Hamiltonian Cycles in Kneser Graphs*

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Abstract. *The Kneser graph $K(n, k)$ is the graph whose vertices are all the subsets with k elements of a set that has n elements, and two vertices are joined by an edge if the corresponding pair of k-sets is disjoint. The odd graph O_k is the special case of the Kneser graph when $n = 2k + 1$. A long-standing conjecture due to Lovász claims that O_k has a hamiltonian path for $k \geq 1$. Previously, Lovász's conjecture had been proved for all $k \leq 13$. We have improved these values by showing that O_k has a hamiltonian path for $14 \leq k \leq 17$. Additionally, we have established how close the odd graphs are to being hamiltonian: O_k has a closed spanning walk or trail in which every vertex appears at most twice.*

1. Introduction

A spanning cycle in a graph is a *hamiltonian cycle* and a graph that contains such cycle is said to be *hamiltonian*. A *hamiltonian path* is a path that contains every vertex of the graph precisely once. Since its formulation by Hamilton in 1859, the hamiltonian cycle problem has been used in several practical applications such as the *traveling salesman problem*, or TSP for short: given a collection of cities and the cost of travelling between each pair of them, the TSP is to find the cheapest way of visiting all of the cities and returning to your starting point. Note that *TSP* is a variation of the hamiltonian cycle since each city is represented by a vertex in a graph.

Determining if a graph G has a hamiltonian cycle is an NP-Complete problem [Karp 1972], even if restricted to bipartite graphs [Krishnamoorthy 1975], planar 3-connected cubic graphs [Garey et al. 1976], or if a hamiltonian path is given as part of the instance [Papadimitriou and Steiglitz 1976]. The hamiltonian path problem is NP-Complete as well [Garey and Johnson 1979].

In this paper, we study hamiltonian cycles and paths in a graph class called *Kneser graphs*. Let n, k be integers such that $n \geq k \geq 1$. The *Kneser graph* $K(n, k)$ has as vertices the subsets of $\{1, 2, \ldots, n\}$ that have cardinality k. Two vertices are adjacent if their corresponding k-subsets are disjoint. The Kneser graph $K(2k - 1, k - 1)$ is also called the *odd graph* O_k for $k \geq 2$. For simplicity, we refer to O_k as $K(2k+1, k)$, $k \geq 1$. With this definition, the graph O_1 is a triangle, and O_2 is the Petersen graph (see Figures 1 (a) e 1 (b)).

Kneser graphs have been extensively studied, especially because of their high degree of symmetry. Biggs mentions the following conjecture:

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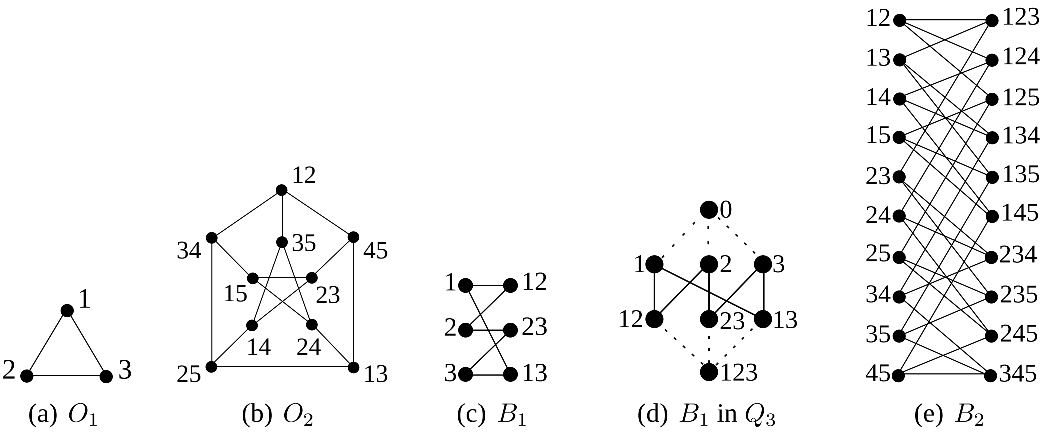

Figure 1. Odd graphs and bipartite Kneser graphs for $k = 1, 2$.

