X} \geq 2$. Let a' be a neighbor of a in G_X that is distinct from M. Since $\{M, a, b, a'\}$ cannot induce a square, b is not adjacent to a', and hence (since G is cubic) a' has two neighbors in $G_X \setminus \{a, b, M\}$. If $deg_{G_X}(a) = 2$ then $\{a', b\}$ is a proper 2-cutset of G, contradicting the assumption that G_X is basic. Hence $deg_{G_X}(a) \geq 3$, and by symmetry $deg_{G_X}(b) \geq 3$. Observe that each of the other vertices – different from a, b and M – has degree $\Delta(G)$. In other words, G_X is a graph with exactly one vertex of degree 2, and each of the other vertices has degree 3. But there is no graph in $\mathcal{C}_B$ with this property, and we have a contradiction to the fact that G_X is basic. So, G is basic and the statement of the lemma clearly holds. □

Theorem 10. *Let $G \in \mathcal{C}'$ be a connected cubic graph. Then G is Class 1 if and only if G is biconnected and is not isomorphic to the Petersen graph.*

Proof:

If G is not biconnected, then, by Lemma 7, G is Class 2. If G is biconnected, then, by Lemma 8, G is isomorphic to the Petersen graph P or to the Heawood graph H or is a complete graph K_4 on four vertices. Remark that H is Class 1, because it is bipartite, and K_4 is Class 1, because it is a complete graph with even number of vertices. Hence, G is Class 2 if and only if it is isomorphic to the Petersen graph. □

5.2 6-hole-free graphs of $\mathcal{C}'$

In the present section, we prove the polynomiality of the edge-colouring problem restricted to 6-hole-free graphs of $\mathcal{C}'$. This is a consequence of Lemma 9, a variation for 3-edge-colouring of Lemma 6.

Lemma 9. *Let $G \in \mathcal{C}'$ be a graph of maximum degree at most 3 and (X, Y, a, b) be a split of a proper 2-cutset, in such a way that G_X is basic but not isomorphic to P^*. If G_Y is 3-edge-colourable, then G is 3-edge-colourable.*

Proof:

Assume G_Y is 3-edge-colourable. Denote by M the marker vertex of G_X and let $\tilde{G_Y}$ be obtained from G_Y by removing its marker if this marker is not a real vertex of G. Since $\tilde{G_Y}$ is a subgraph of G_Y, graph $\tilde{G_Y}$ is 3-edge-colourable. Let π_Y be a 3-edge-colouring of $\tilde{G_Y}$, i.e. a partial-edge-colouring of G, and let F_a and F_b be the sets of the free colours of a and b, respectively, with respect to the partial edge-colouring π_Y. We show how to extend the partial edge-colouring π_Y to G, as described in Observation 2, that is, by colouring the edges of $G_X \setminus M$. Since a and b are not adjacent, G_X is not a complete graph. Moreover, the block G_X cannot be isomorphic to the Petersen graph or to the Heawood graph, because these graphs are cubic and G_X has a marker vertex M of degree 2. Also, by assumption, block G_X is not isomorphic to P^*. So, G_X is isomorphic to a proper induced subgraph of P^*, or to an induced subgraph of H^*, or to a strongly 2-bipartite graph, or is a hole.

Case 1. G_X is a strongly 2-bipartite graph.

Similar to the Case 1 of the proof of Lemma 6, which also works for $\Delta = 3$.

Case 2. G_X is a hole.

Similar to the Case 2 of the proof of Lemma 6, where at most three colours are used in the edges of $G_X \setminus M$.

Case 3. G_X is an induced subgraph of H^*.

First, observe that $deg_{G_X \setminus M}(a) = 2$ and $deg_{G_X \setminus M}(b) = 2$, otherwise G_X has a decomposition by a 1-cutset or a proper 2-cutset and is not basic. Observe, also, that there are at least two colours c_{a_1}, c_{a_2} in F_a and two colours c_{b_1}, c_{b_2} in F_b, and that $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 1$ or 2. We consider each case next.

If $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 1$, we must exhibit a 3-edge-coloring π of $G_X \setminus M$ such that the free colors at a and b are different. If M is a real node of G, then G_X is an induced subgraph of G, and hence $\Delta(G_X) \leq 3$. If M is not a real node of G, then by definition of proper 2-cutset both a and b have a neighbor in Y, and hence $\Delta(G_X) \leq 3$. So $\Delta(G_X) \leq 3$. Since G_X is bipartite, G_X has a 3-edge-colouring π'. So, let π be the restriction of π' to $G_X \setminus M$.

If $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 2$, we must exhibit a coloring of $G_X \setminus M$ such that the free colors at a and b are the same. We exhibit these colourings for each possible induced subgraph of the Heawood graph. First, consider the case $G_X = H^*$, whose coloring is given in Figure 8.

Now, observe that each non-basic proper subgraph of H^* is a subgraph of the graph H_1 of Figure 9, which is obtained from H^* by removing a vertex of degree 2. Graph H_2 of Figure 9 is obtained from H_1 by removing one of the four vertices of degree 2 (any choice yields the same graph up to an isomorphism). Finally, the last non-basic proper subgraph of H^* is the

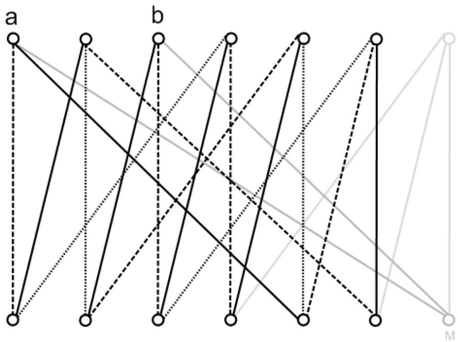

Figure 8: A 3-edge-colouring of $H^* \setminus M$ such that the sets of the colours incident to vertex a and vertex b are the same.

graph H_3 of Figure 9. Observe that there is only one possible choice M for

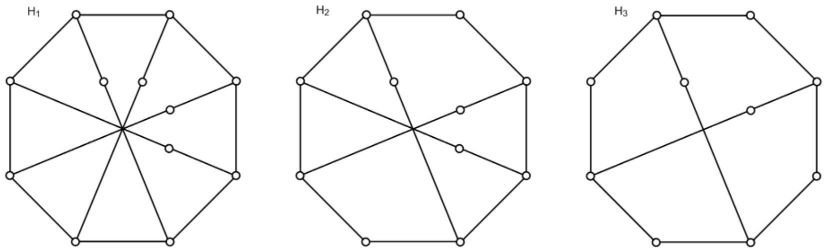

Figure 9: Non-basic proper induced subgraphs of H^*

the marker when $G_X = H_1$, in the sense that, for any other choice $\tilde{M}$, we have $G_X \setminus \tilde{M} = G_X \setminus M$. If $G_X = H_2$, there are two possible choices M' and M'' for the marker, in the sense that, for any other choice $\tilde{M}'$, we have $G_X \setminus \tilde{M}' = G_X \setminus M'$ or $G_X \setminus \tilde{M}' = G_X \setminus M''$. We show, in Figure 10, one edge-colouring of $H_1 \setminus M$, and two edge-colourings of $H_2 \setminus M$, one for each possible choice of marker M. We don't consider here that case $G_X = H_3$ because H_3 is a strongly 2-bipartite graph, considered in Case 1.

Case 4. G_X is a proper subgraph of P^*.

As we already discussed in Case 4 of Lemma 6, except for graph P^{**} shown on the left of Figure 7, each of the other proper induced subgraphs of P^* either has a 1-cutset or a proper 2-cutset, and we do not consider because G_X is basic, or is a hole, which are considered in Case 2. There is only one possible choice of marker M_1 for the case $G_X = P^{**}$, in the sense that for

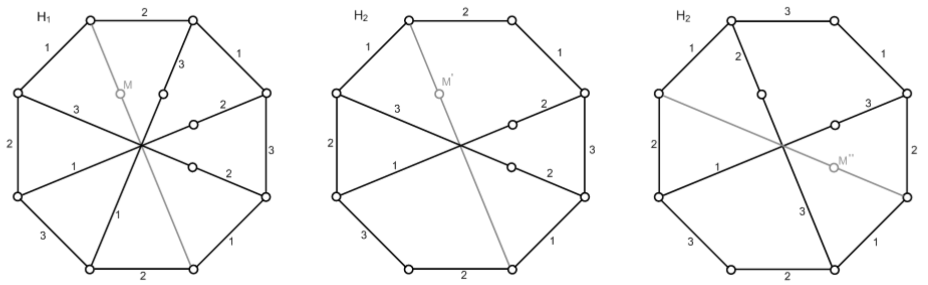

Figure 10: 3-edge-colourings of H_1 and H_2, for each possible choice of marker.

any other choice of marker M_1', we have $G_X \setminus M_1' = G_X \setminus M_1$. Observe, also, that there are at least two colours c_{a_1}, c_{a_2} in F_a and two colours c_{b_1}, c_{b_2} in F_b, and that $|\{c_{a_1}, c_{a_2}\} \cap \{c_{b_1}, c_{b_2}\}| = 1$ or 2. These two possibilities are considered in the first two colourings of Figure 7. □

Remark that the NP-Complete gadget $\tilde{P}$ of Figure 2 is constructed from P^*. The NP-completeness of edge-colouring graphs in $\mathcal{C}'$ is obtained as a consequence of $P^* \in \mathcal{C}'$. Using Lemma 9, we can prove that if the special graph P^* does not appear as a leaf in the decomposition tree, i.e., as a basic block when we recursively apply the proper 2-cutset decomposition to a biconnected graph $G \in \mathcal{C}'$ of maximum degree 3, then G is Class 1.

Theorem 11. *Let $G \in \mathcal{C}'$ be a connected graph of maximum degree 3. If G does not contain a 6-hole all of whose nodes are of degree 3, then G is Class 1.*

Proof: Assume the theorem does not hold and let G be a counterexample with fewest number of nodes. So G is a connected graph of $\mathcal{C}'$ of maximum degree 3, it does not contain a 6-hole all of whose nodes are of degree 3, and it is not 3-edge-colourable. By Theorem 6 either G is basic, or it has a 1-cutset, or it is biconnected and has a proper 2-cutset.

Suppose G is basic. G cannot be strongly 2-bipartite nor an induced subgraph of Heawood graph, since bipartite graphs are Class 1 [26]. Graph G cannot be a complete graph on four vertices, because such a graph is 3-edge-colourable. G cannot be a hole since it has maximum degree 3. So G must be an induced subgraph of the Petersen graph. G cannot be isomorphic to P nor P^*, because both of these graphs contain a 6-hole all of whose nodes are of degree 3. But all the other induced subgraphs of the Petersen graph are in fact 3-edge-colourable [6]. Therefore G cannot be basic.

Now suppose that G has a 1-cutset with split (X, Y, v). Note that blocks of decomposition are induced subgraphs of G, and hence both are connected graphs of $\mathcal{C}'$ that do not contain a 6-hole all of whose nodes are of degree 3. If $\Delta(G_X) = 3$ then since G is a minimum counterexample, G_X is 3-edge-colourable. If $\Delta(G_X) \leq 2$ then G_X is 3-edge-colourable by Vizing's Theorem. So G_X is 3-edge-colourable, and similarly so is G_Y. But then by Observation 1, G is also 3-edge-colourable, a contradiction.

Therefore G is biconnected and has a proper 2-cutset. Let (X, Y, a, b) be a split of a proper 2-cutset such that block G_X is basic (note that such a cutset exists by Lemma 5). By Lemma 4 both of the blocks G_X and G_Y are biconnected graphs of $\mathcal{C}'$. Since the marker node M is of degree 2 in both G_X and G_Y, and $G_X \setminus M$ and $G_Y \setminus M$ are both induced subgraphs of G, it follows that neither G_X nor G_Y can contain a 6-hole all of whose nodes are of degree 3. If M is a real node of G, then G_X and G_Y are both induced subgraphs of G, and hence $\Delta(G_X) \leq 3$ and $\Delta(G_Y) \leq 3$. If M is not a real node of G, then by definition of proper 2-cutset both a and b have a neighbor in both X and Y, and hence $\Delta(G_X) \leq 3$ and $\Delta(G_Y) \leq 3$. Since both G_X and G_Y have fewer nodes than G, it follows either from minimality of counterexample G or by Vizing's Theorem that both G_X and G_Y are 3-edge-colourable. Since G_X does not contain a 6-hole all of whose nodes are of degree 3, G_X is not isomorphic to P^*, and hence by Lemma 9, G is 3-edge-colourable, a contradiction. □

Corollary 2. *Every connected 6-hole-free graph of $\mathcal{C}'$ with maximum degree 3 is Class 1.*

A natural question in connection with Theorem 12 is whether forbidding 6-holes would make it easier to edge-colour graphs of $\mathcal{C}'$, and the answer is **no**. By observing graph G' of the proof of Theorem 2, one can easily verify that this graph has no 6-hole, so that the following theorem holds:

Theorem 12. *For each $\Delta \geq 3$, `CHRIND`(Δ-regular 6-hole-free graph in $\mathcal{C}$) is NP-complete.*

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References

[1] M. M. Barbosa, C. P. de Mello, and J. Meidanis. *Local conditions for edge-colouring of cographs.* Congr. Numer. **133** (1998) 45–55.

[2] H. L. Bodlaender. *Polynomial algorithms for graph isomorphism and chromatic index on partial k-trees.* J. Algorithms **11** (1990) 631–643.

[3] V. A. Bojarshinov. *Edge and total colouring of interval graphs.* Discrete Appl. Math. **114** (2001) 23–28.

[4] O. V. Borodin, A. V. Kostochka, and D. R. Woodall. *List edge and list total colourings of multigraphs.* J. Combin. Theory Ser. B **71** (1997) 184–204.

[5] L. Cai and J. A. Ellis. *NP-Completeness of edge-colouring some restricted graphs.* Discrete Appl. Math. **30** (1991) 15–27.

[6] D. Cariolaro and G. Cariolaro. *Colouring the petals of a graph.* Electron. J. Combin. **10** (2003) #R6.

[7] B. L. Chen, H.-L. Fu, and M. T. Ko. *Total chromatic number and chromatic index of split graphs.* J. Combin. Math. Combin. Comput. **17** (1995) 137–146.

[8] C. M. H. de Figueiredo, J. Meidanis, C. P. de Mello, and C. Ortiz. *Decompositions for the edge colouring of reduced indifference graphs.* Theoret. Comput. Sci. **297** (2003) 145–155.

[9] C. M. H. de Figueiredo, J. Meidanis, and C. P. de Mello. *Local conditions for edge-coloring.* J. Combin. Math. Combin. Comput. **32** (2000) 79–91.

[10] C. De Simone and C. P. de Mello. *Edge-colouring of join graphs.* Theoret. Comput. Sci. **355** (2006) 364–370.

[11] C. De Simone and A. Galluccio. *Edge-colouring of regular graphs of large degree.* Theoret. Comput. Sci. **389** (2007) 91–99.

[12] A. Gyárfás. *Problems from the world surrounding perfect graphs.* Zastos. Mat. **19** (1987) 413–441.

[13] I. Holyer. *The NP-completeness of edge-colouring.* SIAM J. Comput. **10** (1982) 718–720.

[14] D. S. Johnson. *The NP-completeness column: an ongoing guide.* J. Algorithms **6** (1985) 434–451.

[15] A. J. W. Hilton and Cheng Zhao. *On the edge-colouring of a graph whose core has maximum degree two.* J. Combin. Math. Combin. Comput. **21** (1996) 97–108.

[16] D. G. Hoffman and C. A. Rodger. *The chromatic index of complete multipartite graphs.* J.Graph Theory **16** (1992) 159–163.

[17] D. Leven and Z. Galil. *NP-completeness of finding the chromatic index of regular graphs.* J. Algorithms **4** (1983) 35–44.

[18] R. C. S. Machado and C. M. H. de Figueiredo. *Decompositions for edge-coloring join graphs and cobipartite graphs.* To appear in Discrete Appl. Math.

[19] C. Ortiz, N. Maculan, and J. L. Szwarcfiter. *Characterizing and edge-colouring split-indifference graphs.* Discrete Appl. Math. **82** (1998) 209–217.

[20] B. Randerath and I. Schiermeyer. *Vertex colouring and forbidden subgraphs—a survey.* Graphs Combin. **20** (2004) 1–40.

[21] D. P. Sanders and Y. Zhao. *Planar graphs of maximum degree seven are class I.* J. Combin. Theory Ser. B **83** (2001) 201–212.

[22] E. Steffen. *Classifications and characterizations of snarks.* Discrete Math. **188** (1998) 183–203.

[23] N. Trotignon and K. Vušković. *A structure theorem for graphs with no cycle with a unique chord and its consequences.* To appear in Journal of Graph Theory. (Available online at http://www.comp.leeds.ac.uk/vuskovi/chord.ps.)

[24] V. G. Vizing. *On an estimate of the chromatic class of a p-graph* (in Russian). Diskret. Analiz **3** (1964) 25–30.

[25] V. G. Vizing. *Critical graphs with given chromatic index* (in Russian). Diskret. Analiz **5** (1965) 9–17.

[26] D. B. West. *Introduction to Graph Theory.* Second edition. Prentice Hall (2001).

Capítulo 13

Anexo: Manuscrito "Total-chromatic number of unichord-free graphs"

Total-chromatic number of unichord-free graphs

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Abstract

A *unichord* is an edge that is the unique chord of a cycle in a graph. The class $\mathcal{C}$ of unichord-free graphs — that is, graphs that do not contain, as an induced subgraph, a cycle with a unique chord — was recently studied by Trotignon and Vušković [38], who proved for these graphs strong structure results and used these results to solve the recognition and vertex-colouring problems. Machado, Figueiredo and Vušković [29] determined the complexity of the edge-colouring problem in the class $\mathcal{C}$ and in the subclass $\mathcal{C}'$ obtained from $\mathcal{C}$ by forbidding squares. In the present work, we prove that the total colouring problem is NP-complete when restricted to graphs in $\mathcal{C}$. For the subclass $\mathcal{C}'$, we establish the validity of the Total Colouring Conjecture by proving that every non-complete {square,unichord}-free graph of maximum degree at least 4 is Type 1.

Keywords: cycle with a unique chord, decomposition, recognition, Petersen graph, Heawood graph, edge-colouring, total-colouring.

1 Introduction

In the present paper we deal with simple connected graphs. A graph G has vertex set $V(G)$ and edge set $E(G)$. An *element* of G is one of its vertices or edges and the set of elements of G is denoted $S(G) = V(G) \cup E(G)$. Two vertices $u, v \in V(G)$ are *adjacent* if $uv \in E(G)$; two edges $e_1, e_2 \in E(G)$ are *adjacent* if they share a common endvertex; a vertex u and an edge e are *incident* if u is an endvertex of e. The *degree of a vertex* v in G, denoted $deg_G(v)$, is the number of edges of G incident to v. We use the standard notation of K_n, C_n and P_n for complete graphs, cycle-graphs and path-graphs, respectively.

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A *total-colouring* is an association of colours to the elements of a graph in such a way that no adjacent or incident elements receive the same colour. The *total chromatic number* of a graph G, denoted $\chi_T(G)$, is the least number of colours sufficient to total-colour this graph. Clearly, $\chi_T(G) \geq \Delta(G)+1$, where $\Delta(G)$ denotes the maximum degree of a vertex in G. The Total Colouring Conjecture (TCC) states that every graph G can be total-coloured with $\Delta(G)+2$ colours. By the TCC only two values would be possible for the total chromatic number of a graph: $\chi_T(G) = \Delta(G)+1$ or $\Delta(G)+2$. If a graph G has total chromatic number $\Delta(G)+1$, then G is said to be *Type 1*; if G has total chromatic number $\Delta(G)+2$, then G is said to be *Type 2*. The TCC has been verified in restricted cases, such as graphs with maximum degree $\Delta \leq 5$ [24, 25, 34, 40], but the general problem is open since 1964, exposing how challenging the problem of total-colouring is.

It is NP-complete to determine whether the total chromatic number of a graph G is $\Delta(G)+1$ [35] (remark that a bipartite graph G is trivially $\Delta+2$-total-colourable, since we can use $\Delta(G)$ colours to colour the edges of G and 2 additional colours to colour the vertices of G). In fact, the problem remains NP-complete when restricted to r-regular bipartite inputs [30], for each fixed $r \geq 3$. The total-colouring problem is know to be polynomial — and the TCC is valid — for few very restricted graph classes, some of which we enumerate next:

- a cycle-graph G has $\chi_T(G) = \Delta(G)+1 = 3$ if $|V(G)| = 0 \mod 3$, and $\chi_T(G) = \Delta(G)+2 = 4$ otherwise [46];
- a complete graph G has $\chi_T(G) = \Delta(G)+1$ if $|V(G)|$ is odd, and $\chi_T(G) = \Delta(G)+2$ otherwise [46];
- a complete bipartite graph $G = K_{m,n}$ has $\chi_T(G) = \Delta(G)+1 = \max\{m,n\}+1$ if $m \neq n$, and $\chi_T(G) = \Delta(G)+2 = m+2 = n+2$ otherwise [46];
- a grid $G = P_m \times P_n$ has $\chi_T(G) = \Delta(G)+2$ if $G = P_2$ or $G = C_4$, and $\chi_T(G) = \Delta(G)+1$ otherwise [6];
- a series-parallel graph G has $\chi_T(G) = \Delta(G)+2$ if $G = P_2$ or $G = C_n$ with $n = 0 \mod 3$, and $\chi_T(G) = \Delta(G)+1$ otherwise [22, 43, 45].

The computational complexity of the total-colouring problem is unknown for several important and well studied graph classes. The complexity of total-colouring planar graphs is unknown; in fact, even the TCC has not yet been settled for the class [42]. The complexity of total-colouring is open for the class of chordal graphs, and the partial results for the related classes of interval graphs [3], split graphs [9] and dually chordal graphs [13] expose the interest in the total-colouring problem restricted to chordal graphs. Another class for which the complexity of total-colouring is unknown is the class of join graphs: the results found in the literature consider very restricted subclasses of join graphs, such as the join between a complete inequibipartite graph and a path [19] and the join between a complete bipartite graph and a cycle [20], all of which are Type 1.

