Instituto Alberto Luiz Coimbra de
Pós-Graduação e Pesquisa de Engenharia

CONVEXIDADES EM GRAFOS: INTERMEDIAÇÕES, PARÂMETROS E CONVERSÕES

Vinícius Fernandes dos Santos

Tese de Doutorado apresentada ao Programa de Pós-graduação em Engenharia de Sistemas e Computação, COPPE, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Engenharia de Sistemas e Computação.

Orientadores: Jayme Luiz Szwarcfiter
Dieter Rautenbach

Rio de Janeiro
Janeiro de 2013

CONVEXIDADES EM GRAFOS: INTERMEDIAÇÕES, PARÂMETROS E CONVERSÕES

Vinícius Fernandes dos Santos

TESE SUBMETIDA AO CORPO DOCENTE DO INSTITUTO ALBERTO LUIZ COIMBRA DE PÓS-GRADUAÇÃO E PESQUISA DE ENGENHARIA (COPPE) DA UNIVERSIDADE FEDERAL DO RIO DE JANEIRO COMO PARTE DOS REQUISITOS NECESSÁRIOS PARA A OBTENÇÃO DO GRAU DE DOUTOR EM CIÊNCIAS EM ENGENHARIA DE SISTEMAS E COMPUTAÇÃO.

Examinada por:

Prof. Jayme Luiz Szwarcfiter, Ph.D.

Prof. Mitre Costa Dourado, D.Sc.

Prof. Nivio Ziviani, Ph.D.

Profa. Sulamita Klein, D.Sc.

Profa. Yoshiko Wakabayashi, Dr. rer. nat.

RIO DE JANEIRO, RJ – BRASIL
JANEIRO DE 2013

Santos, Vinícius Fernandes dos

Convexidades em Grafos: Intermediações, Parâmetros e Conversões/Vinícius Fernandes dos Santos. – Rio de Janeiro: UFRJ/COPPE, 2013.

X, 168 p. 29, 7cm.

Orientadores: Jayme Luiz Szwarcfiter

Dieter Rautenbach

Tese (doutorado) – UFRJ/COPPE/Programa de Engenharia de Sistemas e Computação, 2013.

Referências Bibliográficas: p. 61 – 66.

1. Convexidade em grafos. 2. Intermediação. 3. Conversão em grafos. 4. Número de envoltória. 5. Número de Carathéodory. 6. Número de Radon. I. Szwarcfiter, Jayme Luiz *et al.* II. Universidade Federal do Rio de Janeiro, COPPE, Programa de Engenharia de Sistemas e Computação. III. Título.

Aos meus pais, Maria José e
Sebastião, e à Debora

Agradecimentos

Gostaria de agradecer a diversas pessoas que influenciaram de alguma forma o doutorado, seja moldando a forma como ele se desenrolou ou, ainda mais fundamentalmente, influenciaram, mesmo que indiretamente, a decisão de cursar um doutorado ou a escolha da área de algoritmos.

Primeiramente, gostaria de agradecer aos meus pais, sem os quais eu nunca teria tido uma educação de qualidade, por propiciarem as diversas oportunidades que tive, ainda que eu imagine que em determinados momentos minha decisão de estudar por mais quatro anos ao invés de buscar um emprego regular não tenha feito muito sentido para eles. De fato, tal decisão faz pouco sentido para a maioria das pessoas que conheço. Ainda assim, nunca me desencorajaram a seguir em frente e sempre me deram total liberdade de escolha, desde a escolha da carreira no vestibular, até a decisão de me engajar ou não no doutorado. Agradeço também ao meu irmão mais velho, André, cujo interesse por ciência ainda na adolescência despertou também a minha curiosidade por entender como as coisas funcionavam.

À Debora, gostaria de agradecer principalmente pela compreensão, das incontáveis horas em que estive ausente por conta do doutorado, e das também muitas horas perdidas no deslocamento entre Rio de Janeiro e Belo Horizonte. Ainda que tenha sido necessário ficar a centenas de quilômetros de distância na maior parte dos quase quatro anos de doutorado, ela nunca me desencorajou a buscar meus objetivos, por entender que essa formação seria fundamental para minha carreira. Sua tolerância foi necessária sobretudo durante a realização do doutorado-sanduíche, época na qual viagens aos finais de semana para nos vermos não eram possíveis.

Ao professor Claudson Bornstein, com quem fiz a disciplina “Estruturas de Dados” ainda na graduação, primeira disciplina onde as noções de complexidade e eficiência são de fato discutidas, e quem, pela primeira vez, despertou meu interesse pela área de algoritmos.

À Maratona de programação e seus organizadores, evento que sem dúvida foi fundamental para incentivar meu interesse pela linha de algoritmos. Em particular, a Pedro Demasi, então técnico da UFRJ na competição, através de quem eu pude de fato conhecer a mesma.

Aos meus orientadores, Jayme Szwarcfiter e Dieter Rautenbach inicialmente por aceitarem me orientar. É, sem dúvida, uma honra ter sido orientado por dois pesquisadores brilhantes, que tantas vezes compartilharam sua experiência comigo. Meus agradecimentos vão, entretanto, muito além da orientação. Ainda que eu discorresse por várias páginas sobre eles, provavelmente não faria justiça a tudo que fizeram por mim, mas tentarei em poucas palavras ressaltar alguns pontos. Em particular, quero agradecê-los por me ajudarem muito além das atribuições de orientadores. Ao Jayme, agradeço por ser um exemplo de pesquisador, cuja generosidade e humildade pode ser atestada nas diversas homenagens recebidas de outros pesquisadores do Brasil e do mundo. Ainda assim, esteve sempre presente, sempre encontrando tempo para me ajudar nas mais diversas questões, ainda que sua agenda estivesse sempre ocupada. Ao Dieter por sua orientação não só durante as suas visitas ao Brasil mas também por toda sua ajuda para que o doutorado-sanduíche se tornasse uma realidade agradável e proveitosa, tanto pela convivência quanto pelo suporte em todas as formalidades burocráticas.

Aos professores do grupo de grafos e algoritmos da UFRJ, por compartilharem sua vivência ao longo do período do doutorado, principalmente nos seminários da linha de pesquisa, sem os quais minha visão do que é a nossa área de pesquisa seria apenas uma pequena fração do que é atualmente.

Aos amigos que fiz ao longo dos últimos anos, por tornarem o decorrer do doutorado muito mais suave e também mais divertido e interessante, com diversas horas gastas no Laboratório de Algoritmos e Combinatória discutindo sobre problemas e resultados da nossa área. Concluo esta etapa da minha formação com a certeza de que fiz amigos que levarei para o resto da minha vida. Evitarei citá-los nominalmente, para não cometer injustiças, mas aqueles a quem me refiro se identificarão ao ler as frases anteriores. Ainda assim, gostaria de agraceder a Philipp Schäfer, com quem tive o prazer de conviver e trabalhar em diversos problemas, tanto em suas visitas ao Brasil acompanhando Dieter, como durante minha visita à Ulm, na Alemanha.

Agradeço também aos membros da banca por aceitar o convite de participar da minha defesa e pelas sugestões que sem dúvida contribuíram melhoria neste texto.

Finalmente, gostaria de agradecer ao CNPq, à Capes e à Faperj por terem financiado meus estudos, tanto no Brasil quanto na Alemanha, financiamento sem o qual teria sido impossível desenvolver os trabalhos de forma satisfatória.

Resumo da Tese apresentada à COPPE/UFRJ como parte dos requisitos necessários para a obtenção do grau de Doutor em Ciências (D.Sc.)

CONVEXIDADES EM GRAFOS: INTERMEDIAÇÕES, PARÂMETROS E CONVERSÕES

Vinícius Fernandes dos Santos

Janeiro/2013

Orientadores: Jayme Luiz Szwarcfiter
Dieter Rautenbach

Programa: Engenharia de Sistemas e Computação

Inspirados no conceito de convexidade, da geometria euclideana, diversos trabalhos vêm sendo feitos nas últimas décadas envolvendo convexidades abstratas. Nesta tese é considerado o caso particular de convexidades em grafos, o qual pode ser utilizado para modelar diversas aplicações, como influências em redes sociais, sistemas distribuídos e automata celular, dentre outras.

São abordados problemas envolvendo intermediações, o número de envoltória, o número de Radon, o número de Carathéodory e conversões com limite de tempo em grafos. Os resultados apresentados compreendem caracterizações, algoritmos eficientes para a determinação de parâmetros, provas de NP-completude e limites superiores e inferiores.

Abstract of Thesis presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Doctor of Science (D.Sc.)

GRAPH CONVEXITIES: BETWEENNESS, PARAMETERS AND CONVERSIONS

Vinícius Fernandes dos Santos

January/2013

Advisors: Jayme Luiz Szwarcfiter
Dieter Rautenbach

Department: Systems Engineering and Computer Science

Motivated by the concept of convexity, from Euclidean geometry, much work has been done in recent decades on abstract convexities. In this thesis, the particular case of graph convexity is considered, which can be used to model many applications, such as influence on social networks, distributed systems and cellular automata, among others.

We address problems involving betweenesses, the hull number, the Radon number, the Carathéodory number and conversions with deadlines in graphs. The results shown in this thesis include characterizations, efficient algorithms for determining parameters, NP-completeness proofs, and upper and lower bounds.

Sumário

Capítulo 1

Introdução

Tá na hora, tá na hora, Tá na hora de brincar.

Cid Guerreiro - Ilariê

1.1 Convexidade e grafos

Grafos modelam relações ou conexões entre objetos de um determinado conjunto. Estes objetos podem ser cidades em um mapa, átomos de uma molécula ou computadores de uma rede, por exemplo. De fato, por sua natureza discreta e versatilidade de representação de problemas de diferentes origens, os grafos são um ingrediente fundamental para qualquer profissional de computação, visto que seu uso é frequente em diversas áreas, como inteligência artificial, banco de dados, redes de computadores, projeto de algoritmos, dentre outros.

Em aplicações envolvendo grafos, muitas vezes estamos interessados em caminhos entre um par vértices e também em quais os possíveis nós intermediários podem fazer parte destes caminhos. Isso motiva uma conexão com os conceitos matemáticos de intervalo e convexidade.

O conceito de convexidade tem sua origem na geometria e remonta aos trabalhos de Euclides e Arquimedes. Um conjunto convexo no espaço euclideado é aquele no qual todo segmento ligando dois pontos do conjunto permanece dentro do conjunto. Para um determinado conjunto de pontos, a envoltória convexa é o conjunto de todos os pontos que podem ser obtidos como combinação convexa dos pontos originais. Note que dois conjuntos distintos podem ter a mesma envoltória. De fato, até conjuntos com cardinalidades diferentes podem ter a mesma envoltória. Assim, quando considerando os conceitos correspondentes em grafos, um primeiro problema interessante é determinar o menor conjunto cuja envoltória é um conjunto dado. Em particular, o conjunto dado pode ser o conjunto de vértices do grafo em questão.

Além desta analogia, alguns outros resultados clássicos de Carathéodory [13] e Radon [59] no espaço euclideano de dimensão d também possuem correspondências em outras convexidades. O Teorema de Carathéodory afirma que se um ponto p pertence à envoltória convexa de um conjunto P de pontos em $\mathbb{R}^d$, então existe um subconjunto P' de P com no máximo $d+1$ tal que p pertence à envoltória convexa de P'. Já o Teorema de Radon afirma que qualquer conjunto R de $d+2$ pontos em $\mathbb{R}^d$ pode ser particionado em conjuntos R_1 e R_2 de forma que as envoltórias de R_1 e R_2 se interceptem. Para convexidades de caminhos induzidos em grafos, por exemplo, Duchet determinou os números de Carathéodory, Helly e Radon [31].

Este trabalho aborda alguns destes parâmetros e também outros problemas relacionados. Os resultados obtidos são de diversas naturezas compreendendo:

- Algoritmos eficientes para a determinação de parâmetros e verificação de propriedades;
- Resultados de intratabilidade de certos problemas, ou seja, provas de que alguns problemas fazem parte da classe dos problemas NP-completos, que é uma classe de problemas sabidamente difíceis;
- Caracterizações descrevendo a estrutura de grafos ou conjuntos que satisfazem determinadas propriedades;
- Limites inferiores e superiores para alguns parâmetros.

1.2 Aplicações

If the facts don't fit the theory,
change the facts.

Albert Einstein

Os problemas de intermediação foram primeiramente estudados como uma generalização natural de problemas de ordenação total [55] e encontram aplicações em psicometria [10] e em problemas de ordenação em biologia molecular [20, 43].

O conceito de convexidade em grafos, por sua vez, encontra aplicações em diversas áreas, ainda que em versões particulares ou generalizações. Apesar das particularidades de cada aplicação e das diferentes nomenclaturas, como conversões ou percolação, problemas relacionados foram estudados em diversos contextos, como influência social [39, 48, 49, 58], redes de expressão gênica [47], sistemas imune [1], automata celular [1, 2, 66], percolação [3], estratégias de marketing [21, 49], e computação distribuída [7, 37, 50, 54, 57].

1.2.1 Um exemplo com redes sociais

De forma a ilustrar os conceitos de convexidade, apresentamos agora uma analogia dos parâmetros de convexidade estudados com o contexto de redes sociais. Esta analogia pode ser facilmente transportada para outros contextos, como redes de computadores, por exemplo. Considere a situação em que uma pessoa é convencida a fazer parte de um grupo (ou comprar um produto, ou visitar um site) sempre que pelo menos k de seus amigos também fizerem parte deste grupo. O caso em que $k = 2$ é modelado fielmente na convexidade P_3.

Chamemos de *convertidos* aqueles indivíduos que fazem parte do grupo. Estes podem ter sido adicionados ao grupo em um momento inicial (após alguma campanha publicitária, por exemplo) ou podem ter entrado após constatarem a presença de amigos no grupo.

Neste contexto, o número de envoltória equivale ao problema de encontrar a menor quantidade de pessoas que devem ser convencidas inicialmente a fazer parte do grupo de forma que, uma vez que tempo suficiente tenha transcorrido, todos os membros da rede social sejam convertidos.

Para compreender o conceito do número de Carathéodory, considere a seguinte situação. Um determinado conjunto S de usuários de uma rede social R está acessível a uma entidade interessada em divulgar um grupo. Seja T o conjunto de pessoas convertidas a partir da persuação inicial de S. Suponha também que esta entidade esteja interessada particularmente que um determinado usuário u, que faz parte de T, esteja dentro do grupo. Uma vez que persuadir as pessoas de S pode ter um custo ou esforço associado, deseja-se minimizar o número de pessoas persuadidas inicialmente, mas ainda assim garantindo a presença do usuário u no grupo final. O número de Carathéodory equivale ao número máximo de pessoas de S que deve ser persuadida inicialmente para se atingir o objetivo desejado, dentre todas as combinações de conjuntos S de R e usuários u possíveis.

Finalmente, considere a existência de dois grupos rivais. Qual o menor número n de forma a garantir que, se um conjunto S com n usuários for selecionado, então sempre existe uma forma de distribuir os usuários de S entre dois subconjuntos disjuntos S_1 e S_2, de forma a garantir que algum usuário, não necessariamente em S, faz parte ao mesmo tempo dos indivíduos convertidos por S_1 e daqueles convertidos por S_2? Este número é chamado o número de Radon.

1.2.2 Um exemplo com redes de computadores

Um segundo exemplo de aplicação é na área de redes de computadores. Consideremos uma rede em que, por conta de uma vulnerabilidade recém descoberta, seus computadores estão vulneráveis a um ataque de negação de serviço (*Denial of*

Service, ou DoS) que, após um ataque, possa ser controlado pelo atacante.

Vamos supor que um potencial atacante saiba o quão sensível cada computador seja em cada instante de tempo, isto é, quantos computadores atacantes são necessários para efetuar um ataque com sucesso. Note que este valor pode variar com o tempo, em função do número de usuários legítimos acessando um servidor, por exemplo. Suponhamos também que, para computador, se conheça uma estimativa de quanto tempo seu administrador levará para atualizar o software responsável pela vulnerabilidade, ou seja, quanto tempo disponível o atacante tem para realizar o ataque.

O problema de determinar a menor quantidade de computadores necessários inicialmente para dominar uma rede de computador está diretamente relacionado ao problema de conversões com limite de tempo.

1.3 Sobre esta tese

Nesta tese estudamos problemas associados à noção de convexidade em grafos. Ela reúne boa parte dos resultados obtidos ao longo de aproximados 3 anos e meio de doutorado iniciados em junho de 2009, dos quais 4 meses se passaram na cidade de Ulm, Alemanha, durante doutorado-sanduíche com o professor Dieter Rautenbach.

Embora esta tese consista de diversos resultados distintos abordando problemas de convexidade, alguns deles merecem destaque. Acreditamos que a prova de NP-completude do problema do número de envoltória da convexidade P_3 seja uma contribuição importante uma vez que pode servir como ferramenta na demonstração de NP-completude para outros problemas na convexidade P_3. Já o problema de conversões com limite de tempo, além de estabelecer relações com diversos outros parâmetros de grafos, possui uma forte motivação prática e pode encontrar aplicações reais. Finalmente, destacamos também o algoritmo de tempo linear obtido para a resolução do problema do número de Radon em árvores, uma vez que este é um problema complexo, sendo NP-completo mesmo para grafos split.

Optamos por omitir parte das demonstrações do texto, para possibilitar uma leitura mais fluida. Estas demonstrações podem ser encontradas nos apêndices, onde se encontram anexados os artigos que contêm resultados relacionados a esta tese, alguns já publicados e outros submetidos, aguardando revisão.

Iniciamos apresentando definições necessárias à compreensão do restante do texto no Capítulo 2.

Os primeiros resultados obtidos durante o doutorado, na sua maior parte já presentes no texto do exame de qualificação, são sobre intermediações em grafos e em famílias de conjuntos. Estes resultados são apresentados no Capítulo 3.

Os três capítulos que seguem são relacionados a três parâmetros de convexidade, o número de envoltória, número de Carathéodory e número de Radon, respectivamente. Em particular, o conceito do número de envoltória, apresentado no Capítulo 2 e abordado no Capítulo 4 é essencial para os capítulos seguintes.

Todos os resultados apresentados no Capítulos 3 a 6 são originais e compõem as contribuições desta tese.

O Capítulo 7 conclui este texto fazendo algumas considerações finais e indicando possíveis direções de continuação dos trabalhos desenvolvidos nesta tese.

Após as Referências Bibliográficas, apresentamos apêndices onde podem ser encontrados todos os trabalhos desenvolvidos ao longo do doutorado, a maior parte deles contendo resultados apresentados no texto desta tese. Os trabalhos apresentados são:

- *On subbetweennesses of trees: hardness, algorithms, and characterizations* [61];
- *Characterization and representation problems for intersection betweennesses* [60];
- *On minimal and minimum hull sets* [6];
- *Irreversible conversion processes with deadlines* [62];
- *On the Carathéodory number of interval and graph convexities* [27];
- *Algorithmic and structural aspects of the P_3-Radon number* [28];
- *An upper bound on the P_3-Radon number* [25];
- *Characterization and recognition of Radon-independent sets in split graphs* [26];
- *On the Radon number for P_3-convexity* [29];
- *On clique graphs of chordal comparability graphs* [44].

Nos apêndices correspondentes podem ser encontradas maiores informações sobre a situação atual de cada um dos trabalhos, ou seja, se já foram publicados, apresentados ou submetidos aguardando revisão.

Capítulo 2

Preliminares

Winter is coming.

Eddard Stark - A Game of Thrones

2.1 Complexidade computacional

Embora não abordemos em detalhes questões envolvendo classes de complexidade, utilizamos ao longo do texto as noções usuais de que um algoritmo cujo tempo de execução é limitado por um polinômio no tamanho da entrada é um algoritmo eficiente e que um problema que pertence à classe dos problemas NP-difíceis é um problema provavelmente intratável. Assim, de modo geral, sempre que considerarmos aspectos computacionais dos problemas envolvidos, estaremos buscando um algoritmo eficiente ou uma prova de NP-completude.

2.2 Definições básicas

Esta seção apresenta as definições e as notação utilizada neste trabalho.

Para um conjunto V e um inteiro k denotamos por $\binom{V}{k}$ o conjunto de todos os subconjuntos de tamanho k de V e por 2^V o conjunto das partes de V. Denotamos por $[k]$ o conjunto $\{1, 2, \ldots, k\}$, por $\mathbb{N}$ o conjunto infinito $\{1, 2, \ldots\}$ e por $\mathbb{N}_0$ o conjunto $\mathbb{N} \cup \{0\}$.

Um grafo simples G é um par ordenado $(V(G), E(G))$, onde $V(G)$, o conjunto de *vértices*, é um conjunto finito não vazio e $E(G) \subseteq \binom{V}{2}$, o conjunto de *arestas*, é um conjunto cujos elementos são subconjuntos de cardinalidade dois de $V(G)$. Quando não há ambiguidade, utilizamos simplesmente $G = (V, E)$. Por simplicidade, denotamos uma aresta $\{u, v\}$ por uv, ou, equivalentemente, por vu.

Se $e = uv \in E$, então dizemos que os vértices u e v são *adjacentes* ou *vizinhos*, e que há uma aresta *entre* os vértices u e v. Dizemos ainda que u e v são incidentes a e e que e é incidente a u e v.

A *ordem* de G é o tamanho de seu conjunto de vértices V. Para simplificar a notação, frequentemente utilizamos $|V| = n$ e $|E| = m$. Um grafo é dito *trivial* se $n = 1$.

Um *subgrafo* $G' = (V', E')$ de $G = (V, E)$ é um grafo com $V' \subseteq V$ e $E' \subseteq E \cap \binom{V'}{2}$. Se $V' \subseteq V$, denotamos por $G[V']$ o subgrafo $G' = (V', E')$ *induzido* por V', onde $E' = \{uv \in E \mid u, v \in V'\} = E \cap \binom{V'}{2}$.

A *vizinhança* de um vértice v, denotada por $N_G(v)$, é o conjunto dos vértices adjacentes a v em G, ou seja

$$N_G(v) = \{w \in V \mid vw \in E(G)\}.$$

Quando não houver ambiguidade, denotaremos a vizinhança de v simplesmente por $N(v)$. De forma análoga, frequentemente omitimos subscritos de outros parâmetros quando estes estiverem claros pelo contexto. O *grau* de um vértice v, denotado por $d(v)$, é a quantidade de vizinhos de v, assim, temos $d(v) = |N(v)|$. Um grafo é *regular* se todo vértice possui o mesmo grau, e é *cúbico* se todo vértice possui grau 3.

Para um grafo $G = (V, E)$, a *contração* de uma aresta uv é a operação que transforma o grafo G em um novo grafo $G' = (V', E')$ onde $V' = \big(V \setminus \{u, v\}\big) \cup \{w\}$ e

$$E' = \big\{xy \in E \mid x, y \in V \setminus \{u, v\}\big\} \cup \big\{wx \mid x \in V \setminus \{u, v\} \wedge (ux \in E \ \vee \ vx \in E)\big\}.$$

Uma *subdivisão* de uma aresta uw em $G = (V, E)$ é a operação que transforma G em um novo grafo $G' = (V', E')$, com um novo vértice v, com conjunto de vértices $V' = V \cup \{v\}$ e conjunto de arestas

$$E' = \big(E \setminus \{uw\}\big) \cup \{uv, vw\}.$$

Dizemos que um grafo H é uma subdivisão de um grafo G, se existe uma sequência de subdivisões de arestas que transforma G em H.

Dois grafos G e H são *isomorfos* se existe uma bijeção f de $V(G)$ em $V(H)$ tal que $uv \in E(G) \iff f(u)f(v) \in E(H)$.

Um *caminho* P de *tamanho* k, ou de *ordem* $k + 1$ entre dois vértices v_0 e v_k é uma sequência de vértices distintos $v_0, v_1, ..., v_k$, onde $v_i v_{i+1} \in E$, para $0 \leq i < k$. Dizemos, nesse caso, que v_0 e v_k são as *extremidades* desse caminho e que v_i, para $0 < i < k$ são os *vértices internos* de P. Um *ciclo* é uma sequência de vértices $v_0, v_1, ..., v_k$, com $v_i \neq v_j$, $0 \leq i < j < k$, $k \geq 3$, $v_0 = v_k$ e $v_i v_{i+1} \in E$, para $0 \leq i < k$. Um grafo é dito *acíclico* se não possui ciclos. A *distância* entre dois vértices u e v, denotada por $\mathrm{dist}_G(u, v)$, é o tamanho do menor caminho entre u e

v. Um *caminho induzido* de tamanho k é um caminho no qual não existe aresta entre v_i e v_j para $0 \leq i < j - 1 < k$. Um grafo é *conexo* quando existe um caminho entre cada par de seus vértices e *desconexo* em caso contrário. Uma *componente conexa*, ou simplesmente *componente*, de um grafo G é um subgrafo conexo maximal. Denotamos por $V(C)$ os vértices de uma componente conexa C.

Se G é um grafo e u é um vértice de G, denotamos por $G - u$ o grafo obtido pela remoção de u e de todas arestas incidentes a u de G. Se G é um grafo conexo e não existe u tal que $G - u$ é desconexo, G é dito *biconexo*. Uma *ponte* é uma aresta cuja remoção desconecta G.

Uma *árvore* é um grafo conexo e acíclico. Um resultado amplamente conhecido diz que toda árvore com n vértices tem $n - 1$ arestas. Além disso, em uma árvore existe exatamente um único caminho entre cada par de vértices. Em uma árvore, os vértices de grau 1 são denominados *folhas*. Uma *árvore enraizada* é uma árvore $T_v = (V, E)$ com um vértice especial v denominado a *raiz* de T_v. Em uma árvore enraizada T_v, dizemos que um vértice u é ancestral de um vértice w se $u \neq w$ e u pertence ao caminho de v para w em T_v. Um grafo acíclico é também chamado de *floresta*, uma vez que cada componente conexa é uma árvore.

Uma *subárvore* de uma árvore T é um subgrafo conexo de T. Seja V um conjunto finito. Uma família $\mathcal{M} = (M_v)_{v \in V}$ de subconjuntos de um conjunto V satisfaz a *propriedade de Helly* se $V' \subseteq V$ e $M_u \cap M_v \neq \emptyset$ para todo $u, v \in V'$ implica que $\bigcap_{v \in V'} M_v \neq \emptyset$. Uma família de subárvores de uma árvore satisfaz a propriedade de Helly [42].

Um *grafo completo* é um grafo com conjunto de arestas

$$E = \{uv \mid u, v \in V\},$$

ou seja, existe uma aresta entre cada par de vértices. Denota-se por K_n um grafo completo de ordem n. Uma clique $S \subseteq V$ é um conjunto de vértices tal que $G[S]$ é um subgrafo induzido completo. O *complemento* $\overline{G} = (V, \overline{E})$ de um grafo $G = (V, E)$, ou o *grafo complementar* de $G = (V, E)$, é um grafo com mesmo conjunto de vértices e conjunto de arestas $\overline{E} = \{uv \mid u, v \in V, uv \notin E\}$. Um *conjunto independente* é um conjunto de vértices S tal que $\overline{G[S]}$ é um grafo completo, ou seja, S é um conjunto de vértices no qual não há vértices adjacentes em G. O tamanho do maior conjunto independente de um grafo G é denotado por $\alpha(G)$.

O *número de dominação* de um grafo G é o tamanho de um conjunto de vértices S de tamanho mínimo tal que todo vértice em $V(G) \backslash S$ possui pelo menos um vizinho em S [45]. Similarmente, o *número de k-dominação* é o tamanho de um conjunto de vértices S de tamanho mínimo tal que todo vértice em $V(G) \setminus S$ possui pelo menos k vizinhos em S [36]. Já o *número de k-distância-dominação* é o tamanho de um conjunto de vértices S de tamanho mínimo tal que todo vértice em $V(G) \setminus S$ está à distância no máximo k de algum elemento de S [46].

2.3 Classes de grafos

Uma *classe* de grafos é um conjunto de grafos. Normalmente uma classe é definida por propriedades que os seus membros satisfazem. *Grafos cúbicos*, *árvores* e *grafos completos* são exemplos de classes interessantes presentes na literatura. Algumas classes de grafos foram extensivamente estudadas ao longo dos anos por possuírem aplicações interessantes ou estruturas que possibilitam algoritmos mais eficientes para problemas que, em geral, são difíceis. Definiremos agora classes que serão abordadas nos capítulos seguintes, como os grafos *bipartidos*, *cordais*, *split*, *cografos* e grafos *distância-hereditários*. Informações sobre estas e diversas outras classes de grafos, bem como suas relações, como interseções e inclusões, podem ser encontradas em [8].

Um grafo *bipartido* $G = (V, E)$ é um grafo no qual o conjunto de vértices pode ser particionado em dois conjuntos A e B de forma que cada aresta tenha uma extremidade em A e outra em B. Um grafo *bipartido completo* é um grafo bipartido onde o conjunto das arestas é exatamente

$$E(G) = \{ab \mid a \in A \wedge b \in B\}.$$

Denotamos um grafo bipartido completo com partições com tamanho $|A| = m$ e $|B| = n$ por $K_{m,n}$. Uma *garra* é um $K_{1,3}$.

Uma *corda* é uma aresta entre dois vértices não consecutivos de um ciclo. Um grafo é *cordal* se cada um dos seus ciclos com 4 ou mais vértices possui pelo menos uma corda. Um grafo *split* é um grafo cujos vértices podem ser particionados em conjuntos K e I, onde $G[K]$ é um grafo completo e $\overline{G[I]}$ é um grafo completo. Todo grafo split é também cordal.

O *grafo de interseção* de uma família de conjuntos $\mathcal{M} = (M_v)_{v \in V}$ é o grafo G com conjunto de vértices $\mathcal{M}$ e uma aresta $M_u M_v$ pertence a $E(G)$ se e somente se $M_u \cap M_v \neq \emptyset$. É sabido que um grafo é cordal se e somente se ele é o grafo de interseção de subárvores de uma árvore [9, 40, 51, 67]. Uma *árvore clique* T de um grafo cordal G é a árvore cujos vértices são as cliques maximais de G e o conjunto de cliques que contém um vértice v de G induz uma subárvore de T, para todo v em $V(G)$ [42, 51].

Um grafo *distância-hereditário* é um grafo no qual as distâncias entre qualquer par de vértices é a mesma em qualquer subgrafo induzido conexo. Um *cografo* é um grafo que não possui um caminho com 4 vértices como subgrafo induzido. Todo cografo ou é um K_1 ou pode ser obtido através da união disjunta e complementação de outros cografos. Todo cografo e toda árvore são grafos distância-hereditários.

2.4 Convexidade

Seja V um conjunto. Uma família $\mathcal{C} \subset 2^V$ é uma *convexidade* [64] em V se:

(C_1) O conjunto vazio $\emptyset$ e o conjunto universo V estão em $\mathcal{C}$;

(C_2) $\mathcal{C}$ é fechada por interseção, e

(C_3) A união de uma cadeia de elementos ordenada por inclusão de $\mathcal{C}$ está em $\mathcal{C}$.

Para convexidades finitas, C_3 é desnecessária, uma vez que é trivialmente satisfeita, já que a união de toda cadeia finita ordenada por inclusão é igual ao seu último elemento. Dizemos que um conjunto S é *convexo* se $S \in \mathcal{C}$.

Exemplo 2.1. *Se* $V = \{1, 2, 3, 4\}$ *e* $\mathcal{C} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3, 4\}\}$*, então* $\mathcal{C}$ *é uma convexidade, uma vez que satisfaz as condições* C_1, C_2 *e* C_3*.*

Quando consideramos convexidades em grafos, as convexidades com estruturas interessantes são motivadas por propriedades que os conjuntos convexos possuam. O tipo de convexidade considerado na maior parte deste trabalho é a *convexidade de caminhos.*

Seja $\mathcal{F}$ uma família de caminhos de um grafo $G = (V, E)$. O *intervalo fechado* $I_{\mathcal{F}}(u, v)$, $u, v \in V$, consiste de u, v e de todos os vértices que são vértices internos em caminhos entre u e v de $\mathcal{F}$. Omitiremos o índice $\mathcal{F}$ quando este estiver implícito. Nesta definição, o conceito de intervalo é análogo à ideia de intervalo presente na geometria, onde o intervalo de dois pontos contém os pontos intermediários presentes no segmento ligando estes dois pontos, ou seja, os pontos obtidos através das combinações convexas das extremidades.

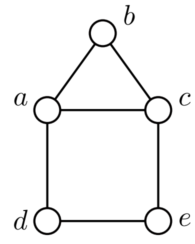

Figura 2.1: Um exemplo de grafo $G = (V, E)$ com conjunto de vértices $V = \{a, b, c, e, d\}$ e conjunto de arestas $E = \{ab, ac, ad, bc, ce, de\}$.

Exemplo 2.2. *Um exemplo de família de caminhos para o grafo da Figura 2.1 seria* $\mathcal{F} = \{abc, ace, ade, bac, bad, bce, cad, ced\}$*. Neste caso, o intervalo fechado de* $\{a, e\}$ *seria* $I(\{a, e\}) = \{a, c, d, e\}$*.*

Seja $S \subseteq V$ um conjunto de vértices. Denotamos por $I(S)$ o intervalo fechado de S, dado por

$$I(S) = \bigcup_{u,v \in S} I(u, v).$$

Seja $I^0(S) = S$. Denotamos por $I^k(S)$ o conjunto $I(I^{k-1}(S))$, para $k > 0$.

Teorema 2.3. *Seja* $G = (V, E)$ *um grafo e* $\mathcal{F}$ *uma família de caminhos em* G*. A família* $\mathcal{C}(\mathcal{F}) = \{S \subseteq V \mid I_{\mathcal{F}}(S) = S\}$ *é uma convexidade em* V*.*

Demonstração. A condição C_1 é trivialmente verdadeira. Como V é um conjunto finito de vértices de um grafo, a condição C_3 é trivialmente satisfeita para qualquer família. Dessa forma, consideraremos apenas a condição C_2.

Seja $\mathcal{C}$ e $\mathcal{F}$ como descritos acima. Suponha, por contradição, que C_2 seja falsa. Então, existem conjuntos S_1, S_2 e S_3 tais que $S_1 \in \mathcal{C}$, $S_2 \in \mathcal{C}$, $S_3 \notin \mathcal{C}$ e $S_1 \cap S_2 = S_3$. Seja $w \in I(S_3) \setminus S_3$. Então, por definição, existem vértices $u, v \in S_3$ tais que w é um vértice interno de um caminho de $\mathcal{F}$ entre u e v e $w \notin S_3$. Como $u, v \in S_1$ e S_1 é convexo, $w \in S_1$. Analogamente, $w \in S_2$ e, portanto, $w \in S_1 \cap S_2$, uma contradição. $\square$

Exemplo 2.4. *Para a família* $\mathcal{F}$ *definida no Exemplo 2.2, os conjuntos convexos de* $\mathcal{C}(\mathcal{F})$ *seriam* $\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}$ *e* $\{a, b, c, d, e\}$*.*

2.4.1 Algumas convexidades relevantes

No caso em que $\mathcal{F}$ é composta exatamente pelos caminhos mínimos de G, conhecida como convexidade *geodética*, diversos estudos foram feitos nas últimas décadas como em [11, 33, 35, 41, 53].

Já quando $\mathcal{F}$ é o conjunto de caminhos induzidos de G, obtemos a convexidade *monofônica* estudada, por exemplo, em [31, 34].