Conjecture 1 ([Biggs 1979])**.** *The odd graph O_k is hamiltonian for all $k > 2$.*

The *bipartite Kneser graph* $B(n, k)$ has $\binom{[n]}{k} \cup \binom{[n]}{n-k}$ as its vertex set and its edges represent the inclusion between two such subsets. The vertex set of $B(n, k)$ can be seen as two (symmetric) layers of the n-dimensional cube. If we consider the two layers in the middle of the cube (see Figure 1 (d)), then the corresponding bipartite Kneser graph $B(2k + 1, k)$ is called the *middle-levels graph*, and we denote it by B_k (see examples in Figures 1 (c) and 1 (e)). We are now ready to state the conjecture that has been attributed to Dejter, Erdos, Trotter, and various other mathematicians, but was most probably originated with Havel.

Conjecture 2 ([Havel 1983])**.** *The middle-levels graph B_k is hamiltonian for all $k \geq 1$.*

In fact, both Conjectures 1 and 2 are strongly related to a notorious conjecture due to Lovász [Lovász 1970] that every connected vertex-transitive graph has a hamiltonian path. The odd graphs O_k and the bipartite Kneser graphs B_k form a well-studied family of connected, $(k + 1)$-regular, vertex-transitive graphs. Therefore, the study of hamiltonian paths in these graphs may provide more evidence to support Lovász's conjecture, or offer a counterexample for it.

However, a direct computation of hamiltonian paths or cycles in O_k and B_k is not feasible for large values of k, because O_k has $\binom{2k+1}{k}$ vertices and B_k has $2\binom{2k+1}{k}$ vertices (see Table 1 in Section 3). Previous verifications of Conjecture 2 for $k \leq 17$ [Shields and Savage 1999, Shields et al. 2009] and Conjecture 1 for $k \leq 13$ [Shields and Savage 2004] relied heavily on computational methods.

A j-*factor* of a graph G is a j-regular spanning subgraph of G. For instance, an 1-*factor* is a perfect matching. A graph G is j-*factorable* if G is the union of disjoint j-factors. Two different 1-factorizations of B_k were found in [Kierstead and Trotter 1988] and [Duffus et al. 1994] hoping that the union of two suitable 1-factors would provide a hamiltonian cycle of B_k. Unfortunately, it turned out not to be the case for the given two 1-factorizations. However, those 1-factorizations were used to find a 2-factorization of O_k [Johnson and Kierstead 2004].

Biggs' conjecture has been extensively studied, thanks to the motivation brought by Lovász's conjecture. Due to the difficulty of proving Biggs' conjecture, researchers try to prove Havel's conjecture instead, because it is expected to be a simpler problem, for B_k is bipartite. However, both conjectures are still open. Hence, one option is to show these graphs are "close" to being hamiltonian, where the word "close" has been interpreted in several different ways. Firstly, long cycles in B_k and O_k have been sought. At the moment, the best result of this type is due to Johnson [Johnson 2004] who showed that B_k contains a cycle of length $(1-o(1))\,|B_k|$ and O_k contains a cycle of length $(1-o(1))\,|O_k|$ where the error term $o(1)$ is of the form $\frac{c}{\sqrt{k}}$ for some constant c. [Chen 2003] showed that the Kneser graph $K(n,k)$ and the bipartite Kneser graph $B(n,k)$ are hamiltonian for $n \geq 2.62k+1$. Note that, for fixed n, the smaller the parameter k is, the denser both the Kneser graph $K(n,k)$ and the bipartite Kneser graph $B(n,k)$ are. Thus, the graphs O_k and B_k are the sparsest among all of these graphs. The density of a graph G is the ratio $\frac{|E(G)|}{\left|\binom{V(G)}{2}\right|}$.