In the present work we consider total-colouring restricted to unichord-free graphs. A *unichord* is an edge that is the unique chord of a cycle in a graph. The class $\mathcal{C}$ of unichord-free graphs — that is, graphs that do not contain (as an induced subgraph) a cycle with a unique chord — was recently studied by Trotignon and Vušković [38]. The main motivation to investigate the class is the existence of a structure theorem for it, a kind of strong result that is not frequent in the literature and that can be used to develop algorithms in the class. Basically, this structure result states that every graph in $\mathcal{C}$ can be built starting from a restricted set $\mathcal{C}_B$ of basic graphs and applying a series of known "gluing" operations, denoted in [38] by $\mathcal{O}_0$, $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$. Another motivation for the class is the concept of χ-boundness, introduced by Gyárfás [16] as a natural extension of perfect graphs. A family of graphs $\mathcal{G}$ is χ-*bounded* with χ-*binding function* f if, for every induced subgraph G' of $G \in \mathcal{G}$, it holds $\chi(G') \leq f(\omega(G'))$, where $\chi(G')$ denotes the chromatic number of G' and $\omega(G')$ denotes the size of a maximum clique in G'. The research of χ-bounded graphs is mainly devoted to understanding for what choices of forbidden induced subgraphs, the resulting family of graphs is χ-bounded (see [33] for a survey). Note that the class perfect graphs is a χ-bounded family with χ-binding function $f(x) = x$, and perfect graphs are characterized by excluding odd holes and their complements. Also, by Vizing's Theorem, the class of line graphs of simple graphs is a χ-bounded family with χ-binding function $f(x) = x + 1$ (this special upper bound is known as the *Vizing bound*) and line graphs are characterized by nine forbidden induced subgraphs [44]. The class $\mathcal{C}$ is also χ-bounded with the Vizing bound [38]. The following results are obtained in [38] for unichord-free graphs: an $O(nm)$ recognition algorithm, an $O(nm)$ algorithm for optimal vertex-colouring, an $O(n+m)$ algorithm for maximum clique, and the NP-completeness of the maximum stable set problem.

Machado, Figueiredo and Vušković [29] investigated whether the structure results of [38] could be applied to obtain a polynomial-time algorithm for the edge-colouring problem in $\mathcal{C}$. The authors obtain a negative answer, by establishing the NP-completeness of the edge-colouring problem restricted to unichord-free graphs. The authors investigate also the complexity of the edge-colouring in the subclass $\mathcal{C}'$ of {square,unichord}-free graphs. The class $\mathcal{C}'$ can be viewed as the class of the graphs that can be constructed from the same set $\mathcal{C}_B$ of basic graphs as in $\mathcal{C}$, but using one less operation (the join operation $\mathcal{O}_2$ of [38] is forbidden). For inputs in $\mathcal{C}'$, an interesting dicothomy is proved in [29]: if the maximum degree is not 3, the edge-colouring problem is polynomial, while for inputs with maximum degree 3, the problem is NP-complete.

It is a natural step to investigate the complexity of total-colouring restricted to classes for which the complexity of edge-colouring is already established. This approach is observed, for example, in the classes of outerplanar [47] graphs, series-parallel [43] graphs, and some subclasses of planar [42] graphs and join [19, 20] graphs. One important motivation for this approach is the search for "separating" classes, that is, classes for which the complexities of edge-colouring and total-colouring differ. All known separating classes, in this sense, are classes for which edge-colouring is polynomial and total-colouring is NP-complete, such as the case of bipartite graphs. In other words, there is no known example of a class for which edge-colouring is NP-complete and total-colouring is polynomial, an evidence that "total-colouring might be 'harder' than edge-colouring".

Another natural line of investigation is to consider validity of the Total Colouring Conjecture in graph classes — again, special attention is given to classes for which the edge-colouring problem is better understood. This approach is observed, for example, in the results for power of cycles [7], subclasses of planar graphs [46] and graphs with fixed maximum degree [46].

Considering the recent interest in colouring problems restricted to unichord-free graphs, it is natural to investigate the total-colouring problem for the classes $\mathcal{C}$ and $\mathcal{C}'$. In the present work, we obtain computational complexity results for the total-colouring problem restricted to unichord-free graphs. Moreover, we settle the validity of the TCC in $\mathcal{C}'$ by proving that every non-complete {square,unichord}-free graph of maximum degree at least 4 is Type 1. Table 1 summarizes the current status of colouring problems restricted to $\mathcal{C}$ and to $\mathcal{C}'$.

Problem \ Class	$\mathcal{C}$	$\mathcal{C}'$, $\Delta \geq 4$	$\mathcal{C}'$, $\Delta = 3$
vertex-colouring	Polynomial [38]	Polynomial [38]	Polynomial [38]
edge-colouring	NP-complete [29]	Polynomial [29]	NP-complete [29]
total-colouring	NP-complete*	Polynomial*	?
TCC	?	Settled*	Settled*

Table 1: Current status of colouring problems in $\mathcal{C}$ and $\mathcal{C}'$ — stars indicate results established in the present paper.

We observe that, while the complexity of total-colouring restricted to unichord-free graphs and to {square,unichord}-free graphs with maximum degree at least 4 are the same as the complexity of edge-colouring, the complexity of total-colouring {square,unichord}-free graphs with maximum degree 3 is not yet established as NP-complete, as is the case of edge-colouring. In fact, as we discuss in Section 5, there are evidences that $\mathcal{C}'$ could be a "special separating class", in the sense that it would be a class for which total-colouring is polynomial and edge-colouring is NP-complete. We remark that is frequent, in the total-colouring literature, the existence of classes for which the case of maximum degree exactly 3 presents great difficulty, representing a special challenge. This is the case, for instance, of series-parallel graphs [22, 45] and partial-grids [6, 28].

In Section 2 we prove the NP-completeness of determining the total chromatic number of graphs in $\mathcal{C}$. In Section 3 we state the structure results that are applied in Section 4 to obtain results on the total chromatic number of graphs in $\mathcal{C}'$. Section 5 contains further discussions on the TCC in class $\mathcal{C}$ and on the difficulty of determining the complexity of total-colouring restricted to {square,unichord}-free graphs with maximum degree 3.

2 NP-completeness result

In the present section, we prove the NP-completeness of the total-colouring problem restricted to unichord-free graphs. In fact, we prove that total-colouring is NP-complete for regular graphs of $\mathcal{C}$ with fixed degree $\Delta \geq 3$. The proof is inspired in the work of McDiarmid and Sánchez-Arroyo [30, 35], but has some critical differences to avoid cycles with a unique chord.

We use the term `TOTCHR`(P) to denote the problem of determining the total chromatic number restricted to graph inputs with property P. For example:

`TOTCHR`(graph of $\mathcal{C}$)
INSTANCE: a graph G of $\mathcal{C}$.
QUESTION: is $\chi_T(G) = \Delta(G) + 1$?

Theorem 1 [30, 35] establishes the NP-completeness of determining the total chromatic number of Δ-regular bipartite graphs of fixed degree $\Delta \geq 3$:

Theorem 1. (McDiarmid and Sanchez-Arroyo [30, 35]) *For each $\Delta \geq 3$, `TOTCHR`(Δ-regular bipartite graph) is NP-complete.*

We prove the NP-completeness of total-colouring restricted to unichord free graphs by a reduction from edge-colouring. The term CHRIND(P) denotes the problem of determining the chromatic index restricted to graph inputs with property P. For example:

CHRIND(graph of $\mathcal{C}$)
INSTANCE: a graph G of $\mathcal{C}$.
QUESTION: is $\chi'(G) = \Delta(G)$?

Theorem 2 [17, 26] establishes the NP-completeness of determining the chromatic index of Δ-regular graphs of fixed degree $\Delta \geq 3$:

Theorem 2. (Holyer [17]; Leven and Galil [26]) *For each $\Delta \geq 3$, CHRIND(Δ-regular graph) is NP-complete.*

Please refer to Figure 1. Graph S_t, for $t \geq 3$, is obtained from the

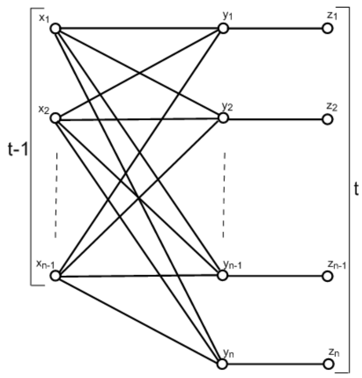

Figure 1: Graph S_t

complete bipartite graph $K_{t-1,t}$ by adding t pendant edges to the t vertices of degree $t-1$. Observe that S_t has $3t-1$ vertices, t of which have degree 1 and the remaining vertices having degree t. Graph S_t has the following property:

Lemma 1. (McDiarmid and Sánchez-Arroyo [30]) *Consider the graph S_t, where $t \geq 3$.*

1. *There is a $(t+1)$-total-colouring of S_t in which each of the vertices $y_1, y_2, ..., y_t$ is coloured differently.*

2. In any $(t+1)$-total-colouring of S_t each pendant edge has the same colour.

Graph S_t is a basic piece to construct the components used in the NP-completeness proof of the present section. We construct the bipartite graph $H_{n,t}$, for $n \geq 2$ and $t \geq 3$, by putting together two copies of S_t and identifying $t-n$ pendant edges of the first copy with $t-n$ edges of the second copy. Note that, except for the $2n$ pendant vertices, each of the other $4t-2$ vertices of $H_{n,t}$ has degree t. Graph $H_{n,t}$ is shown in Figure 2.

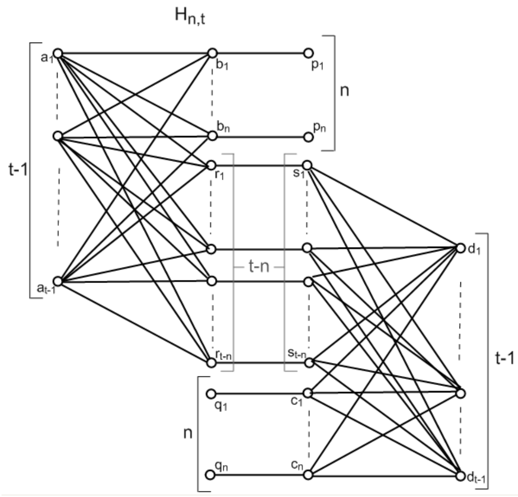

Figure 2: Graph $H_{n,t}$.

Lemma 2. *Consider graph $H_{n,t}$ with $t \geq 5$ and $n = \lceil (t+1)/2 \rceil$.*

1. *Consider a partial $(t+1)$-total-colouring π' of $H_{n,t}$ in which the pendant edges are coloured the same and the pendant vertices are also coloured (and nothing else is coloured). Then, this partial $(t+1)$-total-colouring extends to a $(t+1)$-total-colouring of $H_{n,t}$.*

2. *In any $(t+1)$-total-colouring of $H_{n,t}$, the pendant edges have the same colour.*

Proof:

Part 2 is immediate from Lemma 1. We prove part 1 next.

Claim 1: There is a colouring π_1 of vertices $b_1, ..., b_n$ using only colours in the set $\{1, ..., t\}$ such that $\pi_1(b_i) \neq \pi'(p_i)$, for $i = 1, ..., n$.

Case 1: $|\pi'(\{p_1, ..., p_n\})| \leq t - n$. Then $|\{1, ..., t\} \setminus \pi'(\{p_1, ..., p_n\})| \geq n$. So, just let the colours of $b_1, ..., b_n$ be n different colours in $\{1, ..., t\} \setminus \pi'(\{p_1, ..., p_n\})$.

Case 2: $|\pi'(\{p_1, ..., p_n\})| \geq t - n + 1$.

Case 2.1: t is odd. Then $|\pi'(p_1, ..., p_n)| \geq t-n+1 = t-\lceil (t+1)/2 \rceil +1 = (t+1)/2 = \lceil (t+1)/2 \rceil = n$. That is, each vertex in $c_1, ..., c_n$ has a different colour. So, just let $\pi_1(b_i) := \pi'(p_{i+1})$ for $i = 1, ..., n-1$ and $\pi_1(b_n) := \pi'(p_1)$.

Case 2.2: t is even. Then $|\pi'(p_1, ..., p_n)| \geq t - n + 1 = t - \lceil (t+1)/2 \rceil + 1 = t/2 = \lceil (t+1)/2 \rceil - 1 = n - 1$. That is, at most two vertices, say c_{n-1} and c_n can have the same colour. So, just let $\pi_1(b_i) := \pi'(p_{i+1})$ for $i = 1, ..., n-2$, let $\pi_1(b_{n-1}) := \pi'(p_1)$, and let the colour of b_n be any colour in $\{1, ..., t\} \setminus \pi'(p_1, ..., p_n)$.

Claim 2: There is a colouring π_2 of vertices $c_1, ..., c_n$ using only colours in the set $\{1, ..., t\}$ such that $\pi(c_i) \neq \pi'(q_i)$, for $i = 1, ..., n$. The proof is analogous to that of Claim 1.

Claim 3: If R and S are sets of colours such that $|R| = |S| = t - n \geq 2$, then there is a colouring π_3 of vertices $r_1, ..., r_{t-n}, s_1, ..., s_{t-n}$ such that $\pi_3(r_i) \in R$, $\pi_3(s_i) \in S$, and $\pi_3(r_i) \neq \pi_3(s_i)$, for $i = 1, ..., t-n$. Let $I := R \cap S$ and let the vertices $r_1, ..., r_{|I|}$ be coloured by π_3 with $|I|$ different colours of I. Now let $\pi_3(s_i) := \pi(r_{i+1})$, for $i = 1, ..., |I| - 1$ and $\pi_3(s_{|I|}) := \pi(r_1)$. Finally, let the colours of $r_{|I|+1}, ..., r_{t-n}$ be the $|R| - |I|$ different colours in $R \setminus I$ and let the colours of $s_{|I|+1}, ..., s_{t-n}$ be the $|S| - |I|$ different colours in $S \setminus I$.

Now we are ready to establish the lemma. In the following, assume that the pendant edges $b_1p_1, ..., b_np_n$, $r_1s_1, ..., r_ns_n, c_1q_1, ..., c_nq_n$ are coloured $t+1$ and that the pendant vertices $p_1, ..., p_n, q_1, ..., q_n$ are coloured different of $t+1$.

By Claim 1 we can colour $b_1, ..., b_n$ with n different colours in $\{1, ..., t\}$ creating no conflicts with the colours of $p_1, ..., p_n$. By Claim 2 we can colour $c_1, ..., c_n$ with n different colours in $\{1, ..., t\}$ creating no conflicts with the colours of $q_1, ..., q_n$. Since $t \geq 5$, we have $t-n = t-\lceil (t+1)/2 \rceil \geq t-(t+1)/2 = (t-1)/2 \geq 2$. So, by Claim 3, we can colour $r_1, ..., r_{t-n}$ with the colours in $\{1, ..., t\}$ not used in $b_1, ..., b_{t-n}$ and colour $s_1, ..., s_{t-n}$ with the colours in $\{1, ..., t\}$ not used in $c_1, ..., c_{t-n}$ in such a way that there are no conflicts between each b_i and c_i, for $i = 1, ..., t-n$. Finally, by Lemma 1, we can extend this partial $(t+1)$-total-colouring to each copy of S_t, colouring, then, each element of $H_{n,t}$. $\square$

Lemma 3 considers the cases of $H_{n,t}$ with $n = \lceil (t+1)/2 \rceil$ not covered by Lemma 2.

Lemma 3. *Consider graph $H_{n,t}$ with $n = 2$ and $t = 3$ or $n = 3$ and $t = 4$.*

1. *Consider a partial $(t+1)$-total-colouring π' of the graph $H_{n,t}$ in which the pendant edges are coloured the same and the pendant vertices are also coloured not all the same (and nothing else is coloured). Then this extends to a $(t+1)$-total-colouring of $H_{n,t}$.*

2. *In any $(t+1)$-total-colouring of $H_{n,t}$, the pendant edges have the same colour.*

Proof:

The case $H_{2,3}$ is proved in Lemma 2.2 of [30]. So, we consider the case $H_{n,t} = H_{3,4}$. Part 2 is immediate from Lemma 1. So, we prove part 1.

Assume that the pendant edges are coloured $t+1$ and let $X := \pi'(\{p_1, p_2, p_3\})$ and $Y := \pi'(\{q_1, q_2, q_3)\}$. Observe that both X and Y have size at most 3 and are contained in the set $\{1, 2, 3, 4\}$.

Case 1: $|X| = 3$ and $|Y| = 3$.

Case 1.1: $|X \cap Y| = 3$.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 3, and q_1, q_2 and q_2 are coloured, resp., 1, 2 and 3.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 3, and 1, and let the colours of c_1, c_2 and c_3 be, resp., 2, 3 and 4.

Case 1.2: $|X \cap Y| = 2$.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 3, and q_1, q_2 and q_2 are coloured, resp., 2, 3 and 4.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 3, and 1, and let the colours of c_1, c_2 and c_3 be, resp., 3, 4 and 2.

Case 2: $|X| = 3$ and $|Y| = 2$.

Case 2.1: $|X \cap Y| = 2$.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 3, and q_1, q_2 and q_2 are coloured, resp., 1, 2 and 2.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 3, and 1, and let the colours of c_1, c_2 and c_3 be, resp., 2, 3 and 4.

Case 2.2: $|X \cap Y| = 1$.

Case 2.2.1: The unique colour in $X \cap Y$ appears once in q_1, q_2, q_2.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 3, and q_1, q_2 and q_2 are coloured, resp., 1, 4 and 4.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 3, and 1, and let the colours of c_1, c_2 and c_3 be, resp., 4, 1 and 2.

Case 2.2.2: The unique colour in $X \cap Y$ appears twice in q_1, q_2, q_2.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 3, and q_1, q_2 and q_2 are coloured, resp., 1, 1 and 4.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 3, and 1, and let the colours of c_1, c_2 and c_3 be, resp., 4, 3 and 2.

Case 3: $|X| = 3$ and $|Y| = 1$.

Case 3.1: $|X \cap Y| = 1$.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 3, and q_1, q_2 and q_2 are coloured 1.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 3, and 1, and let the colours of c_1, c_2 and c_3 be, resp., 2, 3 and 4.

Case 3.2: $|X \cap Y| = 0$.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 3, and q_1, q_2 and q_2 are coloured 4.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 3, and 4, and let the colours of c_1, c_2 and c_3 be, resp., 1, 2 and 3.

Case 4: $|X| = 2$ and $|Y| = 2$.

Case 4.1: $|X \cap Y| = 2$.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 2, and q_1, q_2 and q_2 are coloured, resp., 1, 2 and 2 or, resp., 1, 1 and 2.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 1, and 3, and let the colours of c_1, c_2 and c_3 be, resp., 4, 3 and 1.

Case 4.2: $|X \cap Y| = 1$.

Case 4.2.1: The unique colour in $X \cap Y$ appears once in q_1, q_2, q_2.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 2, and q_1, q_2 and q_2 are coloured, resp., 1, 3 and 3.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 1, and 3, and let the colours of c_1, c_2 and c_3 be, resp., 3, 1 and 4.

Case 4.2.2: The unique colour in $X \cap Y$ appears twice in q_1, q_2, q_2.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 3 and 3, and q_1, q_2 and q_2 are coloured, resp., 2, 3 and 3.

Let the colours of b_1, b_2 and b_3 be, resp., 3, 1, and 2, and let the colours of c_1, c_2 and c_3 be, resp., 3, 1 and 4.

Case 4.3: $|X \cap Y| = 0$.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 2, and q_1, q_2 and q_2 are coloured, resp., 3, 4 and 4.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 1, and 3, and let the colours of c_1, c_2 and c_3 be, resp., 4, 3 and 1.

Case 5: $|X| = 2$ and $|Y| = 1$.

Case 5.1: $|X \cap Y| = 1$.

Case 5.1.1: The unique colour in $X \cap Y$ appears once in p_1, p_2, p_2.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 2, and q_1, q_2 and q_2 are coloured 1.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 1, and 3, and let the colours of c_1, c_2 and c_3 be, resp., 4, 2 and 3.

Case 5.1.2: The unique colour in $X \cap Y$ appears twice in p_1, p_2, p_2.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 1 and 2, and q_1, q_2 and q_2 are coloured 1.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 4, and 1, and let the colours of c_1, c_2 and c_3 be, resp., 4, 2 and 3.

Case 5.2: $|X \cap Y| = 0$.

W.l.o.g. p_1, p_2 and p_2 are coloured, resp., 1, 2 and 2, and q_1, q_2 and q_2 are coloured 3.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 1, and 3, and let the colours of c_1, c_2 and c_3 be, resp., 4, 1 and 2.

Case 6: $|X| = 1$ and $|Y| = 1$.

Remark that, by hypothesis, the pendant vertices are not coloured all the same. So, w.l.o.g. p_1, p_2 and p_2 are coloured 1, and q_1, q_2 and q_2 are coloured 2.

Let the colours of b_1, b_2 and b_3 be, resp., 2, 3, and 4, and let the colours of c_1, c_2 and c_3 be, resp., 1, 3 and 4.