Outro tipo de convexidade particularmente interessante é a convexidade P_3, definida pela família $\mathcal{F}$ contendo todos os caminhos com exatamente 3 vértices. Note que no Exemplos 2.2, a convexidade $\mathcal{C}(\mathcal{F})$ é a convexidade P_3, uma vez que a família $\mathcal{F}$ consiste de todos os caminhos de ordem 3 de G. A convexidade P_3 foi originalmente estudada para uma classe de grafos direcionados conhecidos como torneios [32, 52, 56, 65] e mais recentemente para grafos gerais [5, 14, 15]. Para grafos não direcionados, a convexidade P_3 possui uma característica particularmente interessante por conta da sua natureza local: para determinar se um conjunto é convexo, basta checar, se nenhum vértice fora do conjunto possui mais de um vizinho dentro do conjunto. Essa característica local da convexidade P_3 fornece propriedades estruturais interessantes, tendo também consequências algorítmicas.

Outros tipos de convexidade em grafos também foram estudados, como em [16–18, 30]. A seguir apresentamos uma generalização da convexidade de intervalo, para a qual as convexidades citadas acima são casos particulares. Esta generalização será abordada no Capítulo 5.

2.4.2 Convexidade de intervalo generalizado

De acordo com a definição de Calder [12], uma convexidade $(V, \mathcal{C})$ é uma *convexidade de intervalo* se existe uma função de intervalo $I : \binom{V}{2} \to 2^V$ tal que um subconjunto C de V é convexo, se e somente se $I(\{x, y\}) \subseteq C$ para quaisquer dois elementos x e y de C. É fácil ver que as convexidade de caminho citadas anteriormente são também convexidades de intervalo.

Generalizando este conceito, dizemos que uma convexidade $(V, \mathcal{C})$ é uma *convexidade de r-intervalo* para algum inteiro $r \geq 2$ se existir uma *função de r-intervalo* $I : \binom{V}{r} \to 2^V$ tal que um subconjunto C de V pertence a $\mathcal{C}$ se e somente se $I(U) \subseteq C$ para todo $U \in \binom{C}{r}$. Assumiremos sempre que $U \subseteq I(U)$ para todo $U \in \binom{C}{r}$.

2.4.3 Parâmetros de convexidade

Diversos parâmetros relacionados à convexidade têm sido estudados na literatura para convexidades abstratas [64], normalmente a partir de analogias com resultados clássicos no espaco euclideano.

Alguns parâmetros são de particular interesse para este trabalho. Para $G = (V, E)$, uma famíla de caminhos $\mathcal{F}$ e um conjunto $S \subset V$, denotamos por $H(S)$ a *envoltória* de S, definida pela interseção de todos os conjuntos de $\mathcal{C}$ contendo S. Equivalentemente $H(S)$ é o menor conjunto convexo contendo S. Note que, para convexidades de caminhos, $I^\infty(S) = H(S)$. A envoltória, em inglês *hull*, é muitas vezes chamada *fecho*, mas adotaremos o termo envoltória neste texto.

Um conjunto satisfazendo $H(S) = V$ é dito um *conjunto de envoltória*. O *número de envoltória* de G, $h_{\mathcal{F}}(G)$, é o tamanho de um conjunto de envoltória mínimo, isto é, um menor conjunto S tal que $H(S) = V$. Exemplos de conjuntos de envoltória mínimos para as convexidades P_3 e geodética podem ser vistos na Figura 2.2. Note, entretanto, que estes conjuntos não são únicos, ou seja, pode haver mais de um conjunto de envoltória mínimo.

Uma *partição de Radon* de um conjunto R é uma partição de R em dois conjuntos disjuntos R_1 e R_2 com $H(R_1) \cap H(R_2) \neq \emptyset$. Um conjunto R é um *conjunto anti-Radon de G* se ele não possui uma partição de Radon. O *número de Radon* $r(G)$

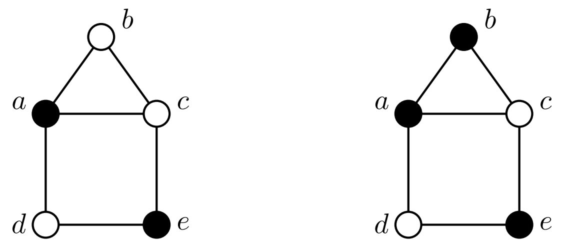

Figura 2.2: Os conjuntos $\{a,e\}$ e $\{a,b,e\}$ são conjuntos de envoltória mínimos para as convexidades P_3 e geodética, respectivamente. Note que $\{a,e\}$ não é um conjunto de envoltória na convexidade geodética uma vez que neste caso $H(\{a,e\}) = \{a,c,d,e\}$, já que b não é um vértice interno de nenhum caminho mínimo de G.

de G é o menor inteiro r tal que qualquer conjunto com pelo menos r vértices de G possui uma partição de Radon. De forma equivalente, o número de Radon de G é o tamanho de um conjunto anti-Radon máximo mais um, isto é,

$$r(G) = \max\{|R| \mid R \text{ é um conjunto anti-Radon de } G\} + 1.$$

O número de Radon é uma generalização do teorema de Radon para convexidades no espaço euclideano [59]. Conjuntos anti-Radon são também conhecidos na literatura como *conjuntos Radon independentes*.

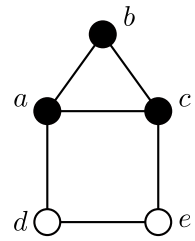

Figura 2.3: O conjunto $\{a,b,c\}$ é um conjunto anti-Radon para a convexidade geodética. Assim, temos que $r(G) \geq 4$ nesta convexidade. Como qualquer conjunto com 4 vértices contém pelo menos 3 vértices no ciclo de tamanho 4, é fácil ver que não há um conjunto anti-Radon de tamanho 4, e portanto, $r(G) = 4$.

O *número de Carathéodory* de $(V, \mathcal{C})$ é o menor inteiro k tal que para todo subconjunto U de V e todo elemento u de $H(U)$, existe um subconjunto F de U com $|F| \leq k$ e $u \in H(F)$. Um subconjunto U de V é um *conjunto de Carathéodory* se o conjunto

$$\partial H(U) = H(U) \setminus \bigcup_{u \in U} H(U \setminus \{u\})$$

não é vazio. Com este conceito, podemos utilizar uma definição equivalente de número de Carathéodory como sendo o tamanho do maior conjunto de Carathéodory. Conjuntos de Carathéodory são também chamados *conjuntos Carathéodory independentes*. Assim como no caso do número de Radon, o número de Carathéodory é uma generalização do resultado de Carathéodory para o espaço euclideano[13].

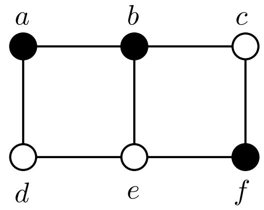

Figura 2.4: O conjunto $\{a, b, f\}$ é um conjunto de Carathéodory para a convexidade P_3. Note que $H(U) = \{a, b, c, d, e, f\}$ e portanto $\partial H(U) = H(U) \setminus \big(H(\{a, b\}) \cup H(\{a, f\}) \cup H(\{b, f\})\big) = \{a, b, c, d, e, f\} \setminus \big(\{a, b\} \cup \{a, f\} \cup \{b, c, e, f\}\big) = \{d\}$.

Mais recentemente, aspectos algorítmicos de conceitos de convexidade e seus parâmetros vêm sendo abordados. A determinação do intervalo de um conjunto de vértices, que pode ser considerada a operação mais básica das convexidades que consideraremos, pode trivialmente ser feita em tempo polinomial para as convexidade P_3 e geodética, porém já se mostra NP-difícil para a convexidade monofônica [23]. Parâmetros como o número de envoltória, número de Radon, número de Carathéodory, dentre outros, também foram estudados do ponto de vista algorítmico [5, 15, 22–24].

2.5 Intermediações

Para um conjunto V, denotaremos por V^3 o produto cartesiano $V \times V \times V$. Uma *intermediação*, em inglês *betweenness*, é um conjunto $\mathcal{B} \subseteq V^3$, para algum conjunto V. O conceito de intermediação, assim como o conceito de intervalo, generaliza propriedades de pontos no espaço. Se uma tripla $(u, v, w) \in \mathcal{B}$, dizemos que v está entre u e w. O conceito de "estar entre dois elementos" pode abstrair propriedades de diversas estruturas. Neste trabalho estudaremos, em particular, intermediações em grafos e famílias de subconjuntos. A menos de menção em contrário, consideraremos que $(u, v, w) \in \mathcal{B} \iff (w, v, u) \in \mathcal{B}$, ou seja, se v está entre u e w, também está entre w e u.

Se $u \neq v \neq w \neq u$ dizemos que (u, v, w) é uma tripla *estrita*. Dizemos que $\mathcal{B}$ é uma intermediação *estrita* se todas as suas triplas são estritas. Para fins de clareza, denotaremos uma intermediação estrita por $\mathcal{B}_s$. Analogamente, denotaremos por V_s^3 como o conjunto de todas as triplas estritas sobre V. Nos resultados apresentados no próximo capítulo, consideraremos apenas intermediações estritas, exceto quando explicitado de forma contrária.

As conexões entre intervalos e intermediações não se restringem à sua motivação geométrica comum. De fato, para um grafo $G = (V, E)$ e uma família de caminhos $\mathcal{F}$, definimos $\mathcal{B}(G)$ como

$$\mathcal{B}(G) = \{(u, v, w) \in V^3 \mid v \in I(u, w)\}.$$

Para uma intermediação estrita, definimos $\mathcal{B}_s(G)$ como

$$\mathcal{B}_s(G) = \{(u, v, w) \in V_s^3 \mid v \in I(u, w)\}.$$

Uma *intermediação de árvore* ou *intermediação arbórea* é o caso particular em que $G = (V, E)$ é uma árvore e $\mathcal{F}$ é o conjunto de todos os caminhos da árvore, ou seja, um caminho para cada par de vértices. Denotaremos intermediações arbóreas por $\mathcal{B}(T)$, onde T é uma árvore, e por $\mathcal{B}_s(T)$, no caso de intermediação estrita.

Exemplo 2.5. *Se T é uma garra com conjunto de vértices $\{v_1, v_2, v_3, v_4\}$, onde v_1 é o vértice de grau 3, então temos $\mathcal{B}_s(T) = \{(v_2, v_1, v_3), (v_2, v_1, v_4), (v_3, v_1, v_4), (v_4, v_1, v_3), (v_4, v_1, v_2), (v_3, v_1, v_2)\}$.*

Burigana introduziu, em [10], o conceito de intermediação baseado em uma família de conjuntos. Para uma família $\mathcal{M} = (M_v)_{v \in V}$, a *intermediação de interseção* $\mathcal{B}(\mathcal{M})$ induzida por $\mathcal{M}$ consiste de todas as triplas (u, v, w) tais que $M_u \cap M_w \subseteq M_v$, ou seja,

$$\mathcal{B}(\mathcal{M}) = \{(u, v, w) \in V^3 \mid M_u \cap M_w \subseteq M_v\}.$$

Analogamente, a *intermediação de interseção estrita* $\mathcal{B}_s(\mathcal{M})$ induzida por $\mathcal{M}$ é dada por

$$\mathcal{B}_s(\mathcal{M}) = \mathcal{B}(\mathcal{M}) \cap V_s^3 = \{(u, v, w) \in V_s^3 \mid M_u \cap M_w \subseteq M_v\}.$$

Exemplo 2.6. *Seja $V = \{u, v, w\}$ com $M_u = \{1, 2, 3\}$, $M_v = \{2, 3, 4\}$, $M_w = \{3, 4, 5\}$. Então temos $\mathcal{B}(\mathcal{M}) = \{(u, v, w), (w, v, u)\}$. Veja a Figura 2.5.*

Burigana também estudou o relacionamento entre intermediação de interseção e intermediação arbórea e problemas relacionados. Exploraremos algumas dessas características no próximo capítulo.

Se $G = (V, E)$ é um grafo simples não direcionado, definimos a *intermediação de caminhos mínimos estrita* de G como $\mathcal{B}_s(G) = \{(u, v, w) \in V_s^3 \mid v$ é um vértice interno de um caminho mínimo entre u e $w\}$. Analogamente, a *intermediação de caminhos induzidos estrita* de G como $\mathcal{B}_s(G) = \{(u, v, w) \in V_s^3 \mid v$ é um vértice interno de um caminho induzido entre u e $w\}$.

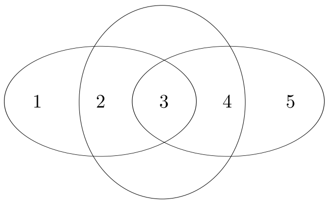

Figura 2.5: $M_u = \{1, 2, 3\}$, $M_v = \{2, 3, 4\}$, $M_w = \{3, 4, 5\}$

Capítulo 3

Intermediações em grafos

With great power comes great responsibility.

Tio Ben - Spider-Man, o filme

O conceito de intermediação (em inglês, *betweenness*) busca representar a noção de um elemento estar entre dois outros. Tal representação pode ser adaptada para uma série de estruturas. Neste capítulo consideramos problemas envolvendo grafos, com atenção especial a árvores e florestas, e famílias, utilizando o conceito de intermediação de interseção, definido por Burigana em [10]. Os resultados apresentados neste capítulo foram publicados em [60, 61] e podem ser encontrados nos Apêndices A e B.

3.1 Subintermediação Arbórea

Nesta seção abordaremos os seguintes problemas de decisão:

SUBINTERMEDIAÇÃO DE UMA ÁRVORE
Entrada: Um conjunto estrito $\mathcal{B} \subseteq V_s^3$.
Pergunta: Existe uma árvore T tal que $\mathcal{B} \subseteq \mathcal{B}_s(T)$?

SUBINTERMEDIAÇÃO INDUZIDA DE UMA ÁRVORE
Entrada: Um conjunto estrito $\mathcal{B} \subseteq V_s^3$.
Pergunta: Existe uma árvore T tal que $\mathcal{B} = \mathcal{B}_s(T) \cap V_s^3$?

Embora semelhantes, o problema de SUBINTERMEDIAÇÃO INDUZIDA DE UMA ÁRVORE se mostra muito mais simples que a versão mais geral, como veremos a seguir, devido à forte estrutura necessária à instância de entrada para que a resposta seja SIM. A seguir, determinamos a complexidade destes problemas.

Subintermediação de uma árvore pode ser visto como uma generalização natural do problema de ordenação total, que é NP-Completo [55]:

Ordenação total
Entrada: Um conjunto $\mathcal{B} \subseteq V_s^3$.
Pergunta: Existe um caminho P tal que $\mathcal{B} \subseteq \mathcal{B}_s(P)$?

Antes de considerar diretamente os dois problemas, apresentaremos um resultado que nos será útil em ambos os casos.

Lema 3.1 ([61])**.** *Seja T uma árvore e $V \subseteq V(T)$.*

(i) Se $d(v) \leq 2$, para $v \in V(T) \setminus V$, então a árvore T' resultante da remoção de v e a união de seus vizinhos por uma nova aresta satisfaz $\mathcal{B}_s(T) \cap V_s^3 = \mathcal{B}_s(T') \cap V_s^3$.

(ii) Se $uv \in E(T)$ com $u, v \in V(T) \setminus V$, a contração de uv resulta em uma árvore T' com $\mathcal{B}_s(T) \cap V_s^3 = \mathcal{B}_s(T') \cap V_s^3$.

Demonstração. (i) A remoção de um vértice v com $d(v) = 1$ claramente não altera $\mathcal{B}_s(T)$. No caso onde $d(v) = 2$, note que qualquer caminho que contenha v como vértice interno, também conterá seus dois vizinhos. Assim, a remoção de v e a inclusão de uma aresta entre seus vizinhos afetará apenas caminhos que passam por v, mas sem alterar a ordem em que os vértices de V ocorrem nestes caminhos.

(ii) Seja w o vértice criado a partir da contração de uv. Quaisquer caminhos entre vértices de V que passassem apenas por vértices em $\{u, v\}$ passarão agora por w. Da mesma forma, caminhos que passem por w em T', possuem caminhos equivalentes em T, passando por u, por v ou por ambos. □

Teorema 3.2 ([61])**.** Subintermediação de uma árvore *é NP-Completo.*

Demonstração. Primeiro, mostraremos que Subintermediação de uma árvore está em NP. Suponha que existe T tal que $\mathcal{B} \subseteq \mathcal{B}_s(T)$. É fácil ver que T é um certificado para a resposta Sim, que pode ser verificado em tempo polinomial no tamanho de T. Note que a partir do Lema 3.1 sabemos que T tem tamanho polinomial em $\mathcal{B}$, uma vez que podemos supor que os vértices de $V(T) \setminus V$ formam um conjunto independente no qual cada vértice possui grau pelo menos 3 e, portanto, $|V(T) \setminus V| \leq |V|$.

Para mostrar que Subintermediação de uma árvore é NP-Difícil, faremos a redução a partir de Ordenação total. A partir de uma instância $\mathcal{B} \subseteq V_s^3$ de Ordenação total, construimos uma instância $\mathcal{B}' \subseteq V_s'^3$, $V' = V \cup \{x, y\}$, onde $x \neq y$ e $x, y \notin V$ e

$$\mathcal{B}' = \mathcal{B} \cup \{(x, v, y) \mid v \in V\}.$$

Note que T é uma solução para a instância $\mathcal{B}'$ se e só se T é um caminho, com extremidades x e y e o caminho formado pelos vértices internos é um certificado para a instância $\mathcal{B}$ de Ordenação total. □

Se por um lado Subintermediação de uma árvore é um problema difícil, Subintermediação induzida de uma árvore pode ser resolvido em tempo polinomial, como veremos adiante. Antes de exibir o algoritmo, apresentamos um lema que nos diz como reconstruir T a partir de $\mathcal{B}$.

Lema 3.3 ([61])**.** *Seja T uma árvore e $V \subseteq V(T)$ tais que $V(T) \setminus V$ é um conjunto independente de T com cada vértice possuindo grau pelo menos 3 em T. Seja $\mathcal{B} = \mathcal{B}_s(T) \cap V_s^3$. Seja $\mathcal{T}$ o conjunto de todos os conjuntos $S \in \binom{V}{3}$ tais que $(u,v,w),(u,w,v),(v,u,w) \notin \mathcal{B}$ e que não existe $x \in V$ e $u,v \in S$ com $(u,x,v) \in \mathcal{B}$. Seja $G = (\mathcal{T}, E)$ o grafo no qual existe aresta entre dois vértices $s,t \in \mathcal{T}$ se e somente se $|s \cap t| = 2$.*

(i) $V(T) \setminus V$ contém um vértice v com vizinhaça N se e somente se existe alguma componente conexa C de G com $N = \bigcup\limits_{t \in V(C)} t$.

(ii) Dois vértices $u,v \in V$ são vizinhos se e somente se não existe $t \in \mathcal{T}$ com $u,v \in t$ e não existe $x \in V$ com $(u,x,v) \in \mathcal{B}$.

Com o Lema 3.3 podemos formular um algoritmo eficiente para resolver Subintermediação induzida de uma árvore.

Algoritmo 1 Algoritmo para Subintermediação induzida de uma árvore

Entrada: $\mathcal{B} \subseteq V_s^3$
Saída: Uma árvore T ou a resposta Não, caso não exista uma árvore

$V(T) \leftarrow V$
$E(T) \leftarrow \emptyset$
Construa $\mathcal{T}$ e G como descrito no Lema 3.3
para toda componente C de G **faça**
  $N \leftarrow \emptyset$
  para todo $S \in C$ **faça**
    $N \leftarrow N \cup S$
  Crie novo vértice v
  $V(T) \leftarrow V(T) \cup \{v\}$
  para todo $w \in N$ **faça**
    $E(T) \leftarrow E(T) \cup \{vw\}$
para todo $\{u,v\} \in \binom{V}{2}$ **faça**
  se $(\nexists S \in \mathcal{V} : u,v \in S) \wedge (\nexists x \in V : (u,v,w) \in \mathcal{B})$ **então**
    $E(T) \leftarrow E(T) \cup \{u,v\}$
se T não é uma árvore **então**
  retorne Não
Construa $\mathcal{B}_s(T)$
se $\mathcal{B} \neq \mathcal{B}_s(T) \cap V_s^3$ **então**
  retorne Não
senão
  retorne T

Teorema 3.4 ([61]). *O Algoritmo 1 resolve o problema* Subintermediação induzida de uma Árvore *corretamente em tempo polinomial.*

Note que uma vez que uma vez que o tamanho de $\mathcal{B} \subseteq V^3$ é $O\left(|V|^3\right)$, o algoritmo é também polinomial em $|V|$.

3.2 Intermediação de Florestas

Em [10], Burigana caracteriza axiomaticamente intermedição de árvores. Rautenbach e Schäfer fornecem uma prova mais curta e fortalecem o resultado em [19].

Para $\mathcal{B} \subseteq V^3$ e $u, v, w \in V$, denotamos por $N(u, v, w)$ a seguinte expressão

$$(u \neq v \neq w \neq u) \wedge ((u, v, w), (v, w, u), (w, u, v) \notin \mathcal{B}),$$

ou seja, usamos $N(u, v, w)$ para denotar que não há nenhuma tripla relacionando u, v e w. Burigana apresenta os seguintes axiomas e resultado.

(T_1) $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (w, v, u) \in \mathcal{B}$

(T_2) $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (v, u, w) \notin \mathcal{B}$

(T_3) $\forall u, v, w, z \in V : (u, v, w), (v, w, z) \in \mathcal{B} \Rightarrow (u, w, z) \in \mathcal{B}$

(T_4) $\forall u, v, w, z \in V : (u, v, w), (u, w, z) \in \mathcal{B} \Rightarrow (v, w, z) \in \mathcal{B}$

(T_5) $\forall u, v, w \in V : N(u, v, w) \Rightarrow \exists c \in V : (u, c, v), (u, c, w), (v, c, w) \in \mathcal{B}$

Teorema 3.5 (Burigana [10]). *Seja V um conjunto finito e $\mathcal{B} \subseteq V^3$. Existe uma árvore T tal que $\mathcal{B}_s(T) = \mathcal{B}$ se e somente se $\mathcal{B}$ satisfaz (T_1), (T_2), (T_3), (T_4) e (T_5).*

Para uma floresta F, os axiomas continuam valendo para cada uma das componentes conexas, mas (T_5) falha no caso geral. Portanto, torna-se necessário modificar o conjunto de axiomas fornecidos por Burigana para a caracterização de intermediação de floresta.

Consideraremos que todas as componentes conexas possuem pelo menos 3 vértices, uma vez que componentes menores podem ser descartadas, já que não contribuem com nenhuma tripla para $\mathcal{B}(F)$. Se três vértices u, v e w pertencem à mesma componente de F mas não pertencem a um caminho então a menor árvore induzida contendo u, v e w é isomorfa a uma subdivisão de uma garra $K_{1,3}$ e tem folhas u, v e w. Além disso, dois vértices pertencem a uma mesma componente de F se e somente se existe um caminho em F com pelo menos 3 vértices contendo ambos.

Para algum $\mathcal{B} \subseteq V^3$ e $u, v \in V$, utilizaremos $u \sim_{\mathcal{B}} v$ para representar a seguinte expressão

$$(u = v) \vee (\exists x \in V : (x, u, v) \in \mathcal{B} \vee (u, x, v) \in \mathcal{B} \vee (u, v, x) \in \mathcal{B}),$$

ou seja, $u = v$ ou existe um caminho com pelo menos 3 vértices contendo u e v.

Considerando as observações acima, um intermediação de floresta satisfaz a seguinte versão modificada de (T_5).

(F_5) $\forall u, v, w \in V : (N(u, v, w) \wedge (u \sim_{\mathcal{B}} v) \wedge (u \sim_{\mathcal{B}} w)) \Rightarrow \exists c \in V : (u, c, v), (u, c, w), (v, c, w) \in \mathcal{B}$

Apresentamos, então, o seguinte resultado auxiliar:

Lema 3.6 ([61])**.** *A relação $\sim_{\mathcal{B}}$ é transitiva.*

Demonstração. Suponha, por contradição, u, v, e w em V tais que $u \sim_{\mathcal{B}} v$ e $v \sim_{\mathcal{B}} w$ mas $u \not\sim_{\mathcal{B}} w$. Claramente, isso implica que u, v e w são distintos entre si e também $N(u, v, w)$.

Porém, por (F_5), existe algum $c \in V$ com (u, c, w), o que contradiz $u \sim_{\mathcal{B}} w$. □

Podemos, então, apresentar o principal resultado desta seção:

Teorema 3.7 ([61])**.** *Seja V um conjunto finito. Se $\mathcal{B} \subseteq V_s^3$ é um conjunto de triplas estritas, então existe uma floresta F tal que $\mathcal{B}_s(F) = \mathcal{B}$ se e somente se $\mathcal{B}$ satisfaz (T_1), (T_2), (T_3), (T_4) e (F_5).*

Demonstração. Como exposto acima, se existe F satisfazendo $\mathcal{B}_s(F) = \mathcal{B}$ então (T_1), (T_2), (T_3), (T_4) e (F_5) são satisfeitos.

Seja $\mathcal{B} \subseteq V_s^3$ satisfazendo (T_1), (T_2), (T_3), (T_4), e (F_5). Por definição e por (T_1), $\sim_\mathcal{B}$ é reflexivo e simétrico. Pelo Lema 3.6, $\sim_\mathcal{B}$ é também transitivo. Assim, $\sim_\mathcal{B}$ é uma relação de equivalência.

Seja $V = V_1 \cup V_2 \cup \cdots \cup V_k$ a partição de V em classes de equivalência de $\sim_\mathcal{B}$. Para $1 \leq i \leq k$, seja $\mathcal{B}_i = \mathcal{B} \cap (V_i)_s^3$. Note que dois elementos distintos de V aparecem juntos em uma tripla de $\mathcal{B}$ se e somente se eles pertencem a mesma classe de equivalência. Assim, $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$. Para $1 \leq i \leq k$, $\mathcal{B}_i$ satisfaz (T_1), (T_2), (T_3), (T_4), e (T_5). Pelo Teorema 3.5, para cada $1 \leq i \leq k$, existe uma árvore T_i com $\mathcal{B}_s(T_i) = \mathcal{B}_i$. Claramente, $\mathcal{B}_s(T_1 \cup T_2 \cup \cdots \cup T_k) = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k = \mathcal{B}$. □

3.3 Caracterização de Intermediação de Interseção

Burigana [10] apresenta os seguintes axiomas, e afirma que eles valem para toda intermediação de interseção estrita $\mathcal{B}$.

$$
\begin{array}{lll}
(I_1) & \forall u, v, w \in V: & (u,v,w) \in \mathcal{B} \Rightarrow (w,v,u) \in \mathcal{B} \\
(I_2) & \forall u, v, w, z \in V: & (u,v,w), (u,z,v) \in \mathcal{B} \Rightarrow (u,z,w) \in \mathcal{B} \\
(I_3) & \forall u, v, w, t, z \in V: & (t,u,z), (t,w,z), (u,v,w) \in \mathcal{B} \Rightarrow (t,v,z) \in \mathcal{B}
\end{array}
$$

(I_1) claramente vale para toda intermediação de interseção estrita. Entretanto, as propriedades (I_2) e (I_3) são problemáticas por possivelmente implicar na existência de triplas não estritas. Para garantir que a tripla (u,z,w), cuja existência é garantida por (I_2), é estrita, é necessário adicionar a condição $w \neq z$. Da mesma forma, para garantir que a tripla (t,v,z), cuja existência é garantida por (I_3), é estrita, devemos adicionar a restrição adicional $t \neq v \neq z$. Assim, obtemos as seguintes versões modificadas de (I_2) e (I_3).

$$
\begin{array}{lll}
(I_2^s) & \forall u, v, w, z \in V: & (u,v,w), (u,z,v) \in \mathcal{B} \text{ e } w \neq z \Rightarrow (u,z,w) \in \mathcal{B} \\
(I_3^s) & \forall u, v, w, t, z \in V: & (t,u,z), (t,w,z), (u,v,w) \in \mathcal{B} \text{ e } t \neq v \neq z \Rightarrow (t,v,z) \in \mathcal{B}
\end{array}
$$

Para intermediações de interseção não estritas, os três axiomas originais, (I_1), (I_2) e (I_3), valem devido a propriedades de básicas interseção de conjuntos. Entretanto, neste caso, triplas da forma (u, u, w) e (u, w, w) sempre estão presentes. Dessa forma, torna-se necessário uma propriedade adicional:

(I_4) $\quad \forall u, w \in V : \quad (u, u, w) \in \mathcal{B}$

Note que (I_4) juntamente com (I_3) implica (I_2), fazendo $t = u$ em (I_3).

Mostraremos que os axiomas acima, além de valerem para qualquer intermediação de interseção, também nos dão uma caracterização para as mesmas. Além disso, mostraremos também que para uma intermediação de interseção $\mathcal{B} \subseteq V^3$, estrita ou não, é sempre possível construir uma família de conjuntos $\mathcal{M}$ com $\mathcal{B}(\mathcal{M}) = \mathcal{B}$ com tamanho de $\bigcup\limits_{v \in V} M_v$ quadrático no tamanho de V.

Teorema 3.8 ([60])**.** *Seja V um conjunto finito e $\mathcal{B} \subseteq V^3$.*

(i) Se existe uma família de subconjuntos $\mathcal{M} = (M_v)_{v \in V}$ com intermediação de interseção não estrita $\mathcal{B}(\mathcal{M}) = \mathcal{B}$, então $\mathcal{B}$ satisfaz (I_1), (I_3) e (I_4).

(ii) Se $\mathcal{B}$ satisfaz (I_1), (I_3) e (I_4), então existe uma família de conjuntos $\mathcal{M} = (M_v)_{v \in V}$ com intermediação de interseção não estrita $\mathcal{B}(\mathcal{M}) = \mathcal{B}$ e $\left| \bigcup\limits_{v \in V} M_v \right| \leq \binom{|V|}{2}$, que pode ser construída em tempo polinomial.

Demonstração. Como (i) é imediata, prosseguiremos com a prova de (ii). Seja $\mathcal{B}$ satisfazendo (I_1), (I_3) e (I_4). Para $v \in V$, seja $M_v \subseteq \{\{u, w\} \mid u, w \in V\}$ tal que $\{u, w\} \in M_v$ se e somente se $(u, v, w) \in \mathcal{B}$. Note que M_v pode facilmente ser construído em tempo polinomial (veja o Algoritmo 2). Note também que $\{u, w\} = \{w, u\}$. Os M_v são bem definidos se e somente se $(u, v, w) \in \mathcal{B}$ vale se e somente $(w, v, u) \in \mathcal{B}$ vale, o que é garantido por (I_1). Mostraremos que $\mathcal{B} = \mathcal{B}(\mathcal{M})$ para a família de conjuntos $\mathcal{M} = (M_v)_{v \in V}$. Seja, então, $(u, v, w) \in V^3$. Por (I_1) e (I_4), $(u, u, w), (u, w, w) \in \mathcal{B}$ e então $\{u, w\} \in M_u \cap M_w$. Suponha $(u, v, w) \notin \mathcal{B}$. Por definição, isso implica $\{u, w\} \in (M_u \cap M_w) \setminus M_v$ e então $(u, v, w) \notin \mathcal{B}(\mathcal{M})$. Assuma, então, $(u, v, w) \in \mathcal{B}$. Por contradição, assuma que $(u, v, w) \notin \mathcal{B}(\mathcal{M})$. Isso implica a existência de $\{t, z\} \in (M_u \cap M_w) \setminus M_v$. Por definição, temos $(t, u, z), (t, w, z) \in \mathcal{B}$. Como $(u, v, w) \in \mathcal{B}$, (I_3) implica $(t, v, z) \in \mathcal{B}$ e então, por definição, $\{t, z\} \in M_v$, uma contradição. Note que $\bigcup\limits_{v \in V} M_v \subseteq \binom{V}{2}$, o que completa a prova. $\square$

Estendemos este resultado a intermediações de interseção estritas. Para o caso estrito, pode ser feita uma prova similar a anterior. Entretanto, apresentaremos uma prova alternativa, que relaciona intermediações estritas a intermediações não estritas que as contém. Além disso, esta prova alternativa mostra que um único algoritmo, o Algoritmo 2, pode ser utilizado para construir a família $\mathcal{M}$ em ambos os casos. Começaremos apresentando um resultado auxiliar.

Lema 3.9 ([60]). *Se V é um conjunto finito e $\mathcal{B}_s \subseteq V_s^3$ é uma intermediação de interseção estrita, então*

$$\mathcal{B} = \mathcal{B}_s \cup \{(u, u, w) \mid u, w \in V\} \cup \{(u, w, w) \mid u, w \in V\}$$

é uma intermediação de interseção não estrita.

Demonstração. Seja $\mathcal{M} = (M_v)_{v \in V}$ a família de conjuntos tal que $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$. Seja $(u_v)_{v \in V}$ uma coleção de $|V|$ elementos distintos dois a dois que não pertençam a $\bigcup\limits_{v \in V} M_v$. Seja $M'_v = M_v \cup \{u_v\}$ para $v \in V$ e seja $\mathcal{M}' = (M'_v)_{v \in V}$. Como $M'_u \not\subseteq M'_v$ e $M'_u \cap M'_v = M_u \cap M_v$ para cada u e v distintos de V, nós obtemos $\mathcal{B}_s(\mathcal{M}') = \mathcal{B}_s(\mathcal{M}) = \mathcal{B}_s$ e $\mathcal{B}(\mathcal{M}') = \mathcal{B}$, o que completa a prova. $\square$

Prosseguimos com o análogo ao Teorema 3.8 para intermediações de interseção estritas.

Teorema 3.10 ([60]). *Seja V um conjunto finito e $\mathcal{B}_s \subseteq V_s^3$.*

(i) Se existe uma família de conjuntos $\mathcal{M} = (M_v)_{v \in V}$ com $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$, então $\mathcal{B}_s$ satisfaz (I_1), (I_2^s), e (I_3^s).

(ii) Se $\mathcal{B}_s$ satisfaz (I_1), (I_2^s) e (I_3^s), então existe uma família de conjuntos $\mathcal{M} = (M_v)_{v\in V}$ com $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$ e $\left|\bigcup_{v\in V} M_v\right| \leq \binom{|V|}{2}$, que pode ser construído em tempo polinomial.

Demonstração. Como (i) é imediata, prosseguiremos com a prova de (ii). Seja $\mathcal{B}_s$ satisfazendo (I_1), (I_2^s), e (I_3^s). Seja $\mathcal{B}$ definido como no Lema 3.9. Claramente, $\mathcal{B}$ satisfaz (I_1) e (I_4) e não contém nenhuma tripla da forma (u, v, u) com $u \neq v$. Mostraremos que $\mathcal{B}$ também satisfaz (I_3). Sejam u, v, w, t e z em V tais que $(t, u, z), (t, w, z), (u, v, w) \in \mathcal{B}$. Precisamos mostrar que $(t, v, z) \in \mathcal{B}$.

Se $u = v$ ou $w = v$ então, trivialmente, $(t, v, z) \in \mathcal{B}$. Podemos então assumir $u \neq v \neq w$. Como $(u, v, w) \in \mathcal{B}$, isso também implica $u \neq w$, ou seja, (u, v, w) é estrita.