Yet another interpretation of "close" to being hamiltonian is provided by [Jackson and Wormald 1990] where a hierarchy of graphs is introduced. A closed spanning walk where each vertex is traversed at most q times is called a q*-walk* and a spanning tree of maximum degree q is a q*-tree*. Thus, in this terminology, a hamiltonian cycle is a 1-walk, and a hamiltonian path is a 2-tree. The authors proved that any graph with a q-tree has a q-walk, and that a q-walk guarantees the existence of a $(q+1)$-tree. These results give the following hierarchy among families of graphs:

$$\begin{aligned} \text{1-walk (Hamiltonian cycle)} &\implies \text{2-tree (Hamiltonian path)} \implies \text{2-walk} \\ &\implies \text{3-tree} \implies \text{3-walk} \implies \dots \end{aligned}$$

The prism over a graph G is the Cartesian product $G\square K_2$ of G with the complete graph on two vertices. Hence, the prism over G consists of two copies of G with a 1-factor joining the corresponding vertices. It was shown in [Kaiser et al. 2007] that the property of having a hamiltonian prism is "sandwiched" between the existence of a 2-tree and the existence of a 2-walk. Thus,

$$\text{2-tree} \implies \text{Hamiltonian prism} \implies \text{2-walk}$$

This means that graphs having a hamiltonian prism are close to being hamiltonian, even closer than graphs having a 2-walk. In [Horák et al. 2005] it is proved that for all $k \geq 1$, the prism over the middle-levels graph B_k is hamiltonian. In the present Thesis, we have shown how close the odd graphs are to being hamiltonian:

Theorem 3 ([Bueno and Horák 2009])**.** *Denote a q-trail as a q-walk that does not repeat edges. The prism over the odd graph O_k, $k \geq 2$, k even, is hamiltonian. For k odd, O_k has not only a 2-walk but also a 2-trail.*

We have also found that some odd graphs have indeed a hamiltonian path:

Theorem 4 ([Bueno et al. 2009, Bueno 2009])**.** *The odd graph O_k has a hamiltonian path for $14 \leq k \leq 17$.*

We proved Theorem 4 without the direct aid of a computer. Instead, we use existing results on the middle-levels problem [Shields and Savage 1999, Shields et al. 2009], therefore further relating two fundamental problems: to find a hamiltonian path in the odd graph and to find a hamiltonian cycle in the middle-levels graph.

The present text is meant to be a brief introduction to the basic ideas underlying the proofs of the results contained in the Thesis. Obviously, it does not delve too much into the details due to space constraints. For further details, we refer to the papers [Bueno et al. 2009, Bueno and Horák 2009].

2. On Hamiltonian Cycles in the Prism over the Odd Graphs

In order to prove the first part of Theorem 3, we constructed a spanning cubic subgraph H of O_k for k even. Then we proved that H is 3-connected. Since [Paulraja 1993] showed that every 3-connected cubic graph has a hamiltonian prism, O_k has a hamiltonian prism for k even.

For k odd, we cannot apply the technique used when k is even. In fact, for some odd values of k, under no circumstances it is possible to find a spanning cubic subgraph of O_k, because O_k has an odd number of vertices. Through the two 1-factorizations of B_k found by [Kierstead and Trotter 1988] and [Duffus et al. 1994], a 2-factorization of O_k has been found by [Johnson and Kierstead 2004]. Using two of these 2-factorizations, we determine a spanning 4-regular subgraph H' of O_k, for k odd. Then we show that H' is connected. Since every 4-regular connected graph is eulerian, H' has a 2-trail.

To better compare hamiltonian prisms and 2-trails, the diagram in Figure 2 illustrates the hierarchy among some families of graphs close to being hamiltonian. Having a 2-walk is a necessary condition – but not sufficient – for having a hamiltonian prism, a 2-trail or a hamiltonian cycle. A hamiltonian prism implies a 2-walk but the converse does not hold in general, so a hamiltonian prism is slightly closer to being hamiltonian than a 2-walk. Since a 2-trail is a 2-walk that does not repeat edges, a 2-trail is slightly better than a 2-walk as well. Therefore a graph with a hamiltonian prism is as close to being hamiltonian as a graph with a 2-trail.

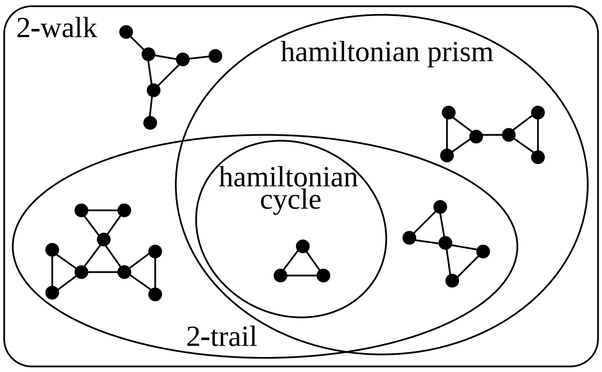

Figure 2. Hierarchy among some families of graphs close to being hamiltonian.