In each of the previous colourings, there is a colour $\alpha \in \{1, 2, 3, 4\} \setminus \pi(\{b_1, b_2, b_3\})$ and a colour $\beta \in \{1, 2, 3, 4\} \setminus \pi(\{c_1, c_2, c_3\})$ such that $\alpha \neq \beta$. Let α be the colour of r_1 and β be the colour of s_1. By Lemma 1, this partial 5-total-colouring extends to each copy of S_t, colouring, then, each element of $H_{3,4}$. □

The original "replacement" graph R of the NP-completeness proof of [30] contains cycles with unique chords. We modify and extend R to a family R_t, $t \geq 3$, of "replacement" graphs in $\mathcal{C}$, as follows. Take $t + 1$ copies of $H_{n,t}$, with $n = \lceil (t+1)/2 \rceil$, and denote these copies by $H^{(1)}, H^{(2)}, ..., H^{(t+1)}$. The "replacement" graph R_t is such that each copy of $H_{n,t}$ in R_t has one pendant edge — which is called *real* — or two pendant edges — one of which is called *real*. For, identify each of t pendant vertices of $H^{(i)}$, $i = 1, 2, ..., t+1$, with a distinct $H^{(j)}$, $j \neq i$, by choosing one of the pendant vertices of $H^{(j)}$. Observe, in Figure 3, the construction of R_3 (resp. R_4) by replacing the vertices of K_4 (resp. K_5) with five copies of $H_{2,3}$ (resp. $H_{3,4}$).

Observe that, if t is even, then there are two pendant edges from each copy of $H_{\lceil (\Delta+1)/2 \rceil, t}$, one of which is called a *real pendant edge* — the other is called *not real*. If t is odd, there is one pendant edge from each copy of $H_{\lceil (\Delta+1)/2 \rceil, t}$, and each of them is called a *real pendant edge*.

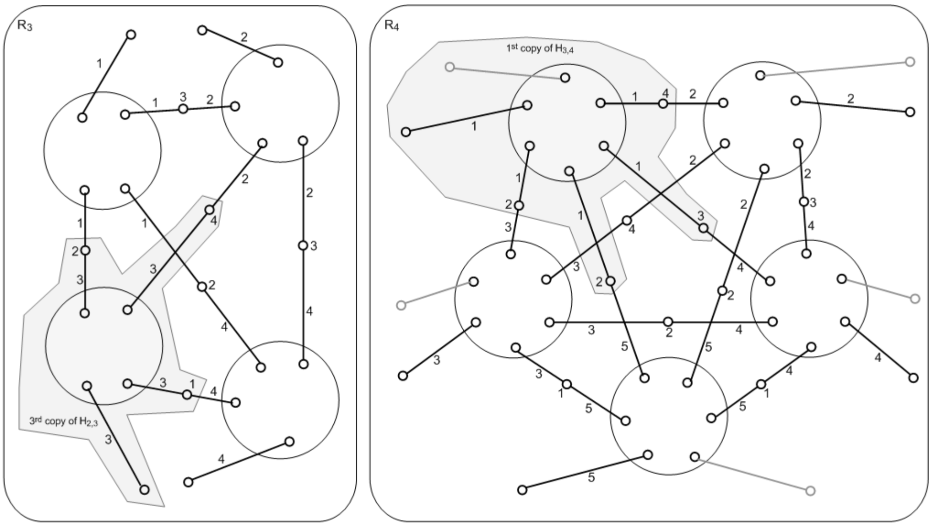

Figure 3: Replacement graphs R_3 and R_4.

Lemma 4. *Consider the replacement graph R_t.*

1. *Any partial $(t+1)$-total-colouring of R_t in which the $t+1$ real pendant edges have different colours and the pendant vertices of real pendant edges are also coloured (and nothing else is coloured) extends to a $(t+1)$-total-colouring of R_t*

2. *In any $(t+1)$-total-colouring of R_t the $t+1$ real pendant edges have all different colours.*

Proof:

Part 2 follows directly from Lemma 2. So, we consider part 1.

W.l.o.g., the colour of the real pendant edge of the i-th copy of $H_{\lceil (t+1)/2 \rceil, t}$, for $i = 1, ..., t+1$, is i. Let the colour of the $2n$ pendant edges of the i-th copy of $H_{\lceil (t+1)/2 \rceil, t}$, for $i = 1, ..., t+1$, be i. If $t = 3$ or $t = 4$ let the colours of the vertices of degree 2 be as shown in Figure 3. If $t \geq 5$, let the colours of each vertex of degree 2 be any colour different from the edges incident to it. By Lemmas 2 and 3, this partial $(t+1)$-total-colouring extends to each copy of $H_{\lceil (t+1)/2 \rceil, t}$, colouring, then, each element of R_t. $\square$

The "forcer" graph [30] $F_{n,t}$, for integers $n \geq 2$ and $t \geq 3$, is constructed by linking n copies of the graph $H_{2,t}$, as shown in Figure 4. Observe that graph $F_{n,t}$ has $2n$ pendant vertices and each of the other vertices have degree t.

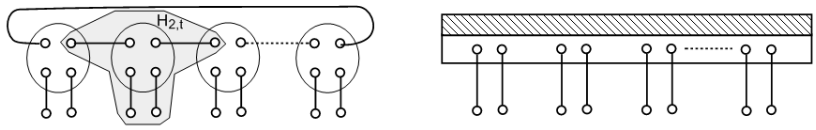

Figure 4: The forcer graph and its schematic representation.

Lemma 5. (McDiarmid and Sánchez-Arroyo [30]) *Consider $F = F_{n,t}$, for integers $n \geq 2$ and $t \geq 3$.*

1. *Consider a partial $(t+1)$-total-colouring of F in which each pendant edge is coloured the same and each pendant vertex is coloured (and nothing else is coloured). Then this extends to a $(t+1)$-total-colouring of F.*

2. *In any $(t+1)$-total-colouring of F each pendant edge has the same colour.*

Theorem 3 proves the NP-completeness of total-colouring Δ-regular graphs that do not contain a cycle with a unique chord, for each fixed degree $\Delta \geq 3$. Before proving Theorem 3 for regular graphs, we prove a different result: Lemma 6 proves the NP-completeness of problem $P_{\Delta,\delta}$ =`TOTCHR`(graph in $\mathcal{C}$ with maximum degree Δ, minimum degree δ, and such that every edge is incident to a maximum-degree vertex) for $\delta = 2$. Theorem 3 obtains a regular graph based on a novel strategy of induction on the minimum degree.

Lemma 6. *For each $\Delta \geq 3$, problem $P_{\Delta,2}$ is NP-complete.*

Proof:

Remark that total-colouring is in NP. Let G be an instance of the NP-complete problem `CHRIND` (Δ-regular graph). We construct an instance G' of problem $P_{\Delta,1}$ satisfying that G' is $(\Delta+1)$-total-colourable if and only if G is Δ-edge-colourable. The construction of graph G' is carried out with the following procedure:

1. Construct a graph G'' by replacing each vertex v of G with a copy of R_Δ, identifying Δ of its $\Delta+1$ real pendant edges with the Δ edges of G incident to v, and leaving pendant the remaining edges. Observe that G'' has $|V(G)|$ pendant edges if Δ is odd — each of which is real — and $(\Delta+2)|V(G)|$ pendant edges if Δ is even — $|V(G)|$ of which are real and $(\Delta+1)|V(G)|$ of which are not real.

2. Construct G' by identifying the $|V(G)|$ real pendant edges of G'' with $|V(G)|$ pendant edges of a forcer graph $F_{\lceil |V(G)|/2 \rceil, \Delta}$ (if $|V(G)|$ is odd the forcer graph will have one pendant edge more than G'').

3. We may assume that G' has no pendant vertices, otherwise, these pendant vertices can be removed without affecting the total chromatic number.

Figure 5 shows the construction of G' in the case where $G = K_4$.

The construction of G' is polynomial-time on the size of G — in fact, it is linear on $|V(G)|$. We claim that G' is $(\Delta + 1)$-total-colourable if and only if G is Δ-edge-colourable.

First, consider a $(\Delta + 1)$-total-colouring of G'. By Part 2 of Lemma 5 the $|V(G)|$ edges connecting the forcer graph to G'' have the same colour, say $\Delta + 1$. Therefore, by Part 2 of Lemma 4, colour $\Delta + 1$ is not used in the edges of G' corresponding to the original edges of G. So, the $(\Delta + 1)$-total-colouring of G' yelds a Δ-edge-colouring of G.

Second, consider a Δ-edge-colouring of G. Let the colours of the edges connecting the forcer graph to G'' be $\Delta + 1$. This yelds a colouring of the corresponding edges of G'. Since the colours of the pendant edges of each copy of the replacement graph are different, by Part 1 of Lemma 4, it is possible to extend the partial-total colouring to them. By Part 1 of Lemma 5 it is possible to extend this colouring to the forcer graph.

Finally, we prove that G' contains no cycle with unique chord and that every edge is incident to a maximum degree vertex. The fact that every edge is adjacent to a maximum degree vertex follows from the fact that this hold for each of the gadgets S_t, $H_{n,t}$, R_t and $F_{n,t}$. Now, observe that no path connecting two pendant vertices of S_t has a unique chord. As a consequence, no path connecting two pendant vertices of $H_{n,t}$ has a unique chord. Therefore, no path connecting two pendant vertices of a replacement graph R_t has a unique chord, in such a way that no cycle of G'' has a unique chord and no path connecting pendant vertices of G'' has a unique chord. Now, we must prove that the "attachment" of the forcer graph to G'' creates no cycle with a unique chord. This holds because: (1) the forcer graph creates no edge between two vertices of G'', and (2) no path connecting pendant vertices of a forcer graph has a unique chord. $\square$

The special graph $H_{1,t}$ is used in Theorem 3 to increase the minimum degree of a graph. Lemma 7 proves the existence of a special total-colouring of $H_{1,t}$.

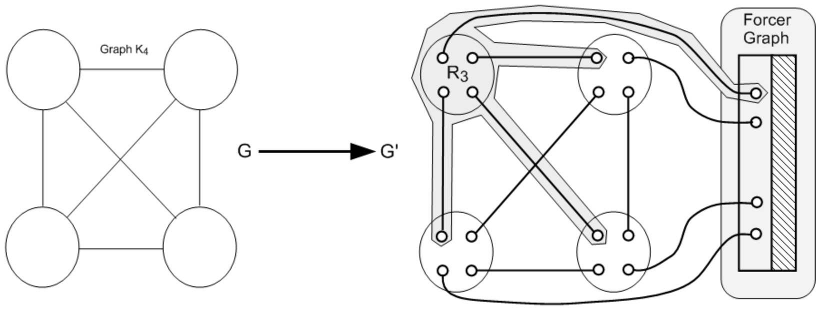

Figure 5: Construction of G' in the case where $G = K_4$.

Lemma 7. *There is a $(t+1)$-total-colouring of graph $H_{1,t}$, for $t \geq 3$, in which the pendant vertices have the same colour and the pendant edges have the same colour.*

Proof:

Please refer to Figure 2. Let π' be a partial $(t+1)$-total-colouring of S_t in which

- the colour of the two pendant vertices is 1;
- the colour of the two pendant edges is $t+1$;
- the colour of each edge $r_i s_i$, for $i = 1, ..., t-1$, is $t+1$;
- the colour of b_1 and c_1 is t;
- the colours of vertices $r_1, r_2, ..., r_{t-1}$ are, respectively, $1, 2, ..., t-1$;
- the colours of vertices $s_1, s_2..., s_{t-1}$ are, respectively, $t-1, 1, ..., t-2$.

By Lemma 1, the partial $(t+1)$-total-colouring π' can be extended to each copy of S_t, colouring, then, each element of $H_{1,t}$. □

Theorem 3. *For each $\Delta \geq 3$, `TOTCHR`(Δ-regular graph in $\mathcal{C}$) is NP-complete.*

Proof:

By Lemma 6, the problem $P_{\Delta,1}$ is NP-complete. Assume, as induction hypothesis, that the problem $P_{\Delta,k}$, $k < \Delta$, is NP-complete. We prove the theorem by induction on k. Let G be an instance of the problem $P_{\Delta,k}$ and construct an instance G' of problem $P_{\Delta,k+1}$ as follows.

1. Let G_1 and G_2 be two graphs isomorphics to G.

2. Let $H_1, ..., H_x$ as many graphs isomorphic to $H_{1,\Delta}$ as there are non-maximum-degree vertices in G.

3. Denote the non-maximum-degree vertices of G_1 (resp. G_2) by $v_1, ..., v_x$ (resp. by $w_1, ..., w_x$).

4. Construct G' by taking graphs G_1 and G_2 and, for each H_i, identifying one of the pendant vertices with v_i and the other pendant vertex with w_i.

Observe that G' has minimum degree $k+1$ and maximum degree Δ, and is constructed in polynomial time from G.

If G' is $(\Delta+1)$-total-colourable, then so is G — just restrict the $(\Delta+1)$-total-colouring of G to G'.

If G is $(\Delta+1)$-total-colourable, then so is G' — we show how to colour G' in the following. Let the colours of the elements of G_1 and G_2 be the same as in a $(\Delta+1)$-total-colouring of G. Let the colour of the pendant edges of H_i be a free colour at v_i. Finally, extend the colouring to each copy of H_i.

Finally, $G' \in \mathcal{C}$ as a consequence of the fact that there is no path with unique chord between the pendant vertices of each copy of $H_{1,\Delta}$ and that no two copies of $H_{1,\Delta}$ are connected to adjacent vertices. Moreover, each of the edges of $H_{1,t}$ and of G is adjacent to a vertex of degree k, so that each edge of G' is adjacent to a maximum degree vertex. So, $P_{\Delta,k+1}$ is NP-complete and the Theorem follows by induction. □

Remark our proposed inductive strategy is not used in [30]. The gadgets constructed in [30] are regular, while the proposed gadgets in $\mathcal{C}$ are not. So, while [30] uses an induction on the maximum degree, we use an induction on the minimum degree.

Class $\mathcal{C}$ has a strong structure [38], yet, it is NP-complete for total-colouring. Following the approach of [29], we manage in Section 4 to define a subclass of $\mathcal{C}$ where total-colouring is solvable in polynomial time. Consider the class $\mathcal{C}'$ as the subset of the graphs of $\mathcal{C}$ that do not have a square. The structure of graphs in $\mathcal{C}'$ is stronger than that of graphs in $\mathcal{C}$, and is described in detail in Section 3. We prove, in Section 4, that total-colouring is polynomial when restricted to inputs in $\mathcal{C}'$ with maximum degree not 4. The case of maximum degree 3 remains open.

3 Structure of graphs in $\mathcal{C}$ and $\mathcal{C}'$

In the present section we review decomposition results of unichord-free graphs and {square,unichord}-free graphs. These results are of the following form: every graph in $\mathcal{C}$ or in $\mathcal{C}'$ either belongs to a basic class or has a cutset. Before we can state these decomposition theorems, we define the basic graphs and the cutsets used in the decompositions.

The *Petersen graph* is the graph on vertices $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ so that both $a_1a_2a_3a_4a_5a_1$ and $b_1b_2b_3b_4b_5b_1$ are chordless cycles, and such that the only edges between some a_i and some b_i are a_1b_1, a_2b_4, a_3b_2, a_4b_5, a_5b_3. We denote by P the Petersen graph and by P^* the graph obtained from P by removal of one vertex. Observe that $P \in \mathcal{C}$.

The *Heawood graph* is a cubic bipartite graph on vertices $\{a_1, \ldots, a_{14}\}$ so that $a_1a_2 \ldots a_{14}a_1$ is a cycle, and such that the only other edges are a_1a_{10}, a_2a_7, a_3a_{12}, a_4a_9, a_5a_{14}, a_6a_{11}, a_8a_{13}. We denote by H the Heawood graph and by H^* the graph obtained from H by removal of one vertex. Observe that $H \in \mathcal{C}$.

A graph is *strongly 2-bipartite* if it is square-free and bipartite with bipartition (X, Y) where every vertex in X has degree 2 and every vertex in Y has degree at least 3. A strongly 2-bipartite graph is in $\mathcal{C}$ because any chord of a cycle is an edge between two vertices of degree at least three, so every cycle in a strongly 2-bipartite graph is chordless.

For the purposes of the present work, a graph G is called *basic*[1] if

1. G is a complete graph, a hole with at least five vertices, a strongly 2-bipartite graph, or an induced subgraph (not necessarily proper) of the Petersen graph or of the Heawood graph; and

2. G has no 1-cutset, proper 2-cutset or proper 1-join (all defined next).

We denote by $\mathcal{C}_B$ the set of the basic graphs. Observe that $\mathcal{C}_B \subseteq \mathcal{C}$.

A *cutset* S of a connected graph G is a set of elements, vertices and/or edges, whose removal disconnects G. A decomposition of a graph is the removal of a cutset to obtain smaller graphs, called the *blocks* of the decompositions, by possibly adding some nodes and edges to connected components of $G \setminus S$. The goal of decomposing a graph is trying to solve a problem on the whole graph by combining the solutions on the blocks. For a graph $G = (V, E)$ and $V' \subseteq V$, $G[V']$ denotes the subgraph of G induced by V'. The following cutsets are used in the decomposition theorems of class $\mathcal{C}$ [38]:

- A *1-cutset* of a connected graph $G = (V, E)$ is a node v such that V can be partitioned into sets X, Y and $\{v\}$, so that there is no edge between X and Y. We say that (X, Y, v) is a *split* of this 1-cutset.

[1] By the definition of [38], a basic graph is not, in general, indecomposable. However, our slightly different definition helps simplifying some of our proofs.

- A *proper 2-cutset* of a connected graph $G = (V, E)$ is a pair of non-adjacent nodes a, b, both of degree at least three, such that V can be partitioned into sets X, Y and $\{a, b\}$ so that: $|X| \geq 2$, $|Y| \geq 2$; there is no edge between X and Y, and both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain an ab-path. We say that (X, Y, a, b) is a *split* of this proper 2-cutset.

- A *1-join* of a graph $G = (V, E)$ is a partition of V into sets X and Y such that there exist sets A, B satisfying:

  - $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq Y$;
  - $|X| \geq 2$ and $|Y| \geq 2$;
  - there are all possible edges between A and B;
  - there is no other edge between X and Y.

  We say that (X, Y, A, B) is a *split* of this 1-join.

  A *proper 1-join* is a 1-join such that A and B are stable sets of G of size at least two.

We can now state a decomposition result for graphs in $\mathcal{C}$:

Theorem 4. (Trotignon and Vušković [38]) *If $G \in \mathcal{C}$ is connected then either $G \in \mathcal{C}_B$ or G has a 1-cutset, or a proper 2-cutset, or a proper 1-join.*

The *block* G_X (resp. G_Y) of a graph G with respect to a 1-cutset with split (X, Y, v) is $G[X \cup \{v\}]$ (resp. $G[Y \cup \{v\}]$).

The *block* G_X (resp. G_Y) of a graph G with respect to a 1-join with split (X, Y, A, B) is the graph obtained by taking $G[X]$ (resp. $G[Y]$) and adding a node y complete to A (resp. x complete to B). Nodes x, y are called *markers* of their respective blocks.

The *blocks* G_X and G_Y of a graph G with respect to a proper 2-cutset with split (X, Y, a, b) are defined as follows. If there exists a node c of G such that $N_G(c) = \{a, b\}$, then let $G_X = G[X \cup \{a, b, c\}]$ and $G_Y = G[Y \cup \{a, b, c\}]$. Otherwise, block G_X (resp. G_Y) is the graph obtained by taking $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) and adding a new node c adjacent to a, b. Node c is called the *marker* of the block G_X (resp. G_Y).

The blocks with respect to 1-cutsets, proper 2-cutsets and proper 1-joins are constructed in such a way that they remain in $\mathcal{C}$, as shown by Lemma 8.

Lemma 8. (Trotignon and Vušković [38]) *Let G_X and G_Y be the blocks of decomposition of G with respect to a 1-cutset, a proper 1-join or a proper 2-cutset. Then $G \in \mathcal{C}$ if and only if $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$.*

Observe that the Petersen graph and the Heawood graph may appear as a block of decomposition with respect to a proper 1-join, as shown in Figure 6. However, these graphs cannot appear as a block of decomposition with respect to proper 2-cutset, because they have no vertex of degree 2 to play the role of a marker.

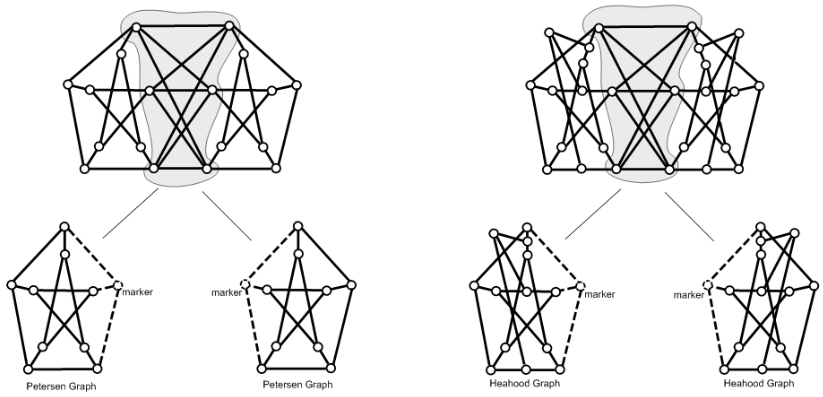

Figure 6: Decomposition trees with respect to proper 1-joins. In the graph on the left, the basic blocks of decomposition are two copies of the Petersen graph. In the graph on the right, the basic blocks of decomposition are two copies of the Heawood graph.

Despite the fact that the Petersen graph and the Heawood graph do not appear as a block of decomposition with respect to proper 2-cutset, they must be listed as a basic graph, because these graphs, themselves, are in $\mathcal{C}'$. So, the Petersen graph (resp. the Heawood graph) appears as a leaf of exactly one decomposition tree, namely, the decomposition tree of the Petersen graph (resp. the Heawood graph) — which is, actually, a trivial decomposition tree. Observe that graphs P^* and H^* may appear as decomposition block with respect to proper 2-cutset, as shown in Figure 7.