Se $t = v$ ou $z = v$, então, por construção, $(t, v, z) \in \mathcal{B}$. Logo, podemos assumir $t \neq v \neq z$.

Se $t = u$, então $(u, w, z), (u, v, w) \in \mathcal{B}$. Como $u \neq w$, isso implica $u \neq z$. Se $w = z$, então $(t, v, z) = (u, v, w) \in \mathcal{B}$. Logo, podemos assumir $w \neq z$, ou seja, (u, w, z) é estrita. Como $v \neq z$, (I_2^s) implica $(t, v, z) = (u, v, z) \in \mathcal{B}$. Então, podemos assumir $t \neq u$ e, por argumentos análogos, $t \neq w$, $z \neq u$ e $z \neq w$. Desta forma, as três triplas (t, u, z), (t, w, z), e (u, v, w) são estritas. Assim, como $t \neq v \neq z$, (I_3^s) implica que $(t, v, z) \in \mathcal{B}$. Pelo Teorema 3.8, existe uma família de conjuntos $\mathcal{M} = (M_v)_{v\in V}$ com $\mathcal{B} = \mathcal{B}(\mathcal{M})$ e $\left|\bigcup_{v\in V} M_v\right| \leq \binom{|V|}{2}$, que pode ser construída em tempo polinomial (veja Algoritmo 2). Como $\mathcal{B}_s = \mathcal{B} \cap V_s^3$, temos que $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$. □

3.4 Intermediação de Interseção em Árvores

É trivial verificar que toda intermediação arbórea estrita satisfaz (I_1), (I_2^s) e (I_3^s). Portanto, o Teorema 3.10 nos diz que toda intermediação arbórea estrita é uma intermediação de interseção estrita.

O resultado a seguir nos fornece uma prova mais simples na qual a família de conjuntos $\mathcal{M} = (M_v)_{v\in V(T)}$ é derivada diretamente a partir de uma árvore T de uma

Algoritmo 2 Representação de uma intermediação de interseção

Entrada: Um conjunto $\mathcal{B} \subseteq V^3$ que seja uma intermediação de interseção, estrita ou não.

Saída: Uma família $\mathcal{M} = (M_v)_{v \in V}$ satisfazendo $\left| \bigcup_{v \in V} M_v \right| \leq \binom{|V|}{2}$ de forma que se $\mathcal{B}$ é uma intermediação de interseção não estrita, então $\mathcal{B} = \mathcal{B}(\mathcal{M})$, e se $\mathcal{B}$ é uma intermediação de interseção estrita, então $\mathcal{B} = \mathcal{B}_s(\mathcal{M})$.

para $\{u, w\} \in \binom{V}{2}$ **faça**
  $\mathcal{B} \leftarrow \mathcal{B} \cup \{(u,u,w),(u,w,w),(u,u,u),(w,w,w)\}$
para $v \in V$ **faça**
  $M_v \leftarrow \emptyset$
para $(u,v,w) \in \mathcal{B}$ **faça**
  $M_v \leftarrow M_v \cup \{\{u,w\}\}$
$\mathcal{M} \leftarrow (M_v)_{v \in V}$
retorne $\mathcal{M}$

maneira bastante interessante, sem a necessidade da construção de $\mathcal{B}_s(T)$, que seria necessária para o uso do Algoritmo 2, e com quantidade $O\big(|V(T)|\big)$ de elementos em $\left| \bigcup_{v \in V(T)} M_v \right|$ em contraste com o limite de $O\big(|V(T)|^2\big)$ fornecido pelo Algoritmo 2.

Para algum $\mathcal{B} \subseteq V^3$ e $u, v, w \in V$, usaremos $N(u,v,w)$ para abreviar a expressão a seguir, como feito anteriormente,

$$(u \neq v \neq w \neq u) \wedge ((u,v,w),(v,w,u),(w,u,v) \notin \mathcal{B}).$$

Teorema 3.11 ([60]). *Seja T uma árvore com n vértices e l folhas. Existe uma família de conjuntos $\mathcal{M} = (M_v)_{v \in V(T)}$ tal que $\mathcal{B}_s(T) = \mathcal{B}_s(\mathcal{M})$ e $\left| \bigcup_{v \in V(T)} M_v \right| \leq 2n - l - 2$.*

Demonstração. Seja T uma árvore com l folhas. Para cada $v \in V(T)$, seja M_v o conjunto de todos os pares ordenados (x,y) tais que $xy \in E(T)$, $d_T(x) \geq 2$ e se enraizamos T em y, então o vértice v é igual a x ou um descendente de x, ou seja $M_v = \big\{(x,y) \mid xy \in E(T) \wedge (v,x,y) \in \mathcal{B}_s(T)\big\} \cup \big\{(v,y) \mid vy \in E(T) \wedge \exists x \in V(T) : (x,v,y) \in \mathcal{B}_s(T)\big\}$. Seja $\mathcal{M} = (M_v)_{v \in V(T)}$. Note que $U = \bigcup_{v \in V(T)} M_v$ consiste de todos os pares ordenados (x,y) tais que $xy \in E(T)$ and $d_G(x) \geq 2$. Logo, $|U| = 2|E(T)| - l = 2n - l - 2$.

Para mostrar que $\mathcal{B}_s(T) \subseteq \mathcal{B}_s(\mathcal{M})$, seja $(u,v,w) \in \mathcal{B}_s(T)$ e seja $(x,y) \in M_u \cap M_w$. Por construção, se enraizamos T em y, então, cada um dos vértices u e w deve ser ou igual a x ou um descendente de x. Como v pertence ao caminho em T entre u e w, isso implica também que v também igual a x ou um descendente de x. Por construção, $(x, y) \in M_v$. Logo $M_u \cap M_w \subseteq M_v$ e assim $(u, v, w) \in \mathcal{B}_s(\mathcal{M})$.

Para mostrar que $\mathcal{B}_s(\mathcal{M}) \subseteq \mathcal{B}_s(T)$, seja $(u, v, w) \in \mathcal{B}_s(\mathcal{M})$. Assuma, por contradição, $(u, v, w) \notin \mathcal{B}_s(T)$, o que implica em $N(u, v, w)$, ou $(u, w, v) \in \mathcal{B}_s(T)$, ou ainda $(v, u, w) \in \mathcal{B}_s(T)$. Se $N(u, v, w)$, então existe algum $c \in V(T)$ que pertence aos caminhos em T entre quaisquer dois dos vértices u, v e w. Seja c' o vizinho de c no caminho em T entre c e v, então, por construção, $(c, c') \in (M_u \cap M_w) \setminus M_v$, o que implica uma contradição em $(u, v, w) \in \mathcal{B}_s(\mathcal{M})$. Se $(u, w, v) \in \mathcal{B}_s(T)$, seja w' o vizinho de w no caminho em T entre w e v, então, por construção, $(w, w') \in (M_u \cap M_w) \setminus M_v$, que implica uma contradição em $(u, v, w) \in \mathcal{B}_s(\mathcal{M})$. O último caso $(v, u, w) \in \mathcal{B}_s(T)$ leva a uma contradição similar, o que completa a prova. $\square$

Capítulo 4

A envoltória convexa de um conjunto

My precious

Gollum - The Lord of the Rings

Neste capítulo consideramos problemas relacionados à envoltória convexa de um conjunto de vértices, como a determinação do número de envoltória, a relação entre conjuntos de envoltória mínimos e minimais, e ainda uma generalização do problema de determinação do número de envoltória, que modela a disseminação de informações, bens ou infecções em uma rede. Na literatura, problemas similares foram estudados sob diversos aspectos, dentre eles um caso particular de percolação (chamado, em inglês, de *bootstrap percolation*), veja por exemplo [4]. Voltaremos à questão de trabalhos relacionados na Seção 4.3, onde abordamos uma versão mais semelhante aos demais resultados encontrados na literatura.

Exceto no Teorema 4.2, quando explicitamos de forma contrária, todos os demais resultados de convexidade deste capítulo são referentes à convexidade P_3 e, portanto, a referência à convexidade será omitida.

Parte dos resultados deste capítulo podem ser encontrados em [6, 62] que se encontram também nos Apêndices C e D.

4.1 O número de envoltória

Como visto no Capítulo 2 o número de envoltória de um grafo G é o tamanho de um conjunto de vértices mínimo cuja envoltória convexa é igual a $V(G)$. O problema de decisão associado é NÚMERO DE ENVOLTÓRIA, com sua entrada sendo composta por um grafo G e um inteiro k, com resposta SIM se e somente se existe um conjunto de envoltória de tamanho menor ou igual a k. Este problema é NP-completo [15], o que motiva o estudo para classes mais restritas, em busca de famílias de instâncias que sejam tratáveis.

Uma classe de grafos de estrutura bastante interessante para aplicações reais é a classe dos grafos bipartidos. Assim, é uma das classes de grafos mais estudadas quando buscamos algoritmos eficientes para problemas que são difíceis no caso geral. Para diversos problemas NP-difíceis de convexidade em grafos, a versão restrita a grafos bipartidos continua NP-difícil. Infelizmente, como provamos a seguir, o mesmo acontece para NÚMERO DE ENVOLTÓRIA.

Teorema 4.1. NÚMERO DE ENVOLTÓRIA *é NP-completo mesmo para grafos bipartidos.*

Demonstração. Como o problema para grafos bipartidos é uma restrição do problema para grafos gerais, que é NP-completo, então o problema claramente também pertence a NP. Para mostrar que o problema é NP-difícil, faremos uma redução da versão geral do problema.

Seja $G = (V, E)$ um grafo com conjunto de vértices $v_1, \ldots, v_n$ e k um inteiro, para os quais se deseja descobrir se o número de envoltória de G é no máximo k. Construímos um novo grafo F da seguinte forma.

- $V(F) = V(G) \cup E(G) \cup \{a, b, x\}$;
- $E(F) = E_1 \cup E_2 \cup E_3$, onde
  - $E_1 = \{uv \mid u \in V(G), v \in E(G), u \text{ incidente a } v\}$;
  - $E_2 = \{xv \mid v \in E(G)\}$;

– $E_3 = \{ax, bx\}$.

Note que F é o grafo obtido através da subdivisão de todas as arestas de G, juntamente com a adição dos vértices a, b e x, de uma aresta de x para cada vértice criado na subdivisão e de arestas ax e bx. Note que F é bipartido com bipartições $V(G) \cup \{x\}$ e $E(G) \cup \{a, b\}$.

Provaremos agora que a resposta para Número de envoltória é Sim para G, k se e somente se a respota é Sim para $F, k + 2$. Primeiramente, suponha que a resposta para a entrada G, k seja Sim e seja $S \in V(G)$ um certificado para essa resposta, ou seja S é um conjunto de envoltória de G com no máximo k vértices. Mostraremos que $S' = S \cup \{a, b\}$ é um conjunto de envoltória para G. Claramente $x \in H_F(S')$ porque axb forma um P_3. Como cada vértice $v \in E(G)$ de F tem exatamente dois vizinhos dentro de $V(G)$, é suficiente mostrar que $V(G) \subset H_F(S')$.

Seja $I^i_G(S)$ a função de intervalo iterada i vezes no conjunto S em G com $I^0_G(S) = S$. Mostraremos que $I^i_G(S) \subset I^{2i+1}_F(S')$ por indução em i. Note que $x \in I^1_F(S')$ e portanto cada vértice de F em $E(G)$ possui pelo menos um vizinho em $I^1_F(S')$. Elementos v de $E(G)$ com $v \cap S \neq \emptyset$ terão pelo menos dois vizinhos em $I^1_F(S')$ e portanto pertencerão a $I^2_F(S')$. Note que todo vértice em $I^1_G(S) \setminus I^0_G(S)$ possui ao menos dois vizinhos em S e, em F, os vértices correspondentes às arestas que os conectam em a vértices de S estarão em $I^2_F(S')$ pela observação anterior. Isso conclui a base, $I^1_G(S) \in I^3_F(S')$. Note que, com argumento análogo é possível provar o resultado para todo i: $I^{2i}_F(S')$ contém os vértices referentes a arestas de G incidentes a vértices em $I^i_G(S)$ e portanto os vértices que possuem dois vizinhos em $I^i_G(S)$ em G possuirão ao menos dois vizinhos em $I^{2i+1}_F(S')$ em H.

Para provar a outra direção, note primeiramente que a e b fazem parte de qualquer conjunto de envoltória, pois possuem grau 1. Por outro lado, x não faz parte de nenhum conjunto de envoltória, pois está em um P_3 com extremidades a e b. Seja então S' um conjunto de envoltória de F de tamanho no máximo $k + 2$. Seja $S'' = S' \setminus \{a, b\}$. Finalmente, seja $S = (S'' \cap V(G)) \cup \{v_i \mid v_i v_j \in (S'' \cap E(G)), i < j\}$. Note que $|S| \leq |S'| - 2 \leq k + 2$. Além disso, $H_F(S \cup \{a, b\}) = V(F)$, uma vez que $S' \subset I_F^2(S \cup \{a,b\})$. Resta mostrar que $H_G(S) = V(G)$. Para isso, basta notar que $I_F^{2i+1}(S \cup \{a,b\}) \cap V(G) \subseteq I_G^i(S)$, que é trivialmente verdadeira para $i = 0$. Seja $i \geq 1$ e seja $v \in (I_F^{2i+1}(S \cup \{a,b\}) \cap V(G)) \setminus I_F^{2i}(S \cup \{a,b\})$. Pela definição da função de intervalo, v possui pelo menos dois vizinhos em $I_F^{2i}(S \cup \{a,b\})$. Como, por construção, os vizinhos de v fazem parte de $E(G)$, então cada cada vizinho u de v em F possui ao menos dois vizinhos em $I_F^{2i-1}(S \cup \{a,b\})$, um deles x e um deles um vértice w de $V(G)$. Mas, por construção, w era originalmente um vizinho de v em G e, por indução, $w \in I_G^{i-1}(S)$. Aplicando este argumento para um segundo vizinho de v em $I_F^{2i}(S \cup \{a,b\})$, concluímos a prova. $\square$

A Figura 4.1 ilustra a construção feita na redução de NP-completude na prova do Teorema 4.1.

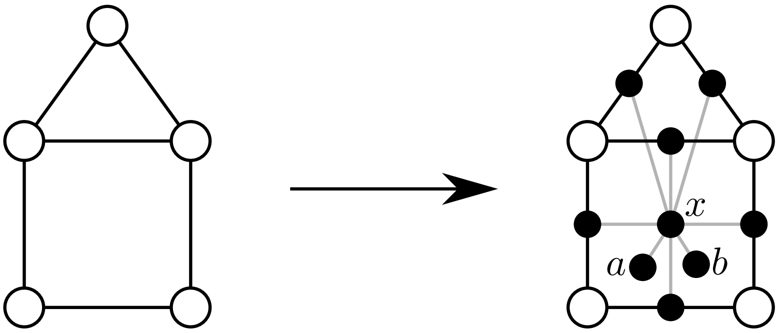

Figura 4.1: Exemplo de construção de F a partir de G, como na prova do Teorema 4.1

4.2 Conjuntos de envoltória mínimos e minimais

Nosso interesse nesta seção é investigar conjuntos de envoltória minimais e sua relação com conjuntos de envoltória mínimos. As provas dos resultados apresentados nesta seção podem ser consultada no Apêndice C.

4.2.1 Um limite superior para grafos gerais

Como mencionado anteriormente, determinar o tamanho de um conjunto de envoltória mínimo é um problema NP-difícil. Entretanto, determinar se um dado conjunto é minimal pode ser feito em tempo polinomial, simplesmente testando, para cada vértice, se sua remoção pode ser feita sem afetar a propriedade de que o conjunto é um conjunto de envoltória.

Essa observação nos leva à seguinte pergunta: quão longe pode estar um conjunto de envoltória minimal de um conjunto de envoltória mínimo?

Com o objetivo de limitar o tamanho máximo de um conjunto de envoltória em função do número de envoltória, isto é, do tamanho de um conjunto de envoltória mínimo, provamos o seguinte limite superior, válido não apenas para a convexidade P_3 mas para qualquer convexidade de grafos.

Teorema 4.2 ([6]). *Se U é um conjunto de envoltória minimal, então $|U| \leq h(G)c(G)$, onde $h(G)$ é o número de envoltória e $c(G)$ é o número de Carathéodory.*

A ideia da prova se baseia no fato de que, se U é um conjunto de envoltória minimal e S é um conjunto mínimo, é possível encontrar um subconjunto de U de tamanho $c(G)$ cuja envoltória contenha um dos elementos de S. Encontrando S subconjuntos, e fazendo sua união, deve-se obter o próprio conjunto U, pela minimalidade.

Para o caso particular da convexidade P_3, este limite é ótimo exceto talvez por um fator de 2. Em particular, em [6] (Apêndice C), apresentamos uma construção onde, para h fixo, construímos um grafo G com $h(g) = 2^h$ e $c(G) = 2$, contendo um conjunto de envoltória minimal de tamanho 2^h.

4.2.2 Grafos cúbicos

Para grafos cúbicos, os conjuntos de envoltória coincidem com *conjunto de vértices de retroalimentação* (em inglês, *feedback vertex set*), que são conjuntos de vértices cuja remoção deixa o grafo sem ciclos. Assim, do ponto de vista algorítmico, este problema está resolvido, uma vez que conjuntos de vértices de retroalimentação de tamanho mínimo podem ser computados em tempo polinomial [63]. A seguir apresentamos limites superior e inferior ótimos para o tamanho de conjuntos de envoltória de grafos cúbicos.

Teorema 4.3 ([6]). *Se G é um grafo cúbico de ordem n e U é um conjunto de envoltória minimal de G, então $\frac{n}{4} < |U| \leq \frac{n}{2}$.*

Para ver que estes limites são ótimos, basta notar que os conjuntos do exemplo da Figura 4.2 podem também ser estendidos juntamente com o grafo de forma que tenhamos $\frac{|U|}{n} < \frac{1}{4} + \epsilon$ e $\frac{|U|}{n} > \frac{1}{2} - \epsilon$ para qualquer ϵ escolhido. Em particular, esses mesmos conjuntos são exemplos de casos extremos do seguinte corolário do Teorema 4.3.

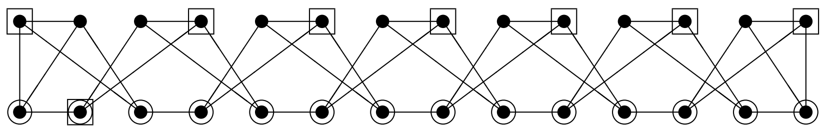

Figura 4.2: Os conjuntos U_1 e U_2 indicados com quadrados e círculos são conjuntos de envoltória minimais.

Corolário 4.4 ([6]). *Se G é um grafo cúbico de ordem n e U_1 e U_2 são conjuntos de envoltória minimais de G, então $\frac{|U_1|}{|U_2}$ < 2.*

Note que, comparado ao caso geral, onde tínhamos uma razão de, no máximo, o número de Carathéodory do grafo, quando nos restringimos a grafos cúbicos, um conjunto de envoltória minimal de cardinalidade máxima está, no pior caso, a uma razão de 2 do número de envoltória.

4.2.3 Grafos cordais e cografos

Duas classes de grafos particularmente bem estudadas e com diversas caracterizações são as classes dos grafos cordais e dos cografos. Devido às suas fortes estruturas, são classes interessantes de serem abordadas, pois permitem uma melhor compreensão do comportamento de propriedades que, em grafos gerais, podem não ser bem comportadas.

Para o caso de conjuntos de envoltória minimais, mostraremos que, para duas classes de grafos, a dos cordais biconexos e dos cografos, para um grafo G fixo, todos os conjuntos minimais possuem o mesmo tamanho e, portanto, são mínimos.

Um resultado conhecido útil para provar o Teorema 4.5 diz que, em um grafo cordal biconexo, quaisquer dois vértices com um vizinho em comum formam um conjunto de envoltória [15].

Teorema 4.5 ([6])**.** *Se G é cordal biconexo com $V(G) \geq 2$, então todo conjunto de envoltória tem tamanho* 2*.*

Demonstração. Suponha, por contradição, que U seja minimal com $|U| > 2$. Note que existem vértices $u, v \in U$ tais que $N(u) \cap N(v) \neq \emptyset$ e, portanto, segue de [15] que $\{u, v\}$ é um conjunto de envoltória. □

Teorema 4.6 ([6])**.** *Se G é cografo conexo e U é um conjunto de envoltória minimal, então U é mínimo.*

A prova do Teorema 4.6 consiste de análise de casos e pode ser encontrada no Apêndice C.

Note que, como consequência direta dos teoremas acima, podemos obter, através de um algoritmo guloso, conjuntos de envoltória de tamanho mínimo para grafos cordais biconexos e para cografos em tempo polinomial.

4.3 Conversões em grafos com limite de tempo

Abordaremos agora um problema que pode ser considerado uma generalização do número de envoltória de um grafo. Para um conjunto S, podemos interpretar a computação da envoltória de S denotada por $H(S)$ da seguinte forma. Inicialmente, temos um conjunto de vértices infectados com alguma doença. A partir daí, se existir algum vértice não infectado adjacente a pelo menos dois outros vértices infectados, ele também fica infectado. Prosseguimos este processo até que todos os vértices sejam infectados ou que todos os vértices não infectados tenham menos de dois vizinhos infectados. Tal processo pode ser generalizado, para um número arbitrário de vizinhos infectados, em vez de dois. Na verdade, não é nem mesmo necessário que todos os vértices precisem da mesma quantidade de vizinhos infectados. Tal processo já foi estudado em diversos trabalhos, como por exemplo em [15]. Um caso particular bastante estudado na literatura, chamado de monopólios dinâmicos é o caso em que um vértice precisa que a maioria dos seus vizinhos estejam infectados para também ser infectado [7, 37, 57].

Como mencionado no Capítulo 1, existem diversas aplicações para processos de difusão, como influência social [39, 48, 49, 58], redes de expressão gênica [47], sistemas imune [1], automata celular [1, 2, 66], percolação [3], estratégias de marketing [21, 49], e computação distribuída [7, 37, 50, 54, 57].

Uma restrição natural a ser feita neste tipo de problema, diz respeito a tempo. Embora, em geral, seja desejável minimizar o número de vértices infectados inicialmente, por outro lado, é natural supor que não é desejável que o processo de infecção do grafo seja muito lento. Para o caso em que o número de vizinhos infectados é a maioria simples, essa relação entre tempo e número de vértices infectados inicialmente foi considerado em [38].

Consideraremos aqui um cenário bem geral, descrito a seguir. Os dados de entrada são

- um grafo G,
- uma *função de limite de tempo* $t_d : \mathcal{D}(t_d) \to \mathbb{N}_0$ com $V(G) \subseteq \mathcal{D}(t_d)$, ou seja, cujo domínio contém os vértices de G e
- uma *função de limiar* $f : \mathcal{D}(f) \to \mathbb{N}_0$ with

$$\{(u, t) : u \in V(G), t \in \mathbb{N}_0, 0 \le t \le t_d(u) - 1\}\} \subseteq \mathcal{D}(f).$$

Um *processo irreversível em* (G, t_d, f) é uma sequência $\mathcal{C} = (c_t)_{t \in \mathbb{N}_0}$ tal que

- $c_t : V(G) \to \{0, 1\}$ para todo $t \in \mathbb{N}_0$ com
- $c_t(u) = 1$ se e somente se
  - $c_{t-1}(u) = 1$

– ou $|\{v \in N_G(u) : c_{t-1}(v) = 1\}| \geq f(u, t-1)$ e $t \leq t_d$

para todo $t \in \mathbb{N}$ e todo $u \in V(G)$.

Se $\mathcal{C}$ é um processo em (G, t_d, f), dizemos que um vértice u é *convertido* no tempo t se $c_{t-1}(u) = 0$ e $c_t(u) = 1$. Se todos os vértices são convertidos, dizemos que o conjunto de vértices u com $c_0(u) = 1$ é um *conjunto de conversão*. O tamanho do melhor conjunto de conversão para uma tripla (G, t_d, f) é denotado $irr(G, t_d, f)$.

Uma diferença em relação às abordagens anteriores, além dos limites de tempo para infecção, é o fato da função de limiar f variar com o tempo. Além desta suposição ser interessante para modelar cenários realistas (em uma aplicação de marketing, por exemplo, um cliente pode estar mais suscetível a comprar um produto em determinada época do ano, ou a visitar um site em horário do dia), este cenário também facilita alguns particularidades técnicas dos resultados, por permitir reduzir o problema original para subproblemas.

Outra caraterística interessante é que, com escolhas apropriadas de f e t_d, este problema pode ser reduzido a diversos outro problemas estudados anteriormente. Se por um lado isso mostra que o problema é de fato difícil, por outro lado resultados para este cenário podem ter implicações diretas em problemas mais restritos. Alguns exemplos de tais conexões são:

- Se $t_d(u) = 1$ e $f(u, 0) = 1$ para todo vértice u, então $irr(G, t_d, f)$ é o número de dominação de G;

- Se $t_d(u) = k$ e $f(u, t) = 1$ para todo vértice u e todo $t \in \mathbb{N}_0$ com $0 \leq t \leq k-1$, então $irr(G, t_d, f)$ coincide com o número de k-distância dominação;

- Se $t_d(u) = 1$ e $f(u, 0) = k$ para todo u, então $irr(G, t_d, f)$ é igual ao número de k-dominação;

- Se $f(u, t) = d_G(u)$ para todo u e todo t e $t_d(u) \geq 1$ para todo u, então $irr(G, t_d, f)$ é igual a $n - \alpha(G)$ onde $\alpha(G)$ é tamanho do maior conjunto independente de G;

- Finalmente, se $t_d(u) \geq n$ e $f(u,t) = 2$ para todo u e t em $0 \leq t \leq t_d(u)$, então $irr(G, t_d, f)$ é o número de envoltória na convexidade P_3.

Como $irr(G, t_d, f)$ generaliza diversos parâmetros cuja determinação é NP-difícil, concluímos que computá-lo também é NP-difícil. Assim, no restante desta seção descreveremos como resolver dois casos particulares. As provas de corretude dos algoritmos bem como limites para $irr(G, t_d, f)$ podem ser encontrados em [62] (Apêndice D).

4.3.1 Conversão de Florestas

Apresentaremos agora um algoritmo eficiente para determinar $irr(G, t_d, f)$ em florestas. A principal estratégia do algoritmo é selecionar uma folha arbitrariamente, decidir se ela fará parte do conjunto de conversão em construção pelo algoritmo, removê-la do grafo e ajustar o grafo restante. Ao final do algoritmo ele encontra $irr(G, t_d, f)$.

Um observação fundamental a ser feita é que, se uma folha não precisa estar em um conjunto de conversão, então é sempre melhor colocar seu vizinho no conjunto do que a folha. Assim, a cada folha considerada, basta analisar se ela deve pentencer ao conjunto de conversão em questão e, caso não seja necessário, a função de limite de tempo de seu pai é atualizada de forma a garantir sua conversão a tempo. Utilizando este princípio, podemos prosseguir removendo vértices da árvore e ajustando a árvore resultante a cada passo.

Teorema 4.7 ([62])**.** *Algoritmo 3 está correto e funciona em tempo polinomial.*

4.3.2 Conversão de Cliques

O caso de cliques, ou seja, de grafos completos, embora seja trivial para diversos problemas, incluindo os casos particulares citados anteriormente, ele se mostra mais interessante quando as funções f e t_d não são constantes.

Algoritmo 3 Algoritmo polinomial para encontrar um conjunto de conversão de tamanho $irr(G, t_d, f)$ em florestas

Entrada: Uma tripla (G, t_d, f) tal que G é uma floresta, t_d é uma função de limite de tempo e f uma função de limiar.

Saída: Um conjunto de conversão C de (G, t_d, f) de cardinalidade $irr(G, t_d, f)$.

$C = \emptyset$;
enquanto $V(G) \neq \emptyset$ **faça**
    enquanto $\exists u \in V(G) : \forall t \in \mathbb{N}_0 : (0 \leq t < t_d(u)) \Rightarrow (f(u,t) > d_G(u))$ **faça**
        para $v \in N_G(u)$ e $0 \leq t < t_d(v)$ **faça**
            $f(v,t) = f(v,t) - 1$;
        $C = C \cup \{u\}$;
        $G = G - u$;
    $G = G - \{u \in V(G) : d_G(u) = 0\}$;
    se $V(G) \neq \emptyset$ **então**
        Seja $u \in V(G)$ tal que $d_G(u) = 1$;
        Seja $v \in V(G)$ o vizinho de u;
        se $\exists t \in \mathbb{N}_0 : 0 \leq t < t_d(u)$ e $f(u,t) = 0$ **então**
            $t_0 = \min\{t \in \mathbb{N}_0 : 0 \leq t < t_d(u) \text{ e } f(u,t) = 0\}$;
            para $t_0 + 1 \leq t < t_d(v)$ **faça**
                $f(v,t) = f(v,t) - 1$;
        senão
            $t_1 = \max\{t \in \mathbb{N}_0 : 0 \leq t < t_d(u) \text{ e } f(u,t) = 1\}$;
            $t_d(v) = \min\{t_d(v), t_1\}$;
        $G = G - u$;
retorne C;

O algoritmo se baseia em uma espécie de simulação e recebe como entrada, além do grafo G e das funções f e t_d, um inteiro k. Inicialmente ele assume que k vértices estão convertidos e através de uma simulação ele decide se existe um conjunto de conversão de tamanho k. Em caso afirmativo, os elementos a fazer parte do conjunto de conversão C retornado são definidos ao final do algoritmo.

Algoritmo 4 Algoritmo polinomial para decidir se existe um conjunto de conversão de tamanho $irr(G, t_d, f)$ para cliques

---

Entrada: Uma tripla (G, t_d, f) tal que G é um grafo completo, t_d é uma função de limite de tempo e f é uma função de limiar; um inteiro k com $0 \leq k \leq |V(G)|$.
Saída: Um conjunto de conversão C de (G, t_d, f) de tamanho k se tal conjunto existe ou NÃO em caso contrário.

$c_0 = k$;
$t = 1$;
$T = \max\{t_d(u) : u \in V(G)\}$;
enquanto $t \leq T$ **faça**
    $S_t = \{u \in V(G) : t \leq t_d(u)$ e $f(u, t-1) \leq c_{t-1}\}$;
    $c_t = c_{t-1} + |S_t|$;
    Denote os elementos de S_t por $u_{c_{t-1}+1}, u_{c_{t-1}+2}, \ldots, u_{c_t}$;
    $V(G) = V(G) \setminus S_t$;
    $t = t + 1$;
se $|V(G)| \leq k$ **então**
    $C = V(G) \cup \{u_{c_T - i} : 0 \leq i \leq (k - |V(G)|) - 1\}$;
    retorne C;
senão
    retorne NÃO;

Lema 4.8 ([62]). *Se o Algoritmo 4 retorna um conjunto C, então C é um conjunto de conversão de tamanho k.*

Lema 4.9 ([62]). *Se o Algoritmo 4 retorna* NÃO, *então não existe um conjunto de conversão de tamanho k.*

Como o Algoritmo 4 claramente roda em tempo polinomial, temos como consequência o Teorema a seguir.

Teorema 4.10 ([62]). *Algoritmo 4 está correto e pode ser executado em tempo polinomial.*

Assim, para determinar $irr(G, t_d, f)$ basta executar o Algoritmo 4 para diferentes valores de k. Isto pode ser feito, por exemplo, com uma busca binária, com $O(\log n)$ chamadas ao Algoritmo 4.

Capítulo 5

Número de Carathéodory

Do or do not... there is no try.

Mestre Yoda - Star Wars:

The Empire Strikes Back

O número de Carathéodory é inspirado no teorema de Carathéodory [13], que diz que todo ponto na envoltória convexa de um conjunto de pontos em $\mathbb{R}^d$ está também na envoltória convexa de um subconjunto de ordem no máximo $d+1$.

O número de Carathéodory foi recentemente considerado em [5], onde foi demonstrado que determiná-lo para convexidade P_3 é um problema NP-completo. Em [27] demonstramos que o problema é também NP-completo para a convexidade geodética e que até mesmo uma restrição natural do problema, o número de Carathéodory local, é também NP-completo. Neste capítulo, apresentamos resultados de [27], relacionados a uma generalização de convexidade de caminhos em grafos, à convexidade P_3 e à geodética. Provas dos resultados apresentados podem ser encontrados no Apêndice E.

5.1 Convexidades de intervalo generalizadas

Teorema 5.1 ([27])**.** *Se $(V, \mathcal{C})$ é uma convexidade de r-intervalo finita para algum $r \geq 2$ e C é um conjunto de Carathéodory com $|C| \geq r$, então $H(C)$ contém um conjunto de Carathéodory C' com $|C| - (r-1) \leq |C'| \leq |C| - 1$.*

O Teorema 5.1 possui implicações algorítmicas para o problema de determinar o número de Carathéodory. Se $(V, \mathcal{C})$ é uma convexidade de r-intervalo então existe um conjunto de Carathéodory de tamanho pelo menos k se e somente se existe um conjunto de Carathéodory de algum tamanho em $\{k, k+1, \ldots, k+r-2\}$. Assim, para determinar o número de Carathéodory para um valor fixo k basta considerar subconjuntos de tamanhos entre k e $k+r-1$, o que pode ser feito em tempo polinomial se a função de intervalo puder ser computada em tempo polinomial, sendo k e r constantes.

O seguinte corolário do Teorema 5.1 está implícito em [31].

Corolário 5.2 ([27]). *Toda convexidade de* 2*-intervalo com número de Carathéodory* c *tem um conjunto de Carathéodory de todos os tamanhos entre* 1 *e* c*.*

Uma *árvore* r*-ária estrita* é uma árvore enraizada na qual todo vértice interno tem exatamente r filhos. O resultado a seguir generaliza um limite apresentado em [5] para a convexidade P_3.

Teorema 5.3 ([27]). *Se* $(V, \mathcal{C})$ *é uma convexidade de* r*-interval para algum* $r \geq 2$ *com número de Carathéodory* c*, então*

$$c \leq \frac{(r-1)|V|+1}{r}. \tag{5.1}$$

A igualdade vale em (5.1) se e somente se para todo conjunto de Carathéodory S *de tamanho* c *existe exatamente um vértice em* $\partial H(S)$*, digamos* v*, e existe uma árvore* r*-ária estrita* T *com raiz* v*, conjunto de vértices* V *e conjunto de folhas* S *tal que para cada* u *de* T *o conjunto de folhas de* T *que são* u *ou descendentes de* u *são o único conjunto minimal* S_u *de* S *com* $u \in H(S_u)$*.*

5.2 Convexidade geodética

Embora determinar o número de Carathéodory seja um problema NP-completo na convexidade geodética, para grafos com forte estrutura ele pode ser um parâmetro bem comportado, como mostramos no resultado abaixo.

Teorema 5.4 ([27]). *O número de Carathéodory de grafos split na convexidade geodética é no máximo* 3.

Vale ressaltar que para algumas classes, convexidades distintas coincidem. Para grafos distância-hereditários, classe que inclui árvores e cografos, suas famílias de caminhos induzidos e mínimos coincidem, portanto segue de [31] que o número de Carathéodory de grafos distância-hereditários na convexidade geodética é 1, caso todas as componentes sejam cliques, e 2, caso contrário.