3. Hamiltonian Paths in Odd Graphs

Let $\mathbb{Z}_n$ denote the set $\{1, \ldots, n\}$ with numbers taken modulo n, plus 1. We consider the vertices of O_k and B_k to be subsets of $\mathbb{Z}_n$ and $n = 2k+1$. We define two special k-subsets of $\mathbb{Z}_n$, which are $r_1 = \{1, \ldots, k\}$ and $r_2 = \{2, 4, 6, \ldots, n-1\}$.

Given a set $v \subseteq \mathbb{Z}_n$, let $v + \delta$ denote the set $\{a + \delta \ : \ a \in v\}$ and $\overline{v}$ denote the complement of v with respect to $\mathbb{Z}_n$. We say that $u, v \subset \mathbb{Z}_n$ satisfy $u \sim v$ if either (i) $u = v + \delta$ or (ii) $\overline{u} = v + \delta$ for some $\delta \in \mathbb{Z}_n$. It is easy to verify that $\sim$ is an equivalence relation. We refer to the equivalence class of v under $\sim$ as $\sigma(v)$.

Given a graph G, we define the *quotient graph* $\widetilde{G}$ as the graph obtained from G by identifying vertices that are equivalent according to $\sim$. More precisely, the vertices of $\widetilde{G}$ are the equivalence classes $\sigma(v)$ for $v \in V(G)$, and if $uv \in E(G)$ then $\sigma(u)\sigma(v) \in E(\widetilde{G})$. Note that if $uv \in E(G)$ satisfies $u \sim v$, then the vertex $\sigma(u) \in V(\widetilde{G})$ has a loop. The quotient graph $\widetilde{B_k}$ is called the *reduced graph*. The graphs O_3 and $\widetilde{O_3}$ are illustrated in Figure 3.

[Shields and Savage 1999] showed that each equivalence class $\sigma(v)$ of $\widetilde{B_k}$ consists of exactly $n = 2k + 1$ k-subsets and n $(k + 1)$-subsets. As a consequence, the reduced graph $\widetilde{B_k}$ has $2n$ times fewer vertices than B_k (see Table 1). For example, B_{17} has $9{,}075{,}135{,}300$ vertices, while $\widetilde{B_{17}}$ has $129{,}644{,}790$ vertices, which is 70 times smaller, but still quite large.

Furthermore, [Shields and Savage 1999] proved that the existence of a hamiltonian path in the reduced graph $\widetilde{B_k}$, starting at the vertex $\sigma(r_1)$ and ending at the vertex $\sigma(r_2)$ implies that B_k is hamiltonian. We refer to a hamiltonian path starting at $\sigma(r_1)$ and ending at $\sigma(r_2)$ as a *useful path*. Using heuristics, [Shields and Savage 1999, Shields et al. 2009] determined useful paths in $\widetilde{B_k}$ for $1 \leq k \leq 17$.

We prove Theorem 4 by showing that if there is a useful path $P = (p_1, \ldots, p_m)$ in $\widetilde{O_k}$, then there is a hamiltonian path in O_k. We use some interesting properties about $\widetilde{O_k}$, all of them proved in [Bueno et al. 2009, Bueno 2009] and exhibited in $\widetilde{O_3}$ (Figure 3). First, we notice that the quotient graphs $\widetilde{O_k}$ and $\widetilde{B_k}$ are equal. Moreover, if there is an edge $\sigma(u)\sigma(v)$ in $\widetilde{O_k}$, then there is a perfect matching between the vertices of $\sigma(u)$ and the vertices of $\sigma(v)$ in O_k. Consequently, if there is a path $P = (p_1, \ldots, p_m)$ in $\widetilde{O_k}$, then O_k has n disjoint paths $(q_1 + (i - 1), \ldots, q_m + (i - 1))$, for $1 \leq i \leq n$, such that $q_j \in p_j$, for $1 \leq j \leq m$. Finally, the subgraph of O_k induced by $\sigma(r_1)$ is the cycle $r_1,\ r_1 + k,\ r_1 + 2k,\ \ldots,\ r_1 + (n - 1)k$ and the subgraph of O_k induced by $\sigma(r_2)$ is the cycle $r_2,\ r_2 + 1,\ r_2 + 2,\ \ldots,\ r_2 + (n - 1)$. Basically, we traverse all the n disjoint paths in O_k and carefully pick edges from the cycles induced by $\sigma(r_1)$ and $\sigma(r_2)$ in order to connect n paths into a single hamiltonian path.