We reviewed results that show how to decompose a graph of $\mathcal{C}$ into basic blocks: Theorem 4 states that each graph in $\mathcal{C}$ has a 1-cutset, a proper 2-cutset or a proper 1-join, while Lemma 8 states that the blocks generated with respect to any of these cutsets are still in $\mathcal{C}$. We now obtain similar results for $\mathcal{C}'$. These results are not explicit in [38], but they can be obtained as consequences of results in [38] and by making minor modifications in its proofs. As we discuss in the following observation [4], for the goal of total-colouring, we only need to consider the **biconnected** graphs of $\mathcal{C}'$.

Observation 1. *Let G be a connected graph with a 1-cutset with split*

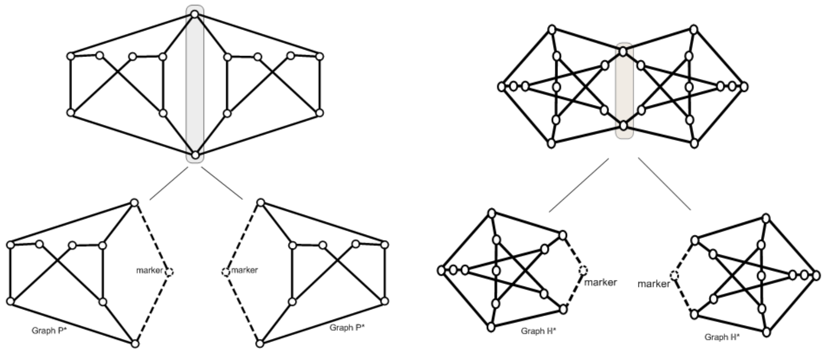

Figure 7: Decomposition trees with respect to proper 2-cutsets. In the graph on the left, the basic blocks of decomposition are two copies of P^*. In the graph on the right, the basic blocks of decomposition are two copies of H^*.

(X, Y, v). *The chromatic index of* G *is* $\chi'(G) = \max\{\chi'(G_X), \chi'(G_Y), \Delta(G)+1\}$.

By Observation 1, if both blocks G_X and G_Y are $\Delta(G)+1$-total-colourable, then so is G. That is, once we know the total chromatic number of the biconnected components of a graph, it is easy to determine the total chromatic number of the whole graph. So, we may focus our investigation on the biconnected graphs of $\mathcal{C}'$.

Theorem 5. (Trotignon and Vušković [38]) *If* $G \in \mathcal{C}'$ *is biconnected, then either* $G \in \mathcal{C}_B$ *or* G *has a proper 2-cutset.*

Theorem 5 is an immediate consequence of Theorem 4: since G has no 4-hole, G cannot have a proper 1-join, and since G is biconnected, G cannot have a 1-cutset.

Next, in Lemma 9, we show that the blocks of decomposition of a biconnected graph of $\mathcal{C}'$ with respect to a proper 2-cutset, are also biconnected graphs of $\mathcal{C}'$. The proof of Lemma 9 is similar to that of Lemma 5.2 of [38]. For the sake of completeness, the proof, which uses the result of Theorem 6 below, is included here.

Theorem 6. (Trotignon and Vušković [38]) *Let* $G \in \mathcal{C}$ *be a connected graph. If* G *contains a triangle then either* G *is a complete graph, or some vertex of the maximal clique that contains this triangle is a 1-cutset of* G.

Lemma 9. (Machado, Figueiredo and Vušković [29]) *Let* $G \in \mathcal{C}'$ *be a biconnected graph and let* (X, Y, a, b) *be a split of a proper 2-cutset of* G. *Then both* G_X *and* G_Y *are biconnected graphs of* $\mathcal{C}'$.

Proof:

We first prove that G is triangle-free. Suppose G contains a triangle. Then, by Theorem 6, either G is a complete graph, which contradicts the assumption that G has a proper 2-cutset, or G has a 1-cutset, which contradicts the assumption that G is biconnected. So G is triangle-free, and hence by construction, both of the blocks G_X and G_Y are triangle-free.

Now we show that G_X and G_Y are square-free. Suppose w.l.o.g. that G_X contains a square C. Since G is square-free, C contains the marker node M, which is not a real node of G, and $C = MazbM$, for some node $z \in X$. Since M is not a real node of G, we have $deg_G(z) > 2$, otherwise, z would be a marker of G_X. Let z' be a neighbor of z distinct of a and b. Since G is triangle-free, z' is not adjacent to a nor b. Since z is not a 1-cutset, there exists a path P in $G[X \cup \{a, b\}]$ from z' to $\{a, b\}$. We choose z' and P subject to the minimality of P. So, w.l.o.g., $z'Pa$ is a chordless path. Note that b is not adjacent to the neighbor of a along P because G is triangle-free and square-free, so that z is the unique common neighbor of a and b in G. So, by the minimality of P, vertex b does not have a neighbor in P. Now let Q be a chordless path from a to b whose interior is in Y. So, $bzz'PaQb$ is a cycle of G with a unique chord (namely az), contradicting the assumption that $G \in \mathcal{C}$.

By Lemma 8, G_X and G_Y both belong to $\mathcal{C}$, and since G_X and G_Y are both square-free, it follows that G_X and G_Y both belong to $\mathcal{C}'$.

Finally we show that G_X and G_Y are biconnected. Suppose w.l.o.g. that G_X has a 1-cutset with split (A, B, v). Since G is biconnected and $G[X \cup \{a, b\}]$ contains an ab-path, we have that $v \neq M$, where M is the marker of G_X. Suppose $v = a$. Then, w.l.o.g., $b \in B$, and $(A, B \cup Y, a)$ is a split of a 1-cutset of G, with possibly M removed from $B \cup Y$, if M is not a real node of G, contradicting the assumption that G is biconnected. So $v \neq a$ and by symmetry $v \neq b$. So $v \in X \setminus \{M\}$. W.l.o.g. $\{a, b, M\} \subset B$. Then $(A, B \cup Y, v)$ is a split of a 1-cutset of G, with possibly M removed from $B \cup Y$ if M is not a real node of G, contradicting the assumption that G is biconnected. $\square$

Observe that Lemma 8 is somehow stronger than Lemma 9. While Lemma 8 states that a graph is in $\mathcal{C}$ **if and only if** the blocks with respect to any cutset are also in $\mathcal{C}$, Lemma 9 establishes only one direction: **if** a graph is a biconnected graph of $\mathcal{C}'$, **then** the blocks with respect to any cutset are also biconnected graph of $\mathcal{C}'$. For our goal of edge-colouring, there is no need of establishing the "only if" part. Anyway, it is possible to verify that, if both blocks G_X and G_Y generated with respect to a proper 2-cutset of a graph G are biconnected graphs of $\mathcal{C}'$, then G itself is a biconnected graph of $\mathcal{C}'$.

Next lemma shows that every non-basic biconnected graph in $\mathcal{C}'$ has a decomposition such that one of the blocks is basic.

Lemma 10. (Machado, Figueiredo and Vušković [29]) *Every biconnected graph $G \in \mathcal{C}' \setminus \mathcal{C}_B$ has a proper 2-cutset such that one of the blocks of decomposition is basic.*

Proof:

By Theorem 5, graph G has a proper 2-cutset. Consider all possible 2-cutset decompositions of G and pick a proper 2-cutset S that has a block of decomposition B whose size is smallest possible. By Lemma 9, $B \in \mathcal{C}'$ and is biconnected. So by Theorem 5, either B has a proper 2-cutset or it is basic. We now show that in fact B must be basic.

Let (X, Y, a, b) be a split with respect to S. Let M be the marker node of G_X, and assume w.l.o.g. that $B = G_X$. Suppose G_X has a proper 2-cutset with split (X_1, X_2, u, v). By minimality of $B = G_X$, $\{a, b\} \neq \{u, v\}$. Assume w.l.o.g. $b \notin \{u, v\}$. Note that since $deg_{G_X}(u) \geq 3$ and $deg_{G_X}(v) \geq 3$, it follows that $M \notin \{u, v\}$. Suppose $a \notin \{u, v\}$. Then w.l.o.g. $\{a, b, M\} \subseteq X_1$, and hence $(X_1 \cup Y, X_2, u, v)$, with M removed if M is not a real node of G, is a proper 2-cutset of G whose block of decomposition G_{X_2} is smaller than G_X, contradicting the minimality of $G_X = B$. Therefore $a \in \{u, v\}$. Then w.l.o.g. $\{b, M\} \subseteq X_1$, and hence $(X_1 \cup Y, X_2, u, v)$, with M removed if M is not a real node of G, is a proper 2-cutset of G whose block of decomposition G_{X_2} is smaller than G_X, contradicting the minimality of $G_X = B$. Therefore G_X does not have a proper 2-cutset, and hence it is basic. □

4 Total Colouring Conjecture in $\mathcal{C}'$

In the present section we investigate the total chromatic number of graphs in $\mathcal{C}'$. We prove that non-complete {square-unichord}-free graphs of maximum degree at least 4 are Type 1. As a consequence, we settle the validity of the Total Colouring Conjecture in $\mathcal{C}'$.

We describe a technique to total-colour a graph in $\mathcal{C}'$ by combining total-colourings of its blocks with respect to a proper 2-cutset. Remark that the decomposition blocks are not necessarily subgraphs of the original graph:

possibly they are constructed by the addition of a marker vertex. This is illustrated in the example of Figure 8, where G is P^*-free, yet, graph P^* appears as a block with respect to a proper 2-cutset of G.

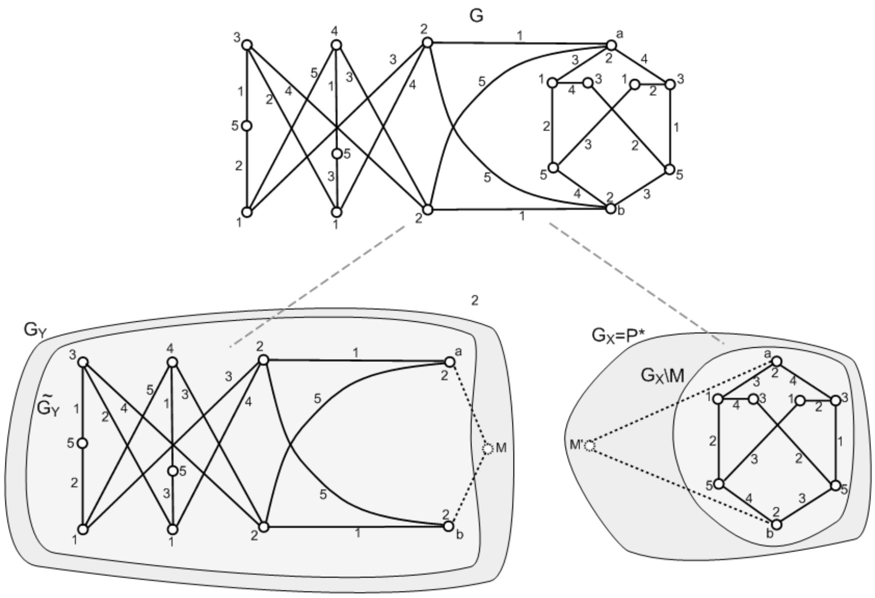

Figure 8: Example of decomposition with respect to a proper 2-cutset $\{a, b\}$. Observe that the marker vertices and their incident edges — identified by dashed lines — do not belong to the original graph.

Observation 2. *Consider a graph $G \in \mathcal{C}'$ with the following properties:*

- (X, Y, a, b) *is a split of a proper 2-cutset of* G*;*
- $\tilde{G_Y}$ *is obtained from* G_Y *by removing its marker if this marker is not a real vertex of* G*;*
- π_Y *is a* $\Delta(G) + 1$*-total-colouring of* $\tilde{G_Y}$*;*
- F_a *(resp.* F_b*) is the set of the colours in* $\{1, 2, ..., \Delta + 1\}$ *not used by* π_Y *in* a *(resp.* b*) nor in any edge of* $\tilde{G_Y}$ *incident to* a *(resp.* b*).*

If there exists a $\Delta(G) + 1$*-total-colouring* π_X *of* $G_X \setminus M$*, where* M *is the marker vertex of* G_X*, such that* $\pi_X(a) = \pi_Y(a)$ *(rep.* $\pi_X(b) = \pi_Y(b)$*) and each colour used in an edge incident to* a *(resp.* b*) is in* F_a *(resp.* F_b*), then* G *is* $\Delta + 1$*-total-colourable.*

The above observation shows that, in order to extend a $\Delta(G)+1$-total-colouring of $\tilde{G_Y}$ to a $\Delta(G)+1$-total-colouring of G, one must colour the elements of $G_X \setminus M$ in such a way that the colours of a, b, and the edges incident to them create no conflicts with the colours of the elements of $\tilde{G_Y}$. Moreover, there is no need to colour the edges incident to the marker M of G_X: if this marker is a vertex of G, its incident edges are already coloured by π_Y, otherwise, these edges are not real edges of G.

In the example of Figure 8, we exhibit a 5-total-colouring $\tilde{\pi_Y}$ of $\tilde{G_Y}$. In the notation of Observation 2, $F_a = \{3,4\}$ and $F_b = \{3,4\}$. We exhibit, also, a 5-total-colouring of $G_X \setminus M$ such that the colours of a and b are the same as in $\tilde{\pi_Y}$ of $\tilde{G_Y}$, the colours of the edges incident to a are $\{3,4\} \subset F_a$ and the colours of the edges incident to b are $\{3,4\} \subset F_b$. So, by Observation 2, we can combine the 5-total-colourings $\tilde{\pi_Y}$ and π_X in a 5-total-colouring of G, as it is done in Figure 8.

Before we proceed and show how to determine the total chromatic number of graphs in $\mathcal{C}'$ with maximum degree $\Delta \geq 4$, we need to introduce additional tools and concepts.

A *partial k-total-colouring* of a graph $G = (V, E)$ is a colouring of a subset E' of E, that is, a function $\pi : E' \rightarrow \{1, 2, ..., k\}$ such that no adjacent or incident elements of E' receive the same colour. The *set of free-colours* at vertex u with respect to a partial-total-colouring $\pi : E' \rightarrow \mathbf{C}$ is the set $\mathbf{C} \setminus \pi(\{u\} \cup \{uv | uv \in E'\})$.

The *list-edge-colouring* problem is described next. Let $G = (V, E)$ be a graph and let $\mathcal{L} = \{L_e\}_{e \in E}$ be a collection which associates to each edge $e \in E$ a set of colours L_e called the *list* relative to e. It is asked whether there is an edge-colouring π of G such that $\pi(e) \in L_e$ for each edge $e \in E$. Theorem 7 is a result on list-edge-colouring which is applied, in the present work, to colour the edges of some basic graphs: strongly 2-bipartite graphs, Heawood graph and its subgraphs, and holes.

Theorem 7. (Borodin, Kostochka, and Woodall [4]) *Let $G = (V, E)$ be a bipartite graph and $\mathcal{L} = \{L_e\}_{e \in E}$ be a collection of lists of colours which associates to each edge $uv \in E$ a list L_{uv} of colours. If, for each edge $uv \in E$, $|L_{uv}| \geq \max\{deg_G(u), deg_G(v)\}$, then there is an edge-colouring π of G such that, for each edge $uv \in E$, $\pi(uv) \in L_{uv}$.*

We investigate, now, how to $(\Delta(G)+1)$-total-colour a graph $G \in \mathcal{C}'$ by combining $(\Delta(G)+1)$-total-colourings of its blocks with respect to a proper 2-cutset. More precisely, Lemma 11 shows how this can be done if one of the blocks is basic. Subsequently, we obtain, in Theorem 8 and its Corollary 1, a characterization of Class 2 graphs in $\mathcal{C}'$ with maximum degree at least 4 which establishes the polynomiality of determining the total chromatic number of these graphs.

Lemma 11. *Let $G \in \mathcal{C}'$ be a graph of maximum degree $\Delta \geq 4$ and let (X, Y, a, b) be a split of proper 2-cutset, in such a way that G_X is basic. If G_Y is $(\Delta+1)$-total-colourable, then G is $(\Delta+1)$-total-colourable.*

Proof:

Denote by M the marker vertex of G_X and let $\tilde{G_Y}$ be obtained from G_Y by removing its marker if this marker is not a real vertex of G. Since $\tilde{G_Y}$ is a subgraph of G_Y, graph $\tilde{G_Y}$ is $(\Delta+1)$-total-colourable. Let π_Y be a $(\Delta+1)$-total-colouring of $\tilde{G_Y}$ — that is, a partial-total-colouring of G — and let F_a and F_b be the sets of the free colours of a and b, respectively, with respect to the partial-total-colouring π_Y. We show how to extend the partial-total-colouring π_Y to G, as described in Observation 2, that is, by colouring the elements of $G_X \setminus M$. Since a and b are not adjacent, G_X is not a complete graph. Moreover, the block G_X cannot be isomorphic to the Petersen graph or to the Heawood graph, because these graphs are cubic and G_X has a marker vertex M of degree 2. So, G_X is isomorphic to an induced subgraph of P^*, or to an induced subgraph of H^*, or to a strongly 2-bipartite graph, or to a cycle-graph.

Case 1. G_X is a strongly 2-bipartite graph.

Since $deg_{G_X}(M) = 2$, vertex M belongs to the bipartition of G_X whose vertices have degree 2. So, vertices a and b belong to the bipartition of G_X whose vertices have degrees larger than 2, and $|F_a| \geq 2$ and $|F_b| \geq 2$. Associate to each edge of $G_X \setminus M$ incident to a (resp. b) a list of colours equal to F_a (resp. F_b). To each of the other edges of $G_X \setminus M$, associate list $\{1, ...\Delta\}$. Now, to each edge uv of $G_X \setminus M$, it is associated a list of colours whose size is not smaller than $\max\{deg_{G_X \setminus M}(u), deg_{G_X \setminus M}(v)\}$ and, by Theorem 7, there is a colouring of the edges of $G_X \setminus M$ from these lists. Now, colour each of the vertices which belongs to the same bipartition of a and b with colour $\Delta + 1$. Finally, colour each of the vertices of degree 3 with some free colour, which can be done because there are at least $\Delta(G) + 1 \geq 5$ free-colours and each of these vertices has two incident edges and two adjacent vertices coloured.

Case 2. G_X is a hole.

First colour the edges of $G_X \setminus M$ with some free colour. Now, observe that each of the uncoloured elements of $G_X \setminus M$ has four incident or adjacent elements, which are two edges and two vertices. Since $\Delta(G) + 1 \geq 5$, these elements can be coloured sequentially in any order by setting, at each step, the colour of an element to be a colour which is free in that element.

Case 3. G_X is an induced subgraph of the Heawood graph.

Figure 9 exhibits the possible restrictions at the proper 2-cutset imposed by the total-colouring of G_Y. Figure 10 exhibits the total-colourings of H^* subject to each possible restriction. Total-colourings of the proper subgraphs of H^* can be obtained from the total-colourings of Figure 10.

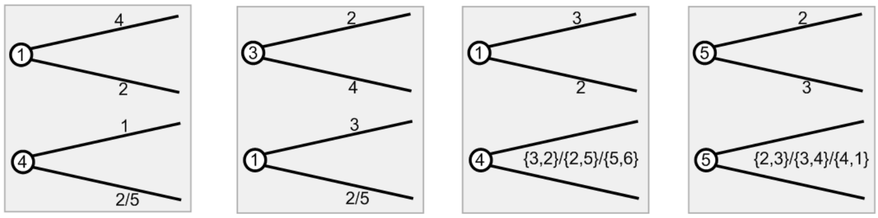

Figure 9: The possible colouring restrictions at the proper 2-cutset.

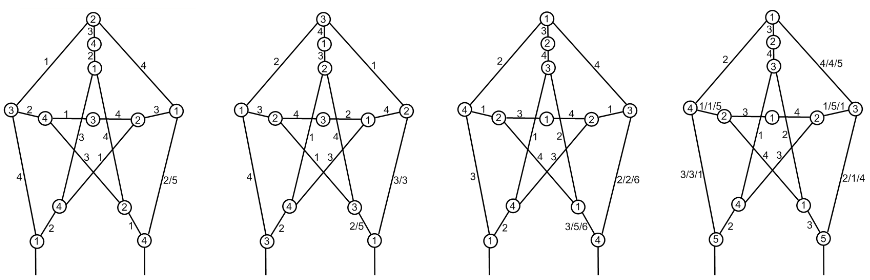

Figure 10: The total-colourings subject to each restriction.

Case 4. G_X is an induced subgraph of the Petersen graph.

Since G_X has a marker of degree 2, G_X is not the Petersen graph, so that G_X is a proper subgraph of the Petersen graph. So, there is a 6-cycle or path $Q = v_1, ..., v_\beta$ in $G_X \setminus M$ such that:

1. $(G_X \backslash M) \backslash \{v_1, ..., v_\beta\}$ is an independent set of vertices denoted $\{s_1, ..., s_k\}$;

2. $\{a, b\} \subset \{s_1, ..., s_k\}$;

3. each $s_1, ..., s_k$ has degree at most 2;

4. no two edges incident to different vertices s_i ans s_j are adjacent.

So, we can colour $G_X \setminus M$ as follows. First, colour each $s_1, ..., s_k$ — except for a and b, which are already coloured — and its incident edges. Now, associate to each element of the cycle/path Q — vertices and edges — a list containing its free colours. Observe that each of the elements of Q is incident/adjacent to at most two elements already coloured, so that each of the lists have size at least 3. Since Q is a 6-cycle or a path, it is 3-total-choosable[2], so that it can be total-coloured from those lists. □

Using Lemma 11 we can determine in polynomial time the total chromatic number of {square,unichord}-free graphs of maximum degree at least 4, as we show in Theorem 8 and its Corollary 1.

Theorem 8. *If λ is an integer at least 4 and G is a connected non-complete graph of $\mathcal{C}'$ with maximum degree $\Delta(G) \leq \lambda$, then G is $\lambda+1$-total-colourable.*

Proof: We prove the theorem by induction on the number of vertices of the graph satisfying the hypothesis. Let $G \in \mathcal{C}'$ be a connected graph with k vertices such that $\Delta(G) \leq \lambda$ and G is not a complete graph. By Theorem 5 either G is basic, or G has a 1-cutset, or G is biconnected and has a proper 2-cutset.