5.3 Convexidade P_3

Como dito anterioremente, em [5] foi demostrada a NP-completude do número de Carathéodory na convexidade P_3. Assim como o problema na convexidade geodética permanece NP-completo mesmo na sua versão local, demonstramos que o mesmo ocorre na convexidade P_3.

A versão local é definida da seguinte forma. Além de um grafo G e um inteiro k, também é dado um subconjunto U de $V(G)$ e um vértice $u \in H(U)$. O problema consiste em decidir se existe um subconjunto F de U com $u \in H(F)$ e $|F| \leq k$. Note que o número de Carathéodory de um grafo é no máximo k se a saída de Número de Carathéodory Local é Sim para todo $U \subseteq V(G)$ e todo $u \in H(U)$, para G e k fixos.

Teorema 5.5 ([27]). Número de Carathéodory Local *é NP-completo na convexidade* P_3.

Demonstração. Como a envoltória convexa de um conjunto de vértices de um grafo pode ser computada eficientemente na convexidade P_3, Número de Carathéodory Local está em NP.

Para completar a prova, descreveremos uma redução de Número de envoltória na convexidade P_3 que é um problema NP-difícil [15].

Seja (G, k) uma instância de Número de envoltória. Possivelmente adicionando vértices isolados a G e aumentando o valor de k pela quantidade de vértices adicionados, podemos assumir que a ordem n de G é uma potência de 2, isto é, $n = 2^p$ para algum inteiro p. Seja G' o grafo obtido a partir de G adicionando-se $n - 1$ novos vértices e $2(n - 1)$ novas arestas tais que G' contém um subgrafo que é uma árvore binária completa T de altura p cujas folhas são os vértices de G e seja r a raiz de T. Então, para um subconjunto F de $V(G)$ é fácil ver que F é um conjunto de envoltória de G se e somente se a envoltória de F contém r. Então, considerando a instância $(G', V(G), r, k)$ de Número de Carathéodory Local, temos a redução de NP-completude. □

Capítulo 6

Número de Radon

A child of five would understand this. Send someone to fetch a child of five.

Groucho Marx

O Teorema de Radon [59] diz que todo conjunto de pelo menos $d+2$ pontos em $\mathbb{R}^d$ pode ser particionado em dois conjuntos não vazios de forma que a interseção das envoltórias convexas dos dois não seja vazia. O problema análogo para grafos consiste em determinar, para um grafo G, qual o menor inteiro k tal que todo conjunto de pelo menos k vértices possa ser particionado em dois conjuntos de forma que a interseção das envoltórias convexas dos dois não seja vazia. Em [28], provamos que na convexidade P_3 o problema de decisão associado é NP-difícil mesmo para grafos split. Para que o problema seja NP-completo, é necessário descrever um certificado cuja verificação possa ser feita em tempo polinomial, mas o certificado natural, que seria um conjunto anti-Radon, não pode ser reconhecido em tempo polinomial, uma vez que o problema de reconhecimento de um conjunto anti-Radon é, também, NP-completo [28]. Por outro lado, este certificado pode ser verificado em tempo polinomial para grafos split, como mostramos em [26].

Essa dificuldade de se resolver o problema e também a dificuldade de se verificar o certificado mais natural, que seria um conjunto anti-Radon, são evidências da dificuldade do problema. De fato, mesmo para árvores, o problema se mostrou complexo, mas para esta classe conseguimos encontrar um algoritmo polinomial, como mostraremos a seguir. Na verdade, resolvemos um problema mais geral, para encontrar o maior conjunto anti-Radon que é um subconjunto de um conjunto de entrada S. No caso particular em que $S = V(G)$, temos o maior conjunto anti-Radon, que, como visto no Capítulo 2 é uma unidade menor que o número de Radon.

Problemas cuja resolução eficiente não é possível motivam a busca por limites justos para os parâmetros associados. Mostramos um limite superior para o número de Radon em [25]. Provas dos resultados deste capítulo podem ser consultadas no Apêndice F. Outros resultados relacionados ao número de Radon na convexidade P_3 estão disponíveis nos Apêndices G e H.

6.1 O número de Radon para árvores na convexidade P_3

Começamos apresentando um princípio geral relacionando conjunto anti-Radon de um grafo conexo G possuindo um vértice u tal que todas as arestas de G incidentes a u sejam pontes de G, com conjuntos anti-Radon das componentes de $G - u$.

A aplicação principal desde princípio é um algoritmo eficiente para computador o número de Radon e conjuntos anti-Radon máximos em árvores.

6.1.1 Notação complementar

Seja G um grafo e seja u um vértice de G.

Ao longo deste capítulo, denotaremos por $G^{u \leftarrow x}$ o grafo resultante da adição de um novo vértice x e uma nova aresta ux a G.

$$\begin{aligned}
\mathcal{R}_+(G,u) &= \{R \mid R \text{ anti-Radon de } G \text{ e } u \in H_G(R)\},\\
\mathcal{R}_-(G,u) &= \{R \mid R \text{ anti-Radon de } G \text{ e } u \notin H_G(R)\},\\
\mathcal{R}'_+(G,u) &= \{R \mid R \subseteq V(G),\, \{x\} \cup R \text{ anti-Radon de } G^{u\leftarrow x} \text{ e } u \in H_G(R)\},\\
\mathcal{R}'_-(G,u) &= \{R \mid R \subseteq V(G), \{x\} \cup R \text{ anti-Radon de } G^{u\leftarrow x} \text{ e } u \notin H_G(R)\}.
\end{aligned}$$

Além disso, para um conjunto S, sejam

$$\begin{aligned}
r_+(G,S,u) &= \max\{|R| \mid R \in \mathcal{R}_+(G,u) \text{ e } R \subseteq S\},\\
r_-(G,S,u) &= \max\{|R| \mid R \in \mathcal{R}_-(G,u) \text{ e } R \subseteq S\},\\
r'_+(G,S,u) &= \max\{|R| \mid R \in \mathcal{R}'_+(G,u) \text{ e } R \subseteq S\},\\
r'_-(G,S,u) &= \max\{|R| \mid R \in \mathcal{R}'_-(G,u) \text{ e } R \subseteq S\}.
\end{aligned}$$

Segue direto destas definições que o número de Radon pode ser calculado da seguinte forma:

$$r(G) = \max\{r_+(G,V(G),u), r_-(G,V(G),u)\} + 1. \tag{6.1}$$

6.1.2 Conjuntos anti-Radon

A seguir são apresentados 4 lemas caracterizando os conjuntos $\mathcal{R}_+(G,u)$, $\mathcal{R}_-(G,u)$, $\mathcal{R}'_+(G,u)$ e $\mathcal{R}'_-(G,u)$. Estes lemas implicam o Corolário 6.5, que por sua vez leva a um algoritmo linear para o problema de determinar um conjunto anti-Radon máximo.

Seja G um grafo conexo e seja u um vértice de G tal que todas as arestas de G incidentes a u sejam pontes de G. Sejam $G_1,\ldots,G_k$ as componentes de $G-u$ e seja u_i o único vizinho de u em $V(G_i)$ para $i \in [k]$. Seja R um conjunto de vértices de G e seja $R_i = R \cap V(G_i)$ para $i \in [k]$. Seja S um conjunto.

Lema 6.1 ([28]). *Sejam G, u, R, G_i, u_i, R_i para $i \in [k]$ e S como acima.*

R pertence a $\mathcal{R}_+(G,u)$ e $R \subseteq S$ se e somente se

- *$u \in R$ só se $u \in S$,*
- *$R_i \subseteq S$ para $i \in [k]$,*
- *e um dos seguintes casos ocorre.*

  (i) $u \in R$ e $R_i \in \mathcal{R}'_-(G_i,u_i)$ para $i \in [k]$.

  (ii) $u \in R$ e existe um índice $i_1 \in [k]$ tal que $R_{i_1} \in \mathcal{R}'_+(G_{i_1},u_{i_1})$ e $R_i \in \mathcal{R}'_-(G_i,u_i)$ para $i \in [k] \setminus \{i_1\}$.

  (iii) $u \notin R$ e existem dois vértices distintos $i_1, i_2 \in [k]$ tais que $R_{i_j} \in \mathcal{R}_+(G_{i_j},u_{i_j})$ para $j \in [2]$ e $R_i \in \mathcal{R}'_-(G_i,u_i)$ para $i \in [k] \setminus \{i_1, i_2\}$.

  (iv) $u \notin R$ E existem três vértices distintos $i_1, i_2, i_3 \in [k]$ tais que $R_{i_j} \in \mathcal{R}'_+(G_{i_j},u_{i_j})$ para $j \in [3]$ e $R_i \in \mathcal{R}'_-(G_i,u_i)$ para $i \in [k] \setminus \{i_1, i_2, i_3\}$.

Lema 6.2 ([28])**.** *Sejam G, u, R, G_i, u_i, R_i para $i \in [k]$, e S como acima.*

R pertence a $\mathcal{R}_-(G,u)$ e $R \subseteq S$ se e somente se

- *$u \notin R$,*
- *$R_i \subseteq S$ para $i \in [k]$,*
- *e um dos seguintes casos ocorre.*

  (i) $R_i \in \mathcal{R}_-(G_i,u_i)$ para $i \in [k]$.

  (ii) Existe um índice $i_1 \in [k]$ tal que $R_{i_1} \in \mathcal{R}_+(G_{i_1},u_{i_1})$ e $R_i \in \mathcal{R}_-(G_i,u_i)$ para $i \in [k] \setminus \{i_1\}$.

Lema 6.3 ([28])**.** *Sejam G, u, R, G_i, u_i, R_i para $i \in [k]$ e S como acima.*

R pertence a $\mathcal{R}'_+(G,u)$ e $R \subseteq S$ se e somente se

- *$u \in R$ só se $u \in S$,*
- *$R_i \subseteq S$ para $i \in [k]$,*
- *e um dos seguintes casos ocorre.*

(i) $u \in R$ *e* $R_i \in \mathcal{R}'_-(G_i, u_i)$ *para* $i \in [k]$.

(ii) $u \notin R$ *e existem dois índices distintos* $i_1, i_2 \in [k]$ *tal que* $R_{i_j} \in \mathcal{R}'_+(G_{i_j}, u_{i_j})$ *para* $j \in [2]$ *e* $R_i \in \mathcal{R}'_-(G_i, u_i)$ *para* $i \in [k] \setminus \{i_1, i_2\}$.

Lema 6.4 ([28]). *Sejam* G, u, R, G_i, u_i, R_i *para* $i \in [k]$ *e* S *como acima.*

R pertence a $\mathcal{R}'_-(G, u)$ *e* $R \subseteq S$ *se e somente se*

- $u \notin R$,
- $R_i \subseteq S$ *para* $i \in [k]$,
- *e um dos seguintes casos ocorre.*

  (i) $R_i \in \mathcal{R}_-(G_i, u_i)$ *para* $i \in [k]$.

  (ii) Existe algum índice $i_1 \in [k]$ *tal que* $R_{i_1} \in \mathcal{R}_+(G_{i_1}, u_{i_1})$ *e* $R_i \in \mathcal{R}'_-(G_i, u_i)$ *para* $i \in [k] \setminus \{i_1\}$.

Para G e S como acima, os Lemas 6.1 a 6.4, juntamente com a observação de que, para cada um dos conjuntos $\mathcal{R}_+(G, u)$, $\mathcal{R}_-(G, u)$, $\mathcal{R}'_+(G, u)$ e $\mathcal{R}'_-(G, u)$, precisamos apenas saber o tamanho do maior elemento, imediatamente implicam as seguintes relações de recorrência.

Corolário 6.5 ([28]). *Sejam* G, u, R, G_i, u_i, R_i *para* $i \in [k]$ *e* S *como acima.*

(i) Se $u \in S$, *então* $r_+(G, S, u)$ *é o máximo dentre as seguintes expressões.*

$$1 + \sum_{i \in [k]} r'_-(G_i, S, u_i),$$

$$\max_{i_1 \in [k]} \left(1 + r'_+(G_{i_1}, S, u_{i_1}) + \sum_{i \in [k] \setminus \{i_1\}} r'_-(G_i, S, u_i) \right),$$

$$\max_{i_1, i_2 \in [k]} \left(r_+(G_{i_1}, S, u_{i_1}) + r_+(G_{i_2}, S, u_{i_2}) + \sum_{i \in [k] \setminus \{i_1, i_2\}} r'_-(G_i, S, u_i) \right) \text{ e}$$

$$\max_{i_1, i_2, i_3 \in [k]} \left(r'_+(G_{i_1}, S, u_{i_1}) + r'_+(G_{i_2}, S, u_{i_2}) + r'_+(G_{i_3}, S, u_{i_3}) + \sum_{i \in [k] \setminus \{i_1, i_2, i_3\}} r'_-(G_i, S, u_i) \right).$$

(ii) Se $u \notin S$, então $r_+(G, S, u)$ é o máximo dentre as seguintes expressões.

$$\max_{i_1,i_2 \in [k]} \left(r_+(G_{i_1}, S, u_{i_1}) + r_+(G_{i_2}, S, u_{i_2}) + \sum_{i \in [k] \setminus \{i_1,i_2\}} r'_-(G_i, S, u_i) \right) \text{ e}$$

$$\max_{i_1,i_2,i_3 \in [k]} \left(r'_+(G_{i_1}, S, u_{i_1}) + r'_+(G_{i_2}, S, u_{i_2}) + r'_+(G_{i_3}, S, u_{i_3}) + \sum_{i \in [k] \setminus \{i_1,i_2,i_3\}} r'_-(G_i, S, u_i) \right).$$

(iii) $r_-(G, S, u) =$

$$\max \left\{ \sum_{i \in [k]} r_-(G_i, S, u_i), \quad \max_{i_1 \in [k]} \left(r_+(G_{i_1}, S, u_{i_1}) + \sum_{i \in [k] \setminus \{i_1\}} r_-(G_i, S, u_i) \right) \right\}.$$

(iv) Se $u \in S$, então $r'_+(G, S, u)$ é o máximo dentre as seguintes expressões.

$$1 + \sum_{i \in [k]} r'_-(G_i, S, u_i) \text{ e}$$

$$\max_{i_1,i_2 \in [k]} \left(r'_+(G_{i_1}, S, u_{i_1}) + r'_+(G_{i_2}, S, u_{i_2}) + \sum_{i \in [k] \setminus \{i_1,i_2\}} r'_-(G_i, S, u_i) \right).$$

(v) Se $u \notin S$, então $r'_+(G, S, u) =$

$$\max_{i_1,i_2 \in [k]} \left(r'_+(G_{i_1}, S, u_{i_1}) + r'_+(G_{i_2}, S, u_{i_2}) + \sum_{i \in [k] \setminus \{i_1,i_2\}} r'_-(G_i, S, u_i) \right).$$

(vi) $r'_-(G, S, u) =$

$$\max \left\{ \sum_{i \in [k]} r_-(G_i, S, u_i), \quad \max_{i_1 \in [k]} \left(r_+(G_{i_1}, S, u_{i_1}) + \sum_{i \in [k] \setminus \{i_1\}} r'_-(G_i, S, u_i) \right) \right\}.$$

Como caso base, se G é um grafo com exatamente um vértice u e S é um conjunto, então $u \in S$ implica

$$r_+(G, S, u) = r'_+(G, S, u) = 1 \text{ e } r_-(G, S, u) = r'_-(G, S, u) = 0.$$

Caso contrário

$$r_+(G, S, u) = r'_+(G, S, u) = r_-(G, S, u) = r'_-(G, S, u) = 0.$$

Estes valores iniciais, juntamente com as recorrências do Corolário 6.5 resultam, como veremos adiante, em um algoritmo eficiente para resolver o seguinte problema em árvores.

Subconjunto Anti-Radon Máximo

Entrada:	Um grafo G e um conjunto S de vértices de G.
Saída:	Determinar um conjunto anti-Radon R de G de máxima cardinalidade que seja um subconjunto de S.

Note que Subconjunto Anti-Radon Máximo pode ser considerada uma generalização do problema de reconhecimento de um conjunto anti-Radon e do problema do conjunto anti-Radon máximo. De fato, um conjunto S é um conjunto anti-Radon de um grafo G exatamente se Subconjunto Anti-Radon Máximo retorna $R = S$ para e entrada (G, S). Além disso, para o caso do conjunto anti-Radon máximo, basta resolver Subconjunto Anti-Radon Máximo com entrada $(G, V(G))$ e então o número de Radon será dado por $|R| + 1$.

Uma análise cuidadosa do Corolário 6.5 nos leva ao seguinte resultado.

Teorema 6.6 ([28]). *Existe um algoritmo linear para resolver* Subconjunto Anti-Radon Máximo.

Para mostrar que o problema pode ser resolvido em tempo linear, inicialmente enraizamos a árvore em um vértice arbitrário v. Note que, utilizando programação dinâmica, cada um dos quatro parâmetros r_+, r_-, r'_+ e r'_- só precisa ser calculado uma vez para cada vértice de G. Se para cada vértice gastarmos tempo proporcional ao seu grau, o problema é resolvido em tempo linear, uma vez que o somatório dos graus de uma árvore é $2n - 2$. Para isso, basta determinar como resolver cada uma das recorrências do Corolário 6.5 em tempo linear no grau do vértice u. Ilustraremos com o caso mais complexo. Os demais casos podem ser resolvidos com raciocínio similar.

Considere a seguinte equação.

$$f(u) = \max_{i_1,i_2,i_3 \in [k]} \left(r'_+(G_{i_1}, S, u_{i_1}) + r'_+(G_{i_2}, S, u_{i_2}) + r'_+(G_{i_3}, S, u_{i_3}) + \sum_{i \in [k] \setminus \{i_1,i_2,i_3\}} r'_-(G_i, S, u_i) \right),$$

onde u é um vértice com $N(u) = \{u_1, \ldots, u_k\}$, de grau k. Note que a equação anterior pode ser reescrita como

$$f(u) = \max_{i_1,i_2,i_3 \in [k]} \left(\sum_{i \in [k]} r'_-(G_i, S, u_i) - \sum_{i \in \{i_1,i_2,i_3\}} \big(r'_+(G_i, S, u_i) - r'_-(G_i, S, u_i) \big) \right).$$

Uma vez que o primeiro somatório não depende dos índices i_1, i_2, e i_3, ele só precisa ser computado uma vez, em tempo $O(k)$ para o vértice u. Resta então determinar os índices que maximizam $\sum_{i \in \{i_1,i_2,i_3\}} \big(r'_+(G_{i_1}, S, u_{i_1}) - r'_-(G_i, S, u_i) \big)$, o que pode ser feito trivialmente em tempo linear após calcular-se diferenças $\big(r'_+(G_i, S, u_i) - r'_-(G_i, S, u_i) \big)$ para todo $i \in [k]$.

Capítulo 7

Conclusões

A culpa é minha e eu a coloco em quem eu quiser!

Homer Simpson - The Simpsons

Ao longo da tese foram apresentados resultados obtidos durante o doutorado. A ordem apresentada foi semelhante à ordem em que os resultados foram obtidos, exceto por algumas alterações para facilitar a compreensão dos resultados. Inicialmente apresentamos os resultados de intermediação, envolvendo árvores, florestas, intermediações de interseção e relações entre elas.

A partir do Capítulo 4, apresentamos resultados de convexidade. Iniciamos com o número de envoltória, inicialmente demonstrando a NP-completude do problema de decisão associado à determinação do número de envoltória em grafos bipartidos. Em seguida, analisando a relação entre o número de envoltória e o tamanho de conjuntos de envoltória minimais, apresentamos limites envolvendo o número de envoltória, o número de Carathéodory e o tamanho de conjuntos de envoltória minimais, e apresentamos resultados específicos para grafos cúbicos, cordais e cografos. Na última seção do Capítulo 4 abordamos o problema de conversões com limite de tempo, que pode ser considerado uma generalização da convexidade P_3 bem como de outros parâmetros clássicos de grafos e possui diversas aplicações práticas.

Os Capítulos 5 e 6 abordaram os números de Carathéodory e Radon, respectivamente. Sobre o número de Carathéodory, apresentamos um resultado para convexidades de r-intervalo, que possibilitou generalizar resultados obtidos anteriormente. Em seguida mostramos um limite superior para o número de Carathéodory para grafos split na convexidade geodésica e provamos a NP-completude do número de Carathéodory local na convexidade P_3. Para o número de Radon, mostramos um algoritmo polinomial para determiná-lo em árvores na convexidade P_3, que pode ser implementado em tempo linear utilizando-se programação dinâmica.

Todos os resultados apresentados na da tese encontram-se submetidos ou publicados, com exceção do resultado apresentado na Seção 4.1, cujo artigo está em fase de elaboração.

7.1 Trabalhos futuros

Como visto no Capítulo 6, o número de Radon na convexidade P_3 é NP-difícil. Por outro lado, não foi possível provar sua NP-completude, uma vez que não conseguimos um certificado verificável em tempo polinomial para demonstrar que o problema pertence a NP. Por outro lado, é fácil ver que o problema pertence à classe Σ_2^P. Uma pergunta natural a ser respondida é se o problema é Σ_2^P-completo. Intuitivamente, esta parece ser uma possibilidade razoável, uma vez que a própria definição do problema contém os quantificadores "para todo" e "existe".

Outro possível caminho a ser seguido em relação ao número de Radon é no sentido de encontrar algoritmos eficientes para casos particulares. Classes candidatas a serem analisadas seriam subclasses de grafos split, como os grafos de limiar (em inglês, *threshold*), ou superclasses de árvores, como grafos bloco. De forma análoga, no estudo dos outros parâmetros relativos à convexidade abordados, a busca pelo limite entre as instâncias para as quais podemos encontrar algoritmos polinomiais e aquelas cuja resolução é difícil também pode ser continuada.

Outros parâmetros de convexidade considerados para convexidades abstratas também podem ser considerados para grafos. Dentre eles, podemos citar o *posto* (*rank*), o *grau gerador* (*generating degree*), o *número de intercâmbio* (*exchange number*) e o *número de Helly* (*Helly number*) [64].

Finalmente, existem desigualdades relacionando diversos parâmetros de convexidades abstratas. Um problema interessante seria analisar estas desigualdades e utilizar particularidades das propriedades em grafos para buscar limites mais justos.

Referências Bibliográficas

[1] Z. Agur. Fixed points of majority rule cellular automata with application to plasticity and precision of the immune system. *Complex Syst.*, v. 5, pp. 351–357, 1991.

[2] J.-P. Allouche, M. Courbage e G. Skordev. Notes on cellular automata. *Cubo Mat. Educ.*, v. 3, pp. 213–244, 2001.

[3] P. Balister, B. Bollobás, J. Johnson e M. Walters. Random majority percolation. *Random Struct. Algorithms*, v. 36, pp. 315–340, 2010.

[4] J. Balogh, B. Bollobás e R. Morris. Bootstrap percolation in three dimensions. *Ann. Probab*, v. 37, pp. 1329–1380, 2009.

[5] R. Barbosa, E. Coelho, M. Dourado, D. Rautenbach e J. Szwarcfiter. On the Carathéodory number for the convexity of paths of order three. *SIAM Journal on Discrete Mathematics*, v. 26, n. 3, pp. 929–939, 2012.

[6] R. Barbosa, D. Rautenbach, V. F. dos Santos e J. L. Szwarcfiter. On minimal and minimum hull sets, 2013. Manuscrito, submetido ao *VII Latin-American Algorithms, Graphs and Optimization Symposium (LAGOS)*.

[7] J.-C. Bermond, J. Bond, D. Peleg e S. Perennes. The power of small coalitions in graphs. *Discrete Appl. Math.*, v. 127, pp. 399–414, 2003.

[8] A. Brandstädt, V. B. Le e J. P. Spinrad. *Graph Classes: a Survey.* Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1999.

[9] P. Buneman. A characterisation of rigid circuit graphs. *Discrete Mathematics*, v. 9, pp. 205–212, 1974.

[10] L. Burigana. Tree representations of betweenness relations defined by intersection and inclusion. *Mathematics and Social Sciences*, v. 185, pp. 5–36, 2009.

[11] J. Cáceres, C. Hernando, M. Mora, I. Pelayo, M. Puertas e C. Seara. On geodetic sets formed by boundary vertices. *Discrete Math.*, v. 306, pp. 188–198, 2006.

[12] J. Calder. Some elementary properties of interval convexities. *J. London Math. Soc.*, v. 3, pp. 422–428, 1971.

[13] C. Carathéodory. Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. *Rend. Circ. Mat. Palermo*, v. 32, pp. 193–217, 1911.

[14] C. Centeno, S. Dantas, M. Dourado, D. Rautenbach e J. Szwarcfiter. Convex partitions of graphs induced by paths of order three. *Discrete Mathematics and Theoretical Computer Science*, v. 12, pp. 175–184, 2010.

[15] C. Centeno, M. Dourado, L. Penso, D. Rautenbach e J. Szwarcfiter. Irreversible conversion of graphs. *Theor. Comput. Sci.*, v. 412, pp. 3693–3700, 2011.

[16] M. Changat e J. Mathew. On triangle path convexity in graphs. *Discrete Mahtematics*, v. 206, pp. 91–95, 1999.

[17] M. Changat, S. Klavžar e H. Mulder. The all-paths transit function of a graph. *Czech. Math. J.*, v. 51, pp. 439–448, 2001.

[18] M. Changat, G. Prasanth e J. Mathews. Triangle path transit functions, betweenness and pseudo-modular graphs. *Discrete Math.*, v. 309, pp. 1575–1583, 2009.

[19] V. Chvátal, D. Rautenbach e P. M. Schäfer. Finite Sholander trees, trees, and their betweenness. *Discrete Mathematics*, v. 311, n. 20, pp. 2143 – 2147, 2011.

[20] D. Cox, M. Burmeister, E. Price, S. Kim e R. Myers. Radiation hybrid mapping: a somatic cell genetic method for constructing high-resolution maps of mammalian chromosomes. *Science*, v. 250, pp. 245–250, 1990.

[21] P. Domingos e M. Richardson. Mining the network value of customers, In: *Proceedings of the seventh ACM SIGKDD international conference on Knowledge discovery and data mining*, KDD '01, pp. 57–66, New York, NY, USA, August. ACM, 2001.

[22] M. Dourado, F. Protti, D. Rautenbach e J. Szwarcfiter. On the hull number of triangle-free graphs. *SIAM J. Discrete Math*, v. 23, pp. 2163–2172, 2010.

[23] M. C. Dourado, F. Protti e J. L. Szwarcfiter. Algorithmic aspects of monophonic convexity. *Electronic Notes in Discrete Mathematics*, v. 30, pp. 177 – 182, 2008. The IV Latin-American Algorithms, Graphs, and Optimization Symposium.

[24] M. C. Dourado, J. G. Gimbel, J. Kratochvíl, F. Protti e J. L. Szwarcfiter. On the computation of the hull number of a graph. *Discrete Mathematics*, v. 309, n. 18, pp. 5668 – 5674, 2009.

[25] M. C. Dourado, D. Rautenbach, V. F. dos Santos, P. M. Schäfer, J. L. Szwarcfiter e A. Toman. An upper bound on the P_3-Radon number. *Discrete Mathematics*, v. 312, n. 16, pp. 2433 – 2437, 2012.

[26] M. C. Dourado, D. Rautenbach, V. F. dos Santos e J. L. Szwarcfiter. Characterization and recognition of Radon-independent sets in split graphs. *Information Processing Letters*, v. 112, n. 24, pp. 948 – 952, 2012.

[27] M. C. Dourado, D. Rautenbach, V. F. dos Santos, P. M. Schäfer e J. L. Szwarcfiter. On the Carathéodory number of interval and graph convexities, 2013. Manuscrito, submetido ao periódico *Theoretical Computer Science*.

[28] M. C. Dourado, D. Rautenbach, V. F. dos Santos, P. M. Schäfer, J. L. Szwarcfiter e A. Toman. Algorithmic and structural aspects of the P_3-Radon number, 2013. Manuscrito, aceito para publicação no periódico *Annals of Op. Research*.

[29] M. C. Dourado, D. Rautenbach, V. F. dos Santos, P. M. Schäfer, J. L. Szwarcfiter e A. Toman. On the radon number for p 3-convexity., In: *LATIN*, v. 7256, *Lecture Notes in Computer Science*, pp. 267–278. Springer, 2012.

[30] F. Dragan, F. Nicolai e A. Brandstädt. Convexity and HHD-free graphs. *SIAM J. Discrete Math.*, v. 12, pp. 119–135, 1999.

[31] P. Duchet. Convex sets in graphs. II: Minimal path convexity. *J. Comb. Theory, Ser. B*, v. 44, pp. 307–316, 1988.

[32] P. Erdős, E. Fried, A. Hajnal e E. Milner. Some remarks on simple tournaments. *Algebra Univers.*, v. 2, pp. 238–245, 1972.

[33] M. Everett e S. Seidman. The hull number of a graph. *Discrete Math.*, v. 57, pp. 217–223, 1985.

[34] M. Farber e R. E. Jamison. Convexity in graphs and hypergraphs. *SIAM J. Algebraic Discrete Methods*, v. 7, pp. 433–444, 1986.

[35] M. Farber e R. Jamison. On local convexity in graphs. *Discrete Math.*, v. 66, pp. 231–247, 1987.

[36] J. F. Fink e M. S. Jacobson. n-domination in graphs. In: Y. Alavi, G. Chartrand, D. R. Lick, C. E. Wall e L. Lesniak (Eds.), *Graph Theory with Applications to Algorithms and Computer Science*, John Wiley & Sons, Inc., pp. 283–300, New York, NY, USA, 1985.

[37] P. Flocchini, F. Geurts, e N. Santoro. Optimal irreversible dynamos in chordal rings. *Discrete Appl. Math.*, v. 113, pp. 23–42, 2001.

[38] P. Flocchini, R. Královič, A. Roncato, P. Ružička e N. Santoro. On time versus size for monotone dynamic monopolies in regular topologies. *J. Discrete Algorithms*, v. 1, pp. 129–150, 2003.

[39] J. French. A formal theory of social power. *Psychology Rev.*, v. 63, pp. 181–194, 1956.

[40] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *Journal of Combinatorial Theory, Series B*, v. 16, n. 1, pp. 47 – 56, 1974.

[41] J. G. Gimbel. Some remarks on the convexity number of a graph. *Graphs and Combinatorics*, v. 19, pp. 357–361, 2003.

[42] M. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*. Elsevier, 2004. Annals of Discrete Mathematics 57.

[43] S. Goss e H. Harris. New methods for mapping genes in human chromosomes. *Nature*, v. 255, pp. 680–684, 1975.

[44] M. Habib, D. Julien, R. M. McConnell, V. F. dos Santos e J. L. Szwarcfiter. On clique graphs of chordal comparability graphs, In: *Latin American Workshop on Cliques in Graphs*, Buenos Aires. 2012.

[45] T. W. Haynes, S. Hedetniemi e P. Slater. *Fundamentals of Domination in Graphs (Pure and Applied Mathematics (Marcel Dekker))*. CRC, jan. 1998.

[46] M. A. Henning. Distance domination in graphs. In: *Domination in Graphs: Advanced Topics*, pp. 335–365, Marcel Dekker, 1998.

[47] S. Huang. Gene expression profiling, genetic networks, and cellular states: an integrating concept for tumorigenesis and drug discovery. *Journal of Molecular Medicine*, v. 77, pp. 469–480, 1999.

[48] P. D. Jr. e F. Roberts. Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion. *Discrete Appl. Math.*, v. 157, pp. 1615–1627, 2009.

[49] D. Kempe, J. Kleinberg e E. Tardos. Maximizing the spread of influence through a social network, In: *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*, KDD '03, pp. 137–146, New York, NY, USA, August. 2003.

[50] S. Kutten e D. Peleg. Fault-local distributed mending. *J. Algorithms*, v. 30, pp. 144–165, 1999.

[51] T. McKee e F. McMorris. *Topics in Intersection Graph Theory.* Society for Industrial and Applied Mathematics, 1999. SIAM Monographs on Discrete Mathematics and Applications 2.

[52] J. Moon. Embedding tournaments in simple tournaments. *Discrete Math.*, v. 2, pp. 389–395, 1972.

[53] H. Mulder. The interval function of a graph. *Mathematical Centre tracts*, v. 132, pp. 191pp, 1980.

[54] N. Mustafa e A. Pekeč. Listen to your neighbors: How (not) to reach a consensus. *SIAM J. Discrete. Math.*, v. 17, pp. 634–660, 2004.

[55] J. Opatrny. Total ordering problem. *SIAM Journal on Computing*, v. 8, n. 1, pp. 111–114, 1979.

[56] D. Parker, R. Westhoff e M. Wolf. On two-path convexity in multipartite tournaments. *European J. Combin.*, v. 29, pp. 641–651, 2008.

[57] D. Peleg. Local majorities, coalitions and monopolies in graphs: A review. *Theor. Comput. Sci.*, v. 282, pp. 231–257, 2002.

[58] S. Poljak e M. Suura. On periodical behaviour in societies with symmetric influences. *Combinatorica*, v. 3, pp. 119–121, 1983.

[59] J. Radon. Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. *Math. Ann.*, v. 83, pp. 113–115, 1921.

[60] D. Rautenbach, V. F. dos Santos, P. M. Schäfer e J. L. Szwarcfiter. Characterization and representation problems for intersection betweennesses. *Discrete Applied Mathematics*, v. 159, n. 5, pp. 389 – 395, 2011.

[61] D. Rautenbach, V. F. dos Santos, P. M. Schäfer e J. L. Szwarcfiter. On subbetweennesses of trees: Hardness, algorithms, and characterizations. *Computers & Mathematics with Applications*, v. 62, n. 12, pp. 4674 – 4681, 2011.

[62] D. Rautenbach, V. F. dos Santos e P. M. Schäfer. Irreversible conversion processes with deadlines, 2013. Manuscrito, submetido ao *Journal of Discrete Algorithms*.

[63] S. Ueno, Y. Kajitani e S. Gotoh. On the nonseparating independent set problem and feedback set problem for graphs with no vertex degree exceeding three. *Discrete Math.*, v. 72, pp. 355–360, 1988.

[64] M. van de Vel. *Theory of Convex Structures*. Elsevier, 1993.

[65] J. Varlet. Convexity in tournaments. *Bull. Soc. R. Sci. Liège*, v. 45, pp. 570–586, 1976.

[66] J. von Neumann. *Theory of Self-Reproducing Automata*. Univ. Illinois Press, Urbana, IL, 1966.

[67] J. Walter. Representations of chordal graphs as subtrees of a tree. *J. Graph Theory*, v. 2, pp. 265–267, 1978.

Apêndice A

On subbetweennesses of trees: Hardness, algorithms, and characterizations

Este apêndice contém o artigo "*On subbetweennesses of trees: Hardness, algorithms, and characterizations*", publicado no periódico *Computers and Mathematics with Applications.*

Computers and Mathematics with Applications 62 (2011) 4674–4681

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On subbetweennesses of trees: Hardness, algorithms, and characterizations

Dieter Rautenbach [a,*], Vinícius Fernandes dos Santos [b], Philipp M. Schäfer [a], Jayme L. Szwarcfiter [b]

[a] Institut für Optimierung und Operations Research, Universität Ulm, D-89069 Ulm, Germany

[b] Instituto de Matemática, NCE, and COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ, Brazil

ARTICLE INFO

Article history:
Received 22 December 2010
Received in revised form 7 September 2011
Accepted 17 October 2011

Keywords:
Betweenness
Total ordering problem
Shortest paths
Trees

ABSTRACT

We study so-called betweennesses induced by trees and forests. Algorithmic problems related to betweennesses are typically hard. They have been studied as relaxations of ordinal embeddings and occur for instance in psychometrics and molecular biology. Our contributions are hardness results, efficient algorithms, and structural insights such as complete axiomatic characterizations.