Given a path Q, we denote by $\overleftarrow{Q}$ the path Q traversed from the last to the first vertex. Given two paths Q_1, Q_2 with no vertices in common and such that the last vertex of Q_1 is adjacent to the first vertex of Q_2, we denote by $Q_1 \circ Q_2$ the path obtained by the vertices of Q_1 followed by the vertices of Q_2.

By the definition of a useful path, $P = (p_1, \ldots, p_m)$ is hamiltonian in $\widetilde{O_k}$, $m = |V(O_k)|/n$, $p_1 = \sigma(r_1)$, and $p_m = \sigma(r_2)$. For $1 \leq i \leq n$, there are n disjoint paths P_i of the following form: $P_i = (q_1 + (i - 1), \ldots, q_m + (i - 1))$ with $q_1 + (i - 1) \in \sigma(r_1)$, $q_m + (i - 1) \in \sigma(r_2)$ and $q_j + (i - 1) \in p_j$.

Because of the cycle induced by $\sigma(r_1)$, and because $n = 2k+1$, we have that q_1+i is adjacent to q_1+i+k. It follows that $\overleftarrow{P_{i+1}} \circ P_{i+k+1}$ is a valid path. Considering the cycle induced by $\sigma(r_2)$, $q_m + i$ is adjacent to $q_m + i + 1$. Therefore, $P_i \circ \overleftarrow{P_{i+1}}$ is a valid path as well. Consequently, the following is a valid path: $Q_i = P_i \circ \overleftarrow{P_{i+1}} \circ P_{i+k+1} \circ \overleftarrow{P_{i+k+2}}$, where $P_i = (q_1 + (i - 1), \ldots, q_m + (i - 1))$, $\overleftarrow{P_{i+1}} = (q_m + i, \ldots, q_1 + i)$, $P_{i+k+1} = (q_1 + i + k, \ldots, q_m + i + k)$, and $\overleftarrow{P_{i+k+2}} = (q_m + i + k + 1, \ldots, q_1 + i + k + 1)$.

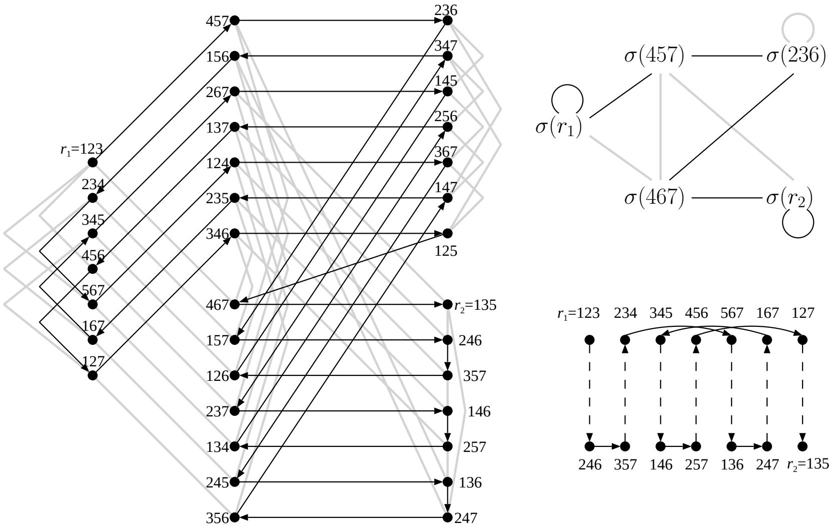

Figure 3. On the top-right corner: Useful hamiltonian path in $\widetilde{O_3}$. Unused edges are gray. On the bottom-right corner: Vertices in $\sigma(r_1)$ and $\sigma(r_2)$, and dashed paths $P_1, P_2, \ldots, P_7$ used in the construction of the hamiltonian path in O_3. On the left: hamiltonian path in O_3 between r_1 and r_2.