Suppose G is basic. If G is a strongly 2-bipartite graph, we can easily colour its elements with $\Delta(G) + 1$ colours as follows. First, colour the edges of G with colours 1,2,...,$\Delta(G)$. Then, colour each of the vertices of degree at least 3 with a colour in 1,2,...,$\Delta(G)$+1 not used in its incident edges. Finally, colour the vertices of degree 2 with a colour in 1,2,...,$\Delta(G) + 1$ not used in the four incident or adjacent elements. If G is not strongly 2-bipartite, then G is a hole or a subgraph of the Petersen graph or of the Heawood graph, so that $\Delta(G) \leq 3 \leq \lambda + 1 - 2$ and G is $\lambda + 1$-total-colourable — in fact, these graphs are Type 1, as shown in Figure 11. Assume, as induction hypothesis, that every connected non-complete graph $G' \in \mathcal{C}'$ with $k' < k$ vertices such that $\Delta(G') \leq \lambda$ is $\lambda + 1$-total-colourable.

Suppose G has a 1-cutset with split (X, Y, v). Note that blocks of decomposition G_X and G_Y are induced subgraphs of G and hence both belong to $\mathcal{C}'$. If G_X (resp. G_Y) is complete, then its maximum degree is at most $\lambda - 1$, so that G_X (resp. G_Y) is $\lambda + 1$-total-colourable. If G_X (resp. G_Y) is not complete, G_X (resp. G_Y) is $\lambda + 1$-total-colourable by the induction hypothesis. In any case, both G_X and G_Y are $\lambda + 1$-total-colourable, and hence by Observation 1, graph G is $\lambda + 1$-total-colourable.

Finally, suppose G is biconnected and has a proper 2-cutset. Let (X, Y, a, b) be a split of a proper 2-cutset such that block G_X is basic (note that such

[2] The reader may refer to [22] for results on total-choosability of cycles.

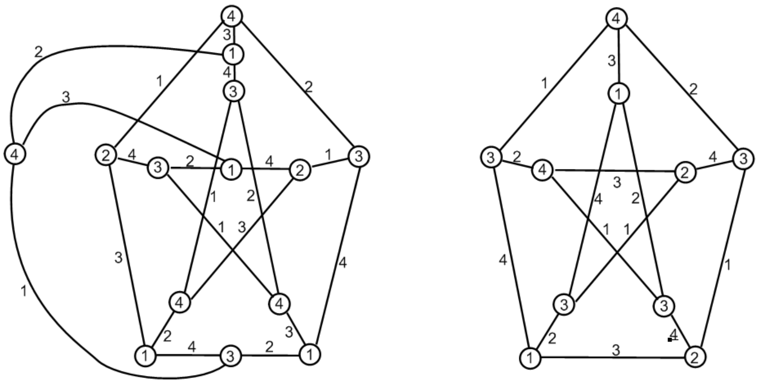

Figure 11: Total-colourings of the Heawood graph and of the Petersen graph.

a cutset exists by Lemma 10). By Theorem 6, block G_X is not a complete graph. By Lemma 9, block G_Y is in $\mathcal{C}'$. By the induction hypothesis, block G_Y is $\lambda+1$-total-colourable. By Lemma 11, graph G is $\lambda+1$-total-colourable. □

Corollary 1. *A connected graph $G \in \mathcal{C}'$ of maximum degree $\Delta \geq 4$ is Type 2 if and only if it is an even order complete graph.*

Proof:

If G is complete, then the result clearly holds. So, we may assume G is not complete. Just choose $\lambda = \Delta$ in Theorem 8 to prove that every connected non-complete graph of $\mathcal{C}'$ with maximum degree $\Delta(G) \geq 4$ is $\lambda+1$-total-colourable, hence Class 1. □

The result of Corollary 1 allows us to settle the validity of the TCC in $\mathcal{C}'$.

Corollary 2. *The Total Colouring Conjecture holds for {square,unichord}-free graphs.*

Proof:

If G is a complete graph or G has maximum degree at most 2, then the TCC holds. So, we may assume that G is not a complete graph and $\Delta(G) \geq 3$. If $\Delta(G) \geq 4$ then, by Corollary 1, graph G is $\Delta+1$-total-colourable. If $\Delta(G) = 3$ then just choose $\lambda = 4$ in Theorem 8 and G is total-colourable with $\lambda + 1 = \Delta(G) + 2$ colors. □

5 Final considerations

The present work represents an important step toward the understanding of the computational complexity of classical colouring problems restricted to unichord-free graphs. As we discussed in the Introduction, it is natural to consider the total-colouring problem restricted to classes for which the edge-colouring problem is solved. Up to now, three kinds of results have been obtained by this approach: either a class is NP-complete for both edge-colouring and total-colouring, as in the case of perfect graphs [5, 35], or a class is polynomial for both edge-colouring and total-colouring, as in the case of series-parallel graphs [22, 45], or a class is polynomial for edge-colouring and NP-complete for total-colouring, as in the case of bipartite graphs [35, 44]. So, it would be natural, after the NP-completeness result in [29], to expect that classes $\mathcal{C}$ and $\mathcal{C}'$ were both NP-complete for total-colouring. What is observed, in the present paper (please, refer to Table 1), is that, in the two classes for which we achieve computational complexity results, the complexities of edge-colouring and total-colouring are similar: NP-completeness for the general case of unichord-free graphs and polynomiality for the case of {square,unichord}-free graphs with maximum degree at least 4 that establishes the validity of the TCC in $\mathcal{C}'$. However, the complexity of total-colouring {square,unichord}-free graphs was not "confirmed" to be NP-complete for the case of maximum degree 3 — neither we have a polynomial algorithm for the case. One source of difficulty to construct an NP-completeness proof is the fact that the basic graphs of $\mathcal{C}'$ with maximum degree 3 are all Type 1. Interestingly, it was the existence of a Class 2 basic graph that allowed Machado, Figueiredo and Vušković [29] to construct an NP-completeness proof for edge-colouring in $\mathcal{C}'$. Table 2 summarizes this discussion by classifying the basic graphs of $\mathcal{C}'$ with respect to colouring problems.

The difficulty in proving the NP-completeness of total-colouring {square, unichord}-free graphs, together with the fact that all basic graphs of the class are Type 1, suggests that, maybe, $\mathcal{C}'$ could be a special separating class, in the sense that it would be a surprising class for which edge-colourig is NP-complete but total colouring is polynomial. Note, however, that, at the same time that is very difficult to establish an NP-completeness proof for total-colouring in $\mathcal{C}'$, the technique of Section 4 could not be applied to graphs of maximum degree 3 (Figure 12 exhibits a 4-total-colouring of

Basic graph \ Problem	vertex-colouring	edge-colouring	total-colouring
Strongly 2-bipartite	2-colourable	Class 1	Type 1
Heawood	2-colourable	Class 1	Type 1
Petersen	3-colourable	Class 2	Type 1
Complexity in $\mathcal{C}'$	Polynomial [38]	NP-complete [29]	?

Table 2: The basic graphs of $\mathcal{C}'$ with maximum degree 3 and their classification with respect to colouring problems. We remark that the basic graphs are 3-vertex-colourable as well as all biconnected graphs in $\mathcal{C}'$, and there is a Class 2 basic graph that is used to construct Class 2 graphs in $\mathcal{C}'$. The lack of Type 2 basic graphs makes it difficult to construct Type 2 graphs in $\mathcal{C}'$ (so far, no such example is known).

G_Y that prevents the extension of this colouring to $G_X \setminus M$). This suggests that, as in the case of previous classes [6, 22, 28, 45], the complexity of total-colouring {square,unichord}-free graphs in the remaining case of maximum degree 3 will be hard to establish.

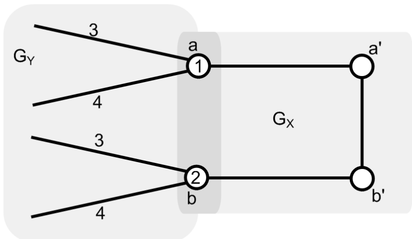

Figure 12: In a 4-total-colouring, the only colour available to colour edge aa' is 2, while the only colour available to colour edge bb' is 1. But, in this case, there will be only two colours (3 and 4) to colour three elements (vertices a' and b', and edge $a'b'$). Hence, this 4-total-colouring of G_Y does not extend to the elements of G_X.

Another topic that deserves some further discussion is the problem of the Total Colouring Conjecture in $\mathcal{C}$. It is not clear whether establishing the TCC for unichord-free graphs would be significantly easier than the general case. Note, however, that the 3-vertex-colourability [38] of every biconnected unichord-free graph G combined with its $\Delta(G) + 1$-edge-colourability [39] allow us to establish an upper bound of "maximum degree plus 4" for the total chromatic number of unichord-free graphs.

References

[1] M. M. Barbosa, C. P. de Mello, and J. Meidanis. *Local conditions for edge-colouring of cographs.* Congr. Numer. **133** (1998) 45–55.

[2] H. L. Bodlaender. *Polynomial algorithms for graph isomorphism and chromatic index on partial k-trees.* J. Algorithms **11** (1990) 631–643.

[3] V. A. Bojarshinov. *Edge and total colouring of interval graphs.* Discrete Appl. Math. **114** (2001) 23–28.

[4] O. V. Borodin, A. V. Kostochka, and D. R. Woodall. *List edge and list total colourings of multigraphs.* J. Combin. Theory Ser. B **71** (1997) 184–204.

[5] L. Cai and J. A. Ellis. *NP-Completeness of edge-colouring some restricted graphs.* Discrete Appl. Math. **30** (1991) 15–27.

[6] C. N. Campos, C. P. Mello. *The total chromatic number of some bipartite graphs.* Ars Combin. **88** (2008) 335–347.

[7] C. N. Campos, C. P. Mello. *A result on the total colouring of powers of cycles.* Discrete Appl. Math. **155** (2007) 585–597.

[8] D. Cariolaro and G. Cariolaro. *Colouring the petals of a graph.* Electron. J. Combin. **10** (2003) #R6.

[9] B. L. Chen, H.-L. Fu, and M. T. Ko. *Total chromatic number and chromatic index of split graphs.* J. Combin. Math. Combin. Comput. **17** (1995) 137–146.

[10] A. G. Chetwynd and A. J. W. Hilton. *Star multigraphs with three vertices of maximum degree.* Math. Proc. Cambridge Philos. Soc. **100** (1986) 303–317.

[11] C. M. H. de Figueiredo, J. Meidanis, C. P. de Mello, and C. Ortiz. *Decompositions for the edge colouring of reduced indifference graphs.* Theoret. Comput. Sci. **297** (2003) 145–155.

[12] C. M. H. de Figueiredo, J. Meidanis, and C. P. de Mello. *Local conditions for edge-coloring.* J. Combin. Math. Combin. Comput. **32** (2000) 79–91.

[13] C. H. de Figueiredo, J. Meidanis, C. P. de Mello. *Total-chromatic number and chromatic index of dually chordal graphs.* Inform. Process. Lett. **70** (1999) 147–152.

[14] C. De Simone and C. P. de Mello. *Edge-colouring of join graphs.* Theoret. Comput. Sci. **355** (2006) 364–370.

[15] C. De Simone and A. Galluccio. *Edge-colouring of regular graphs of large degree.* Theoret. Comput. Sci. **389** (2007) 91–99.

[16] A. Gyárfás. *Problems from the world surrounding perfect graphs.* Zastos. Mat. **19** (1987) 413–441.

[17] I. Holyer. *The NP-completeness of edge-colouring.* SIAM J. Comput. **10** (1982) 718–720.

[18] D. S. Johnson. *The NP-completeness column: an ongoing guide.* J. Algorithms **6** (1985) 434–451.

[19] G. Li, L. Zhang. *Total chromatic number of one kind of join graphs.* Discrete Math. **306** (2006) 1895–1905.

[20] G. Li, L. Zhang. *Total chromatic number of the join of $K_{m,n}$ and C_n.* Chinese Quart. J. Math. **21** (2006) 264–270.

[21] A. J. W. Hilton and Cheng Zhao. *On the edge-colouring of a graph whose core has maximum degree two.* J. Combin. Math. Combin. Comput. **21** (1996) 97–108.

[22] T. J. Hetherington, D. R. Woodall. *Edge and total choosability of near-outerplanar graphs.* Electron. J. Combin. **13** (2006) #R98.

[23] D. G. Hoffman and C. A. Rodger. *The chromatic index of complete multipartite graphs.* J. Graph Theory **16** (1992) 159–163.

[24] A. V. Kostochka. *The total coloring of a multigraph with maximal degree 4.* Discrete Math. **17** (1977) 161–163.

[25] A. V. Kostochka. *The total chromatic number of any multigraph with maximum degree five is at most seven.* Discrete Math. **162** (1996) 161–163.

[26] D. Leven and Z. Galil. *NP-completeness of finding the chromatic index of regular graphs.* J. Algorithms **4** (1983) 35–44.

[27] R. C. S. Machado and C. M. H. de Figueiredo. *Decompositions for edge-coloring join graphs and cobipartite graphs.* To appear in Discrete Appl. Math.

[28] R. C. S. Machado and C. M. H. de Figueiredo. *A decomposition for total-coloring graphs of maximum degree 3.* In: Cologne Twente Workshop 2008 (available online in http://ctw08.dti.unimi.it/CTW2008_Program.pdf). Full paper to appear in Networks.

[29] R. C. S. Machado, C. M. H. de Figueiredo, and K. Vušković. *Chromatic index of graphs with no cycle with unique chord.* In: Proc. of the ALIO/EURO Workshop on Applied Optimization (2008). Full paper submitted to Theoret. Comput. Sci.

[30] C. J. H. McDiarmid, A. Sánchez-Arroyo. *Total colouring regular bipartite graphs is NP-hard.* Discrete Math. **124** (1994) 155-162.

[31] T. Niessen. *How to find overfull subgraphs in graphs with large maximum degree. II.* Electron. J. Combin. **8** (2001) #R7.

[32] M. Plantholt. *The chromatic index of graphs with a spanning star.* J. Graph Theory **5** (1981) 45–53.

[33] B. Randerath and I. Schiermeyer. *Vertex colouring and forbidden subgraphs—a survey.* Graphs Combin. **20** (2004) 1–40.

[34] M. Rosenfeld. *On the total coloring of certain graphs.* Israel J. Math. **9** (1971) 396–402.

[35] A. Sánchez-Arroyo. *Determining the total colouring number is NP-hard.* Discrete Math. **78** (1989) 315–319.

[36] D. P. Sanders and Y. Zhao. *Planar graphs of maximum degree seven are class I.* J. Combin. Theory Ser. B **83** (2001) 201–212.

[37] E. Steffen. *Classifications and characterizations of snarks.* Discrete Math. **188** (1998) 183–203.

[38] N. Trotignon and K. Vušković. *A structure theorem for graphs with no cycle with a unique chord and its consequences.* To appear in J. Graph Theory. (Available online at http://www.comp.leeds.ac.uk/vuskovi/chord.ps.)

[39] V. G. Vizing. *On an estimate of the chromatic class of a p-graph* (in Russian). Diskret. Analiz **3** (1964) 25–30.

[40] N. Vijayaditya. *On the total chromatic number of a graph.* J. London Math. Soc. (2) **3** (1971) 405–408.

[41] V. G. Vizing. *Critical graphs with given chromatic index* (in Russian). Diskret. Analiz **5** (1965) 9–17.

[42] W. Wang. *Total chromatic number of planar graphs with maximum degree ten.* J. Graph Theory **54** (2007) 91–102.

[43] S.-D. Wang and S.-C. Pang. *The determination of the total-chromatic number of series-parallel graphs.* Graphs Combin. **21** (2005) 531–540.

[44] D. B. West. *Introduction to Graph Theory.* Second edition. Prentice Hall (2001).

[45] D. R. Woodall. *Total 4-choosability of series-parallel graphs.* Electron. J. Combin. **13** (2006) #R97.

[46] H. P. Yap. *Total colourings of graphs.* Lecture Notes in Mathematics. Springer-Verlag 1996. Germany.

[47] Z. Zhang, J. Zhang, and J. Wang. *Total chromatic number of some graphs.* Sci. Sinica Ser. A **31** (1988) 1434–1441.

Capítulo 14

Anexo: Manuscrito “Total chromatic number of {square,unichord}-free graphs”

DRAFT: Total chromatic number of {square,unichord}-free graphs

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Abstract

In the present work, we determine a surprising class for which edge-colouring is NP-complete but whose graphs are all Type 1. A *unichord* is an edge that is the unique chord of a cycle in a graph. The class $\mathcal{C}$ of unichord-free graphs was recently studied by Trotignon and Vušković [39], who proved for these graphs strong structure results and used these results to solve the recognition and vertex-colouring problems. Machado, Figueiredo and Vušković [29] established the NP-completeness of edge-colouring in the class $\mathcal{C}$. For the subclass $\mathcal{C}'$ of {square,unichord}-free graphs, an interesting complexity dicothomy holds with respect to edge-colouring [29]: if the maximum degree is 3, the problem is NP-complete, otherwise, the problem is polynomial. Subsequently, Machado and Figueiredo [30] settled the validity of the Total-Colouring Conjecture (TCC) in $\mathcal{C}'$ — in fact the TCC holds as a consequence of the fact that non-complete {square,unichord}-free graphs of maximum degree at least 4 are Type 1. In the present work, we prove that non-complete {square,unichord}-free graphs of maximum degree 3 are Type 1 establishing, then, the polynomiality of total-colouring restricted to {square,unichord}-free graphs. **Keywords**: cycle with a unique chord, decomposition, recognition, Petersen graph, Heawood graph, edge-colouring, total-colouring.

1 Introduction

In the present paper we deal with simple connected graphs. A graph G has vertex set $V(G)$ and edge set $E(G)$. An *element* of G is one of its vertices or edges and the set of elements of G is denoted $S(G) = V(G) \cup E(G)$. Two

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vertices $u, v \in V(G)$ are *adjacent* if $uv \in E(G)$; two edges $e_1, e_2 \in E(G)$ are *adjacent* if they share a common endvertex; a vertex u and an edge e are *incident* if u is an endvertex of e. The *degree of a vertex* v in G, denoted $deg_G(v)$, is the number of edges of G incident to v. We use the standard notation of K_n, C_n and P_n for complete graphs, cycle-graphs and path-graphs, respectively.

A *total-colouring* is an association of colours to the elements of a graph in such a way that no adjacent or incident elements receive the same colour. The *total chromatic number* of a graph G, denoted $\chi_T(G)$, is the least number of colours sufficient to total-colour this graph. Clearly, $\chi_T(G) \geq \Delta(G)+1$, where $\Delta(G)$ denotes the maximum degree of a vertex in G. The Total Colouring Conjecture (TCC) states that every graph G can be total-coloured with $\Delta(G) + 2$ colours. By the TCC only two values would be possible for the total chromatic number of a graph: $\chi_T(G) = \Delta(G)+1$ or $\Delta(G)+2$. If a graph G has total chromatic number $\Delta(G)+1$, then G is said to be *Type 1*; if G has total chromatic number $\Delta(G)+2$, then G is said to be *Type 2*. The TCC has been verified in restricted cases, such as graphs with maximum degree $\Delta \leq 5$ [24, 25, 35, 41], but the general problem is open since 1964, exposing how challenging the problem of total-colouring is.

It is NP-complete to determine whether the total chromatic number of a graph G is $\Delta(G) + 1$ [36]. In fact, the problem remains NP-complete when restricted to r-regular bipartite inputs [31], for each fixed $r \geq 3$ (remark that a bipartite graph G is trivially $\Delta + 2$-total-colourable, since we can use $\Delta(G)$ colours to colour the edges of G and 2 additional colours to colour the vertices of G). The total-colouring problem is know to be polynomial — and the TCC is valid — for few very restricted graph classes, some of which we enumerate next:

- a cycle-graph G has $\chi_T(G) = \Delta(G)+1 = 3$ if $|V(G)| = 0 \mod 3$, and $\chi_T(G) = \Delta(G) + 2 = 4$ otherwise [47];
- a complete graph G has $\chi_T(G) = \Delta(G) + 1$ if $|V(G)|$ is odd, and $\chi_T(G) = \Delta(G) + 2$ otherwise [47];
- a complete bipartite graph $G = K_{m,n}$ has $\chi_T(G) = \Delta(G) + 1 = \max\{m, n\} + 1$ if $m \neq n$, and $\chi_T(G) = \Delta(G) + 2 = m + 2 = n + 2$ otherwise [47];
- a grid $G = P_m \times P_n$ has $\chi_T(G) = \Delta(G) + 2$ if $G = P_2$ or $G = C_4$, and $\chi_T(G) = \Delta(G) + 1$ otherwise [6];
- a series-parallel graph G has $\chi_T(G) = \Delta(G) + 2$ if $G = P_2$ or $G = C_n$ with $n = 0 \mod 3$, and $\chi_T(G) = \Delta(G) + 1$ otherwise [22, 44, 46].

The computational complexity of the total-colouring problem is unknown for several important and well studied graph classes. The complexity of total-colouring planar graphs is unknown; in fact, even the TCC has not yet been settled for the class [43]. The complexity of total-colouring is open for the class of chordal graphs, and the partial results for the related classes of interval graphs [3], split graphs [9] and dually chordal graphs [13] expose the interest in the total-colouring problem restricted to chordal graphs. Another class for which the complexity of total-colouring is unknown is the class of join graphs: the results found in the literature consider very restricted subclasses of join graphs, such as the join between a complete inequibipartite graph and a path [19] and the join between a complete bipartite graph and a cycle [20], all of which are Type 1.