1. Introduction

In the present paper we study so-called betweennesses induced by trees and forests. Betweennesses capture and generalize in an abstract way natural geometric properties of points in $\mathbb{R}^n$ and the axiomatic study of betweenness as a mathematical concept goes back to Huntington and Kline [1] in 1917. Algorithmic problems related to betweennesses have been studied as relaxations of ordinal embeddings [2–4] and occur for instance in psychometrics [5] and as arrangement problems in molecular biology [6,7]. For such betweenness problems, several strong hardness results have been obtained [8–10] and only few positive results are known [8,4,11].

We consider finite, simple, and undirected graphs as well as finite set systems defined over finite ground sets. Let G be a graph. The vertex set of G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. For a vertex u of G, the neighbourhood $N_G(u)$ in G equals $\{v \in V(G) \mid uv \in E(G)\}$ and the degree $d_G(u)$ in G equals $|N_G(u)|$. A path P of length l in G between two vertices v_0 and v_l of G is a sequence $P : v_0v_1 \dots v_l$ of $l+1$ distinct vertices $v_0, v_1, \dots, v_l \in V(G)$ such that $v_{i-1}v_i \in E(G)$ for $1 \le i \le l$. The distance $\mathrm{dist}_G(u, v)$ in G between two vertices u and v of G is the minimum length of a path in G between u and v. A cycle C of length $l \ge 3$ in G is a sequence $C : v_1v_2 \dots v_lv_1$ such that $v_1v_2 \dots v_l$ is a path in G and $v_1v_l \in E(G)$. For a finite set V, V^3 denotes the set of all ordered triples of elements of V. A triple $(u, v, w) \in V^3$ is called strict if u, v, and w are all distinct. Let V_s^3 denote the set of all strict triples in V^3. For $k \in \mathbb{N}_0$, let $\binom{V}{k}$ denote the set of all subsets of V that are of cardinality k.

For a graph G, the *shortest path betweenness* $\mathcal{B}(G)$ of G consists of all triples $(u, v, w) \in V(G)^3$ such that v lies on a shortest path in G between u and w, or equivalently

$$\mathcal{B}(G) = \left\{(u, v, w) \in V(G)^3 \mid \mathrm{dist}_G(u, w) = \mathrm{dist}_G(u, v) + \mathrm{dist}_G(v, w) < \infty\right\}.$$

* Corresponding author.
E-mail addresses: dieter.rautenbach@uni-ulm.de, dieter.rautenbach@gmx.de (D. Rautenbach), vinicius.santos@gmail.com (V.F. dos Santos), philipp.schaefer@gmail.com (P.M. Schäfer), jayme@nce.ufrj.br (J.L. Szwarcfiter).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.10.056

The *strict shortest path betweenness* $\mathcal{B}_s(G)$ of G consists of all strict triples in $\mathcal{B}(G)$, i.e.

$$\mathcal{B}_s(G) = \mathcal{B}(G) \cap V(G)^3_s.$$

Shortest path betweenesses are a special case of betweenesses induced by metrics, which were first studied by Menger in 1928 [12]. Next to metrics also partially ordered sets naturally induce betweennesses, which were first studied by Birkhoff in 1948 [13]. While axiomatic characterizations of the betweennesses induced by partially ordered sets are known and well-studied [14–17], no comparable results exist for betweennesses induced by general metrics [18,19]. Therefore algorithmic approaches and structural insights such as axiomatic descriptions of betweennesses induced by special metrics such as the shortest path metric of some graph are of interest. Note that the shortest path betweenness of a graph G captures essentially the same information as the so-called *interval function* of G. Therefore, Mulder and Nebeský's deep axiomatic results concerning the interval function of arbitrary graphs [20–22] yield axiomatic characterizations of shortest path betweennesses. The shortest path betweennesses of trees have received special attention and several different axiomatic characterizations were proposed [15,23,5,24]. For a tree/forest T, we call $\mathcal{B}(T)$ the *tree/forest betweenness of* T and $\mathcal{B}_s(T)$ the *strict tree/forest betweenness of* T.

Our results are as follows. In Section 2, we relate subbetweennesses of strict tree betweennesses to the well-known NP-complete total ordering problem [9] and establish a corresponding hardness result. Furthermore, we show that some of the few positive results concerning the total ordering problem carry over to subbetweennesses of strict tree betweennesses. In Section 3, we describe an efficient algorithm for the recognition of induced subbetweennesses of strict tree betweennesses. In Section 4, we give a complete axiomatic description of induced subbetweennesses of strict tree betweennesses. Finally, in Section 5, we prove an axiomatic characterization of strict forest betweennesses and pose a related conjecture.

2. Algorithmic aspects of subbetweennesses of trees

In this section, we consider the following problem.

Subbetweenness of a tree
Instance: A finite set V and a set $\mathcal{B} \subseteq V^3_s$ of strict triples.
Question: Is there a tree T such that $\mathcal{B} \subseteq \mathcal{B}_s(T)$?

We prove that Subbetweenness of a tree is NP-complete. Subbetweenness of a tree is a variation of well-studied constraint satisfaction problems [3], which occur for instance in computational biology [6,7].

The next lemma captures properties of special trees solving the above problem.

Lemma 1. *Let T be a tree and let $V \subseteq V(T)$.*

(i) *If $d_T(v) \le 2$ for some $v \in V(T) \setminus V$, then deleting v and joining all pairs of distinct neighbours of v by a new edge results in a tree T' with $\mathcal{B}_s(T) \cap V^3_s = \mathcal{B}_s(T') \cap V^3_s$.*
(ii) *If there is some edge uv of T with $u, v \in V(T) \setminus V$, then contracting the edge uv results in a tree T'' with $\mathcal{B}_s(T) \cap V^3_s = \mathcal{B}_s(T'') \cap V^3_s$.*

Proof. (i) Deleting a vertex $v \in V(T) \setminus V$ of degree at most 1 clearly has no influence on $\mathcal{B}_s(T) \cap V^3_s$. Furthermore, if $d_T(v) = 2$ for some $v \in V(T) \setminus V$, then every path in T between vertices in V contains either both or none of the two edges incident with v. Therefore, contracting one of these two edges has no influence on $\mathcal{B}_s(T) \cap V^3_s$.
(ii) If c denotes the vertex that arises from the contraction of uv, then the paths in T that are between two vertices of V and contain a vertex in $\{u, v\}$ are in one to one correspondence with the paths in T'' that are between two vertices of V and contain the vertex c. This easily implies (ii). □

We proceed to the hardness result for Subbetweenness of a tree, which is obtained by a reduction from the well-known Total ordering [9].

Theorem 2. Subbetweenness of a tree *is NP-complete.*

Proof. In order to prove that Subbetweenness of a tree is in NP, we assume that $\mathcal{B} \subseteq V^3_s$ is such that there is some tree T with $\mathcal{B} \subseteq \mathcal{B}_s(T)$. By Lemma 1, we may assume that $V(T) \setminus V$ is an independent set of vertices of degree at least 3 in T. Hence T contains at least $|V(T) \setminus V|$ endvertices, which all belong to V. This implies that the order of T is polynomially bounded in terms of $|V|$ and hence Subbetweenness of a tree is in NP. In order to prove that Subbetweenness of a tree is NP-complete, we reduce the following NP-complete problem [9] to it.

Total ordering
Instance: A finite set V and a set $\mathcal{B} \subseteq V^3_s$ of strict triples.
Question: Is there a path P such that $\mathcal{B} \subseteq \mathcal{B}_s(P)$?

Let V and $\mathcal{B} \subseteq V_s^3$ be an instance of TOTAL ORDERING. Let $V' = V \cup \{x, y\}$ for two distinct elements $x, y \notin V$ and let $\mathcal{B}' = \mathcal{B} \cup \{(x, u, y) \mid u \in V\}$. It is easy to see that there is a path P such that $\mathcal{B} \subseteq \mathcal{B}_s(P)$ if and only if there is a tree T' such that $\mathcal{B}' \subseteq \mathcal{B}_s(T')$. This equivalence completes the proof. □

Note that TOTAL ORDERING is even hard to approximate [8,10]. Unfortunately, our proof does not seem to imply similar hardness results for SUBBETWEENNESS OF A TREE.

There are some few positive results concerning TOTAL ORDERING [8,11,4]. Let V and $\mathcal{B} \subseteq V_s^3$ be an instance of TOTAL ORDERING. Let $V = \{1, 2, \dots, n\}$. If $\pi \in S_n$ is a random permutation, then the expected number of triples $(u, v, w) \in \mathcal{B}$ with

$$\pi^{-1}(u) < \pi^{-1}(v) < \pi^{-1}(w) \quad \text{or} \quad \pi^{-1}(u) > \pi^{-1}(v) > \pi^{-1}(w)$$

is exactly $\frac{|\mathcal{B}|}{3}$, i.e. the path $P : \pi(1)\pi(2)\dots\pi(n)$ satisfies $\frac{|\mathcal{B}\cap\mathcal{B}_s(P)|}{|\mathcal{B}|} \geq \frac{1}{3}$. This yields a randomized $\frac{1}{3}$-approximation algorithm, which can easily be derandomized using the standard method of conditional expectation. Therefore there is a polynomial $\frac{1}{3}$-approximation algorithm for the following problem.

Maximization version of SUBBETWEENNESS OF A TREE

Instance: A finite set V and a non-empty set $\mathcal{B} \subseteq V_s^3$ of strict triples.

Task: Determine a tree T such that $\frac{|\mathcal{B}\cap\mathcal{B}_s(T)|}{|\mathcal{B}|}$ is maximum.

In [8] Chor and Sudan describe an efficient algorithm that for a given 'Yes'-instance V and $\mathcal{B} \subseteq V_s^3$ of TOTAL ORDERING, determines a path P with $\frac{|\mathcal{B}\cap\mathcal{B}_s(P)|}{|\mathcal{B}|} \geq \frac{1}{2}$. Their approach relies on semidefinite programming and is quite involved. Very recently, Makarychev [11] devised a surprisingly simple purely combinatorial algorithm, which achieves the same performance. We show that Makarychev's algorithm (cf. Algorithm 1) can be adapted to the maximization version of SUBBETWEENNESS OF A TREE. If $P : u_0 \dots u_l$ is a path and v is a vertex not in $V(P)$, then Pv denotes the path $u_0 \dots u_l v$ and vP denotes the path $vu_0 \dots u_l$.

Theorem 3. *Algorithm 1 works correctly and can be implemented to run in linear time, i.e. there is a linear time $\frac{1}{2}$-approximation algorithm for the maximization version of* SUBBETWEENNESS OF A TREE *restricted to 'Yes'-instances of* SUBBETWEENNESS OF A TREE.

Input: A finite set V and a set $\mathcal{B} \subseteq V_s^3$ such that there is a tree T with $\mathcal{B} \subseteq \mathcal{B}_s(T)$.
Output: A path P with $\frac{|\mathcal{B}\cap\mathcal{B}_s(P)|}{|\mathcal{B}|} \geq \frac{1}{2}$.

1 $n \leftarrow |V|$;
2 $i \leftarrow n$;
3 **while** $i \geq 1$ **do**
4 Select $v \in V$ with $\nexists u, w \in V : (u, v, w) \in \mathcal{B} \cap V_s^3$;
5 $v_i \leftarrow v$;
6 $V \leftarrow V \setminus \{v\}$;
7 $i \leftarrow (i - 1)$;
8 **end**
9 $P \leftarrow v_1 v_2$;
10 **for** $i = 3$ **to** n **do**
11 **if** $|\mathcal{B} \cap \mathcal{B}_s(Pv_i)| \geq |\mathcal{B} \cap \mathcal{B}_s(v_i P)|$ **then**
12 $P \leftarrow Pv_i$;
13 **else**
14 $P \leftarrow v_i P$;
15 **end**
16 **end**
17 **return** P;

Algorithm 1: Algorithm for the maximization version of SUBBETWEENNESS OF A TREE.

Proof. The crucial observation is that a vertex as selected in line 4 of Algorithm 1 always exists. This can be seen as follows. Let T be a tree and let $U \subseteq V(T)$. If we root T in an arbitrary vertex and select a vertex $v \in U$ of maximal depth, then there are no two elements $u, w \in U$ with $(u, v, w) \in \mathcal{B}_s(T)$.

Note that the path P in line 11 has vertex set $V(P) = \{v_1, v_2, \dots, v_{i-1}\}$. Hence, by the choice of v_i in line 4, there are no two elements $v_r, v_s \in V(P)$ with $(v_r, v_i, v_s) \in \mathcal{B}$. Hence for one path Q among Pv_i and v_iP, we obtain that $\mathcal{B}_s(Q)$ contains at least half the triples in $\{(u, v, w) \in \mathcal{B} \mid v_i \in \{u, v, w\} \subseteq \{v_1, v_2, \dots, v_i\}\}$, which completes the correctness proof. For the linear running time, we refer to [11]. □

3. Algorithmic aspects of induced subbetweennesses of trees

In this section, we consider the following problem.

Induced subbetweenness of a tree
Instance: A finite set V and a set $\mathcal{B} \subseteq V_s^3$ of strict triples.
Task: Decide whether there is a tree T such that $\mathcal{B} = \mathcal{B}_s(T) \cap V_s^3$ and construct such a tree if possible.

While Subbetweenness of a tree is NP-complete, we show that Induced subbetweenness of a tree can be solved in polynomial time. The reason for this divergent behaviour is that 'Yes'-instances of Induced subbetweenness of a tree necessarily have a lot of structure, which allows to construct a suitable tree and also to give a complete axiomatic characterization of these instances.

Let V and $\mathcal{B} \subseteq V_s^3$ be an instance of Induced subbetweenness of a tree. If there is a tree T such that $\mathcal{B} = \mathcal{B}_s(T) \cap V_s^3$, then, by Lemma 1, we may assume that $V(T) \setminus V$ is an independent set of vertices of degree at least 3 in T. Therefore, in order to (re)construct T, we only need to determine the sets $N_T(v)$ for $v \in V(T) \setminus V$ as well as the edges between the vertices in V, which can be done using the following lemma.

Lemma 4. *Let T be a tree and let $V \subseteq V(T)$ be such that $V(T) \setminus V$ is an independent set of vertices of degree at least 3 in T. Let $\mathcal{B} = \mathcal{B}_s(T) \cap V_s^3$. Let $\mathcal{T}$ denote the set of all sets $\{u, v, w\} \in \binom{V}{3}$ such that $(u, v, w), (u, w, v), (v, u, w) \notin \mathcal{B}$ and there is no $x \in V$ with $(u, x, v) \in \mathcal{B} \vee (u, x, w) \in \mathcal{B} \vee (v, x, w) \in \mathcal{B}$. Let G be the graph with vertex set $\mathcal{T}$ in which two vertices $s, t \in \mathcal{T}$ are adjacent exactly if $|s \cap t| = 2$.*

(i) *$V(T) \setminus V$ contains a vertex v with neighbourhood N if and only if there is some component C of G with $N = \bigcup_{t \in V(C)} t$.*
(ii) *Two distinct vertices u and v in V are adjacent in T if and only if there is no $t \in \mathcal{T}$ with $u, v \in t$ and there is no $x \in V$ with $(u, x, v) \in \mathcal{B}$.*

Proof. (i) Let $v \in V(T) \setminus V$ and let $N = N_T(v)$. By construction, all elements of $\binom{N}{3}$ belong to one component C of G. If $C \neq \binom{N}{3}$, then there is some $t \in C \setminus \binom{N}{3}$ and some $s \in \binom{N}{3}$ with $|t \cap s| = 2$. Let $t \cap s = \{x, y\}$. By construction, there is some vertex $v' \in V(T) \setminus V$ that is distinct from v such that $x, y \in N_T(v) \cap N_T(v')$. This implies the existence of a cycle $xvyv'x$ in T, which is a contradiction. Hence $C = \binom{N}{3}$.

Conversely, let C be a component of G and let $N = \bigcup_{t \in V(C)} t$. Let $t = \{u, v, w\} \in V(C)$. Since $(u, v, w), (u, w, v), (v, u, w) \notin \mathcal{B}$, the minimal subtree T_t of T that contains u, v, and w is the subdivision of a claw $K_{1,3}$ with endvertices u, v, and w. Since there is no $x \in V$ with $(u, x, v) \in \mathcal{B} \vee (u, x, w) \in \mathcal{B} \vee (v, x, w) \in \mathcal{B}$, the tree T_t is in fact isomorphic to $K_{1,3}$ and the vertex c_t of degree 3 in T_t belongs to $V(G) \setminus V$. Since T has no cycles, if $t, s \in \mathcal{T}$ satisfy $|t \cap s| = 2$, then $c_t = c_s$. By the definition of G, this implies that all vertices in N are adjacent to the same vertex c in $V(G) \setminus V$, i.e. $N \subseteq N_T(c)$. Let $y \in N_T(c)$. Let $t = \{u, v, w\} \in V(C)$. By construction, $\{y, u, v\} \in V(C)$, which implies $y \in N$. Therefore, $N_T(c) \subseteq N$. Hence $N = N_T(c)$, which completes the proof of (i).

(ii) Let u and v be two distinct vertices in V. Clearly, if $uv \in E(T)$, then there is no $t \in \mathcal{T}$ with $u, v \in t$ and there is no $x \in V$ with $(u, x, v) \in \mathcal{B}$. Conversely, if $uv \notin E(T)$, then the path P in T between u and v contains at least one internal vertex. If some internal vertex x of P belongs to V, then $(u, x, v) \in \mathcal{B}$. If no internal vertex of P belongs to V, then, since $V(T) \setminus V$ is independent, P contains exactly one internal vertex $x \in V(T) \setminus V$. Since x has degree at least 3, there is some $w \in N_T(x) \setminus \{u, v\}$. Now $\{u, v, w\} \in \mathcal{T}$, which completes the proof. □

With Lemma 4 at hand, we can now describe an efficient algorithm for Induced subbetweenness of a tree (cf. Algorithm 2).

Theorem 5. *Algorithm 2 correctly solves* Induced subbetweenness of a tree *in polynomial time. (Since the encoding length of $\mathcal{B} \subseteq V^3$ is $O\left(|V|^3\right)$, 'polynomial time' means a running time polynomially bounded in terms of $|V|$.)*

Proof. The correctness follows from Lemmas 1 and 4. Note that $\mathcal{T}$ and G in Lemma 4 are defined using only V and $\mathcal{B}$. Therefore, by Lemma 4(i) and (ii), the tree T is uniquely determined by V and $\mathcal{B}$ up to the naming of the vertices in $V(T) \setminus V$. The steps executed by Algorithm 2 during the construction of T correspond exactly to Lemma 4: Lines 4 to 9 correspond to Lemma 4(i) and lines 10 to 14 correspond to Lemma 4(ii).

Furthermore, note that the task in line 3 can be executed in polynomial time, because it only requires knowledge of $\mathcal{B}$. The polynomiality is obvious. □

4. Structural aspects of induced subbetweennesses of trees

Our result in this section is the axiomatic characterization of the 'Yes'-instances of Induced subbetweenness of a tree. In order to motivate and phrase our result, we recall Burigana's characterization of strict tree betweennesses [5].

Input: A finite set V and a set $\mathcal{B} \subseteq V_s^3$.
Output: A tree T with $\mathcal{B} = \mathcal{B}_s(T) \cap V_s^3$ or the answer "No" if no such tree exists.

$V(T) \leftarrow V$;
$E(T) \leftarrow \emptyset$;
Construct $\mathcal{T}$ and G as in Lemma 4;
for *every component C of G* **do**
  $N \leftarrow \bigcup_{t \in V(C)} t$;
  Let c be a new vertex not in $V(T)$;
  $V(T) \leftarrow V(T) \cup \{c\}$;
  $E(T) \leftarrow E(T) \cup \{cu \mid u \in N\}$;
end
for *every* $\{u, v\} \in \binom{V}{2}$ **do**
  if $(\nexists t \in \mathcal{T} : u, v \in t) \wedge (\nexists x \in V : (u, x, v) \in \mathcal{B})$ **then**
    $E(T) \leftarrow E(T) \cup \{uv\}$;
  end
end
if *T is not a tree* **then**
  return *"No"*;
end
Construct $\mathcal{B}_s(T)$;
if $\mathcal{B} = \mathcal{B}_s(T) \cap V_s^3$ **then**
  return T;
else
  return *"No"*;
end

Algorithm 2: Algorithm for INDUCED SUBBETWEENNESS OF A TREE.

For some $\mathcal{B} \subseteq V^3$ and $u, v, w \in V$, let $N(u, v, w)$ abbreviate the following assertion

$$(u \neq v \neq w \neq u) \wedge ((u, v, w), (v, w, u), (w, u, v) \notin \mathcal{B}) .$$

Burigana proposes the following five axioms and result.

(T_1) $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (w, v, u) \in \mathcal{B}$
(T_2) $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (v, u, w) \notin \mathcal{B}$
(T_3) $\forall u, v, w, z \in V : (u, v, w), (v, w, z) \in \mathcal{B} \Rightarrow (u, w, z) \in \mathcal{B}$
(T_4) $\forall u, v, w, z \in V : (u, v, w), (u, w, z) \in \mathcal{B} \Rightarrow (v, w, z) \in \mathcal{B}$
(T_5) $\forall u, v, w \in V : N(u, v, w) \Rightarrow \exists c \in V : (u, c, v), (u, c, w), (v, c, w) \in \mathcal{B}$

Theorem 6 (*Burigana [5]*). *Let V be a finite set. If $\mathcal{B} \subseteq V_s^3$ is a set of strict triples, then there is a tree T such that $\mathcal{B}_s(T) = \mathcal{B}$ if and only if $\mathcal{B}$ satisfies* (T_1), (T_2), (T_3), (T_4), *and* (T_5).

For a short proof and a strengthening of Theorem 6 cf. Chvátal et al. [23].

Obviously, the crucial axiom is (T_5), which no longer holds for induced subbetweenness of a tree. We propose the following two weakenings of (T_5) and proceed to our result in this section.

(T_5^1) $\forall u, v, w, x \in V : (N(u, v, w) \wedge (u, x, v) \in \mathcal{B}) \Rightarrow ((u, x, w) \in \mathcal{B} \vee (v, x, w) \in \mathcal{B})$
(T_5^2) $\forall u, v, w, x \in V : (N(u, v, w) \wedge (x, u, v) \in \mathcal{B}) \Rightarrow (x, u, w) \in \mathcal{B}$

Theorem 7. *Let V be a finite set. If $\mathcal{B} \subseteq V_s^3$ is a set of strict triples, then there is a tree T such that $\mathcal{B} = \mathcal{B}_s(T) \cap V_s^3$ if and only if $\mathcal{B}$ satisfies* (T_1), (T_2), (T_3), (T_4), (T_5^1), *and* (T_5^2).

Proof. If there is a tree T such that $\mathcal{B} = \mathcal{B}_s(T) \cap V_s^3$, then it is straightforward to verify that $\mathcal{B}$ satisfies (T_1), (T_2), (T_3), (T_4), (T_5^1), and (T_5^2), which implies the "only if"-part of the statement. For the proof of the "if"-part of the statement, let $\mathcal{B} \subseteq V_s^3$ satisfy (T_1), (T_2), (T_3), (T_4), (T_5^1), and (T_5^2). For $u, v, w \in V$, let $N^*(u, v, w)$ abbreviate the following assertion

$$N(u, v, w) \wedge \nexists x \in V : ((u, x, v) \in \mathcal{B} \vee (u, x, w) \in \mathcal{B} \vee (v, x, w) \in \mathcal{B}) .$$

Exactly as in Lemma 4, let $\mathcal{T} = \left\{\{u, v, w\} \in \binom{V}{3} \mid N^*(u, v, w)\right\}$ and let G be the graph with vertex set $\mathcal{T}$ in which two vertices $t, s \in \mathcal{T}$ are adjacent if and only if $|t \cap s| = 2$. Let $\{C_1, \ldots, C_k\}$ be the set of components of G. For a subgraph C of G, let

$$N(C) = \bigcup_{t \in V(C)} t.$$

Claim 1. *If $1 \le i \le k$ and u, v, and w are three distinct elements in $N(C_i)$, then $N^*(u, v, w)$.*

Proof of Claim 1. For every non-empty connected subgraph C of C_i, we prove, by induction over $|V(C)|$, that $N^*(u, v, w)$ for every three distinct elements u, v, and w in $N(C)$.

If $|V(C)| = 1$, the desired statement holds by definition.

Now let $|V(C)| \ge 2$. Let $t' = \{u', v', w'\} \in V(C)$ be such that $C' = G[V(C) \setminus \{t'\}]$ is connected. Note that, since t' has a neighbour in $V(C')$, we obtain, by the construction of G, that $|N(C) \setminus N(C')| \le 1$. Let u, v, and w be three distinct elements of $N(C)$. If $u, v, w \in N(C')$, then the desired statement follows by induction. Hence, we may assume that $u, v \in N(C')$ and $w \notin N(C')$. Hence $w \in t'$ and we may assume that $w = w'$. Note that $|N(C) \setminus N(C')| = 1$. Since $t' \in \mathcal{T}$, we have $N^*(u', v', w)$. For contradiction, we assume that $N^*(u, v, w)$ does not hold.

First, we assume that $u = u'$ and $v \neq v'$, i.e. $N^*(u, v', w)$ and, by induction for C', $N^*(u, v, v')$ hold. If $N(u, v, w)$ does not hold, then $N^*(u, v', w)$ and $N^*(u, v, v')$ imply $(v, u, w) \in \mathcal{B}$. By (T_5^2), $N^*(u, v', w)$ and $(v, u, w) \in \mathcal{B}$ imply $(v, u, v') \in \mathcal{B}$, which is a contradiction to $N^*(u, v, v')$. Hence $N(u, v, w)$. Since $N^*(u, v, w)$ does not hold, $N^*(u, v', w)$ and $N^*(u, v, v')$ imply that there is some $x \in V$ with $(v, x, w) \in \mathcal{B}$. Now, by (T_5^1), $N(u, v, w)$ and $(v, x, w) \in \mathcal{B}$ imply $(u, x, v) \in \mathcal{B}$ or $(u, x, w) \in \mathcal{B}$, which is a contradiction to $N^*(u, v, v')$ or $N^*(u, v', w)$.

Next, we assume that $u \neq u'$ and $v \neq v'$. By the arguments used in the previous case, we obtain that $N^*(u', v', w)$ implies $N^*(u', v, w)$, and that $N^*(u', v, w)$ implies $N^*(u, v, w)$, which completes the proof. $\square$

Let T be the graph with vertex set $V(T) = V \cup \{c_1, \ldots, c_k\}$ where $c_1, \ldots, c_k$ are k distinct elements not contained in V and edge set

$$\begin{aligned} E(T) = {} & \{c_i u \mid (1 \le i \le k) \wedge (u \in N(C_i))\} \cup \\ & \{uv \mid (u, v \in V) \wedge (\nexists w \in V : N(u, v, w)) \wedge (\nexists x \in V : (u, x, v) \in \mathcal{B})\}. \end{aligned}$$

Note that, by Claim 1, if $u, w \in V$ and $c \in V(T) \setminus V$ are such that $u \neq w$ and $cu, cw \in E(T)$, then there is some $v \in V$ such that $N^*(u, v, w)$ and $cv \in E(T)$.

We prove a sequence of claims, which imply that T is a tree and $\mathcal{B} = \mathcal{B}_s(T) \cap V_s^3$.

Claim 2. *If $u, v \in V$ are such that $u \neq v$, $uv \notin E(T)$, and there is no $x \in V$ with $(u, x, v) \in \mathcal{B}$, then there is some $w \in V$ such that $N^*(u, v, w)$.*

Proof of Claim 2. Let u and v be as in the statement of the claim. Since $uv \notin E(T)$, there is, by the definition of $E(T)$, some $w \in V$ such that $N(u, v, w)$. We assume that w is chosen such that

$$Z(w) = \{x \in V \mid (u, x, w) \in \mathcal{B} \vee (v, x, w) \in \mathcal{B}\}$$

has minimum possible cardinality. For contradiction, we assume that $N^*(u, v, w)$ does not hold, i.e. $Z(w)$ is not empty. Since there is no $x \in V$ with $(u, x, v) \in \mathcal{B}$, (T_1) and (T_5^1) imply the existence of some $x \in V$ with $(u, x, w), (v, x, w) \in \mathcal{B}$. If $(u, v, x) \in \mathcal{B}$, then, by (T_1) and (T_3), $(v, x, w) \in \mathcal{B}$ implies $(u, v, w) \in \mathcal{B}$, which is a contradiction. Hence $(u, v, x) \notin \mathcal{B}$ and, by symmetry, $(x, u, v) \notin \mathcal{B}$, which implies $N(u, v, x)$. Let $y \in Z(x)$. If $(u, y, x) \in \mathcal{B}$, then, by (T_4), $(u, x, w) \in \mathcal{B}$ implies $(y, x, w) \in \mathcal{B}$. Now, by (T_1) and (T_3), $(u, y, x) \in \mathcal{B}$ and $(y, x, w) \in \mathcal{B}$ imply $(u, y, w) \in \mathcal{B}$, i.e. $y \in Z(w)$. Similarly, if $(v, y, x) \in \mathcal{B}$, then $(v, y, w) \in \mathcal{B}$, i.e. $y \in Z(w)$. Altogether, $Z(x) \subseteq Z(w)$. Since $x \in Z(w) \setminus Z(x)$, we obtain a contradiction to the choice of w. $\square$

Claim 3. *If $P : u_1 u_2 \ldots u_l$ is a path in T such that $u_1, u_i, u_l \in V$ for some $2 \le i \le l - 1$, then $(u_1, u_i, u_l) \in \mathcal{B}$.*

Proof of Claim 3. We proof the assertion by induction on the cardinality $v(P)$ of $V(P) \cap V$.

First, let $v(P) = 3$. If $l = 3$, then, by construction, $N(u_1, u_2, u_3)$ does not hold. Since $u_1u_2, u_2u_3 \in E(T)$, this implies, again by construction, that $(u_1, u_2, u_3) \in \mathcal{B}$. If $l = 4$, then, by (T_1), we may assume that $u_3 \in V(T) \setminus V$. By construction, there is some component C of G with $u_2, u_4 \in N(C)$. Therefore, by Claim 1, there is some $w \in V$ with $N^*(u_2, u_4, w)$, which implies $(u_2, u_1, u_4) \notin \mathcal{B}$. Since $u_1u_2 \in E(T)$, we obtain that $(u_1, u_4, u_2) \notin \mathcal{B}$ and that $N(u_1, u_2, u_4)$ does not hold. This implies that $(u_1, u_2, u_4) \in \mathcal{B}$. If $l = 5$, we have $u_2, u_4 \in V(T) \setminus V$. Since u_1 and u_3 have a common neighbour in $V(T) \setminus V$, we obtain, by Claim 1, that $(u_1, u_5, u_3) \notin \mathcal{B}$. Similarly, $(u_3, u_1, u_5) \notin \mathcal{B}$. For contradiction, we assume $(u_1, u_3, u_5) \notin \mathcal{B}$, i.e. $N(u_1, u_3, u_5)$. If $(u_1, x, u_5) \in \mathcal{B}$ for some $x \in V \setminus \{u_3\}$, then, by (T_5^1), $(u_1, x, u_3) \in \mathcal{B}$ or $(u_3, x, u_5) \in \mathcal{B}$, which is a contradiction. Hence $N^*(u_1, u_3, u_5)$ holds. Now the construction of G implies the existence of some component C of G with $u_1, u_3, u_5 \in N(C)$, which implies the contradiction $u_2 = u_4$. Since, by construction, $V(T) \setminus T$ is an independent set in T, this concludes the case $v(P) = 3$.

Now let $v(P) \geq 4$. We may assume that there is some j with $2 \leq j \leq i-1$ such that $u_j \in V$. By induction applied to the subpath of P between u_1 and u_i, we obtain $(u_1, u_j, u_i) \in \mathcal{B}$. Similarly, by induction applied to the subpath of P between u_j and u_l, we obtain $(u_j, u_i, u_l) \in \mathcal{B}$. Now (T_3) implies $(u_1, u_i, u_l) \in \mathcal{B}$, which completes the proof of the claim. $\square$

Claim 4. *T has no cycle.*

Proof of Claim 4. For contradiction, we assume that G contains a cycle D. Since $V(T) \setminus T$ is an independent set in T, D contains at least two elements of V. First we assume that D contains exactly two elements v and w of V. If $D : vwcv$ for some $c \in V(T) \setminus V$, then $vw \in E(T)$ implies that there is no $u \in V$ with $N(u, v, w)$, while, by Claim 1, $vc, wc \in E(T)$ imply that there is some $u \in V$ with $N(u, v, w)$, which is a contradiction. If $D : vc_1wc_2v$ for some $c_1, c_2 \in V(T) \setminus V$, then, by Claim 1, there is exactly one component C of G with $\{v, w\} \subseteq N(C)$, which implies the contradiction $c_1 = c_2$. Hence D contains at least three elements u, v, and w of V. Applying Claim 3 to the path in D between u and w with v as an internal vertex implies $(u, v, w) \in \mathcal{B}$, and applying Claim 3 to the path in D between u and v with w as an internal vertex implies $(u, w, v) \in \mathcal{B}$, which contradicts (T_2). This completes the proof. $\square$

Claim 5. *T is connected.*

Proof of Claim 5. Since every vertex in $V(T) \setminus V$ has a neighbour in V, it suffices to show that all elements of V belong to the same component of T. For contradiction, we assume that there are two elements u and v of V that belong to different components of T. We assume that u and v are chosen such that

$$Z((u, v)) = \{x \in V \mid (u, x, v) \in \mathcal{B}\}$$

has minimum possible cardinality. If $Z((u, v))$ is not empty, then let $x \in Z((u, v))$. By symmetry, we may assume that u and x belong to different components of T. If $y \in Z((u, x))$, then (T_1), (T_3), and (T_4) imply $(u, y, v) \in \mathcal{B}$. Therefore $Z((u, x)) \subseteq Z((u, v)) \setminus \{x\}$, which contradicts the choice of u and v. Hence $Z((u, v))$ is empty. Since $uv \notin E(T)$, Claim 2 implies that there is some $w \in V$ with $N^*(u, v, w)$. Hence, by construction, u and v have a common neighbour in $V(T) \setminus V$, which contradicts our assumption. $\square$

Altogether, we obtain that T is a tree. Hence $\mathcal{B}_s(T) \cap V_s^3$ satisfies (T_1), (T_2), (T_3), (T_4), (T_5^1), and (T_5^2). By Claim 3, $\mathcal{B}_s(T) \cap V_s^3 \subseteq \mathcal{B}$. Therefore, the following claim completes the proof.

Claim 6. $\mathcal{B} \subseteq \mathcal{B}_s(T) \cap V_s^3$.

Proof of Claim 6. Let $\mathcal{B}' = \mathcal{B}_s(T) \cap V_s^3$. For contradiction, we assume that $\mathcal{B} \setminus \mathcal{B}'$ is not empty. We choose $(u, v, w) \in \mathcal{B} \setminus \mathcal{B}'$ such that the cardinality of

$$Z((u, v, w)) = \{x \in V \mid (u, x, v) \in \mathcal{B} \vee (v, x, w) \in \mathcal{B}\}$$

is minimum possible.