The idea is to build a hamiltonian path $Q_1 \circ Q_3 \circ Q_5 \circ \ldots$. If k is odd, then we show that $R_{odd} = Q_1 \circ Q_3 \circ \ldots \circ Q_{k-2}$ is a valid path. Because of the cycle induced by $\sigma(r_1)$, we know that the last vertex of Q_i, $q_1 + i + k + 1$, is adjacent to $q_1 + i + 1$, the first vertex of Q_{i+2}, since $P_{i+2} = (q_1 + i + 1, \ldots, q_m + i + 1)$. Also, R_{odd} contains either P_i or $\overleftarrow{P_i}$, for $i \in \{1, \ldots, 2k+1\} \setminus \{k, k+1, 2k+1\}$. To include the missing paths, we define the hamiltonian path in O_k as $H_{odd} = R_{odd} \circ P_k \circ \overleftarrow{P_{k+1}} \circ P_{2k+1}$. The full construction of a hamiltonian path in O_3 is illustrated in Figure 3. We omit the construction of the hamiltonian path for k even, since it is similar to the case k odd.

Since there is a useful path in $\widetilde{B_k}$ for $1 \leq k \leq 17$ [Shields and Savage 1999, Shields et al. 2009], O_k has a hamiltonian path for $1 \leq k \leq 17$.

4. Conclusion and Open Problems

In our thesis, we showed a relationship between the reduced graphs $\widetilde{B_k}$ and $\widetilde{O_k}$, and determined a hamiltonian path in the odd graph O_k by using a useful path in the reduced graph $\widetilde{O_k} = \widetilde{B_k}$. In this way, we determine hamiltonian paths in O_k for k up to 17. Further improved results for the middle-levels problem can be used to find hamiltonian paths in O_k for larger values of k [Bueno et al. 2009, Bueno 2009]. It is natural to ask whether a hamiltonian cycle in O_k can be constructed in a similar manner.

All hamiltonian paths known for the reduced graph $\widetilde{B_k}$ were determined by computational methods, using heuristics. Finding an useful path in the reduced graph $\widetilde{B_{17}}$ [Shields et al. 2009] took more than 20 days of processing on an AMD Athlon 3500+. Further studies in the structure of the reduced graph may help finding useful paths faster, and possibly determine whether all reduced graphs have a useful path. It is important to note that even if the reduced graph does not have a useful path, the corresponding odd graph may still have a hamiltonian path.

Table 1. Number of vertices of the graphs O_k, B_k and $\widetilde{O_k} = \widetilde{B_k}$

k	$n = 2k + 1$	$\|V(B_k)\|$	$\|V(O_k)\|$	$\|V(\widetilde{O_k})\|$
1	3	6	3	1
2	5	20	10	2
3	7	70	35	5
4	9	252	126	14
5	11	924	462	42
6	13	3,432	1,716	132
7	15	12,870	6,435	429
8	17	48,620	24,310	1,430
9	19	184,756	92,378	4,862
10	21	705,432	352,716	16,796
11	23	2,704,156	1,352,078	58,786
12	25	10,400,600	5,200,300	208,012
13	27	40,116,600	20,058,300	742,900
14	29	155,117,520	77,558,760	2,674,440
15	31	601,080,390	300,540,195	9,694,845
16	33	2,333,606,220	1,166,803,110	35,357,670
17	35	9,075,135,300	4,537,567,650	129,644,790

Two different kinds of approximations for hamiltonian cycles in the middle-levels graphs are known. [Savage and Winkler 1995] showed that B_k has a cycle containing at least 86.7% of the graph vertices, for $k \geq 18$. [Horák et al. 2005] showed the middle-levels graph has a closed spanning 2-walk. We proved that, for every k even, O_k has a closed spanning 2-walk. Moreover, for every k odd, O_k has a closed spanning trail in which every vertex appears at most twice [Bueno and Horák 2009, Bueno 2009].

Since vertex-transitive graphs defined by a single parameter, such as the odd graphs and the middle-levels graphs, are not known to have hamiltonian paths, Lovász's conjecture [Lovász 1970] remains an open challenge to this day.

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