In the present work we consider total-colouring restricted to unichord-free graphs. A *unichord* is an edge that is the unique chord of a cycle in a graph. The class $\mathcal{C}$ of unichord-free graphs — that is, graphs that do not contain (as an induced subgraph) a cycle with a unique chord — was recently studied by Trotignon and Vušković [39]. The main motivation to investigate the class is the existence of a structure theorem for it, a kind of strong result that is not frequent in the literature and that can be used to develop algorithms in the class. Basically, this structure result states that every graph in $\mathcal{C}$ can be built starting from a restricted set $\mathcal{C}_B$ of basic graphs and applying a series of known "gluing" operations, denoted in [39] by $\mathcal{O}_0$, $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$. Another motivation for the class is the concept of χ-boundness, introduced by Gyárfás [16] as a natural extension of perfect graphs. A family of graphs $\mathcal{G}$ is *χ-bounded* with *χ-binding function f* if, for every induced subgraph G' of $G \in \mathcal{G}$, it holds $\chi(G') \leq f(\omega(G'))$, where $\chi(G')$ denotes the chromatic number of G' and $\omega(G')$ denotes the size of a maximum clique in G'. The research of χ-bounded graphs is mainly devoted to understanding for what choices of forbidden induced subgraphs, the resulting family of graphs is χ-bounded (see [34] for a survey). Note that the class perfect graphs is a χ-bounded family with χ-binding function $f(x) = x$, and perfect graphs are characterized by excluding odd holes and their complements. Also, by Vizing's Theorem, the class of line graphs of simple graphs is a χ-bounded family with χ-binding function $f(x) = x + 1$ (this special upper bound is known as the *Vizing bound*) and line graphs are characterized by nine forbidden induced subgraphs [45]. The class $\mathcal{C}$ is also χ-bounded with the Vizing bound [39]. The following results are obtained in [39] for unichord-free graphs: an $O(nm)$ recognition algorithm, an $O(nm)$ algorithm for optimal vertex-colouring, an $O(n+m)$ algorithm for maximum clique, and the NP-completeness of the maximum stable set problem.

Machado, Figueiredo and Vušković [29] investigated whether the structure results of [39] could be applied to obtained a polynomial-time algorithm for the edge-colouring problem in $\mathcal{C}$. The authors obtain a negative answer, by establishing the NP-completeness of the edge-colouring problem restricted to unichord-free graphs. The authors investigate also the complexity of the edge-colouring in the subclass $\mathcal{C}'$ of {square,unichord}-free graphs. The class $\mathcal{C}'$ can be viewed as the class of the graphs that can be constructed from the same set $\mathcal{C}_B$ of basic graphs as in $\mathcal{C}$, but using one less operation (the join operation $\mathcal{O}_2$ of [39] is forbidden). For inputs in $\mathcal{C}'$, an interesting dicothomy is proved in [29]: if the maximum degree is not 3, the edge-colouring problem is polynomial, while for inputs with maximum degree 3, the problem is NP-complete.

It is a natural step to investigate the complexity of total-colouring restricted to classes for which the complexity of edge-colouring is already established. This approach is observed, for example, in the classes of outerplanar [48] graphs, series-parallel [44] graphs, and some subclasses of planar [43] graphs and join [19, 20] graphs. One important motivation for this approach is the search for "separating" classes, that is, classes for which the complexities of edge-colouring and total-colouring differ. All known separating classes, in this sense, are classes for which edge-colouring is polynomial and total-colouring is NP-complete, such as the case of bipartite graphs. In other words, there is no known example of a class for which edge-colouring is NP-complete and total-colouring is polynomial[1], an evidence that "total-colouring might be 'harder' than edge-colouring".

Considering the recent interest in colouring problems restricted to unichord-free and {square,unichord}-free graphs, specially the results on total-colouring {square,unichord}-free graphs of maximum degree at least 4, it is natural to investigate the remaining case of the total-colouring problem restricted to {square,unichord}-free graphs of maximum degree 3. In the present work, we prove that, except for the complete graph K_4, every {square,unichord}-free graph of maximum degree 3 is Type 1, settling, then, the polynomiality of total-colouring restricted to the class. Table 1 summarizes the current status of colouring problems restricted to $\mathcal{C}$ and to $\mathcal{C}'$.

Observe the interesting dicothomy with respect to edge-colouring {square,unichord}-free graphs. Since the technique used in [30] to total-colour {square,unichord}-free graphs could only be applied to the case of maximum degree at least 4, a

[1]We are not considering graph classes artificially constructed that have NO answer for the total-colouring problem, such as the class of disjoint union of a cubic graph and the complete graph on four vertices.

Problem \ Class	$\mathcal{C}$	$\mathcal{C}'$, $\Delta \geq 4$	$\mathcal{C}'$, $\Delta = 3$
vertex-colouring	Polynomial [39]	Polynomial [39]	Polynomial [39]
edge-colouring	NP-complete [29]	Polynomial [29]	NP-complete [29]
total-colouring	NP-complete [30]	Polynomial [30]	Polynomial*

Table 1: Computational complexity of colouring problems in $\mathcal{C}$ and $\mathcal{C}'$ — the star indicates the result established in the present paper.

similar dicothomy would be expected for the total-colouring problem. What is shown, in the present work, is that such dicothomy does not exist, and the total-colouring problem is polynomial when restricted to {square,unichord}-free graphs, even in the case of maximum degree 3. Anyway, it is interesting to note that different approaches have to be used for total-colouring in the cases $\Delta \geq 4$ and $\Delta = 3$.

In Section 2 we recall structure results that are applied in Section 3 show that non-complete {square,unichord}-free graphs are Type 1.

2 Structure of graphs in $\mathcal{C}$ and $\mathcal{C}'$

In the present section we review decomposition results of unichord-free graphs and {square,unichord}-free graphs. These results are of the following form: every graph in $\mathcal{C}$ or in $\mathcal{C}'$ either belongs to a basic class or has a cutset. Before we can state these decomposition theorems, we define the basic graphs and the cutsets used in the decompositions.

The *Petersen graph* is the graph on vertices $\{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ so that both $a_1a_2a_3a_4a_5a_1$ and $b_1b_2b_3b_4b_5b_1$ are chordless cycles, and such that the only edges between some a_i and some b_i are a_1b_1, a_2b_4, a_3b_2, a_4b_5, a_5b_3. We denote by P the Petersen graph and by P^* the graph obtained from P by the removal of one vertex. Observe that $P \in \mathcal{C}$.

The *Heawood graph* is a cubic bipartite graph on vertices $\{a_1, \ldots, a_{14}\}$ so that $a_1a_2 \ldots a_{14}a_1$ is a cycle, and such that the only other edges are a_1a_{10}, a_2a_7, a_3a_{12}, a_4a_9, a_5a_{14}, a_6a_{11}, a_8a_{13}. We denote by H the Heawood graph and by H^* the graph obtained from H by the removal of one vertex. Observe that $H \in \mathcal{C}$.

A graph is *strongly 2-bipartite* if it is square-free and bipartite with bipartition (X, Y) where every vertex in X has degree 2 and every vertex in Y has degree at least 3. A strongly 2-bipartite graph is in $\mathcal{C}$ because any chord of a cycle is an edge between two vertices of degree at least three, so every cycle in a strongly 2-bipartite graph is chordless.

For the purposes of the present work, a graph G is called *basic*[2] if

1. G is a complete graph, a hole with at least five vertices, a strongly 2-bipartite graph, or an induced subgraph (not necessarily proper) of the Petersen graph or of the Heawood graph; and
2. G has no 1-cutset, proper 2-cutset or proper 1-join (all defined next).

We denote by $\mathcal{C}_B$ the set of the basic graphs. Observe that $\mathcal{C}_B \subseteq \mathcal{C}$.

A *cutset* S of a connected graph G is a set of elements, vertices and/or edges, whose removal disconnects G. A decomposition of a graph is the removal of a cutset to obtain smaller graphs, called the *blocks* of the decompositions, by possibly adding some nodes and edges to connected components of $G \setminus S$. The goal of decomposing a graph is trying to solve a problem on the whole graph by combining the solutions on the blocks. For a graph $G = (V, E)$ and $V' \subseteq V$, $G[V']$ denotes the subgraph of G induced by V'. The following cutsets are used in the decomposition theorems of class $\mathcal{C}$ [39]:

- A *1-cutset* of a connected graph $G = (V, E)$ is a node v such that V can be partitioned into sets X, Y and $\{v\}$, so that there is no edge between X and Y. We say that (X, Y, v) is a *split* of this 1-cutset.
- A *proper 2-cutset* of a connected graph $G = (V, E)$ is a pair of non-adjacent nodes a, b, both of degree at least three, such that V can be partitioned into sets X, Y and $\{a, b\}$ so that: $|X| \geq 2$, $|Y| \geq 2$; there is no edge between X and Y, and both $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ contain an ab-path. We say that (X, Y, a, b) is a *split* of this proper 2-cutset.
- A *1-join* of a graph $G = (V, E)$ is a partition of V into sets X and Y such that there exist sets A, B satisfying:
  - $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq Y$;
  - $|X| \geq 2$ and $|Y| \geq 2$;
  - there are all possible edges between A and B;
  - there is no other edge between X and Y.

  We say that (X, Y, A, B) is a *split* of this 1-join.

  A *proper 1-join* is a 1-join such that A and B are stable sets of G of size at least two.

[2]By the definition of [39], a basic graph is not, in general, indecomposable. However, our slightly different definition helps simplifying some of our proofs.

We can now state a decomposition result for graphs in $\mathcal{C}$:

Theorem 1. (Trotignon and Vušković [39]) *If $G \in \mathcal{C}$ is connected then either $G \in \mathcal{C}_B$ or G has a 1-cutset, or a proper 2-cutset, or a proper 1-join.*

The *block* G_X (resp. G_Y) of a graph G with respect to a 1-cutset with split (X, Y, v) is $G[X \cup \{v\}]$ (resp. $G[Y \cup \{v\}]$).

The *block* G_X (resp. G_Y) of a graph G with respect to a 1-join with split (X, Y, A, B) is the graph obtained by taking $G[X]$ (resp. $G[Y]$) and adding a node y complete to A (resp. x complete to B). Nodes x, y are called *markers* of their respective blocks.

The *blocks* G_X and G_Y of a graph G with respect to a proper 2-cutset with split (X, Y, a, b) are defined as follows. If there exists a node c of G such that $N_G(c) = \{a, b\}$, then let $G_X = G[X \cup \{a, b, c\}]$ and $G_Y = G[Y \cup \{a, b, c\}]$. Otherwise, block G_X (resp. G_Y) is the graph obtained by taking $G[X \cup \{a, b\}]$ (resp. $G[Y \cup \{a, b\}]$) and adding a new node c adjacent to a, b. Node c is called the *marker* of the block G_X (resp. G_Y).

The blocks with respect to 1-cutsets, proper 2-cutsets and proper 1-joins are constructed in such a way that they remain in $\mathcal{C}$, as shown by Lemma 1.

Lemma 1. (Trotignon and Vušković [39]) *Let G_X and G_Y be the blocks of decomposition of G with respect to a 1-cutset, a proper 1-join or a proper 2-cutset. Then $G \in \mathcal{C}$ if and only if $G_X \in \mathcal{C}$ and $G_Y \in \mathcal{C}$.*

Observe that the Petersen graph and the Heawood graph may appear as a block of decomposition with respect to a proper 1-join, as shown in Figure 1. However, these graphs cannot appear as a block of decomposition with respect to proper 2-cutset, because they have no vertex of degree 2 to play the role of a marker.

Despite the fact that the Petersen graph and the Heawood graph do not appear as a block of decomposition with respect to proper 2-cutset, they must be listed as a basic graph, because these graphs, themselves, are in $\mathcal{C}'$. So, the Petersen graph (resp. the Heawood graph) appears as a leaf of exactly one decomposition tree, namely, the decomposition tree of the Petersen graph (resp. the Heawood graph) — which is, actually, a trivial decomposition tree. Observe that graphs P^* and H^* may appear as decomposition block with respect to proper 2-cutset, as shown in Figure 2.

We reviewed results that show how to decompose a graph of $\mathcal{C}$ into basic blocks: Theorem 1 states that each graph in $\mathcal{C}$ has a 1-cutset, a proper 2-cutset or a proper 1-join, while Lemma 1 states that the blocks generated with respect to any of these cutsets are still in $\mathcal{C}$. We now obtain similar

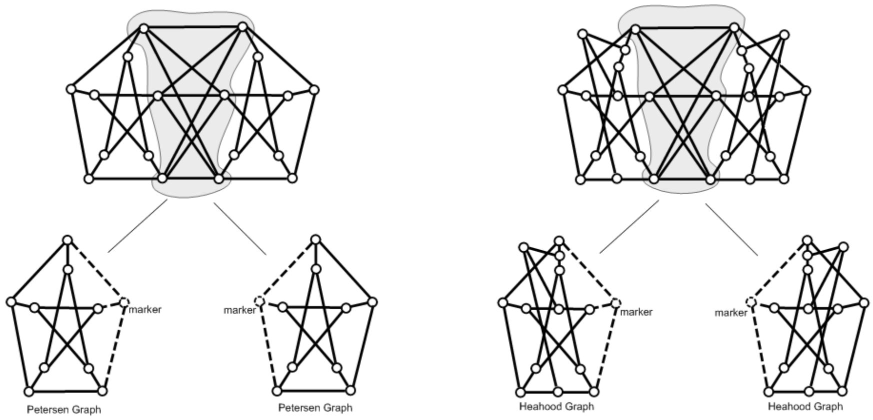

Figure 1: Decomposition trees with respect to proper 1-joins. In the graph on the left, the basic blocks of decomposition are two copies of the Petersen graph. In the graph on the right, the basic blocks of decomposition are two copies of the Heawood graph.

results for $\mathcal{C}'$. These results are not explicit in [39], but they can be obtained as consequences of results in [39] and by making minor modifications in its proofs. As we discuss in the following observation [4], for the goal of total-colouring, we only need to consider the **biconnected** graphs of $\mathcal{C}'$.

Observation 1. *Let G be a connected graph with a 1-cutset with split (X, Y, v). The chromatic index of G is $\chi'(G) = \max\{\chi'(G_X), \chi'(G_Y), \Delta(G)+1\}$.*

By Observation 1, if both blocks G_X and G_Y are $\Delta(G)$+1-total-colourable, then so is G. That is, once we know the total chromatic number of the biconnected components of a graph, it is easy to determine the total chromatic number of the whole graph. So, we may focus our investigation on the biconnected graphs of $\mathcal{C}'$.

Theorem 2. (Trotignon and Vušković [39]) *If $G \in \mathcal{C}'$ is biconnected, then either $G \in \mathcal{C}_B$ or G has a proper 2-cutset.*

Theorem 2 is an immediate consequence of Theorem 1: since G has no 4-hole, G cannot have a proper 1-join, and since G is biconnected, G cannot have a 1-cutset.

Lemma 2 shows that the blocks of decomposition of a biconnected graph of $\mathcal{C}'$ with respect to a proper 2-cutset, are also biconnected graphs of $\mathcal{C}'$.

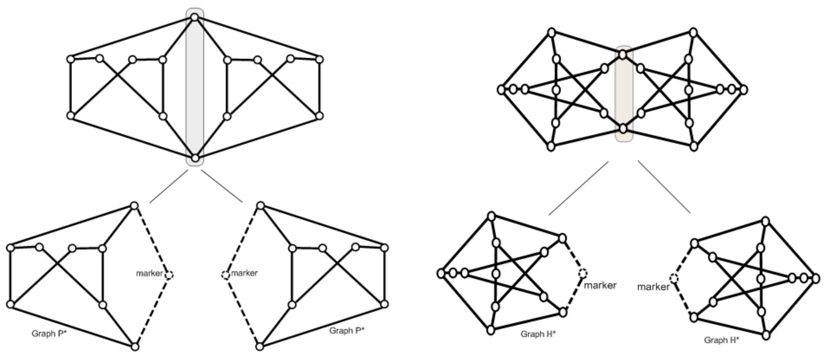

Figure 2: Decomposition trees with respect to proper 2-cutsets. In the graph on the left, the basic blocks of decomposition are two copies of P^*. In the graph on the right, the basic blocks of decomposition are two copies of H^*.

Lemma 2. (Machado, Figueiredo and Vušković [29]) *Let $G \in \mathcal{C}'$ be a biconnected graph and let (X, Y, a, b) be a split of a proper 2-cutset of G. Then both G_X and G_Y are biconnected graphs of $\mathcal{C}'$.*

Observe that Lemma 1 is somehow stronger than Lemma 2. While Lemma 1 states that a graph is in $\mathcal{C}$ **if and only if** the blocks with respect to any cutset are also in $\mathcal{C}$, Lemma 2 establishes only one direction: **if** a graph is a biconnected graph of $\mathcal{C}'$, **then** the blocks with respect to any cutset are also biconnected graph of $\mathcal{C}'$. For our goal of edge-colouring, there is no need of establishing the "only if" part. Anyway, it is possible to verify that, if both blocks G_X and G_Y generated with respect to a proper 2-cutset of a graph G are biconnected graphs of $\mathcal{C}'$, then G itself is a biconnected graph of $\mathcal{C}'$.

Lemma 3 shows that every non-basic biconnected graph in $\mathcal{C}'$ has an extremal decomposition.

Lemma 3. (Machado, Figueiredo and Vušković [29]) *Every biconnected graph $G \in \mathcal{C}' \setminus \mathcal{C}_B$ has a proper 2-cutset such that one of the blocks of decomposition is basic.*

3 Total-colouring {square-unichord}-free graphs of maximum degree 3

In the present section we determine the total chromatic number of graphs in $\mathcal{C}'$. We prove that non-complete {square-unichord}-free graphs of maximum degree 3 are Type 1. Combined with previous results on total-colouring {square-unichord}-free graphs, we can settle the polynomiality of total-colouring in $\mathcal{C}'$.

We describe a technique to total-colour a graph in $\mathcal{C}'$ by extending a total-colouring of one of the decomposition blocks to the other. Remark that the decomposition blocks are not necessarily subgraphs of the original graph: possibly they are constructed by the addition of a marker vertex. This is illustrated in the example of Figure 3, where G is P^*-free, yet, graph P^* appears as a block with respect to a proper 2-cutset of G.

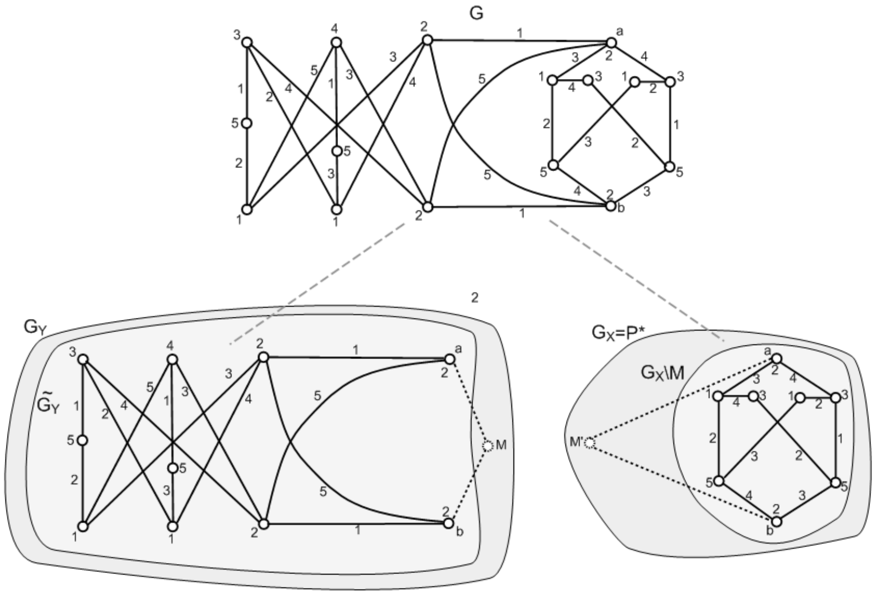

Figure 3: Example of decomposition with respect to a proper 2-cutset $\{a, b\}$. Observe that the marker vertices and their incident edges — identified by dashed lines — do not belong to the original graph.

Observation 2. *Consider a graph $G \in \mathcal{C}'$ with the following properties:*

- (X, Y, a, b) *is a split of a proper 2-cutset of* G*;*

- *$\tilde{G_Y}$ is obtained from G_Y by removing its marker if this marker is not a real vertex of G;*

- *π_Y is a $\Delta(G)+1$-total-colouring of $\tilde{G_Y}$;*

- *F_a (resp. F_b) is the set of the colours in $\{1, 2, ..., \Delta+1\}$ not used by π_Y in a (resp. b) nor in any edge of $\tilde{G_Y}$ incident to a (resp. b).*

If there exists a $\Delta(G)+1$-total-colouring π_X of $G_X \setminus M$, where M is the marker vertex of G_X, such that $\pi_X(a) = \pi_Y(a)$ (rep. $\pi_X(b) = \pi_Y(b)$) and each colour used in an edge incident to a (resp. b) is in F_a (resp. F_b), then G is $\Delta+1$-total-colourable.

The above observation shows that, in order to extend a $\Delta(G)+1$-total-colouring of $\tilde{G_Y}$ to a $\Delta(G)+1$-total-colouring of G, one must colour the elements of $G_X \setminus M$ in such a way that the colours of a, b, and the edges incident to them create no conflicts with the colours of the elements of $\tilde{G_Y}$. Moreover, there is no need to colour the edges incident to the marker M of G_X: if this marker is a vertex of G, its incident edges are already coloured by π_Y, otherwise, these edges are not real edges of G.

In the example of Figure 3, we exhibit a 5-total-colouring $\tilde{\pi_Y}$ of $\tilde{G_Y}$. In the notation of Observation 2, $F_a = \{3, 4\}$ and $F_b = \{3, 4\}$. We exhibit, also, a 5-total-colouring of $G_X \setminus M$ such that the colours of a and b are the same as in $\tilde{\pi_Y}$ of $\tilde{G_Y}$, the colours of the edges incident to a are $\{3, 4\} \subset F_a$ and the colours of the edges incident to b are $\{3, 4\} \subset F_b$. So, by Observation 2, we can combine the 5-total-colourings $\tilde{\pi_Y}$ and π_X in a 5-total-colouring of G, as it is done in Figure 3. Before we proceed and show how to determine the total chromatic number of graphs in $\mathcal{C}'$ with maximum degree $\Delta = 3$, we need to introduce additional tools and concepts.