If $Z((u, v, w))$ is not empty, then, by symmetry, we may assume that there is some $x \in V$ with $(u, x, v) \in \mathcal{B}$. By (T_3) and (T_4), this implies $(x, v, w) \in \mathcal{B}$. As before, we obtain that $Z((u, x, v))$ as well as $Z((x, v, w))$ are proper subsets of $Z((u, v, w))$. By the minimality of $|Z((u, v, w))|$, this implies $(u, x, v), (x, v, w) \in \mathcal{B}'$. By (T_4) for $\mathcal{B}'$, we obtain $(u, v, w) \in \mathcal{B}'$, which is a contradiction. Hence $Z((u, v, w))$ is empty.

By Claim 3, we may assume that $uv \notin E(T)$. By Claim 2, there is some $y_1 \in V$ with $N^*(u, v, y_1)$ and hence there is some $c_1 \in V(T) \setminus V$ with $u, v \in N_T(c_1)$. Since $(u, v, w) \notin \mathcal{B}'$, the existence of c_1 implies $vw \notin E(T)$. Applying the above argument again, we obtain that there is some $c_2 \in V(T) \setminus V$ with $v, w \in N_T(c_2)$. Since $(u, v, w) \in \mathcal{B}$, Claim 1 implies that c_1 and c_2 are distinct. Hence uc_1vc_2w is a path in T and $(u, v, w) \in \mathcal{B}'$, which is a contradiction. $\square$

This concludes the proof of Theorem 7. $\square$

5. Structural aspects of forest betweennesses

In this section, we give an axiomatic characterization of strict forest betweennesses. While the strict forest betweenness of some forest F clearly satisfies (T_1), (T_2), (T_3), and (T_4), the axiom (T_5) only holds within the components of F. Since components of order two do not contribute any triple to the strict forest betweenness of F, we may assume that F has no such components. Three vertices u, v, and w of F belong to the same component of F but do not lie in a common path if and only if u and v as well as u and w belong to the same component of F and the minimal subtree T of F that contains u, v, and w is the subdivision of a claw $K_{1,3}$ with endvertices u, v, and w. Furthermore, two distinct vertices belong to the same component of F if and only if there is a path in F of length at least 2 containing both.

For some $\mathcal{B} \subseteq V^3$ and $u, v \in V$, let $u \sim_{\mathcal{B}} v$ abbreviate the following assertion

$$(u = v) \vee (\exists x \in V : (x, u, v) \in \mathcal{B} \vee (u, x, v) \in \mathcal{B} \vee (u, v, x) \in \mathcal{B})\,.$$

In view of the above observations, strict forest betweennesses satisfy the following modified version of (T_5).

(F_5) $\forall u, v, w \in V : (N(u, v, w) \wedge (u \sim_{\mathcal{B}} v) \wedge (u \sim_{\mathcal{B}} w)) \Rightarrow \exists c \in V : (u, c, v), (u, c, w), (v, c, w) \in \mathcal{B}$

We proceed to the result in this section.

Theorem 8. *Let V be a finite set. If $\mathcal{B} \subseteq V_s^3$ is a set of strict triples, then there is a forest F such that $\mathcal{B}_s(F) = \mathcal{B}$ if and only if $\mathcal{B}$ satisfies* (T_1), (T_2), (T_3), (T_4), *and* (F_5).

Proof. Since the 'only if'-part is obvious from the previous remarks, we proceed to the proof of the 'if'-part. Let $\mathcal{B} \subseteq V_s^3$ satisfy (T_1), (T_2), (T_3), (T_4), and (F_5). By definition and (T_1), $\sim_{\mathcal{B}}$ is reflexive and symmetric.

Claim 7. *$\sim_{\mathcal{B}}$ is transitive.*

Proof of Claim 7. For contradiction, we assume that u, v, and w in V are such that $u \sim_{\mathcal{B}} v$ and $v \sim_{\mathcal{B}} w$ but $u \not\sim_{\mathcal{B}} w$. Clearly, this implies that u, v, and w are all distinct and hence $N(v, u, w)$. Now, by (F_5), there is some $c \in V$ with (u, c, w), which implies the contradiction $u \sim_{\mathcal{B}} w$. $\square$

By the claim, $\sim_{\mathcal{B}}$ is an equivalence relation. Let $V = V_1 \cup V_2 \cup \cdots \cup V_k$ be the partition of V into the equivalence classes of $\sim_{\mathcal{B}}$. For $1 \leq i \leq k$, let $\mathcal{B}_i = \mathcal{B} \cap (V_i)_s^3$. Note that two distinct elements of V appear together in a triple of $\mathcal{B}$ if and only if they belong to the same equivalence class. Therefore, $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$. For $1 \leq i \leq k$, $\mathcal{B}_i$ satisfies (T_1), (T_2), (T_3), (T_4), and (T_5). By Theorem 6, for every $1 \leq i \leq k$, there is a tree T_i with $\mathcal{B}_s(T_i) = \mathcal{B}_i$. Clearly, $\mathcal{B}_s(T_1 \cup T_2 \cup \cdots \cup T_k) = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k = \mathcal{B}$, which completes the proof. $\square$

We believe that Theorem 8 can be strengthened using the following weaker version of (F_5).

(F_5^w) $\forall u, v, w \in V : (N(u, v, w) \wedge (\exists x \in V : (u, x, v) \in \mathcal{B}) \wedge (\exists y \in V : (u, y, w) \in \mathcal{B})) \Rightarrow \exists c \in V : (u, c, v), (u, c, w), (v, c, w) \in \mathcal{B}$

Conjecture 9. *Let V be a finite set. If $\mathcal{B} \subseteq V_s^3$ is a set of strict triples, then there is a forest F such that $\mathcal{B}_s(F) = \mathcal{B}$ if and only if $\mathcal{B}$ satisfies* (T_1), (T_2), (T_3), (T_4), *and* (F_5^w).

Acknowledgments

The authors acknowledge partial support by the CAPES/DAAD Probral project "Cycles, Convexity, and Searching in Graphs". The second author acknowledges support from the Conselho Nacional de Desenvolvimento Cientifico e Tecnologico, CNPq, Brazil.

References

[1] E.V. Huntington, J.R. Kline, Sets of independent postulates for betweenness, American M.S. Trans. 18 (1917) 301–325.
[2] N. Alon, M. Bădoiu, E.D. Demaine, M. Farach-Colton, M. Hajiaghayi, A. Sidiropoulos, Ordinal embeddings of minimum relaxation: general properties, trees, and ultrametrics, ACM Trans. Algorithms 4 (2008) Article No. 46.
[3] A. Krokhin, P. Jeavons, P. Jonsson, Constraint satisfaction problems on intervals and lengths, SIAM J. Discrete Math. 17 (2004) 453–477.
[4] G. Gutin, E.J. Kim, M. Mnich, A. Yeo, Ordinal embedding relaxations parameterized above tight lower bound, arXiv:0907.5427v2 (pre-print).
[5] L. Burigana, Tree representations of betweenness relations defined by intersection and inclusion, Mathematics and Social Sciences 185 (2009) 5–36.
[6] D. Cox, M. Burmeister, E. Price, S. Kim, R. Myers, Radiation hybrid mapping: a somatic cell genetic method for constructing high-resolution maps of mammalian chromosomes, Science 250 (4978) (1990) 245–250.
[7] S. Goss, H. Harris, New methods for mapping genes in human chromosomes, Nature 255 (1975) 680–684.
[8] B. Chor, M. Sudan, A geometric approach to betweenness, SIAM J. Discrete Math. 11 (1998) 511–523.
[9] J. Opatrny, Total ordering problem, SIAM J. Comput. 8 (1979) 111–114.
[10] M. Charikar, V. Guruswami, R. Manokaran, Every permutation csp of arity 3 is approximation resistant, in: Proceedings of the 2009 24th Annual IEEE Conference on Computational Complexity (CCC '09), pp. 62-73.
[11] Y. Makarychev, Simple linear time approximation algorithm for betweenness, Microsoft Research Technical Report MSR-TR-2009-74, June 2009.
[12] K. Menger, Untersuchungen über allgemeine Metrik, Math. Ann. 100 (1928) 75–163.
[13] G. Birkhoff, Lattice Theory, American Mathematical Society (AMS), New York, 1948, Rev. ed, American Mathematical Society Colloquium Publications.
[14] M. Altwegg, Zur Axiomatik der teilweise geordneten Mengen, Comment. Math. Helv. 24 (1950) 149–155.
[15] M. Sholander, Trees, lattices, order, and betweenness, Proc. Amer. Math. Soc. 3 (1952) 369–381.
[16] I. Düntsch, A. Urquhart, Betweenness and comparability obtained from binary relations, Lecture Notes in Comput. Sci. 4136 (2006) 148–161.
[17] V. Chvátal, Antimatroids, betweenness, convexity, in: W.J. Cook, L. Lovász, J. Vygen (Eds.), Research Trends in Combinatorial Optimization, Springer, Berlin, 2009, (Bonn 2008).
[18] V. Chvátal, Sylvester–Gallai theorem and metric betweenness, Discrete Comput. Geom. 31 (2004) 175–195.
[19] R. Mendris, P. Zlatoš, Axiomatization and undecidability results for metrizable betweenness relations, Proc. Amer. Math. Soc. 123 (1995) 873–882.
[20] H.M. Mulder, L. Nebeský, Axiomatic characterization of the interval function of a graph, Eur. J. Comb. 30 (2009) 1172–1185.
[21] L. Nebeský, Intervals and steps in a connected graph, Discrete Math. 286 (2004) 151–156.
[22] L. Nebeský, The interval function of a connected graph and road systems, Discrete Math. 307 (2007) 2067–2073.
[23] V. Chvátal, D. Rautenbach, P.M. Schäfer, Finite sholander trees, trees, and their betweennes, Discrete Math. 311 (2011) 2143–2147.
[24] D. Defays, Tree representations of ternary relations, J. Math. Psych. 19 (1979) 208–218.

Apêndice B

Characterization and representation problems for intersection betweennesses

Este apêndice contém o artigo "*Characterization and representation problems for intersection betweennesses*", publicado no periódico *Discrete Applied Mathematics*.

Discrete Applied Mathematics 159 (2011) 389–395

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Characterization and representation problems for intersection betweennesses

Dieter Rautenbach [a,*], Vinícius Fernandes dos Santos [b], Philipp M. Schäfer [a], Jayme L. Szwarcfiter [b]

[a] Institut für Optimierung und Operations Research, Universität Ulm, Ulm, Germany

[b] Instituto de Matemática, NCE, and COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ, Brazil

ARTICLE INFO

Article history:
Received 3 May 2010
Received in revised form 1 December 2010
Accepted 9 December 2010
Available online 3 January 2011

Keywords:
Betweenness
Shortest paths
Trees

ABSTRACT

For a set system $\mathcal{M} = (M_v)_{v \in V}$ indexed by the elements of a finite set V, the intersection betweenness $\mathcal{B}(\mathcal{M})$ induced by $\mathcal{M}$ consists of all triples $(u, v, w) \in V^3$ with $M_u \cap M_w \subseteq M_v$. Similarly, the strict intersection betweenness $\mathcal{B}_s(\mathcal{M})$ induced by $\mathcal{M}$ consists of all triples $(u, v, w) \in \mathcal{B}(\mathcal{M})$ such that u, v, and w are pairwise distinct. The notion of a strict intersection betweenness was introduced by Burigana [L. Burigana, Tree representations of betweenness relations defined by intersection and inclusion, Math. Soc. Sci. 185 (2009) 5–36]. We provide axiomatic characterizations of intersection betweennesses and strict intersection betweennesses. Our results yield a simple and efficient algorithm that constructs a representing set system for a given (strict) intersection betweenness. We study graphs whose strict shortest path betweenness is a strict intersection betweenness. Finally, we explain how the algorithmic problem related to Burigana's notion of a partial tree representation can be solved efficiently using well-known algorithms.

1. Introduction

In this paper, we study so-called betweennesses induced by graphs as well as set systems. Betweennesses capture and generalize in an abstract way natural geometric properties of points in $\mathbb{R}^n$, and the axiomatic study of betweenness as a mathematical concept goes back to Huntington and Kline [13] in 1917. Algorithmic problems related to betweennesses have been studied as relaxations of ordinal embeddings [1,14,12] and occur for instance in psychometrics [3] and as arrangement problems in molecular biology [6,10]. For such betweenness problems, several strong hardness results have been obtained [5,18,4] and only a few positive results are known [5,12,15].

We consider finite, simple, and undirected graphs as well as finite set systems defined over finite ground sets. Let G be a graph. The vertex set of G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. For a vertex u of G, the neighbourhood $N_G(u)$ in G equals $\{v \in V(G) \mid uv \in E(G)\}$ and the degree $d_G(u)$ in G equals $|N_G(u)|$. A path P of length l in G between two vertices v_0 and v_l of G is a sequence $P{:}v_0v_1 \ldots v_l$ of $l + 1$ distinct vertices $v_0, v_1, \ldots, v_l \in V(G)$ such that $v_{i-1}v_i \in E(G)$ for $1 \leq i \leq l$. The distance $\mathrm{dist}_G(u, v)$ in G between two vertices u and v of G is the minimum length of a path in G between u and v. A cycle C of length l in G is a sequence $C{:}v_1v_2 \ldots v_lv_1$ such that $v_1v_2 \ldots v_l$ is a path in G and $v_1v_l \in E(G)$. For a finite set V, V^3 denotes the set of all ordered triples of elements of V. A triple $(u, v, w) \in V^3$ is called strict if u, v, and w are

* Corresponding address: Institut für Optimierung und Operations Research, Universität Ulm, Helmholtzstrasse 22/Raum E02, 89081 Ulm, Germany. Tel.: +49 731 5023630; fax: +49 731 501223630.

E-mail addresses: dieter.rautenbach@uni-ulm.de, dieter.rautenbach@gmx.de (D. Rautenbach), vinicius.santos@gmail.com (V.F. dos Santos), philipp.schaefer@gmail.com (P.M. Schäfer), jayme@nce.ufrj.br (J.L. Szwarcfiter).

0166-218X/$ – see front matter
doi:10.1016/j.dam.2010.12.009

pairwise distinct. Let V_s^3 denote the set of all strict triples in V^3. For $k \in \mathbb{N}_0$, let $\binom{V}{k}$ denote the set of all subsets of V that are of cardinality k.

For a graph G, the *shortest path betweenness* $\mathcal{B}(G)$ *of* G consists of all triples $(u, v, w) \in V(G)^3$ such that v lies on a shortest path in G between u and w, or equivalently $(u, v, w) \in V(G)^3$ belongs to $\mathcal{B}(G)$ if and only if $\mathrm{dist}_G(u, w) = \mathrm{dist}_G(u, v) + \mathrm{dist}_G(v, w) < \infty$. The *strict shortest path betweenness* $\mathcal{B}_s(G)$ *of* G consists of all strict triples in $\mathcal{B}(G)$, i.e. $\mathcal{B}_s(G) = \mathcal{B}(G) \cap V(G)_s^3$. Shortest path betweennesses are a special case of betweennesses induced by metrics, which were first studied by Menger in 1928 [17]. The shortest path betweennesses of trees have received special attention, and several different axiomatic characterizations have been proposed [3,7,19,20]. For a tree/forest T, we call $\mathcal{B}(T)$ the *tree/forest betweenness of* T and $\mathcal{B}_s(T)$ the *strict tree/forest betweenness of* T.

In [3], Burigana introduces a betweenness notion derived from set systems. For a set system $\mathcal{M} = (M_v)_{v\in V}$ indexed by the elements of a finite set V, the *intersection betweenness* $\mathcal{B}(\mathcal{M})$ *induced by* $\mathcal{M}$ consists of all triples $(u, v, w) \in V^3$ with $M_u \cap M_w \subseteq M_v$, and the *strict intersection betweenness* $\mathcal{B}_s(\mathcal{M})$ *induced by* $\mathcal{M}$ consists of all strict triples $(u, v, w) \in V_s^3$ with $M_u \cap M_w \subseteq M_v$, i.e.

$$\mathcal{B}(\mathcal{M}) = \{(u, v, w) \in V^3 \mid M_u \cap M_w \subseteq M_v\}$$
$$\mathcal{B}_s(\mathcal{M}) = \mathcal{B}(\mathcal{M}) \cap V_s^3 = \{(u, v, w) \in V_s^3 \mid M_u \cap M_w \subseteq M_v\}.$$

For every finite set V, a set $\mathcal{B} \subseteq V^3$ is an *intersection betweenness* if there is some set system $\mathcal{M}$ with $\mathcal{B} = \mathcal{B}(\mathcal{M})$. Similarly, a set $\mathcal{B} \subseteq V_s^3$ is a *strict intersection betweenness* if there is some set system $\mathcal{M}$ with $\mathcal{B} = \mathcal{B}_s(\mathcal{M})$.

Burigana provides some axioms for strict intersection betweennesses. Furthermore, he characterizes the two classes of strict intersection betweennesses that coincide with some strict tree betweenness (betweennesses with a *full tree representation*), and that contain some strict tree betweenness (betweennesses with a *partial tree representation*). A central problem left open in [3] is the (axiomatic) characterization of (strict) intersection betweennesses. Furthermore, the procedure proposed in [3] for the solution of the partial tree representation problem does not lead to an efficient algorithm.

Our results are as follows. In Section 2, we provide axiomatic characterizations of intersection betweennesses and strict intersection betweennesses thus solving the problem left open in [3]. Furthermore, our results yield a simple and efficient algorithm that constructs a representing set system for a given (strict) intersection betweenness. In Section 3, we characterize those graphs whose strict shortest path betweenness is a strict intersection betweenness. Furthermore, we describe representations of strict tree betweennesses as strict intersection betweennesses of set systems over small ground sets. Finally, in Section 4, we explain how the algorithmic problem related to Burigana's partial tree representation can be solved efficiently using well-known algorithms.

2. Characterizing and representing intersection betweennesses

Burigana [3] provides the following three axioms, which he claims to hold for every strict intersection betweenness $\mathcal{B}$.

(I_1) $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (w, v, u) \in \mathcal{B}$.
(I_2) $\forall u, v, w, z \in V : (u, v, w), (u, z, v) \in \mathcal{B} \Rightarrow (u, z, w) \in \mathcal{B}$.
(I_3) $\forall u, v, w, t, z \in V : (t, u, z), (t, w, z), (u, v, w) \in \mathcal{B} \Rightarrow (t, v, z) \in \mathcal{B}$.

These three axioms clearly hold for every intersection betweenness because of elementary properties of set intersection and inclusion. Furthermore, (I_1) also holds for every strict intersection betweenness. Contrary to Burigana's claim, the properties (I_2) and (I_3) are actually problematic for strict intersection betweennesses because they potentially imply the presence of non-strict triples. In order to ensure that the triple (u, z, w), whose existence is guaranteed by (I_2), is strict, one has to add the condition $w \neq z$. Similarly, in order to ensure that the triple (t, v, z), whose existence is guaranteed by (I_3), is strict, one has to add the condition $t \neq v \neq z$. This leads to the following modified versions of (I_2) and (I_3).

(I_2^s) $\forall u, v, w, z \in V : (u, v, w), (u, z, v) \in \mathcal{B}$ and $w \neq z \Rightarrow (u, z, w) \in \mathcal{B}$.
(I_3^s) $\forall u, v, w, t, z \in V : (t, u, z), (t, w, z), (u, v, w) \in \mathcal{B}$ and $t \neq v \neq z \Rightarrow (t, v, z) \in \mathcal{B}$.

While strict intersection betweennesses can be empty, intersection betweennesses always contain all triples of the form (u, u, w) and (u, w, w). Therefore, they necessarily satisfy another property.

(I_4) $\forall u, w \in V : (u, u, w) \in \mathcal{B}$.

Note that (I_4) together with (I_3) actually implies (I_2) by choosing $t = u$ in (I_3).

We now show that the above axioms yield characterizations of intersection betweennesses and strict intersection betweennesses. Furthermore, we also prove that there are always representing set systems over quadratic ground sets, which can be constructed efficiently.

Theorem 1. *Let V be a finite set and let $\mathcal{B} \subseteq V^3$.*

(i) *If there is a set system $\mathcal{M} = (M_v)_{v\in V}$ with $\mathcal{B} = \mathcal{B}(\mathcal{M})$, then $\mathcal{B}$ satisfies* (I_1), (I_3), *and* (I_4).

(ii) *If $\mathcal{B}$ satisfies* (I_1), (I_3), *and* (I_4), *then there is a set system* $\mathcal{M} = (M_v)_{v\in V}$ *with* $\mathcal{B} = \mathcal{B}(\mathcal{M})$ *and* $|\bigcup_{v\in V} M_v| \le \binom{|V|}{2}$, *which can be constructed in polynomial time.*

Proof. Since (i) is obvious, we proceed to the proof of (ii). Let $\mathcal{B}$ satisfy (I_1), (I_3), and (I_4). For $v \in V$, let $M_v = \{\{u, w\} \mid u, w \in V \text{ and } (u, v, w) \in \mathcal{B}\}$. Note that M_v can easily be constructed in polynomial time (see Algorithm 1). Note that $\{u, w\} = \{w, u\}$. Therefore the sets M_v are well defined, because $(u, v, w) \in \mathcal{B}$ holds if and only if $(w, v, u) \in \mathcal{B}$ holds, which is ensured by (I_1). We will show that $\mathcal{B} = \mathcal{B}(\mathcal{M})$ for the set system $\mathcal{M} = (M_v)_{v\in V}$. Therefore, let $(u, v, w) \in V^3$. By (I_1) and (I_4), $(u, u, w), (u, w, w) \in \mathcal{B}$, and hence $\{u, w\} \in M_u \cap M_w$. First, we assume that $(u, v, w) \notin \mathcal{B}$. By definition, this implies that $\{u, w\} \in (M_u \cap M_w) \setminus M_v$, and hence $(u, v, w) \notin \mathcal{B}(\mathcal{M})$. Conversely, we assume that $(u, v, w) \in \mathcal{B}$. For contradiction, we assume that $(u, v, w) \notin \mathcal{B}(\mathcal{M})$. This implies that there is some $\{t, z\} \in (M_u \cap M_w) \setminus M_v$. By definition, this implies that $(t, u, z), (t, w, z) \in \mathcal{B}$. Since $(u, v, w) \in \mathcal{B}$, (I_3) implies that $(t, v, z) \in \mathcal{B}$, and hence, by definition, $\{t, z\} \in M_v$, which is a contradiction. Note that $\bigcup_{v\in V} M_v \subseteq \binom{V}{2}$, which completes the proof. □

We extend this result to strict intersection betweennesses.

Theorem 2. *Let V be a finite set and let $\mathcal{B}_s \subseteq V_s^3$.*

(i) *If there is a set system* $\mathcal{M} = (M_v)_{v\in V}$ *with* $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$, *then* $\mathcal{B}_s$ *satisfies* (I_1), $(\mathrm{I}_2^{\mathrm{s}})$, *and* $(\mathrm{I}_3^{\mathrm{s}})$.

(ii) *If* $\mathcal{B}_s$ *satisfies* (I_1), $(\mathrm{I}_2^{\mathrm{s}})$, *and* $(\mathrm{I}_3^{\mathrm{s}})$, *then there is a set system* $\mathcal{M} = (M_v)_{v\in V}$ *with* $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$ *and* $|\bigcup_{v\in V} M_v| \le \binom{|V|}{2}$, *which can be constructed in polynomial time.*

It is possible to prove Theorem 2 in a similar way as Theorem 1. We present an alternative proof, which relates strict intersection betweennesses to special intersection betweennesses containing them. Furthermore, this alternative proof shows that a single algorithm, Algorithm 1, constructs representing set systems in both cases.

Lemma 3. *If V is a finite set and $\mathcal{B}_s \subseteq V_s^3$ is a strict intersection betweenness, then*

$$\mathcal{B} = \mathcal{B}_s \cup \{(u, u, w) \mid u, w \in V\} \cup \{(u, w, w) \mid u, w \in V\}$$

is an intersection betweenness.

Input: A finite set V and a set $\mathcal{B} \subseteq V^3$ that is an intersection betweenness or a strict intersection betweenness.
Output: A set system $\mathcal{M} = (M_v)_{v\in V}$ such that $\left|\bigcup_{v\in V} M_v\right| \le \binom{|V|}{2}$. Furthermore, if $\mathcal{B}$ is an intersection betweenness, then $\mathcal{B} = \mathcal{B}(\mathcal{M})$, and if $\mathcal{B}$ is a strict intersection betweenness, then $\mathcal{B} = \mathcal{B}_s(\mathcal{M})$.

for $\{u, w\} \in \binom{V}{2}$ **do**
  $\mathcal{B} \leftarrow \mathcal{B} \cup \{(u, u, w), (u, w, w), (u, u, u)\}$;
end
for $v \in V$ **do**
  $M_v \leftarrow \emptyset$;
end
for $(u, v, w) \in \mathcal{B}$ **do**
  $M_v \leftarrow M_v \cup \{\{u, w\}\}$;
end
$\mathcal{M} \leftarrow (M_v)_{v\in V}$;
return $\mathcal{M}$;

Algorithm 1: Algorithm for representing an intersection betweenness.

Proof. Let the set system $\mathcal{M} = (M_v)_{v\in V}$ be such that $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$. Note that $\mathcal{B}(\mathcal{M})$ might contain triples of the form (u, v, u) with $u \ne v$, which are not contained in $\mathcal{B}$ as in the statement. Let $(u_v)_{v\in V}$ be a collection of $|V|$ pairwise distinct elements that do not belong to $\bigcup_{v\in V} M_v$. Let $M'_v = M_v \cup \{u_v\}$ for $v \in V$ and let $\mathcal{M}' = (M'_v)_{v\in V}$. Since $M'_u \not\subseteq M'_v$ and $M'_u \cap M'_v = M_u \cap M_v$ for every two distinct elements u and v of V, we obtain $\mathcal{B}_s(\mathcal{M}') = \mathcal{B}_s(\mathcal{M}) = \mathcal{B}_s$ and $\mathcal{B}(\mathcal{M}') = \mathcal{B}$. This completes the proof. □

We proceed to the proof of Theorem 2.

Proof of theorem 2. Since (i) is obvious, we proceed to the proof of (ii). Let $\mathcal{B}_s$ satisfy (I_1), $(\mathrm{I}_2^{\mathrm{s}})$, and $(\mathrm{I}_3^{\mathrm{s}})$. Let $\mathcal{B}$ be defined as in Lemma 3. Clearly, $\mathcal{B}$ satisfies (I_1) and (I_4) and does not contain any triple of the form (u, v, u) with $u \ne v$. We will show that $\mathcal{B}$ also satisfies (I_3). Therefore, let u, v, w, t, and z be in V such that $(t, u, z), (t, w, z), (u, v, w) \in \mathcal{B}$. We need to prove that $(t, v, z) \in \mathcal{B}$.

If $u = v$ or $w = v$, then trivially $(t, v, z) \in \mathcal{B}$. Hence, we may assume that $u \ne v \ne w$. Since $(u, v, w) \in \mathcal{B}$, this also implies that $u \ne w$, i.e. (u, v, w) is strict.

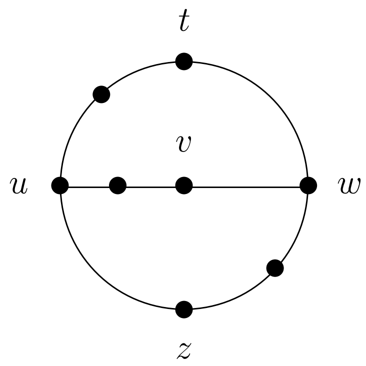

Fig. 1. A distance theta.

If $t = v$ or $z = v$, then, by construction, $(t, v, z) \in \mathcal{B}$. Hence we may assume that $t \neq v \neq z$.

If $t = u$, then $(u, w, z), (u, v, w) \in \mathcal{B}$. Since $u \neq w$, this implies that $u \neq z$. If $w = z$, then $(t, v, z) = (u, v, w) \in \mathcal{B}$. Hence we may assume that $w \neq z$, i.e. (u, w, z) is strict. Since $v \neq z$, $(\mathrm{I}_2^{\mathrm{s}})$ implies that $(t, v, z) = (u, v, z) \in \mathcal{B}$. Hence we may assume that $t \neq u$ and, by symmetric arguments, that $t \neq w, z \neq u$, and $z \neq w$. It follows that all three triples (t, u, z), (t, w, z), and (u, v, w) are strict. Now, since $t \neq v \neq z$, $(\mathrm{I}_3^{\mathrm{s}})$ implies that $(t, v, z) \in \mathcal{B}$. Therefore, $\mathcal{B}$ satisfies (I_3).

By Theorem 1, there is a set system $\mathcal{M} = (M_v)_{v \in V}$ with $\mathcal{B} = \mathcal{B}(\mathcal{M})$ and $|\bigcup_{v\in V} M_v| \leq \binom{|V|}{2}$, which can be constructed in polynomial time (see Algorithm 1). Since $\mathcal{B}_s = \mathcal{B} \cap V_s^3$, it follows that $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$, which completes the proof. □

3. Intersection betweennesses in graphs

In this section, we characterize those graphs whose strict shortest path betweenness is a strict intersection betweenness using forbidden subgraphs.

Let G be a graph. A *distance theta* in G is a subgraph H of G that is the subdivision of the complete bipartite graph with partite sets $\{u, w\}$ and $\{t, v, z\}$ such that the path in H between t and z containing u but not v has length $\mathrm{dist}_G(t, z)$, the path in H between t and z containing w but not v has length $\mathrm{dist}_G(t, z)$, the path in H between u and w containing v has length $\mathrm{dist}_G(u, w)$, and $\mathrm{dist}_G(t, z) < \mathrm{dist}_G(t, v) + \mathrm{dist}_G(v, z)$ (see Fig. 1).

Theorem 4. *Let G be a graph. The strict shortest path betweenness of G is a strict intersection betweenness if and only if G does not contain a distance theta.*

Proof. Let G be a graph. Clearly, $\mathcal{B}_s(G)$ satisfies (I_1). Now we show that $\mathcal{B}_s(G)$ satisfies $(\mathrm{I}_2^{\mathrm{s}})$. Therefore, let $(u, v, w), (u, z, v) \in \mathcal{B}_s(G)$ be such that $w \neq z$. By definition,

$$\begin{aligned}
\mathrm{dist}_G(u, w) &= \mathrm{dist}_G(u, v) + \mathrm{dist}_G(v, w) \\
&= \mathrm{dist}_G(u, z) + \mathrm{dist}_G(z, v) + \mathrm{dist}_G(v, w) \\
&\geq \mathrm{dist}_G(u, z) + \mathrm{dist}_G(z, w) \quad \text{and}
\end{aligned}$$
$$\mathrm{dist}_G(u, w) \leq \mathrm{dist}_G(u, z) + \mathrm{dist}_G(z, w),$$

which implies that $\mathrm{dist}_G(u, w) = \mathrm{dist}_G(u, z) + \mathrm{dist}_G(z, w)$, and hence $(u, z, w) \in \mathcal{B}_s(G)$. Therefore, $\mathcal{B}_s(G)$ satisfies $(\mathrm{I}_2^{\mathrm{s}})$ and, by Theorem 2, $\mathcal{B}_s(G)$ is a strict intersection betweenness if and only if $\mathcal{B}_s(G)$ satisfies $(\mathrm{I}_3^{\mathrm{s}})$. It remains to show that $\mathcal{B}_s(G)$ does not satisfy $(\mathrm{I}_3^{\mathrm{s}})$ if and only if G contains a distance theta.

If G contains a distance theta, then, using the notation as above (see Fig. 1), we have $(t, u, z), (t, w, z), (u, v, w) \in \mathcal{B}_s(G)$ but $(t, v, z) \notin \mathcal{B}_s(G)$, i.e. $\mathcal{B}_s(G)$ does not satisfy $(\mathrm{I}_3^{\mathrm{s}})$.

Conversely, if $\mathcal{B}_s(G)$ does not satisfy $(\mathrm{I}_3^{\mathrm{s}})$, then there are five distinct vertices u_0, v, w_0, t_0, and z_0 in $V(G)$ such that

$$\begin{aligned}
\mathrm{dist}_G(t_0, z_0) &= \mathrm{dist}_G(t_0, u_0) + \mathrm{dist}_G(u_0, z_0) \\
&= \mathrm{dist}_G(t_0, w_0) + \mathrm{dist}_G(w_0, z_0),
\end{aligned}$$
$$\mathrm{dist}_G(u_0, w_0) = \mathrm{dist}_G(u_0, v) + \mathrm{dist}_G(v, w_0), \quad \text{and}$$
$$\mathrm{dist}_G(t_0, z_0) < \mathrm{dist}_G(t_0, v) + \mathrm{dist}_G(v, z_0).$$

Let Q_0 be a shortest path in G between t_0 and z_0 that contains u_0, and let R_0 be a shortest path in G between t_0 and z_0 that contains w_0. If w_0 lies on Q_0, then, by symmetry, we may assume that w_0 lies between u_0 and z_0 on Q_0, and hence

$$\begin{aligned}
\mathrm{dist}_G(t_0, z_0) &= \mathrm{dist}_G(t_0, u_0) + \mathrm{dist}_G(u_0, w_0) + \mathrm{dist}_G(w_0, z_0) \\
&= \mathrm{dist}_G(t_0, u_0) + \mathrm{dist}_G(u_0, v) + \mathrm{dist}_G(v, w_0) + \mathrm{dist}_G(w_0, z_0) \\
&\geq \mathrm{dist}_G(t_0, v) + \mathrm{dist}_G(v, z_0),
\end{aligned}$$

which implies a contradiction to $\text{dist}_G(t_0, z_0) < \text{dist}_G(t_0, v) + \text{dist}_G(v, z_0)$. Hence w_0 does not lie on Q_0 and, by symmetry, u_0 does not lie on R_0. Let C be the shortest cycle in the graph $Q_0 \cup R_0$ that contains u_0, w_0. Note that C contains exactly two vertices t and z that belong to $V(Q) \cap V(R)$. We may assume that t lies between u_0 and t_0 on Q_0. This easily implies that t lies between w_0 and t_0 on R_0. Let Q denote the subpath of Q_0 between t and z, and let R denote the subpath of R_0 between t and z. Let S_0 be a shortest path in G between u_0 and w_0 that contains v. If t lies on S_0, then, by symmetry, we may assume that v lies between t and u_0 on S_0, and hence

$$\begin{aligned}\text{dist}_G(t_0, z_0) &= \text{dist}_G(t_0, t) + \text{dist}_G(t, u_0) + \text{dist}_G(u_0, z_0)\\ &= \text{dist}_G(t_0, t) + \text{dist}_G(t, v) + \text{dist}_G(v, u_0) + \text{dist}_G(u_0, z_0)\\ &\geq \text{dist}_G(t_0, v) + \text{dist}_G(v, z_0),\end{aligned}$$

which is a contradiction. Hence t and, by symmetry, z do not lie on S_0. Furthermore, we obtain, by a similar argument, that v does not lie on Q or R. Let S be the shortest subpath of S_0 that contains v as an internal vertex and contains two vertices from $V(Q) \cup V(R)$. Since $\text{dist}_G(t_0, z_0) < \text{dist}_G(t_0, v) + \text{dist}_G(v, z_0)$, S is a shortest path between an internal vertex, say u, of Q and an internal vertex, say w, on R. Now the five distinct vertices u, v, w, t, and z together with the three paths Q, R, and S yield a distance theta in G, which completes the proof. □

Note that Theorem 4 and its proof yield efficient algorithms for checking whether the strict shortest path betweenness of some given graph G is a strict intersection betweenness, and for finding a distance theta in G, if this is not the case. As seen in the proof of Theorem 4, the strict shortest path betweenness $\mathcal{B}_s(G)$ of G always satisfies (I_1) and (I_2^s). Since, for every three vertices $u, v, w \in V(G)$, we have $(u, v, w) \in \mathcal{B}_s(G)$ if and only if $\text{dist}_G(u, w) = \text{dist}_G(u, v) + \text{dist}_G(v, w)$, in order to check whether $\mathcal{B}_s(G)$ satisfies (I_3^s), one can just consider all polynomially many 5-tupels of vertices and test the suitable distance conditions. If $\mathcal{B}_s(G)$ does not satisfy (I_3^s), this yields a 5-tupel (t, z, u, v, w) with $(t, u, z), (t, w, z), (u, v, w) \in \mathcal{B}_s(G)$ and $(t, v, z) \notin \mathcal{B}_s(G)$, which, following the proof of Theorem 4, easily yields a distance theta in G.