A *partial k-total-colouring* of a graph $G = (V, E)$ is a colouring of a subset E' of E, that is, a function $\pi : E' \to \{1, 2, ..., k\}$ such that no adjacent or incident elements of E' receive the same colour. The *set of free-colours* at vertex u with respect to a partial-total-colouring $\pi : E' \to \mathbf{C}$ is the set $\mathbf{C} \setminus \pi(\{u\} \cup \{uv | uv \in E'\})$.

The *list-edge-colouring* problem is described next. Let $G = (V, E)$ be a graph and let $\mathcal{L} = \{L_e\}_{e \in E}$ be a collection which associates to each edge $e \in E$ a set of colours L_e called the *list* relative to e. It is asked whether there is an edge-colouring π of G such that $\pi(e) \in L_e$ for each edge $e \in E$. Theorem 3 is a result on list-edge-colouring which is applied, in the present work, to colour the edges of some basic graphs: strongly 2-bipartite graphs, Heawood graph and its subgraphs, and holes.

Theorem 3. (Borodin, Kostochka, and Woodall [4]) *Let* $G = (V, E)$ *be a bipartite graph and* $\mathcal{L} = \{L_e\}_{e \in E}$ *be a collection of lists of colours which associates to each edge* $uv \in E$ *a list* L_{uv} *of colours. If, for each edge* $uv \in E$, $|L_{uv}| \geq \max\{deg_G(u), deg_G(v)\}$, *then there is an edge-colouring* π *of* G *such that, for each edge* $uv \in E$, $\pi(uv) \in L_{uv}$.

An edge 2-cutset is a pair of non-adjacent edges whose removal disconnects a graph. The split of an edge 2cutset is $(X, Y, a'a, b'b)$ is no vertex in X (resp. Y) is adjacent to a vertex of in $Y \cup \{a, b\}$ (resp. $X \cup \{a', b'\}$). Observe that every graph of maximum degree 3 that has a proper 2-cutset has an edge 2-cutset. The block G_X (resp. G_Y) with respect to an edge 2-cutset $(X, Y, a'a, b'b)$ is $G[X \cup \{a', b'\}]$ (resp. $G[Y \cup \{a, b\}]$).

A graph is G *extended biconnected* {*square,unichord*}*-free* (resp. *extended basic*) it there is a set of pairs $(a_1'a_1, b_1'b_1), ..., (a_k'a_k, b_k'b_k)$ of pendant edges — where $a_1, ..., a_k$ and $b_1, ..., b_k$ are pendant vertices — such that the graph $\breve{G}$ obtained by the identification of the pendant vertices in each pair is a biconnected {square,unichord}-free graph (resp. basic). We call such pairs *frontier pairs.*

Observe that not every total-colouring of G_Y can be extended to G_X without conflicts — consider, for instance, the total-colouring of G_Y shown in Figure 4. So, we need to define a special total-colouring of G_Y that allows this colouring to be extended to the elements of G_X — we call such a total-colouring a *frontier-colouring.* Let $G \in \mathcal{C}'$ be a biconnected graph and let $C = (X, Y, a'a, b'b)$ be an edge 2-cutset of G. We say that a 4-total-colouring π of G_X *satisfies* C if $pi(a') = \pi(b')$ (*vertex condition*) or $pi(a'a) = \pi(b'b)$ (*edge-condition*), but **not** both. A *frontier-colouring* of a weakly biconnected {square,unichord}-free with pendant pairs $(a_1'a_1, b_1'b_1), ..., (a_k'a_k, b_k'b_k)$ is a 4-total-colouring that satisfies each pendant pair.

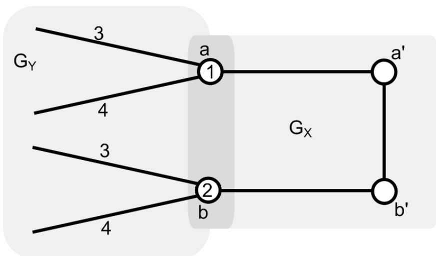

Figure 4: The 4-total-colouring of G_Y cannot be extended to the elements of G_X.

By Lemma 4, when investigating the existence of frontier-colourings in graphs that admits an edge 2-cutset, we may assume that both endvertices of this edge 2-cutset have degree at least 3.

Lemma 4. *Let G be a graph and let $C = (X, Y, a'a, b'b)$ be an edge 2-cutset of G. Let a'' be a degree-2 neighbor of a, in such a way that $C' = (X \cup \{a\}, Y \setminus \{a''\}, aa'', b'b)$ is an edge 2-cutset of G. If G_Y has a frontier-colouring, then so is G.*

Proof:
Just extend the colouring as shown in Figure 5. □

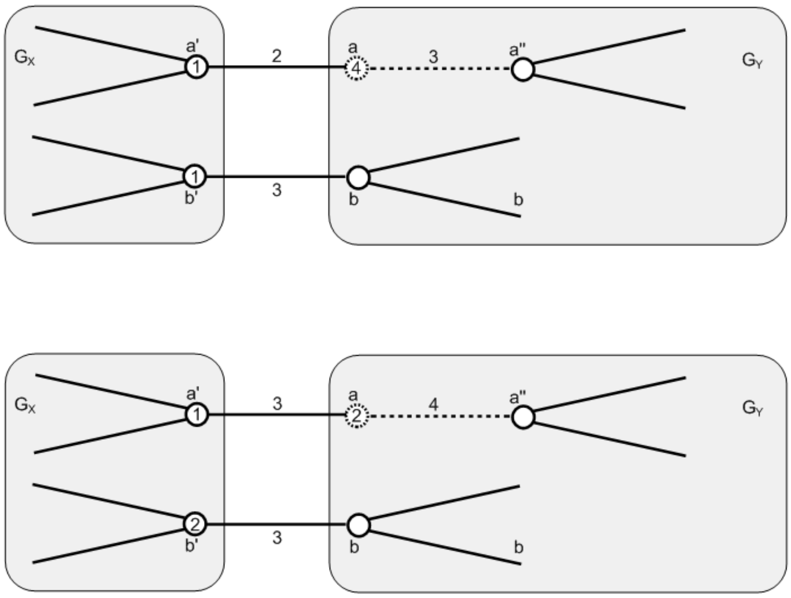

Figure 5: Propagating frontier condition. Observe that in the case in the top, a vertex-condition becomes an edge-condition, while in the case in the botton, an edge-condition becomes a vertex-condition.

Lemma 5. *Let G be a weakly biconnected {square,unichord}-free and let $C = (X, Y, a'a, b'b)$ be an edge 2-cutset of G such that G_X is weakly basic. If G_Y has a frontier-colouring, then so is G.*

Proof:
Denote as $(a'_1a_1, b'_1b_1), ..., (a'_ka_k, b'_kb_k), (a'a, b'b)$ the pendant pairs of G_Y — where the pendant vertices are $a_1, ..., a_k, b_1, b_k$ — and let π_Y be a frontier-colouring of G_Y. Denote by $\breve{G_X}$ the graph obtained from $G[X \cup \{a, a', b, b'\}]$

by identifying the pairs of pendant vertices. We construct a frontier-colouring π of G by setting $\pi(x) = \pi_Y(x)$ if x is an element of G_Y and by colouring the remaining elements of G_X as follows.

Case 1: $\breve{G_X}$ strongly 2-bipartite.

We total-colour G_X from a total-colouring of $\breve{G_X}$. Since $\pi(a') = \pi(b')$ or $\pi(a'a) = \pi(b'b)$, there exists a colour in the set $\{1, 2, 3, 4\}$ — say, w.l.o.g., 1 — that is not used in $pi(a')$, $\pi(b')$, $pi(a'a)$ nor $\pi(b'b)$. So, let 1 be the colour of all vertices that belong to the same bipartition of $\breve{G_X}$ as a and b (which is the bipartition whose vertices have degree at least 3). Associate, to each edge of $\breve{G_X}$ incident to a (resp. b) the set of the two colours available at a (resp. b), and associate the set $\{1, 2, 3, 4\}$ to the remaining edges. By Theorem 3, it is possible to colour the edges of $\breve{G_X}$ from these lists. Finally, colour the remaining degree-2 vertices with some free colour — remark that the two neighbors of these edges are coloured 1, so that there is always some free colour.

Once $\breve{G}$ is 4-total-coloured, its is easy to obtain a frontier-colouring of G by simply spliting the degree-2 vertices of $\breve{G}$ that correspond to the identification of pendant pairs of G_X. Observe that this new frontier colouring satisfies the frontier pairs attached to X with the vertex condition.

Case 2: $\breve{G_X}$ is a cycle.

Let $c_1, d_1), ..., c_t d_t)$ be the pendant pairs of G_X. Colour the elements of $G_X \setminus \{c_1, d_1, ..., c_t, d_t\}$ as in a 4-total-colouring of the graph obtained from $G_X \setminus \{c_1, d_1, ..., c_t, d_t\}$ by the identification of vertices c_1 and d_1, c_2 and d_2, ...,c_t and d_t. Finally, let the pairs of pendant edges receive different colours. Observe that this new frontier colouring satisfies the frontier pairs attached to X with the vertex condition.

Case 3: $\breve{G_X}$ is a subgraph of Petersen graph.

Case 3.1: $\breve{G_X}$ is P^*.

Case 3.1.1: G_X has no child in G. Figure 6.

Case 3.1.2: G_X has one child in G. Figure 7.

Case 3.1.3: G_X has two children in G. Figure 8.

Case 3.2: $\breve{G_X}$ is P^{**}. Can be obtained from 3.1.

Case 4: $\breve{G_X}$ is subgraph of Heawood graph.

Case 4.1: $\breve{G_X}$ is H^*.

Case 4.1.1: G_X has no child in G. Figure 9.

Case 4.1.2: G_X has one child in G. Figure 10.

Case 4.1.3: G_X has two children in G. Figure 11.

Case 4.2: $\breve{G_X}$ is H^{**}.

Case 4.2.1: G_X has no child in G. Figure 12.

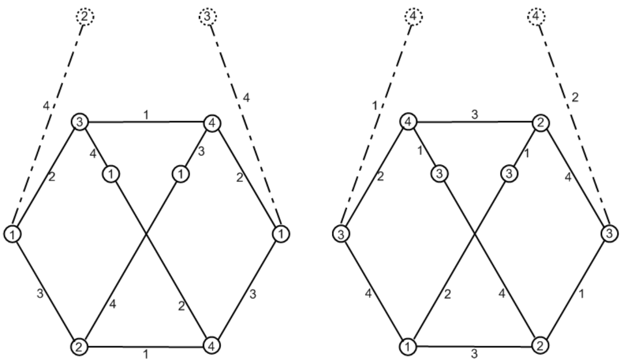

Figure 6: The connection of G_X with G_Y is identified by the two "dash and dot" edges. The colouring on the left considers the case where G_Y satisfies the edge condition. The colouring on the right considers the case where G_Y satisfies the vertex condition.

Case 4.2.2: G_X has one child in G. Either this child attachs to vertices adjacent to the vertices where G_Y is attached — two leftmost graphs of Figure 13 — or not — two rightmost graphs of Figure 13.

Case 4.2.3: G_X has two children in G. Either both children attach to vertices adjacent to the vertices where G_Y is attached — two leftmost graphs of Figure 13 — or not — two rightmost graphs of Figure 13.

Case 4.2.4: G_X has three children in G. Figure 15.

Case 4.3: $\check{G}_X$ is H^{***}. Can be obtained from 4.2.

Theorem 4. *Every biconnected non-complete {square,unichord}-free graph G of maximum degree 3 is 4-total-colourable.*

Proof:

Let G be a biconnected non-complete {square,unichord}-free graph of maximum degree 3. Suppose that G is basic. If G is a subgraph of the Petersen graph or of the Heawood graph, then G is 4-total-colourable, as shown in Figure 16. If G is strongly 2-bipartite, a 4-total-colouring of G is given as follows: let 1 the colour of the vertices of degree at least 3, let the edges be coloured with colours 2, 3 and 4 as in a 3-edge-colouring of G, and let the vertices of degree 2 be coloured with some colour not used in its two incident edges nor its two adjacent vertices.

We exhibit a sequence $\mathcal{G} = G^1, ..., G^k$ such that:

- each G^i is extended biconnected {square,unichord}-free;

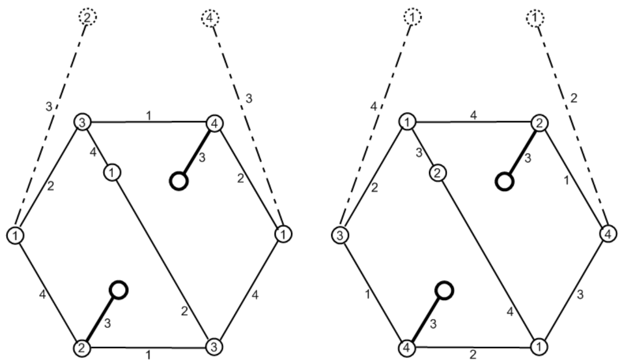

Figure 7: The connection of G_X with G_Y is identified by the two "dash and dot" edges. The colouring on the left considers the case where G_Y satisfies the edge condition. The colouring on the right considers the case where G_Y satisfies the vertex condition. The connection with other blocks is identified by the thicker edges.

- G^1 is extended basic, $G^k = G$;
- each G_i, $i = 2, ..., k$, has an edge 2-cutset $(X, Y, a'a, b'b)$ such that $G^i[X \cup \{a, a', b, b'\}]$ is extended basic and $G^i[Y \cup \{a, a', b, b'\}] = G^{i-1}$;
- each G^i has a frontier colouring.

We construct $\mathcal{G}$ as follows. Let G^1 be an extended basic graph that has only one frontier pair (this is the case, for example, of the graphs in Figures 6, 9 and 12). Graph G^{i+1} is constructed from G^i by the union with an extended basic graph B that shares a frontier pair with G^i. We claim that G^1 hass a frontier colouring. If G^1 is an extended cycle, then G^1 is, actually, a path. So, just colour properly the elements at the frontier pair and extend the colouring to the remaining elements. If G^1 is strongly 2-bipartite, colour G^1 as in the total-colouring constructed in the first paragraph of this proof. Otherwise, G^1 is a subgraph of the Petersen graph or of the Heawood graph — then, choose the right colouring of Figure 6 or the left colouring of Figure 12 or the left colouring of Figure 9, according to the case. □

Corollary 1. *Every non-complete {square,unichord}-free graph of maximum degree 3 is Type 1.*

Corollary 2. *A {square,unichord}-free graph is Type 2 if and only if it is*

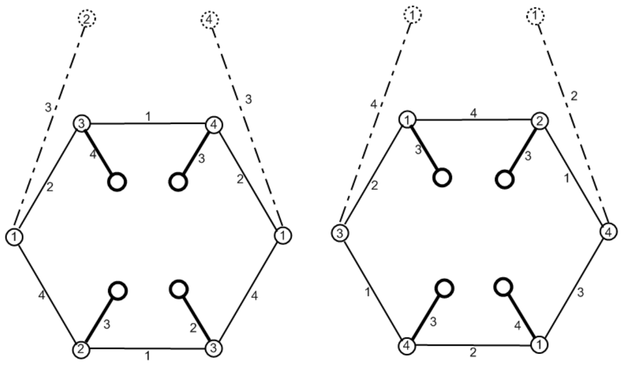

Figure 8: The connection of G_X with G_Y is identified by the two "dash and dot" edges. The colouring on the left considers the case where G_Y satisfies the edge condition. The colouring on the right considers the case where G_Y satisfies the vertex condition. The connection with other blocks is identified by the thicker edges.

a cycle on $n \not\equiv 0 \mod 3$ *vertices or a complete graph on* $n \equiv 0 \mod 2$ *vertices.*

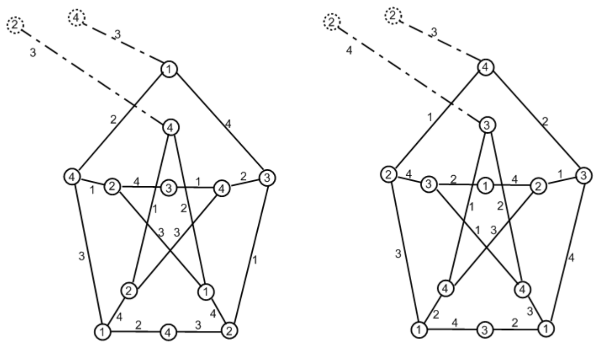

Figure 9: The connection of G_X with G_Y is identified by the two "dash and dot" edges. The colouring on the left considers the case where G_Y satisfies the edge condition. The colouring on the right considers the case where G_Y satisfies the vertex condition.

Basic subgraphs of the Petersen graph and of the Heawood graph

The discussions of the present paper brings the need for an explicit determination of which are the subgraphs of the Petersen graph and of the Heawood graph and that are basic and that are not basic graphs of another type (cycle or strongly 2-bipartite). The goal of the present paper is to prove the following:

Proposition 1. *There are three subgraphs of the Petersen graph (including itself) that are indecomposable and and that are basic and that are not a cycle or strongly 2-bipartite. There are four subgraphs of the Heawood graph (including itself) that are indecomposable and and that are basic and that are not a cycle or strongly 2-bipartite.*

Proof:

We denote by n-th class of subgraphs of G the set of subgraphs obtained from G by the removal on n vertices and that are not decomposable by 1-cutsets or proper 2-cutsets.

Case 1. Petersen graph and subgraphs. The first class of the Petersen graph P is obtained by the removal of one vertex. Since the Petersen graph is vertex-transitive, all graphs in the first class are isomorphic to P^*. The second class is obtained from P^* by the removal of one vertex. We claim

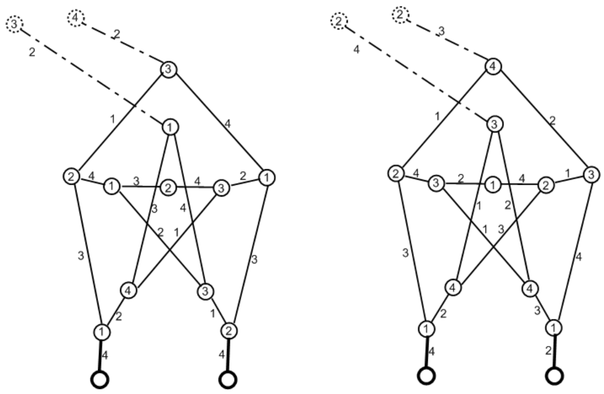

Figure 10: The connection of G_X with G_Y is identified by the two "dash and dot" edges. The colouring on the left considers the case where G_Y satisfies the edge condition. The colouring on the right considers the case where G_Y satisfies the vertex condition. The connection with other blocks is identified by the thicker edges.

that this vertex must be a degree-2 vertex, since the removal of a degree-3 vertex gives origin to a graph that has a 1-cutset. Any choice between the three degree-2 vertices gives origin to the same graph (up to isomorphisms), denoted P^{**}. Any removal of a vertex of P^{**} gives origin to a graph that has a 1-cutset or a proper 2-cutset, and subsequent removals give origin to other decomposable graphs or basic graphs (such as a 6-cycle).

Case 2. Heawood graph and subgraphs. The first class of the Heawood graph H is obtained by the removal of one vertex. Since the Heawood graph is vertex-transitive, all graphs in the first class are isomorphic to H^*. The second class is obtained from P^* by the removal of one vertex. We claim that this vertex must be a degree-2 vertex, since the removal of a degree-3 vertex gives origin to a graph that has a 1-cutset. Any choice between the three degree-2 vertices gives origin to the same graph (up to isomorphisms), denoted P^{**}. Again, we claim that the third class is obtained by the removal of a degree-2 vertex, and that any choice leds to the same graph (up to isomorphisms), denoted P^{***}. Any removal of a vertex of P^{***} gives origin to a graph that has a 1-cutset or a proper 2-cutset, and subsequent removals give origin to other decomposable graphs or basic graphs (such as a 8-cycle or strongly 2-bipartite graph).

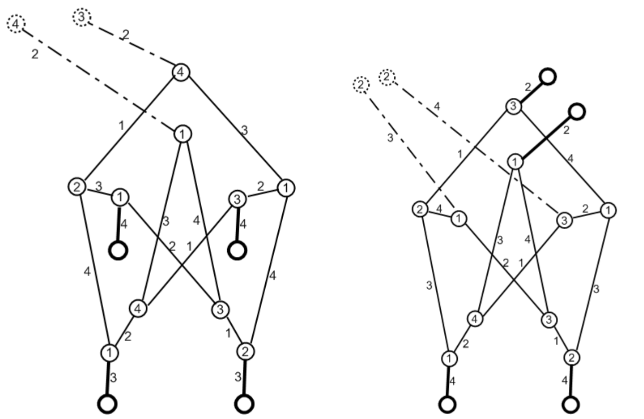

Figure 11: The connection of G_X with G_Y is identified by the two "dash and dot" edges. The colouring on the left considers the case where G_Y satisfies the edge condition. The colouring on the right considers the case where G_Y satisfies the vertex condition. The connection with other blocks is identified by the thicker edges.

References

[1] M. M. Barbosa, C. P. de Mello, and J. Meidanis. *Local conditions for edge-colouring of cographs.* Congr. Numer. **133** (1998) 45–55.

[2] H. L. Bodlaender. *Polynomial algorithms for graph isomorphism and chromatic index on partial k-trees.* J. Algorithms **11** (1990) 631–643.

[3] V. A. Bojarshinov. *Edge and total colouring of interval graphs.* Discrete Appl. Math. **114** (2001) 23–28.

[4] O. V. Borodin, A. V. Kostochka, and D. R. Woodall. *List edge and list total colourings of multigraphs.* J. Combin. Theory Ser. B **71** (1997) 184–204.

[5] L. Cai and J. A. Ellis. *NP-Completeness of edge-colouring some restricted graphs.* Discrete Appl. Math. **30** (1991) 15–27.

[6] C. N. Campos, C. P. Mello. *The total chromatic number of some bipartite graphs.* Ars Combin. **88** (2008) 335–347.

[7] C. N. Campos, C. P. Mello. *A result on the total colouring of powers of cycles.* Discrete Appl. Math. **155** (2007) 585–597.

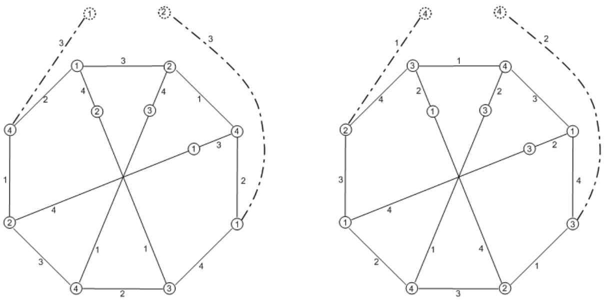

Figure 12: The connection of G_X with G_Y is identified by the two "dash and dot" edges. The colouring on the left considers the case where G_Y satisfies the edge condition. The colouring on the right considers the case where G_Y satisfies the vertex condition.

[8] D. Cariolaro and G. Cariolaro. *Colouring the petals of a graph.* Electron. J. Combin. **10** (2003) #R6.

[9] B. L. Chen, H.-L. Fu, and M. T. Ko. *Total chromatic number and chromatic index of split graphs.* J. Combin. Math. Combin. Comput. **17** (1995) 137–146.

[10] A. G. Chetwynd and A. J. W. Hilton. *Star multigraphs with three vertices of maximum degree.* Math. Proc. Cambridge Philos. Soc. **100** (1986) 303–317.

[11] C. M. H. de Figueiredo, J. Meidanis, C. P. de Mello, and C. Ortiz. *Decompositions for the edge colouring of reduced indifference graphs.* Theoret. Comput. Sci. **297** (2003) 145–155.

[12] C. M. H. de Figueiredo, J. Meidanis, and C. P. de Mello. *Local conditions for edge-coloring.* J. Combin. Math. Combin. Comput. **32** (2000) 79–91.

[13] C. H. de Figueiredo, J. Meidanis, C. P. de Mello. *Total-chromatic number and chromatic index of dually chordal graphs.* Inform. Process. Lett. **70** (1999) 147–152.

[14] C. De Simone and C. P. de Mello. *Edge-colouring of join graphs.* Theoret. Comput. Sci. **355** (2006) 364–370.

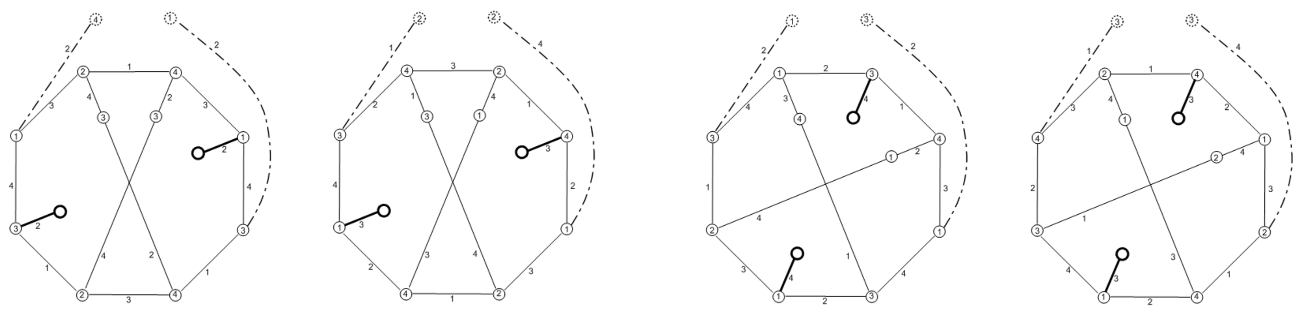

Figure 13: The connection of G_X with G_Y is identified by the two "dash and dot" edges. The colouring on the left considers the case where G_Y satisfies the edge condition. The colouring on the right considers the case where G_Y satisfies the vertex condition. The connection with other blocks is identified by the thicker edges.

[15] C. De Simone and A. Galluccio. *Edge-colouring of regular graphs of large degree*. Theoret. Comput. Sci. **389** (2007) 91–99.

[16] A. Gyárfás. *Problems from the world surrounding perfect graphs.* Zastos. Mat. **19** (1987) 413–441.

[17] I. Holyer. *The NP-completeness of edge-colouring*. SIAM J. Comput. **10** (1982) 718–720.

[18] D. S. Johnson. *The NP-completeness column: an ongoing guide*. J. Algorithms **6** (1985) 434–451.

[19] G. Li, L. Zhang. *Total chromatic number of one kind of join graphs.* Discrete Math. **306** (2006) 1895–1905.

[20] G. Li, L. Zhang. *Total chromatic number of the join of $K_{m,n}$ and C_n.* Chinese Quart. J. Math. **21** (2006) 264–270.

[21] A. J. W. Hilton and Cheng Zhao. *On the edge-colouring of a graph whose core has maximum degree two.* J. Combin. Math. Combin. Comput. **21** (1996) 97–108.

[22] T. J. Hetherington, D. R. Woodall. *Edge and total choosability of near-outerplanar graphs.* Electron. J. Combin. **13** (2006) #R98.

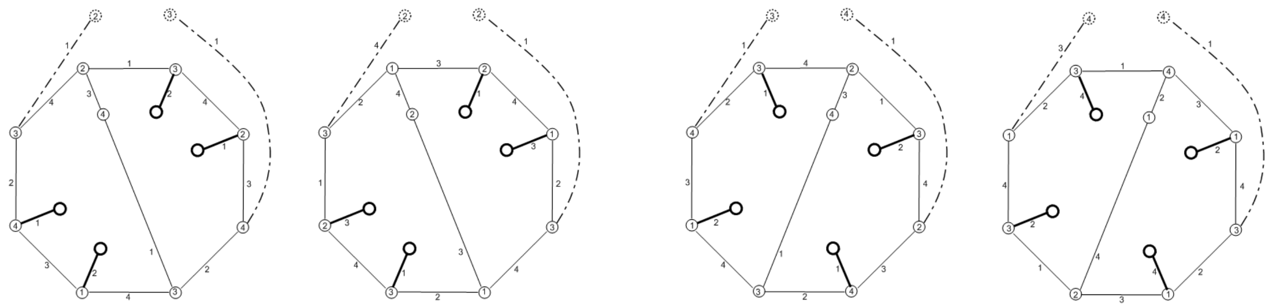

Figure 14: The connection of G_X with G_Y is identified by the two "dash and dot" edges. The colouring on the left considers the case where G_Y satisfies the edge condition. The colouring on the right considers the case where G_Y satisfies the vertex condition. The connection with other blocks is identified by the thicker edges.

[23] D. G. Hoffman and C. A. Rodger. *The chromatic index of complete multipartite graphs.* J. Graph Theory **16** (1992) 159–163.

[24] A. V. Kostochka. *The total coloring of a multigraph with maximal degree 4.* Discrete Math. **17** (1977) 161–163.

[25] A. V. Kostochka. *The total chromatic number of any multigraph with maximum degree five is at most seven.* Discrete Math. **162** (1996) 161–163.

[26] D. Leven and Z. Galil. *NP-completeness of finding the chromatic index of regular graphs.* J. Algorithms **4** (1983) 35–44.

[27] R. C. S. Machado and C. M. H. de Figueiredo. *Decompositions for edge-coloring join graphs and cobipartite graphs.* To appear in Discrete Appl. Math.

[28] R. C. S. Machado and C. M. H. de Figueiredo. *A decomposition for total-coloring graphs of maximum degree 3.* In: Cologne Twente Workshop 2008 (available online in http://ctw08.dti.unimi.it/CTW2008_Program.pdf). Full paper to appear in Networks.

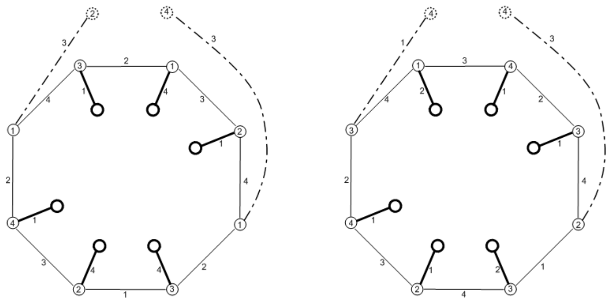

Figure 15: The connection of G_X with G_Y is identified by the two "dash and dot" edges. The colouring on the left considers the case where G_Y satisfies the edge condition. The colouring on the right considers the case where G_Y satisfies the vertex condition. The connection with other blocks is identified by the thicker edges.

[29] R. C. S. Machado, C. M. H. de Figueiredo, and K. Vušković. *Chromatic index of graphs with no cycle with unique chord.* In: Proc. of the ALIO/EURO Workshop on Applied Optimization (2008). Full paper submitted to Theoret. Comput. Sci.

[30] R. C. S. Machado and C. M. H. de Figueiredo. *Total chromatic number of chordless graphs.* In: Proc. of the Cologne-Twente Workshop (2009). Full paper submitted to Discrete Math.

[31] C. J. H. McDiarmid, A. Sánchez-Arroyo. *Total colouring regular bipartite graphs is NP-hard.* Discrete Math. **124** (1994) 155-162.

[32] T. Niessen. *How to find overfull subgraphs in graphs with large maximum degree. II.* Electron. J. Combin. **8** (2001) #R7.

[33] M. Plantholt. *The chromatic index of graphs with a spanning star.* J. Graph Theory **5** (1981) 45–53.

[34] B. Randerath and I. Schiermeyer. *Vertex colouring and forbidden subgraphs—a survey.* Graphs Combin. **20** (2004) 1–40.

[35] M. Rosenfeld. *On the total coloring of certain graphs.* Israel J. Math. **9** (1971) 396–402.

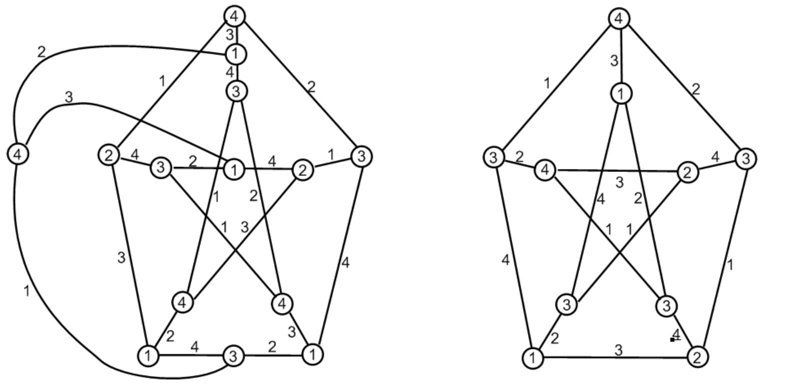

Figure 16: Total-colourings of the Heawood graph and of the Petersen graph.

[36] A. Sánchez-Arroyo. *Determining the total colouring number is NP-hard.* Discrete Math. **78** (1989) 315–319.

[37] D. P. Sanders and Y. Zhao. *Planar graphs of maximum degree seven are class I.* J. Combin. Theory Ser. B **83** (2001) 201–212.

[38] E. Steffen. *Classifications and characterizations of snarks.* Discrete Math. **188** (1998) 183–203.

[39] N. Trotignon and K. Vušković. *A structure theorem for graphs with no cycle with a unique chord and its consequences.* To appear in J. Graph Theory. (Available online at http://www.comp.leeds.ac.uk/vuskovi/chord.ps.)

[40] V. G. Vizing. *On an estimate of the chromatic class of a p-graph* (in Russian). Diskret. Analiz **3** (1964) 25–30.

[41] N. Vijayaditya. *On the total chromatic number of a graph.* J. London Math. Soc. (2) **3** (1971) 405–408.

[42] V. G. Vizing. *Critical graphs with given chromatic index* (in Russian). Diskret. Analiz **5** (1965) 9–17.

[43] W. Wang. *Total chromatic number of planar graphs with maximum degree ten.* J. Graph Theory **54** (2007) 91–102.

[44] S.-D. Wang and S.-C. Pang. *The determination of the total-chromatic number of series-parallel graphs.* Graphs Combin. **21** (2005) 531–540.

[45] D. B. West. *Introduction to Graph Theory.* Second edition. Prentice Hall (2001).

[46] D. R. Woodall. *Total 4-choosability of series-parallel graphs.* Electron. J. Combin. **13** (2006) #R97.

[47] H. P. Yap. *Total colourings of graphs.* Lecture Notes in Mathematics. Springer-Verlag 1996. Germany.

[48] Z. Zhang, J. Zhang, and J. Wang. *Total chromatic number of some graphs.* Sci. Sinica Ser. A **31** (1988) 1434–1441.

Referências Bibliográficas

[1] K. Appel e W. Haken. *Every planar graph is four colorable.* Bull. Amer. Math. Soc. **82** (1976) 711–712.

[2] M. M. Barbosa, C. P. de Mello, and J. Meidanis. *Local conditions for edge-colouring of cographs.* Congr. Numer. **133** (1998) 45–55.

[3] Behzad, M.: 1965, *Graphs and their chromatic numbers.* Doctoral Thesis - Michigan State University.

[4] S. N. Bhatt e S. S. Cosmadakis. *The complexity of minimizing wire length in VLSI layouts.* Inform. Process. Lett. **25** (1987) 263–267.

[5] H. L. Bodlaender. *Polynomial algorithms for graph isomorphism and chromatic index on partial k-trees.* J. Algorithms **11** (1990) 631–643.

[6] V. A. Bojarshinov. *Edge and total colouring of interval graphs.* Discrete Appl. Math. **114** (2001) 23–28.

[7] O. V. Borodin, A. V. Kostochka, and D. R. Woodall. *List edge and list total colourings of multigraphs.* J. Combin. Theory Ser. B **71** (1997) 184–204.

[8] G. Burosch e J.-M. Laborde. *Characterization of grid graphs.* Discrete Math. **87** (1991) 85–88.

[9] L. Cai and J. A. Ellis. *NP-Completeness of edge-colouring some restricted graphs.* Discrete Appl. Math. **30** (1991) 15–27.

[10] C. N. Campos, C. P. Mello. *The total chromatic number of some bipartite graphs.* Ars Combin. **88** (2008) 335–347.

[11] D. Cariolaro and G. Cariolaro. *Colouring the petals of a graph.* Electron. J. Combin. **10** (2003) #R6.

[12] Y. Caspi e E. Dekel. *Edge coloring series-parallel graphs.* J. Algorithms **18** (1995) 296–321.

[13] B. L. Chen, H.-L. Fu, and M. T. Ko. *Total chromatic number and chromatic index of split graphs.* J. Combin. Math. Combin. Comput. **17** (1995) 137–146.

[14] A. G. Chetwynd and A. J. W. Hilton. *Star multigraphs with three vertices of maximum degree.* Math. Proc. Cambridge Philos. Soc. **100** (1986) 303–317.

[15] D. G. Corneil, F. F. Dragan e E. Köhler. *On the power of BFS to determine a graphs diameter.* Networks **42** (2003) 209–222.

[16] C. M. H. de Figueiredo, J. Meidanis, C. P. de Mello, and C. Ortiz. *Decompositions for the edge colouring of reduced indifference graphs.* Theoret. Comput. Sci. **297** (2003) 145–155.

[17] C. M. H. de Figueiredo, J. Meidanis, and C. P. de Mello. *Local conditions for edge-coloring.* J. Combin. Math. Combin. Comput. **32** (2000) 79–91.

[18] C. H. de Figueiredo, J. Meidanis, C. P. de Mello. *Total-chromatic number and chromatic index of dually chordal graphs.* Inform. Process. Lett. **70** (1999) 147–152.

[19] J.-C. Fournier. *Méthode et théorème générale de coloration des arêtes.* J. Math. Pures Appl. **56** (1977) 437–453.

[20] A. Gregori. Unit-length embedding of binary trees on a square grid. Inform. Process. Lett. **31** (1989) 167–173.

[21] A. Gyárfás. *Problems from the world surrounding perfect graphs.* Zastos. Mat. **19** (1987) 413–441.

[22] T. J. Hetherington, D. R. Woodall. *Edge and total choosability of near-outerplanar graphs.* Electron. J. Combin. **13** (2006) #R98.

[23] A. J. W. Hilton and Z. Cheng. *The chromatic index of a graph whose core has maximum degree 2.* Discrete Appl. Math. **101** (1992) 135–147.

[24] A. J. W. Hilton and Cheng Zhao. *On the edge-colouring of a graph whose core has maximum degree two.* J. Combin. Math. Combin. Comput. **21** (1996) 97–108.

[25] D. G. Hoffman and C. A. Rodger. *The chromatic index of complete multipartite graphs.* J.Graph Theory **16** (1992) 159–163.

[26] I. Holyer. *The NP-completeness of edge-colouring.* SIAM J. Comput. **10** (1982) 718–720.

[27] D. S. Johnson. *The NP-completeness column: an ongoing guide.* J. Algorithms **6** (1985) 434–451.

[28] D. Leven and Z. Galil. *NP-completeness of finding the chromatic index of regular graphs.* J. Algorithms **4** (1983) 35–44.

[29] B. Lévêque, F. Maffray e N. Trotignon. *On graphs that do not contains a subdivision of the complete graph on four vertices as an induced subgraph.* Manuscrito (disponível em http://www.liafa.jussieu.fr/ trot/isk4.pdf).

[30] R. C. S. Machado e C. M. H. de Figueiredo. *Decompositions for edge-coloring join graphs and cobipartite graphs.* To appear in Discrete Appl. Math.

[31] R. C. S. Machado e C. M. H. de Figueiredo. *A decomposition for total-colouring partial-grids and list-total-colouring outerplanar graphs.* Aceito para publicação (Networks). Em anexo.

[32] R. C. S. Machado, C. M. H. de Figueiredo e K. Vušković. *Chromatic index of graphs with no cycle with unique chord.* Submetido para publicação (Theoretical Computer Science). Em anexo.

[33] R. C. S. Machado e C. M. H. de Figueiredo. *Total chromatic number of unichord-free graphs.* Submetido para publicação (Discrete Applied Mathematics). Em anexo.

[34] R. C. S. Machado e C. M. H. de Figueiredo. *Total chromatic number of {square,unichord}-free graphs.* Manuscrito. Em anexo.

[35] J. Misra e D. Gries. *A constructive proof of Vizing's theorem.* Inf. Proc. Lett. **41** (1992) 131–133.

[36] T. Niessen. *How to find overfull subgraphs in graphs with large maximum degree. II.* Electron. J. Combin. **8** (2001) #R7.

[37] C. Ortiz, N. Maculan e J. L. Szwarcfiter. *Characterizing e edge-colouring split-indifference graphs.* Discrete Appl. Math. **82** (1998) 209–217.

[38] M. Planthold. *The chromatic index of graphs with a spanning star.* J. Graph Theory **5** (1981) 45–53.

[39] B. Randerath e I. Schiermeyer. *Vertex colouring e forbidden subgraphs—a survey.* Graphs Combin. **20** (2004) 1–40.

[40] V. P. G. de Sá, G. D. da Fonseca, R. C. S. Machado e C. M. H. de Figueiredo. *Complexity dichotomy on degree constrained VLSI layouts with unit length edges.* Manuscrito. Em anexo.

[41] C. J. H. McDiarmid, A. Sánchez-Arroyo. *Total colouring regular bipartite graphs is NP-hard.* Discrete Math. **124** (1994) 155-162.

[42] A. Sánchez-Arroyo. *Determining the total colouring number is NP-hard.* Discrete Math. **78** (1989) 315-319.

[43] D. P. Sanders e Y. Zhao. *Planar graphs of maximum degree seven are class I.* J. Combin. Theory Ser. B **83** (2001) 201–212.

[44] C. De Simone e C. P. de Mello. *Edge-colouring of join graphs.* Theoret. Comput. Sci. **355** (2006) 364–370.

[45] C. De Simone e A. Galluccio. *Edge-colouring of regular graphs of large degree.* Theoret. Comput. Sci. **389** (2007) 91–99.

[46] E. Steffen. *Classifications e characterizations of snarks.* Discrete Math. **188** (1998) 183–203.

[47] N. Trotignon e K. Vušković. *A structure theorem for graphs with no cycle with a unique chord and its consequences*. To appear in Journal of Graph Theory. (Available online at http://www.comp.leeds.ac.uk/vuskovi/chord.ps.)

[48] V. G. Vizing. *On an estimate of the chromatic class of a p-graph* (in Russian). Diskret. Analiz **3** (1964) 25–30.

[49] V. G. Vizing. *Critical graphs with given chromatic index* (in Russian). Diskret. Analiz **5** (1965) 9–17.

[50] D. B. West. *Introduction to Graph Theory.* Second edition. Prentice Hall (2001).

[51] H. P. Yap. *Total colourings of graphs.* Lecture Notes in Mathematics. Springer-Verlag 1996. Germany.

[52] S.-D. Wang e S.-C. Pang. *The determination of the total-chromatic number of series-parallel graphs.* Graphs Combin. **21** (2005) 531-540.

[53] W. Wang. *Total chromatic number of planar graphs with maximum degree ten.* J. Graph Theory **54** (2007) 91–102.

[54] D. R. Woodall. *Total 4-choosability of series-parallel graphs.* Electron. J. Combin. **13** (2006) #R97.

[55] X. Zhou, Y. Matsuo e T. Nishizeki. *List total colorings of series-parallel graphs.* J. Discrete Algorithms **3** (2005) 47–60