There are several ways to show that (strict) tree betweennesses are (strict) intersection betweennesses. First, it is straightforward to verify that all tree betweennesses satisfy (I_1), (I_3), and (I_4) and that all strict tree betweennesses satisfy (I_1), (I_2^s), and (I_3^s). Therefore, Theorems 1 and 2 imply that every (strict) tree betweenness is a (strict) intersection betweenness. Second, if G is a graph all cycles of which are edge disjoint, then, by Theorem 4, the strict shortest path betweenness of G is a strict intersection betweenness. The next result yields a much simpler proof for which the representing set system $\mathcal{M} = (M_v)_{v \in V(T)}$ is derived from the underlying tree T in a very simple way.

For some $\mathcal{B} \subseteq V^3$ and $u, v, w \in V$, let $N(u, v, w)$ abbreviate the following assertion:

$$(u \neq v \neq w \neq u) \wedge ((u, v, w), (v, w, u), (w, u, v) \notin \mathcal{B}).$$

Theorem 5. *Let T be a tree of order n with l leaves.*

(i) *There is a set system $\mathcal{M} = (M_v)_{v \in V(T)}$ such that $\mathcal{B}_s(T) = \mathcal{B}_s(\mathcal{M})$ and $|\bigcup_{v \in V(T)} M_v| \leq 2n - l - 2$.*

(ii) *There is a set system $\mathcal{M} = (M_v)_{v \in V(T)}$ such that $\mathcal{B}(T) = \mathcal{B}(\mathcal{M})$ and $|\bigcup_{v \in V(T)} M_v| \leq 2n - 2$.*

Proof. (i) Let T be a tree with l leaves. For every $v \in V(T)$, let M_v consist of all ordered pairs (x, y) such that $xy \in E(T)$, $d_T(x) \geq 2$, and, if we root T in y, then the vertex v is equal to either x or a descendant of x. Equivalently,

$$M_v = \{(x, y) \mid xy \in E(T) \wedge (v, x, y) \in \mathcal{B}_s(T)\} \cup \{(v, y) \mid vy \in E(T) \wedge \exists x \in V(T) : (x, v, y) \in \mathcal{B}_s(T)\}.$$

Let $\mathcal{M} = (M_v)_{v \in V(T)}$. Note that $U = \bigcup_{v \in V(T)} M_v$ consists of all ordered pairs (x, y) such that $xy \in E(T)$ and $d_G(x) \geq 2$. Hence $|U| = 2|E(T)| - l = 2n - l - 2$.

In order to show that $\mathcal{B}_s(T) \subseteq \mathcal{B}_s(\mathcal{M})$, let $(u, v, w) \in \mathcal{B}_s(T)$ and let $(x, y) \in M_u \cap M_w$. By construction, if we root T in y, then each of the two vertices u and w is equal to either x or a descendant of x. Since v lies on the path in T between u and w, this implies that v is equal to either x or a descendant of x. By construction, $(x, y) \in M_v$. Hence $M_u \cap M_w \subseteq M_v$, and thus $(u, v, w) \in \mathcal{B}_s(\mathcal{M})$.

In order to show that $\mathcal{B}_s(\mathcal{M}) \subseteq \mathcal{B}_s(T)$, let $(u, v, w) \in \mathcal{B}_s(\mathcal{M})$. For contradiction, we assume that $(u, v, w) \notin \mathcal{B}_s(T)$, which implies that either $N(u, v, w)$, or $(u, w, v) \in \mathcal{B}_s(T)$, or $(v, u, w) \in \mathcal{B}_s(T)$. If $N(u, v, w)$, then there is some $c \in V(T)$ that lies on the paths in T between any two of the vertices u, v, and w. If c' is the neighbour of c on the path in T between c and v, then, by construction, $(c, c') \in (M_u \cap M_w) \setminus M_v$, which implies a contradiction to $(u, v, w) \in \mathcal{B}_s(\mathcal{M})$. If $(u, w, v) \in \mathcal{B}_s(T)$ and w' is the neighbour of w on the path in T between w and v, then, by construction, $(w, w') \in (M_u \cap M_w) \setminus M_v$, which implies a contradiction to $(u, v, w) \in \mathcal{B}_s(\mathcal{M})$. Since the last case $(v, u, w) \in \mathcal{B}_s(T)$ leads to a similar contradiction, this completes the proof of (i).

(ii) The proof of (ii) is a simple modification of the proof of (i). Let T be a tree. For every $v \in V(T)$, let M_v consist of all ordered pairs (x, y) such that $xy \in E(T)$, and, if we root T in y, then the vertex v is equal to either x or a descendant of x. Similar arguments as in the proof of (i) show that $|\bigcup_{v \in V(T)} M_v| = 2n - 2$ and $\mathcal{B}(T) = \mathcal{B}(\mathcal{M})$. We leave the details to the reader. □

4. Recognizing partial tree representability

In [3] Burigana considers the following problem.

Partial tree representation

Instance: A finite set V and a strict intersection betweenness $\mathcal{B}_s \subseteq V_s^3$.

Task: Decide whether there is a tree T with vertex set V such that $\mathcal{B}_s(T) \subseteq \mathcal{B}_s$, and construct such a tree if possible.

He shows that the so-called GYO algorithm of Graham [11] and Yu and Ozsoyoglu [22] can be used to solve Partial tree representation and reiterates essentially its entire proof of correctness. The main drawback of Burigana's approach is that he assumes the strict intersection betweenness $\mathcal{B}_s$ to be given by a representing set system $\mathcal{M} = (M_v)_{v\in V}$ with $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$. Since he applies the GYO algorithm to the sets in $\mathcal{M}$, the overall running time can at best be polynomial in the encoding length of $\mathcal{M}$, which might be totally unrelated to the encoding length $O(|V|^3)$ of $\mathcal{B}_s$. In fact, no algorithm for Partial tree representation, which assumes the input $\mathcal{B}_s$ to be given by an arbitrary representing set system, can have a running time polynomially bounded in terms of $|V|$.

In this section, we describe an algorithm for Partial tree representation, which uses the set of triples $\mathcal{B}_s$ of encoding length $O(|V|^3)$ as input and has a running time that is polynomially bounded in terms of $|V|$.

Let V and $\mathcal{B}_s \subseteq V_s^3$ be an instance of Partial tree representation. By Theorem 2, we can construct a set system $\mathcal{M} = (M_v)_{v\in V}$ with $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$ and

$$\left|\bigcup_{v\in V} M_v\right| = O(|V|^2) \tag{1}$$

in polynomial time. Since we consider the strict intersection betweenness induced by $\mathcal{M}$, we may assume that, for every $v \in V$,

$$M_v \setminus \bigcup_{u\in V\setminus\{v\}} M_u \neq \emptyset \tag{2}$$

(see the proof of Lemma 3). Let $X = \bigcup_{v\in V} M_v$ and, for $x \in X$, let $M_x^* = \{v \in V \mid x \in M_v\}$. We assume now that there exists some tree T with vertex set V such that $\mathcal{B}_s(T) \subseteq \mathcal{B}_s$. Consider a path in T between two vertices u and w that contains v as an internal vertex, and consider $x \in M_u \cap M_w$. Since $(u, v, w) \in \mathcal{B}_s(T) \subseteq \mathcal{B}_s$, we obtain that $M_u \cap M_w \subseteq M_v$, and hence $x \in M_v$. Therefore, for every $x \in X$, the set M_x^* induces a subtree T_x of T. It follows from well-known results [2,8,21] that the intersection graph G [16] of the set system $(M_x^*)_{x\in X}$ is chordal. Recall that G has vertex set $V(G) = X$ and that two distinct vertices x and y of G are adjacent if and only if $M_x^* \cap M_y^* \neq \emptyset$. Let C be a maximal clique of G. By definition, for every two elements x and y in C, the sets M_x^* and M_y^* intersect, i.e. the two subtrees T_x and T_y share a vertex. Since subtrees of a tree have the Helly property [9], there is some vertex $v_C \in V$ such that $C \subseteq M_{v_C}$. Since M_{v_C} clearly induces a clique in G, the maximality of C implies that $C = M_{v_C}$. This implies that $\{M_v \mid v \in V\}$ is the collection of all maximal cliques of G. Furthermore, by (2), no two of these maximal cliques are equal, and thus G has exactly $|V|$ many distinct maximal cliques that are in bijective correspondence with the elements of V. Let T_G denote a clique tree [9,16] of G, where we use v to denote the maximal clique M_v for $v \in V$. By the definition of a clique tree, for every $x \in X$, the set M_x^* induces a subtree of T_G. This implies that, if $(u, v, w) \in \mathcal{B}_s(T_G)$ and $x \in M_u \cap M_w$, then $x \in M_v$, i.e. $M_u \cap M_w \subseteq M_v$, and hence $(u, v, w) \in \mathcal{B}_s$. Therefore, T_G satisfies $\mathcal{B}_s(T_G) \subseteq \mathcal{B}_s$ and solves the Partial tree representation.

The above exposition yields the following algorithm (see Algorithm 2).

Input: A finite set V and a strict intersection betweenness $\mathcal{B}_s \subseteq V_s^3$.
Output: A tree T with vertex set V such that $\mathcal{B}_s(T) \subseteq \mathcal{B}_s$ or the answer "No" if no such tree exists.

Construct a set system $\mathcal{M} = (M_v)_{v\in V}$ with $\mathcal{B}_s = \mathcal{B}_s(\mathcal{M})$, (1), and (2);
$X \leftarrow \bigcup_{v\in V} M_v$;
for $x \in X$ **do** $M_x^* \leftarrow \{v \in V \mid x \in M_v\}$;
Construct the intersection graph G of $(M_x^*)_{x\in X}$;
if *G is not chordal* **then**
  return *"No"*;
end
if *$\{M_v \mid v \in V\}$ is not the set of all exactly $|V|$ many maximal cliques of G* **then**
  return *"No"*;
end
Construct a clique tree T_G of G using v to denote the maximal clique M_v for $v \in V$;
if $\mathcal{B}_s(T_G) \not\subseteq \mathcal{B}_s$ **then**
  return *"No"*;
else
  return T_G;
end

Algorithm 2: Algorithm for Partial tree representation.

Theorem 6. *Algorithm* 2 *correctly solves* Partial tree representation *in polynomial time.*

Proof. The correctness follows immediately from the preceding exposition. The task in line 1 can be done in polynomial time using Algorithm 1. Furthermore, the tasks in lines 4, 5, 8, and 11 can be performed in polynomial time using standard methods [9,16]. Therefore, the overall running time is polynomially bounded in terms of $|V|$. $\square$

Acknowledgement

The authors acknowledge partial support by the CAPES/DAAD Probral project "Cycles, Convexity, and Searching in Graphs".

References

[1] N. Alon, M. Bădoiu, E.D. Demaine, M. Farach-Colton, M. Hajiaghayi, A. Sidiropoulos, Ordinal embeddings of minimum relaxation: general properties, trees, and ultrametrics, ACM Trans. Algorithms 4 (2008) Article No. 46.
[2] P. Buneman, A characterisation of rigid circuit graphs, Discrete Math. 9 (1974) 205–212.
[3] L. Burigana, Tree representations of betweenness relations defined by intersection and inclusion, Math. Soc. Sci. 185 (2009) 5–36.
[4] M. Charikar, V. Guruswami, R. Manokaran, Every permutation csp of arity 3 is approximation resistant, in: Proceedings of the 2009 24th Annual IEEE Conference on Computational Complexity, CCC'09, pp. 62–73.
[5] B. Chor, M. Sudan, A geometric approach to betweenness, SIAM J. Discrete Math. 11 (1998) 511–523.
[6] D. Cox, M. Burmeister, E. Price, S. Kim, R. Myers, Radiation hybrid mapping: a somatic cell genetic method for constructing high-resolution maps of mammalian chromosomes, Science 250 (4978) (1990) 245–250.
[7] D. Defays, Tree representations of ternary relations, J. Math. Psych. 19 (1979) 208–218.
[8] F. Gavril, The intersection graphs of subtrees in trees are exactly the chordal graphs, J. Combin Theory Ser. B 16 (1974) 47–56.
[9] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, in: Annals of Discrete Mathematics, vol. 57, Elsevier, Amsterdam, 2004.
[10] S. Goss, H. Harris, New methods for mapping genes in human chromosomes, Nature 255 (1975) 680–684.
[11] M.H. Graham, On the universal relation, Computer Systems Research Group Report, University of Toronto, 1979.
[12] G. Gutin, E.J. Kim, M. Mnich, A. Yeo, Ordinal Embedding Relaxations Parameterized Above Tight Lower Bound, preprint, arXiv:0907.5427v2.
[13] E.V. Huntington, J.R. Kline, Sets of independent postulates for betweenness, Trans. Amer. Math. Soc. 18 (1917) 301–325.
[14] A. Krokhin, P. Jeavons, P. Jonsson, Constraint satisfaction problems on intervals and lengths, SIAM J. Discrete Math. 17 (2004) 453–477.
[15] Y. Makarychev, Simple linear time approximation algorithm for betweenness, Microsoft Research Technical Report MSR-TR-2009-74, June 2009.
[16] T.A. McKee, F.R. McMorris, Topics in intersection graph theory, in: SIAM Monographs on Discrete Mathematics and Applications, vol. 2, Society for Industrial and Applied Mathematics, Philadelphia, 1999.
[17] K. Menger, Untersuchungen über allgemeine Metrik, Math. Ann. 100 (1928) 75–163.
[18] J. Opatrny, Total ordering problem, SIAM J. Comput. 8 (1979) 111–114.
[19] D. Rautenbach, P.M. Schäfer, Finite Sholander Trees, Trees, and their Betweennesses, manuscript 2009.
[20] M. Sholander, Trees, lattices, order, and betweenness, Proc. Amer. Math. Soc. 3 (1952) 369–381.
[21] J.R. Walter, Representations of chordal graphs as subtrees of a tree, J. Graph Theory 2 (1978) 265–267.
[22] C.T. Yu, M.Z. Ozsoyoglu, An algorithm for tree-query membership of a distributed query, in: Proceedings 1979 IEEE COMPSAC, Institute of Electrical and Electronic Engineers, New York; 1979, pp. 306–312.

Apêndice C

On Minimal and Minimum Hull Sets

Este apêndice contém o artigo "*On Minimal and Minimum Hull Sets*" aceito para apresentação no *VII Latin-American Algorithms, Graphs and Optimization Symposium*, a ser realizado no México em abril de 2013.

On Minimal and Minimum Hull Sets

Rommel Barbosa [a,1], Dieter Rautenbach [b,2],
Vinícius Fernandes dos Santos [c,3], Jayme L. Szwarcfiter [d,4]

[a] *Instituto de Informática, Universidade Federal de Goiás, Brazil*

[b] *Institut für Optimierung und Operations Research, Universität Ulm, Germany*

[c] *PESC, COPPE, Universidade Federal do Rio de Janeiro, Brazil*

[d] *Inst Mat., COPPE and NCE, Universidade Federal do Rio de Janeiro, Brazil*

Abstract

A graph convexity $(G, \mathcal{C})$ is a graph G together with a collection $\mathcal{C}$ of subsets of $V(G)$, called *convex sets*, such that $\emptyset, V(G) \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. For a set $U \subseteq V(G)$, the *hull of* U, denoted $H(U)$, is the smallest convex set containing U. If $H(U) = V(G)$, then U is a *hull set* of G. Motivated by the theory of well covered graphs, which investigates the relation between maximal and maximum independent sets of a graph, we study the relation between minimal and minimum hull sets. We concentrate on the P_3 convexity, where convex sets are closed under adding common neighbors of their elements.

Keywords: Graph convexity, P_3 convexity, hull sets, Carathéodory number

---

1 Introduction

In the present paper we study minimal hull sets and the relationship between minimal and minimum hull sets in finite graph convexities. Our motivation are similar studies for other graph notions, as for example, independent sets,

[1] E-mail: `rommel@inf.ufg.br`
[2] E-mail: `dieter.rautenbach@uni-ulm.de`
[3] E-mail: `vinicius.santos@gmail.com`. Supported by FAPERJ, Brazil.
[4] E-mail: `jayme@nce.ufrj.br`. Partially Supported by CNPq, CAPES and FAPERJ, Brazil. Presently, visiting the Instituto Nacional de Metrologia, Qualidade e Tecnologia.

which lead to the theory of well covered graphs introduced in [6]. We focus on the P_3-convexity, although we also present some results for general convexities. The P_3-convexity was first considered for tournaments [4,5] and was proposed as a model of the spread of information or of a disease within a network. Some classical convexity parameters have been considered for P_3-convexities, like the hull number and the Carathéodory number [1,2,3].

We consider only finite, simple and undirected graphs. For a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. For a vertex $u \in V(G)$, the neighborhood $N_G(u)$ is the set $\{v \in V(G) \mid uv \in E(G)\}$ and the degree $d_G(u)$ of u in G equals $|N_G(u)|$. The complement $\overline{G}$ of G is a graph with the same vertex set as G and $E(\overline{G}) = \{uv \mid u, v \in V(G), u \neq v, uv \notin E(G)\}$. For a graph G and a set of vertices $S \subseteq V(G)$, the induced subgraph $G[S]$ is the subgraph H with $V(H) = S$ and $E(H) = \{uv \in E(G) \mid u, v \in S\}$. A cubic graph is a graph where $d_G(u) = 3$ for every u in G. A chordal graph is a graph where every cycle with at least 4 vertices contains a chord. A cograph is a graph that does not contain a path on 4 vertices as an induced subgraph.

A graph convexity $(G, \mathcal{C})$ is a graph G together with a collection $\mathcal{C}$ of subsets of $V(G)$, called *convex sets*, such that $\emptyset, V(G) \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. For a set $U \subseteq V(G)$, the *hull of* U, denoted $H(U)$ is the smallest convex set containing U. If $H(U) = V(G)$, then U is a *hull set* of G. The minimum cardinality of a hull set of G is the *hull number* of G, denoted by $h(G)$. The set U is a *Carathéodory set* of G if $H(U) \setminus \bigcup_{u \in U} H(U \setminus \{u\}) \neq \emptyset$. The *Carathéodory number* $c(G)$ is the maximum size of a Carathéodory set.

In the P_3-convexity a set $U \subseteq V(G)$ is convex precisely if no vertex in $V(G) \setminus U$ has two or more neighbors in U. The P_3-hull of U can be constructed efficiently starting with the set U and iteratively adding further vertices of G that have at least two neighbors in the current set.

The paper is organized as follows. Section 2 presents optimal bounds for the size of minimal hull sets of cubic graphs. Section 3 shows that, for certain graph classes, every minimal hull set is also minimum. In Section 4 we discuss minimal hull sets for general graph convexities and show a bound relating their maximum cardinality to the hull number and the Carathéodory number.

2 Cubic graphs

For cubic graphs, P_3-hull sets coincide with the so-called feedback vertex sets, which are sets of vertices whose removal destroy all cycles. This observation implies that the P_3-hull number of a cubic graph can be computed in polynomial time [7]. In this section we establish optimal bounds for the size of minimal P_3-hull sets of cubic graphs.

Theorem 2.1 *If G is a cubic graph of order n and U is a minimal P_3-hull set of G, then $\frac{n}{4} < |U| \leq \frac{n}{2}$.*

Proof. Let $X_0 = U$. For $i \geq 1$, let X_i denote the set of vertices in $V(G) \setminus (X_0 \cup \ldots \cup X_{i-1})$ that have at least two neighbors in $X_0 \cup \ldots \cup X_{i-1}$. Since U is a P_3-hull set, $V(G) = \bigcup\limits_{i \geq 0} X_i$. For integers i and j with $0 \leq i < j$, let $x_i = |X_i|$ and let $m_{i,j}$ denote the number of edges between X_i and X_j. For every $j \geq 1$, the definition of X_i implies $2x_j \leq \sum\limits_{i=0}^{j-1} m_{i,j}$. Furthermore, since G is cubic, the definition of X_i implies $x_i \geq \sum\limits_{j \geq i+1} m_{i,j}$, where this inequality is strict if i is the maximum index with $x_i > 0$.

Combining these inequalities, we obtain

$$\begin{aligned}\sum_{i\geq 1} x_i &= \sum_{j \geq 1} 2x_j - \sum_{i \geq 1} x_i < \sum_{j \geq 1} \sum_{i=0}^{j-1} m_{i,j} - \sum_{i \geq 1} \sum_{j \geq i+1} m_{i,j} \\ &= \sum_{i \geq 0} \sum_{j \geq i+1} m_{i,j} - \sum_{i \geq 1} \sum_{j \geq i+1} m_{i,j} = \sum_{j \geq 1} m_{0,j} \leq 3x_0,\end{aligned}$$

which implies $n = x_0 + \sum\limits_{i \geq 1} x_i \leq 4x_0 = 4|U|$. This completes the proof of the first inequality and we proceed to the proof of the second.

Let U_2 and U_3 denote the set of vertices in U that have exactly 2 and 3 neighbors in $V(G) \setminus U$, respectively. Similarly, for $i \in \{0, 1, 2, 3\}$, let R_i denote the set of vertices in $V(G) \setminus U$, that have exactly i neighbors in U. Let $u_i = |U_i|$ and $r_i = |R_i|$.

The hypothesis that U is a minimal P_3-hull set of G implies that no vertex u in U has two neighbors in U, no vertex u in U_2 is adjacent to a vertex in R_3, and no vertex u in U_3 has at least two neighbors in R_3. Since G is cubic, these observations imply that the order of U is at most the optimum value of the following integer linear program.

$$(P) \begin{cases} \max\ u_2 + u_3 & \\ \quad u_2 + u_3 + r_0 + r_1 + r_2 + r_3 & = n \\ \quad 2u_2 + 3u_3 & = r_1 + 2r_2 + 3r_3 \\ \quad u_3 & \geq 3r_3 \\ \quad u_2, u_3, r_0, r_1, r_2, r_3 & \in \mathbb{N}_0. \end{cases}$$

Let $(u_2, u_3, r_0, r_1, r_2, r_3)$ be an optimum solution of (P) such that u_3 is mini-

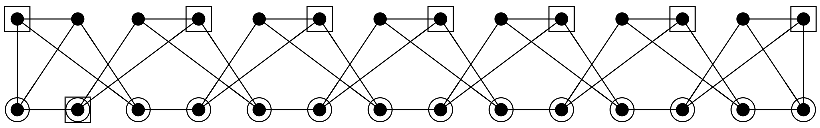

Fig. 1. The two sets indicated by the squares and the cycles, respectively, are minimal P_3-hull sets of the given cubic graph.

mum. If $u_3 > 0$ and i is maximum such that $r_i > 0$, then decreasing u_3 and r_i by 1 and increasing u_2 and r_{i-1} by 1, results in another optimum solution of (P). By the choice of the solution, this implies that $u_3 = 0$ and, by (P), $r_3 = 0$. Adding the first two conditions of (P) and using $r_2 \leq n - u_2$, we obtain $3u_2 = n - r_0 - r_1 + r_2 \leq n + r_2 \leq 2n - u_2$. This implies $|U| = u_2 \leq \frac{n}{2}$ and completes the proof of the second inequality. □

Corollary 2.2 *If G is a cubic graph of order n and U_1 and U_2 are minimal P_3-hull sets of G, then $\frac{|U_1|}{|U_2|} < 2$.*

Extending the graph in Figure 1 in the obvious way, it is easy to show that the factor 2 in Corollary 2.2 is best possible, that is, for every $\epsilon > 0$, there is a cubic graph G and minimal P_3-hull sets U_1 and U_2 of G with $\frac{|U_1|}{|U_2|} > 2 - \epsilon$. Moreover, Figure 1 also gives an example of a graph where the upper bound of Theorem 2.1 is tight. We now show that the lower bound is also tight.

Theorem 2.3 *For every $\epsilon > 0$, there is a cubic graph G of order n with a minimum P_3-hull set U of size $\frac{|U|}{n} < \frac{1}{4} + \epsilon$.*

3 Chordal graphs and cographs

In this section we show that, for some graph classes, every minimal P_3-hull set is also minimum. It is known [2] that if G is a chordal biconnected graph, then every two vertices u and v of G with a common neighbor form a P_3-hull set and hence $h(G) = 2$. This observation easily implies the following.

Theorem 3.1 *If G is a biconnected chordal graph with more than 2 vertices, then every minimal P_3-hull set has size 2.*

Note that this implies that for biconnected chordal graphs every minimal hull set is also a minimum. The same holds for cographs.

Theorem 3.2 *If G is a connected cograph and U is a minimal P_3-hull set, then U is minimum.*

Proof. If G is a star or $|U| \leq 2$, then the statement clearly follows. Hence assume that G is not a star and U is a minimal P_3-hull set with $|U| \geq 3$ that is not minimum. Let $C_1, \ldots, C_k$ be the connected components of $\overline{G}$. If $k \geq 3$, then any two vertices of G form a P_3-hull set, contradicting the minimality of U. Hence $\overline{G}$ has exactly two connected components C_1 and C_2. Without loss of generality, we assume $|C_1| \leq |C_2|$. If $|C_1| \geq 2$, then any subset of size 2 of $C_1 \cap U$ or $C_2 \cap U$ is a P_3-hull set. Since $|U| \geq 3$, such a set exists, contradicting the minimality of U. Finally, it remains the case when $|C_1| = 1$. Let $v \in V(C_1)$ and let $D_1, \ldots, D_l$ be the connected components of $G[C_2]$. If $l = 1$, then any two vertices of G will be a P_3-hull set, a contradiction. Hence any minimal P_3-hull set of G will have $l + 1$ vertices, since $|D_i \cap U|$ must be exactly 1 for every i, otherwise either there would be a component D_j with $|D_j \cap U| = 0$ implying each vertex of D_j would have exactly one neighbor in $H(U)$, namely v, or there would be a component D_j with $|D_j \cap U| > 1$, implying U is not minimal, since the removal of one vertex of $D_j \cap U$ from U would produce another P_3-hull set. This completes the proof. $\square$

4 General bounds

We now present a bound for the maximum size of a minimal hull set on any finite graph convexity.

Theorem 4.1 *If U is a minimal hull set, then $|U| \leq h(G)c(G)$.*

Proof. Let U be a minimal hull set and let $S = \{v_1, \ldots, v_{h(G)}\}$ be a minimum hull set. For $v_i \in S$, let U_i be a set with minimum cardinality satisfying $v_i \in H(U_i)$ and $U_i \subseteq U$. By the definition of the Carathéodory number, $|U_i| \leq c(G)$ for every i. Now let $U' = U_1 \cup \ldots \cup U_{h(G)} \subseteq U$. Note that $S \subseteq H(U')$ and hence $V(G) = H(S) = H(U')$. Since U is a minimal hull set, we have $U' = U$ and hence $|U| = |U'| \leq h(G)c(G)$, which completes the proof. $\square$

Here is an example showing that for P_3-convexity this bound is best possible up to the factor of 2.

For $i \in \{1, 2\}$, let T_i be a binary tree of height h with root r_i and a set $L_i = \{l_{i,1}, \ldots, l_{i,2^h}\}$ of 2^h leaves. Let K be a clique on $2^{h+2} + 2$ vertices such that $V(K) = \{x, y\} \cup \bigcup\limits_{i \in \{1,2\}} \bigcup\limits_{k=0}^{2^h} \{x_{i,k}, y_{i,k}\}$. Let G arise from the disjoint union of T_1, T_2, and K by adding the four edges xr_1, xr_2, yr_1, and yr_2, and for every $i \in \{1, 2\}$ and $k \in \{1, \ldots, 2^h\}$, the two edges $x_{i,k}l_{i,k}$ and $y_{i,k}l_{i,k}$. Note that every vertex in K except for x and y has only one neighbor outside of K. Therefore, the set $U = L_1 \cup L_2$ is a minimal P_3-hull set of G and the set $\{r_1, r_2\}$ is a minimum P_3-hull set of G. Furthermore, x is in the P_3-convex hull of U, while x is not in the P_3-convex hull of any proper subset of U. Hence the P_3-Carathéodory number of G is at least 2^{h+1}. In fact, a small case analysis shows that the P_3-Carathéodory number of G is exactly 2^{h+1} and we obtain $|U| = 2^{h+1} = \frac{1}{2}h(G)c(G)$.

We leave the following conjecture.

Conjecture 4.2 *If U is a minimal P_3-hull set of some graph G, then $|U| \leq \frac{1}{2}h(G)c(G)$.*

References

[1] Barbosa, R.M., E.M.M. Coelho, M.C. Dourado, D. Rautenbach, and J.L. Szwarcfiter, *On the Carathéodory Number for the Convexity of Paths of Order Three*, SIAM Journal on Discrete Mathematics **26** (2012), 929-939.

[2] Centeno, C.C., M.C. Dourado, L.D. Penso, D. Rautenbach, and J.L. Szwarcfiter, *Irreversible conversion of graphs*, Theor. Comput. Sci. **412** (2011), 3693-3700.

[3] Dreyer, P.A., and F.S. Roberts, *Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion*, Discrete Appl. Math. **157** (2009), 1615-1627.

[4] Erdős, P., E. Fried, A. Hajnal, and E.C. Milner, *Some remarks on simple tournaments*, Algebra Univers. **2** (1972), 238-245.

[5] Moon, J.W., *Embedding tournaments in simple tournaments*, Discrete Math. **2** (1972), 389-395.

[6] Plummer, M. D., *Some covering concepts in graphs*, Journal of Combinatorial Theory **8** (1970), 507-512.

[7] Ueno, S., Y. Kajitani, and S. Gotoh, *On the nonseparating independent set problem and feedback set problem for graphs with no vertex degree exceeding three*, Discrete Math. **72** (1988), 355-360.

Apêndice D

Irreversible Conversion Processes with Deadlines

Este apêndice contém o artigo "*Irreversible Conversion Processes with Deadlines*", submetido em dezembro de 2010 para publicação no periódico *Journal of Discrete Algorithms*.

Irreversible Conversion Processes with Deadlines

Dieter Rautenbach[1],
Vinícius Fernandes dos Santos[2],
Philipp M. Schäfer[1]

[1] Institut für Optimierung und Operations Research, Universität Ulm, D-89069 Ulm, Germany, dieter.rautenbach@uni-ulm.de, philipp.schaefer@gmail.com

[2] PESC, COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ, Brazil, vinicius.santos@gmail.com

Abstract

Given a graph G, a deadline $t_d(u)$ and a time-dependent threshold $f(u,t)$ for every vertex u of G, we study sequences $\mathcal{C} = (c_0, c_1, \ldots)$ of 0/1-labelings c_i of the vertices of G such that for every $t \in \mathbb{N}$, we have $c_t(u) = 1$ if and only if either $c_{t-1}(u) = 1$ or at least $f(u,t-1)$ neighbours v of u satisfy $c_{t-1}(v) = 1$. The sequence $\mathcal{C}$ models the spreading of a property/commodity within a network and it is said to converge to 1 on time, if $c_{t_d(u)}(u) = 1$ for every vertex u of G, i.e. if every vertex u has the spreading property/received the spreading good by time $t_d(u)$.

We study the smallest number $irr(G, t_d, f)$ of vertices u with initial label $c_0(u)$ equal to 1 that result in a sequence $\mathcal{C}$ converging to 1 on time. If G is a forest or a clique, we present efficient algorithms computing $irr(G, t_d, f)$. Furthermore, we prove lower and upper bounds relying on counting and probabilistic arguments. For special choices of t_d and f, the parameter $irr(G, t_d, f)$ coincides with well-known graph parameters related to domination and independence in graphs.

Keywords: iterative graph process; dynamic monopoly; domination; independence
MSC 2010 classification: 05C69, 05C85

1 Introduction

The spreading of something like a property or a commodity within a network is a recurrent feature in many contexts such as social influence [8, 12, 16, 20], gene expression networks [15], immune systems [2], cellular automata [3], percolation [4], marketing strategies [7, 16], and distributed computing [5, 10, 17–19].

Motivated by these spreading effects we recently studied irreversible conversion processes on graphs [6]. Such a process is a sequence

$$\mathcal{C} = (c_0, c_1, \ldots) = (c_t)_{t\in\mathbb{N}_0}$$

of 0/1-labelings

$$c_t : V(G) \to \{0, 1\}$$

of the vertices of G and its evolution is governed by a threshold function

$$f : V(G) \to \mathbb{N}_0,$$

which captures the individual sensitivity of each vertex.

For every $t \in \mathbb{N}$, the t-th labeling c_t is such that for every vertex u of G, we have $c_t(u) = 1$ if and only if either $c_{t-1}(u) = 1$ or the number of neighbours v of u with $c_{t-1}(v) = 1$ is at least the threshold value $f(u)$ of u.

A process $(c_t)_{t\in\mathbb{N}_0}$ is said to converge to 1, if there is some $t_0 \in \mathbb{N}_0$ such that $c_{t_0}(u) = 1$ for every $u \in V(G)$.

In [6] we mainly studied algorithmic issues related to the minimum possible cardinality of the set of vertices u with initial label $c_0(u)$ equal to 1 for processes $\mathcal{C} = (c_t)_{t\in\mathbb{N}_0}$ converging to 1. This quantity relates to the minimum size of so-called dynamic monopolies [5, 10, 19]. Such monopolies were studied especially for so-called majority processes in which the threshold values are such that some majority of the neighbours of a vertex is needed to convert it.

Clearly, there is a tradeoff between the number of vertices with initial label 1 and the number of interations/time needed in order to achieve convergence. If an impacient aggressor wants to spread an infection quickly within a network or if the process actually models the spreading of a desirable property/commodity rather than of an infection or of bankruptcy, then it might actually be required that the property/commodity reaches the individual vertices until certain deadlines. In the case of majority processes and very special network topologies, the above tradeoff has been studied in [11]. The purpose of the present paper is to study irreversible conversion processes subject to such deadline restrictions in a more general setting.

Our formal setup is a follows.

We consider finite, simple, and undirected graphs. For a graph G, the vertex set is denoted $V(G)$ and the edge set is denoted $E(G)$. The order of G is denoted $n(G)$. For a vertex u in a graph G, the neighbourhood of u in G is denoted $N_G(u)$ and the degree of u in G is denoted $d_G(u)$.

We are given

- a graph G,
- a *deadline function* $t_d : \mathcal{D}(t_d) \to \mathbb{N}_0$ with $V(G) \subseteq \mathcal{D}(t_d)$, and
- a *time-dependent threshold function* $f : \mathcal{D}(f) \to \mathbb{N}_0$ with

$$\{(u,t) : u \in V(G), t \in \mathbb{N}_0, 0 \le t \le t_d - 1\}\} \subseteq \mathcal{D}(f).$$

An *irreversible process on* (G, t_d, f) is a sequence $\mathcal{C} = (c_t)_{t\in\mathbb{N}_0}$ such that

- $c_t : V(G) \to \{0,1\}$ for every $i \in \mathbb{N}_0$ and
- $c_t(u) = 1$ if and only if
  - either $c_{t-1}(u) = 1$
  - or $|\{v \in N_G(u) : c_{t-1}(v) = 1\}| \ge f(u, t-1)$

  for every $t \in \mathbb{N}$ and every $u \in V(G)$.

Next to the introduction of deadlines, our model differs from [6] because we allow the threshold function to be time-dependent. Apart from allowing a more realistic modelling of sensitivities, the technical reason for this is that such functions are more suited to the reduction principles, which we will apply in order to solve the corresponding algorithmic problems. Note that time-dependent threshold functions allow to eliminate deadlines simply by setting $f(u,t)$ to $d_G(u)+1$ for $u \in V(G)$ and $t \geq t_d(u)$. For notational simplicity, we prefer to keep deadlines in our model.

A natural encoding length of the triple (G, t_d, f) is at least the encoding length of G plus the encoding length of the functions t_d and f, which is something like

$$\Omega\left(\sum_{u \in V(G)} t_d(u)\left(\log\left(\max\{f(u,t) : t \in \mathbb{N}_0, 0 \leq t < t_d(u)\}\right) + 1\right)\right).$$

Let $\mathcal{C} = (c_t)_{t \in \mathbb{N}_0}$ be an irreversible process on (G, t_d, f). We say that a vertex u of G *is converted by* $\mathcal{C}$ *at time* t if $c_t(u) = 1$ and $c_s(u) = 0$ for every $0 \leq s < t$. Furthermore, we say that u *is converted on time by* $\mathcal{C}$ if $c_{t(u)}(u) = 1$, i.e. the time at which u is converted by $\mathcal{C}$ is at most $t_d(u)$. The process $\mathcal{C}$ is said to *converge to* 1 *on time*, if all vertices of G are converted on time by $\mathcal{C}$. If $\mathcal{C}$ converges to 1 on time, then the set $c_0^{-1}(1)$ of vertices u with initial label $c_0(u)$ equal to 1 is called an *irreversible conversion set* of (G, t_d, f). The minimum cardinality of an irreversible conversion set of (G, t_d, f) is denoted by $irr(G, t_d, f)$. Clearly, $V(G)$ is always an irreversible conversion set of (G, t_d, f), which implies that $irr(G, t_d, f)$ is well-defined.

For several special choices of t_d and f, the value of $irr(G, t_d, f)$ coincides with well-known graph parameters. If $t_d(u) = 1$ and $f(u, 0) = 1$ for every vertex u of some graph G, then $irr(G, t_d, f)$ equals the domination number $\gamma(G)$ of G [13]

$$\gamma(G) = \min\{|D| \mid D \subseteq V(G), \forall u \in V(G) \setminus D : N_G(u) \cap D \neq \emptyset\}.$$

Similarly, if $t_d(u) = k$ and $f(u,t) = 1$ for every vertex u and every $t \in \mathbb{N}_0$ with $0 \leq t \leq k-1$, then $irr(G, t_d, f)$ equals the distance-k-domination number [14], and if $t_d(u) = 1$ and $f(u, 0) = k$ for every vertex u, then $irr(G, t_d, f)$ equals the k-domination number [9]. Finally, if $f(u,t) = d_G(u)$ for every vertex u and every relevant t, then $irr(G, t_d, f)$ equals $n(G) - \alpha(G)$ where

$$\alpha(G) = \max\{|I| \mid I \subseteq V(G), \forall u \in I : N_G(u) \cap I = \emptyset\}$$

is the independence number $\alpha(G)$ of G.

The last observations immediately imply that calculating $irr(G, t_d, f)$ is an extremely hard problem even for very restricted instances. Therefore, we focus on efficiently solvable special cases as well as on bounds. In Sections 2 and 3 we describe polynomial time algorithms for forests and cliques. In Section 4 we prove lower and upper bounds using counting and probabilistic arguments.

2 Irreversible Conversion of Forests

In this section we present an efficient algorithm computing $irr(G, t_d, f)$ for forests G. The main ingredients are suitable reduction steps at leaves of G.

Input: A triple (G, t_d, f) such that G is a forest, t_d is a deadline function, and f is a threshold function.
Output: An irreversible conversion set C of (G, t_d, f) of order $irr(G, t_d, f)$.

1 $C = \emptyset$;
2 **while** $V(G) \neq \emptyset$ **do**
3   **while** $\exists u \in V(G) : \forall t \in \mathbb{N}_0 : (0 \leq t < t_d(u)) \Rightarrow (f(u,t) > d_G(u))$ **do**
4   **for** $v \in N_G(u)$ *and* $0 \leq t < t_d(v)$ **do**
5             $f(v,t) = f(v,t) - 1$;
6   **end**
7         $C = C \cup \{u\}$;
8         $G = G - u$;
9   **end**
10     $G = G - \{u \in V(G) : d_G(u) = 0\}$;
11   **if** $V(G) \neq \emptyset$ **then**
12   Let $u \in V(G)$ be such that $d_G(u) = 1$;
13   Let $v \in V(G)$ be the unique neighbour of u;
14   **if** $\exists t \in \mathbb{N}_0 : 0 \leq t < t_d(u)$ *and* $f(u,t) = 0$ **then**
15             $t_0 = \min\{t \in \mathbb{N}_0 : 0 \leq t < t_d(u) \text{ and } f(u,t) = 0\}$;
16   **for** $t_0 + 1 \leq t < t_d(v)$ **do**
17                 $f(v,t) = f(v,t) - 1$;
18   **end**
19   **else**
20             $t_1 = \max\{t \in \mathbb{N}_0 : 0 \leq t < t_d(u) \text{ and } f(u,t) = 1\}$;
21             $t_d(v) = \min\{t_d(v), t_1\}$;
22   **end**
23         $G = G - u$;
24   **end**
25 **end**
26 **return** C;

Algorithm 1: Forest-On-Time-Conversion-Set

Theorem 1. Forest-On-Time-Conversion-Set *(cf. Algorithm 1) is correct and runs in polynomial time.*

Proof. Since in every execution of the **while**-loop in line 2, the order of G is reduced by at least 1 and all manipulations of the functions t_d and f can be implemented to run in polynomial time, the overall running time of Forest-On-Time-Conversion-Set is polynomial.

We proceed to the proof of correctness. Every irreversible conversion set of (G, t_d, f) necessarily contains all vertices u such that $f(u,t) > d_G(u)$ for every t with $0 \leq t < t_d(u)$. This implies the correctness of the reduction applied in the **while**-loop in line 3. Note that when we remove such vertices from G, we have to reduce the threshold values of their neighbours (cf. line 5). After the completion of the **while**-loop in line 3, for every remaining vertex u of G, there is some t with $0 \leq t < t_d(u)$ such that $f(u,t) \leq d_G(u)$.

If some remaining vertex u of G has degree 0, then this implies that $f(u,t) = 0$ for some $0 \leq t < t_d(u)$. Such vertices will always be converted on time and will not help to convert any other vertex. Hence they can just be removed from G (cf. line 10). After line 10, all remaining vertices have degree at least 1.

If G is not empty, then, since G is a forest, we can select a vertex u of degree 1 whose unique neighbour is v. If there is some t with $0 \leq t < t_d(u)$ and $f(u,t) = 0$, then u is guaranteed to be converted at the latest at time $t_0 + 1$ (cf. line 15). If C is a minimum irreversible conversion set of (G, t_d, f) containing u, then $(C \setminus \{u\}) \cup \{v\}$ is also a minimum irreversible conversion set of (G, t_d, f). Therefore, no minimum irreversible conversion set of (G, t_d, f) needs to contain u. We can remove u from G and reduce the threshold value of its unique neighbour v for all t with $t \geq t_0 + 1$ (cf. line 17). In line 19, the vertex u satisfies $f(u,t) \geq 1$ for every $0 \leq t < t_d(u)$ and t_1 defined in line 20 is the largest t with $f(u,t) = 1$. This implies that there is an irreversible conversion set of (G, t_d, f) of order $irr(G, t_d, f)$, which does not contain u. Furthermore, in the processes generated by such sets, the vertex v is converted at the latest at time $\min\{t_d(v), t_1\}$ (cf. line 20).

These observations imply the correctness of the reductions executed by Forest-On-Time-Conversion-Set, which implies the overall correctness. □

3 Irreversible Conversion of Cliques

In this section we present an efficient algorithm computing $irr(G, t_d, f)$ for cliques G. The idea for this algorithm is a suitable simulation.

Let (G, t_d, f) be such that G is a complete graph, t_d is a deadline function, and f is a threshold function.

The algorithm Clique-On-Time-Conversion-Set simulates an irreversible process on an extended instance: A set of k new vertices is added to the complete graph G to form a complete graph G' of order $|V(G)| + k$ and an irreversible process $\mathcal{C}' = (c'_t)_{t \in \mathbb{N}_0}$ on this extended instance (G', t_d, f) is considered where

- the set of vertices with initial label 1 consists exactly of the k new vertices and
- vertices get converted by $\mathcal{C}'$ only if they get converted on time.

Note that while formally the functions t_d and f would have to be extended to larger domains $\mathcal{D}(t_d)$ and $\mathcal{D}(f)$ because of the new vertices, their undefined values are irrelevant because the k new vertices have label 1 at time 0.

T is set to the maximum deadline $\max\{t_d(u) : u \in V(G)\}$ (cf. line 3).

For every t with $1 \leq t \leq T$, the set S_t consists of the vertices of G that are converted by $\mathcal{C}'$ at time t (cf. line 5). Note that a vertex u is added to S_t exactly if

- it was not converted by $\mathcal{C}'$ earlier and hence still belongs to $V(G)$,
- $t \leq t_d(u)$, and

- $f(u, t-1) \le c_{t-1}$,

i.e. as said above vertices get converted by $\mathcal{C}'$ only if they get converted on time.

The variable c_t counts the vertices of G' that are converted by $\mathcal{C}'$ until time t. Therefore, c_0 is exactly the number k of new vertices (cf. line 1) and $c_t = c_{t-1} + |S_t|$ (cf. line 6). The vertices in $S_1 \cup S_2 \cup \ldots \cup S_T$ are ordered in an order $u_{k+1}, u_{k+2}, \ldots, u_{c_T}$ of non-decreasing time of conversion (cf. line 7).

After the completion of the **while**-loop in line 4, the set $V(G)$ contains exactly those vertices of G that have not been converted by $\mathcal{C}'$. If $V(G)$ contains at most k vertices, it is completed with vertices from $S_1 \cup S_2 \cup \ldots \cup S_T$ of maximum possible time of conversion to a set C with exactly k vertices in line 12, which is then returned by the algorithm. Otherwise the algorithm returns NO. It remains to prove that the decisions made by CLIQUE-ON-TIME-CONVERSION-SET in lines 13 and 15 are correct.

Input: A triple (G, t_d, f) such that G is a complete graph, t_d is a deadline function, and f is a threshold function. Furthermore, an integer k with $0 \le k \le |V(G)|$.
Output: An irreversible conversion set C of (G, t_d, f) of size k if such a set exists, or NO, otherwise.

1 $c_0 = k$;
2 $t = 1$;
3 $T = \max\{t_d(u) : u \in V(G)\}$;
4 **while** $t \le T$ **do**
5     $S_t = \{u \in V(G) : t \le t_d(u) \text{ and } f(u, t-1) \le c_{t-1}\}$;
6     $c_t = c_{t-1} + |S_t|$;
7   Denote the elements of S_t by $u_{c_{t-1}+1}, u_{c_{t-1}+2}, \ldots, u_{c_t}$;
8     $V(G) = V(G) \setminus S_t$;
9     $t = t + 1$;
10 **end**
11 **if** $|V(G)| \le k$ **then**
12     $C = V(G) \cup \{u_{c_T - i} : 0 \le i \le (k - |V(G)|) - 1\}$;
13   **return** C;
14 **else**
15   **return** NO;
16 **end**

Algorithm 2: CLIQUE-ON-TIME-CONVERSION-SET

Lemma 2. *If* CLIQUE-ON-TIME-CONVERSION-SET *returns a set C in line 13, then C is an irreversible conversion set of (G, t_d, f) of order k.*

Proof. Let $\mathcal{C} = (c_t)_{t \in \mathbb{N}_0}$ be the irreversible process on (G, t_d, f) with $C = c_0^{-1}(1)$, i.e. C is the set of vertices with initial label 1. It follows easily by an inductive argument that if $v \in S_t \setminus C$ for some $1 \le t \le T$, then $c_t(v) = 1$ and $t \le t_d(v)$. This implies that every vertex of G is converted by $\mathcal{C}$ on time, i.e. C is in fact an irreversible conversion set of (G, t_d, f) of order k. $\square$

Lemma 3. *If* Clique-On-Time-Conversion-Set *returns* No *in line 15, then there is no irreversible conversion set of* (G, t_d, f) *of order* k.

Proof. For contradiction, we assume that $\tilde{C}$ is an irreversible conversion set of (G, t_d, f) of order k. Let $\tilde{\mathcal{C}} = (\tilde{c}_t)_{t \in \mathbb{N}_0}$ be the irreversible process on (G, t_d, f) with $\tilde{C} = \tilde{c}_0^{-1}(1)$. By definition, all vertices of G get converted on time by $\tilde{\mathcal{C}}$.

We prove by induction on t with $1 \leq t \leq T$, that if some vertex u in $V(G) \setminus \tilde{C}$ gets converted by $\tilde{\mathcal{C}}$ at time t, then $u \in S_s$ for some $1 \leq s \leq t$. For $t = 1$, this follows because $\tilde{C}$ has exactly $c_0 = k$ elements. Now let $t > 1$. Let the vertex u be converted by $\tilde{\mathcal{C}}$ at time t. Clearly, we may assume that $u \notin S_s$ for $1 \leq s \leq t-1$. The number of elements in $\tilde{C}$ plus the number x of vertices of G that is converted by $\tilde{\mathcal{C}}$ at some time s with $1 \leq s \leq t-1$ is at least $f(u, t-1)$ and $t \leq t_d(u)$. By induction, $S_1 \cup S_2 \cup \ldots \cup S_{t-1}$ contains at least x elements. This implies that c_{t-1} is at least $c_0 + x = k + x$. Hence $f(u, t-1) \leq c_{t-1}$. Since $t \leq t_d(u)$, we obtain $u \in S_t$, which completes the proof of the induction step.

Since no vertex gets converted too late by $\tilde{\mathcal{C}}$, the set $V(G)$ in line 11 cannot have more than k elements of $\tilde{C}$ and Clique-On-Time-Conversion-Set would not return No in line 15, which is a contradiction. □

Since the polynomial running time of Clique-On-Time-Conversion-Set is obvious, we obtain the following.

Theorem 4. Clique-On-Time-Conversion-Set *(cf. Algorithm 2) is correct and runs in polynomial time.*

Invoking Clique-On-Time-Conversion-Set for all values k with $0 \leq k \leq |V(G)|$ clearly allows to calculate $irr(G, t_d, f)$ for cliques in polynomial time.

4 Bounds on $irr(G, t_d, f)$

In this section we study bounds on $irr(G, t_d, f)$. In order to obtain bounds in closed form it is convenient to restrict our attention to instances (G, t_d, f) where the local structure of G is simple and the deadline function as well as the threshold function are constant.

Theorem 5. *Let $t_d, f, r \in \mathbb{N}$ be such that $r \geq 3$. Let G be a graph of maximum degree at most r and girth at least $2t_d + 1$. Let $t_d(u) = t_d$ and $f(u, t) = f$ for every $u \in V(G)$ and every $t \in \mathbb{N}_0$ with $0 \leq t < t_d$.*

a) If G is r-regular and $f \geq r + 1$, then $irr(G, t_d, f) = n(G)$.

b) If G is r-regular and $f = r$, then $irr(G, t_d, f) = n(G) - \alpha(G)$.

c) If $f = r - 1$, then

$$irr(G, t_d, f) \geq \frac{n(G)}{1 + \frac{r t_d}{r-1}}.$$

d) If $f \leq r - 2$, then

$$irr(G, t_d, f) \geq \frac{n(G)}{1 + \frac{r}{r-1-f}\left(\left(\frac{r-1}{f}\right)^{t_d} - 1\right)}.$$

Proof. Let $\mathcal{C} = (c_t)_{t\in\mathbb{N}_0}$ be an irreversible process on (G, t_d, f) that converges to 1 on time, i.e. $c_{t_d}(u) = 1$ for every $u \in V(G)$.

If $f \geq r+1$, then $c_0(u) = 1$ for every vertex $u \in V(G)$, which implies a).

If $f = r$, then $c_0^{-1}(0)$ must be an independent set of vertices, which implies b).

Now let $f \leq r-1$.

By the girth and degree conditions, for every vertex $u \in V(G)$, the graph G contains a rooted subtree T_u with root u and depth t_d such that

- u has at most r children in T_u,
- every vertex $v \in V(T_u) \setminus \{u\}$ of depth between 1 and $t_d - 1$ in T_u has at most $r-1$ children in T_u, and
- T_u contains all vertices at distance t_d from u.

Let $u \in V(G)$.

Let the *u-weight of a vertex v at time t* be $f^{-\mathrm{dist}_G(u,v)}$ if

- $\mathrm{dist}_G(u,v) \leq t_D$,
- $c_t(v) = 1$, and
- $c_t(w) = 0$ for every vertex w different from v on the unique shortest path in G between u and v.

Otherwise let the u-weight of a vertex v at time t be 0.

Claim 1. *For every $t \in \{t_d, t_d - 1, \ldots, 0\}$, the total u-weight of vertices v at time t with $\mathrm{dist}_G(u,v) \leq t_d - t$ is at least 1.*

Proof. We prove the claim by an inductive argument. For $t = t_d$, this statement is equivalent to $c_{t_d}(u) = 1$. Now let t be such that $0 \leq t < t_d$. Let v be a vertex of positive u-weight at time $t+1$ such that $\mathrm{dist}_G(u,v) \leq t_d - (t+1)$. We obtain that

- either $c_t(v) = 1$ and hence the u-weight of v at time t is the same as the u-weight of v at time $t+1$
- or $c_t(v) = 0$ and hence there are at least f distinct children $v_1, \ldots, v_f$ of v in T_u such that $c_t(v_1) = \ldots = c_t(v_f) = 1$ and the total u-weight of $v_1, \ldots, v_f$ at time t is equal to the u-weight of v at time $t+1$.

This implies that the total u-weight of vertices v at time t with $\mathrm{dist}_G(u,v) \leq t_d - t$ is at least the total u-weight of vertices v at time $t+1$ with $\mathrm{dist}_G(u,v) \leq t_d - (t+1)$, which, by induction, is at least 1. □

For $t = 0$, we obtain that the sum over all vertices u in $V(G)$ of the total u-weight of vertices v at time 0 is at least $n(G)$. Since each of the $|c_0^{-1}(1)|$ many vertices v with $c_0(v) = 1$ contributes at most

$$1 \cdot f^{-0} + r \cdot (r-1)^0 \cdot f^{-1} + r \cdot (r-1)^1 \cdot f^{-2} + \cdots + r \cdot (r-1)^{t_d - 1} \cdot f^{-t_d}$$

to the sum over all vertices u of the total u-weight of vertices at time 0, we obtain

$$\left(1 \cdot f^{-0} + r \cdot (r-1)^0 \cdot f^{-1} + r \cdot (r-1)^1 \cdot f^{-2} + \cdots + r \cdot (r-1)^{t_d - 1} \cdot f^{-t_d}\right) |c_0^{-1}(1)| \geq n(G),$$

which implies c) and d). □

In order to assess the strength of Theorem 5 it is instructive to consider small values for t_d and f. If $f = t_d = 1$ and $r \geq 3$ for instance, then Theorem 5 implies $irr(G, t_d, f) \geq \frac{n(G)}{1+r}$ for every graph G of maximum degree r. As noted in the introduction, for this choice of t_d and f, the value of $irr(G, t_d, f)$ equals the domination number $\gamma(G)$ of G. Since the lower bound $\gamma(G) \geq \frac{n(G)}{1+r}$ for graphs of maximum degree r is clearly best-possible, Theorem 5 give a good bound at least in this case.

Our next goal is an upper bound on $irr(G, t_d, f)$ obtained using the probabilistic method [1].

Let $t_d, f, r \in \mathbb{N}$.

Since most of our estimations will only be acceptable if t_d and f are relatively small compared to r, we may assume that $r \geq \max\{3, f + 2\}$.

Let $T(t_d, r)$ denote the complete $(r-1)$-ary rooted tree of depth t_d with root $u(t_d, r)$, i.e. every vertex of depth between 0 and $t_d - 1$ has exactly $(r-1)$ children in $T(t_d, r)$ and every vertex of depth t_d is a leaf.

For $p \in [0, 1]$, let $c_0 : V(T(t_d, r)) \to \{0, 1\}$ be a random 0/1-labeling of $T(t_d, r)$ that assumes value 1 independently at random with probability p for every vertex and let $\mathcal{C} = (c_t)_{t \in \mathbb{N}_0}$ be the irreversible process on $(T(t_d, r), t_d, f)$ starting with the random 0/1-labeling c_0 as initial labeling where the functions t_d and f are constant as in Theorem 5.

Let $\mathbf{P}(p, t_d, f, r)$ denote the probability that the root $u(t_d, r)$ of $T(t_d, r)$ is not converted by $\mathcal{C}$ on time, i.e.

$$\mathbf{P}(p, t_d, f, r) = \mathbf{P}\left[c_{t_d}(u(t_d, r)) = 0\right].$$

In view of the definition of $\mathcal{C}$ the following recursions are immediate:

$$\begin{aligned}
\mathbf{P}(p, 0, f, r) &= (1-p) \text{ and, for } t_d \geq 1,\\
\mathbf{P}(p, t_d, f, r) &= (1-p) \cdot \sum_{i=0}^{f-1} \binom{r-1}{i} \cdot (1 - \mathbf{P}(p, t_d - 1, f, r))^i \cdot \mathbf{P}(p, t_d - 1, f, r)^{r-1-i}.
\end{aligned}$$

The following lemma gives an upper bound on this probability.

Lemma 6. *If $t_d, f, r \in \mathbb{N}$ are such that $r \geq \max\{3, f + 2\}$ and $p \in [0, 1]$, then*

$$\mathbf{P}(p, t_d, f, r) \leq \left(r^{f-1}\right)^{\frac{\left((r-f)^{t_d}-1\right)}{r-f-1}} \cdot (1-p)^{\frac{(r-f)^{(t_d+1)}-1}{r-f-1}}.$$

Proof. We prove the lemma by induction on t_d. Let $q = 1 - p$.

For $t_d = 0$, the right hand side equals $q = 1 - p$.

Now let $t_d \geq 1$. By the recursion (R) and induction (I), we obtain

$$\begin{aligned}
\mathbf{P}(p, t_d, f, r) &\overset{(R)}{=} (1-p) \cdot \sum_{i=0}^{f-1} \binom{r-1}{i} \cdot (1 - \mathbf{P}(p, t_d - 1, f, r))^i \cdot \mathbf{P}(p, t_d - 1, f, r)^{r-1-i}\\
&\leq q \cdot \mathbf{P}(p, t_d - 1, f, r)^{r-1-(f-1)} \cdot \sum_{i=0}^{f-1} \binom{r-1}{i}\\
&\leq q \cdot r^{f-1} \cdot \mathbf{P}(p, t_d - 1, f, r)^{r-f}\\
&\overset{(I)}{\leq} q \cdot r^{f-1} \cdot \left(\left(r^{f-1}\right)^{\frac{\left((r-f)^{(t_d-1)}-1\right)}{r-f-1}} \cdot q^{\frac{(r-f)^{t_d}-1}{r-f-1}}\right)^{r-f}
\end{aligned}$$

$$
\begin{aligned}
&= r^{f-1}\cdot\left(r^{f-1}\right)^{\left((r-f)^0+(r-f)^1+\cdots+(r-f)^{(t_d-2)}\right)\cdot(r-f)} \\
&\quad \cdot q\cdot q^{\left((r-f)^0+(r-f)^1+\cdots+(r-f)^{(t_d-1)}\right)\cdot(r-f)} \\
&= \left(r^{f-1}\right)^{(r-f)^0+(r-f)^1+\cdots+(r-f)^{(t_d-1)}}\cdot q^{(r-f)^0+(r-f)^1+\cdots+(r-f)^{t_d}} \\
&= \left(r^{f-1}\right)^{\frac{\left((r-f)^{t_d}-1\right)}{r-f-1}}\cdot q^{\frac{(r-f)^{(t_d+1)}-1}{r-f-1}},
\end{aligned}
$$

which completes the proof. $\square$

Theorem 7. *Let $t_d, f, r \in \mathbb{N}$ be such that $r \geq \max\{3, f+2\}$ Let G be a graph of minimum degree at least r and girth at least $2t_d+1$. Let $t_d(u) = t_d$ and $f(u,t) = f$ for every $u \in V(G)$ and every $t \in \mathbb{N}_0$ with $0 \leq t < t_d$.*

$$
irr(G,t_d,f) \leq \frac{1+\ln\left(\frac{(r-f)^{(t_d+1)}-1}{r-f-1}\cdot\left(r^{f-1}\right)^{\frac{\left((r-f)^{t_d}-1\right)}{r-f-1}}\right)}{\frac{(r-f)^{(t_d+1)}-1}{r-f-1}}\cdot n(G).
$$

Proof. For $p \in [0,1]$, let $c_0 : V(G) \to \{0,1\}$ be a random 0/1-labeling of G that assumes value 1 independently at random with probability p for every vertex and let $\mathcal{C} = (c_t)_{t\in\mathbb{N}_0}$ be the irreversible process on (G, t_d, f) starting with the random 0/1-labeling c_0 as initial labeling.

By the girth and degree conditions, for every vertex $u \in V(G)$, the probability that u is not converted on time by $\mathcal{C}$ is at most $\mathbf{P}(p, t_d, f, r)$. Note that the union of

- the set $c_0^{-1}(1)$ of vertices with initial label 1 and
- the set of vertices of G that are not converted on time by $\mathcal{C}$

is an irreversible conversion set of (G, t_d, f) of expected cardinality at most $(p + \mathbf{P}(p, t_d, f, r)) \cdot n(G)$.

Choosing

$$
p = \frac{\ln\left(\frac{(r-f)^{(t_d+1)}-1}{r-f-1}\cdot\left(r^{f-1}\right)^{\frac{\left((r-f)^{t_d}-1\right)}{r-f-1}}\right)}{\frac{(r-f)^{(t_d+1)}-1}{r-f-1}},
$$

the first moment method using Lemma 6 and the estimate $1 - x \leq \exp(-x)$ imply

$$
\begin{aligned}
\frac{irr(G,t_d,f)}{n(G)} &\leq p+\mathbf{P}(p,t_d,f,r) \\
&\leq p+\left(r^{f-1}\right)^{\frac{\left((r-f)^{t_d}-1\right)}{r-f-1}}\cdot(1-p)^{\frac{(r-f)^{(t_d+1)}-1}{r-f-1}} \\
&\leq p+\left(r^{f-1}\right)^{\frac{\left((r-f)^{t_d}-1\right)}{r-f-1}}\cdot\exp\left((-p)\cdot\frac{(r-f)^{(t_d+1)}-1}{r-f-1}\right) \\
&= \frac{1+\ln\left(\frac{(r-f)^{(t_d+1)}-1}{r-f-1}\cdot\left(r^{f-1}\right)^{\frac{\left((r-f)^{t_d}-1\right)}{r-f-1}}\right)}{\frac{(r-f)^{(t_d+1)}-1}{r-f-1}},
\end{aligned}
$$

which completes the proof. (Note that if $p \geq 1$, then $irr(G, t_d, f) \leq p \cdot n(G)$ is trivial.) $\square$

Again it is instructive to consider small values. For $f = t_d = 1$ and $r \geq 3$, Theorem 7 implies

$$\gamma(G) = irr(G, t_d, f) \leq \frac{1 + \ln(r)}{r} n(G),$$

which is only slightly worse than the well-known bound [1]

$$\gamma(G) \leq \frac{1 + \ln(r + 1)}{r + 1} n(G).$$

The reason for this small discrepancy is that - for the sake of simplicity - Lemma 6 considers processes on rooted trees in which all non-leaves have $r - 1$ children, while we could actually assume the root to have r children.

Acknowledgement This paper was written while the second author visited the University of Ulm. We acknowledge support by the CAPES/DAAD Probral project "Cycles, Convexity, and Searching in Graphs".

References

[1] N. Alon and J.H. Spencer, The probabilistic method. With an appendix on the life and work of Paul Erdős. 3rd ed. John Wiley & Sons (2008).

[2] Z. Agur, Fixed points of majority rule cellular automata with application to plasticity and precision of the immune system, *Complex Syst.* **5** (1991), 351-357.

[3] J.-P. Allouche, M. Courbage, and G. Skordev, Notes on cellular automata, *Cubo Mat. Educ.* **3** (2001), 213-244.

[4] P. Balister, B. Bollobás, J.R. Johnson, and M. Walters, Random majority percolation, *Random Struct. Algorithms* **36** (2010), 315-340.

[5] J.-C. Bermond, J. Bond, D. Peleg, and S. Perennes, The power of small coalitions in graphs, *Discrete Appl. Math.* **127** (2003), 399-414.

[6] C.C. Centeno, M.C. Dourado, L.D. Penso, D. Rautenbach, and J.L. Szwarcfiter, Irreversible Conversion of Graphs, manuscript 2010.

[7] P. Domingos and M. Richardson, Mining the Network Value of Customers, in *Proceedings of the 7th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, 2001, 57-66.

[8] P.A. Dreyer Jr. and F.S. Roberts, Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion, *Discrete Appl. Math.* **157** (2009), 1615-1627.

[9] J.F. Fink, M.S. Jacobson, n-domination in graphs, in: Graph Theory with Applications to Algorithms and Computer Science, John Wiley and Sons, New York, 1985, 282-300.

[10] P. Flocchini, F. Geurts, and N. Santoro, Optimal irreversible dynamos in chordal rings, *Discrete Appl. Math.* **113** (2001), 23-42.

[11] P. Flocchini, R. Královič, A. Roncato, P. Ružička, and N. Santoro, On time versus size for monotone dynamic monopolies in regular topologies, *J. Discrete Algorithms* **1** (2003), 129-150.

[12] J. French, A formal theory of social power, *Psychology Rev.* **63** (1956), 181-194.

[13] T. Haynes, S.T. Hedetniemi, and P.J. Slater, Fundamentals of domination in graphs, Marcel Dekker, Inc., New York, 1998.

[14] M.A. Henning, Distance domination in graphs. Domination in Graphs: Advanced Topics, T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds), Marcel Dekker, Inc. New York, 1998, 335-365.

[15] S. Huang, Gene expression profiling, genetic networks, and cellular states: an integrating concept for tumorigenesis and drug discovery, *J. Mol. Med.* **77** (1999), 469-480.

[16] D. Kempe, J. Kleinberg, and É. Tardos, Maximizing the spread of influence through a social network, in *Proceedings of the 9th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, 2003, 137-146.

[17] S. Kutten and D. Peleg, Fault-local distributed mending, *J. Algorithms* **30** (1999), 144-165.

[18] N.H. Mustafa and A. Pekeč, Listen to your neighbors: How (not) to reach a consensus, *SIAM J. Discrete. Math.* **17** (2004), 634-660.

[19] D. Peleg, Local majorities, coalitions and monopolies in graphs: A review, *Theor. Comput. Sci.* **282** (2002), 231-257.

[20] S. Poljak and M. Suura, On periodical behaviour in societies with symmetric influences, *Combinatorica* **3** (1983), 119-121.

Apêndice E

On the Carathéodory Number of Interval and Graph Convexities

Este apêndice contém o artigo "*On the Carathéodory Number of Interval and Graph Convexities*", submetido em abril de 2012 para publicação no periódico *Theoretical Computer Science*.

On the Carathéodory Number of Interval and Graph Convexities

Mitre C. Dourado[1], Dieter Rautenbach[2], Vinícius Fernandes dos Santos[1], Philipp M. Schäfer[2], and Jayme L. Szwarcfiter[1]

[1] Instituto de Matemática, NCE, and COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ, Brazil, Emails: mitre@nce.ufrj.br, vinicius.santos@gmail.com, jayme@nce.ufrj.br

[2] Institut für Optimierung und Operations Research, Universität Ulm, Ulm, Germany, dieter.rautenbach@uni-ulm.de, philipp.schaefer@uni-ulm.de

Abstract

Inspired by a result of Carathéodory (Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen, Rend. Circ. Mat. Palermo 32 (1911), 193-217), the Carathéodory number of a convexity space is defined as the smallest integer k such that for every subset U of the ground set V and every element u in the convex hull of U, there is a subset F of U with at most k elements such that u in the convex hull of F. We study the Carathéodory number for generalized interval convexities and for convexity spaces derived from finite graphs. We establish structural properties, bounds, and hardness results.

Keywords: convexity space; Carathéodory number; interval convexity; geodetic convexity; monophonic convexity; P_3-convexity
AMS subject classification: 05C99; 52A37

1 Introduction

Several classical statements concerning convex sets in $\mathbb{R}^d$ have inspired generalizations for abstract convexity spaces [28]. Here we study problems related to Carathéodory's theorem [5,17], which states that every point in the convex hull of a set of points in $\mathbb{R}^d$ is already in the convex hull of a subset of order at most $d+1$, We consider aspects of Carathéodory's theorem for generalized interval convexities as well as for convexity spaces derived from finite graphs. Before we explain our contributions, we summarize some terminology and previous results.

1.1 Finite convexity spaces

A finite *convexity space* is a pair $(V, \mathcal{C})$ where V is a finite set and $\mathcal{C}$ is a collection of subsets of V such that

- $\emptyset, V \in \mathcal{C}$ and
- $\mathcal{C}$ is closed under intersections.

The sets in $\mathcal{C}$ are considered the *convex sets*. The *convex hull* $H(U)$ of a subset U of V is the intersection of all convex sets containing U. Equivalently, it is the smallest superset of U in $\mathcal{C}$.

The *Carathéodory number* of $(V, \mathcal{C})$ is the smallest integer k such that for every subset U of V and every element u in $H(U)$, there is a subset F of U with $|F| \leq k$ and $u \in H(F)$. Using this terminology, Carathéodory's theorem [5] states that the Carathéodory number of $\mathbb{R}^d$ is at most $d+1$. Note that for finite convexity spaces, the Carathéodory number is always well defined.

A subset U of V is a *Carathéodory set* if the set

$$\partial H(U) := H(U) \setminus \bigcup_{u \in U} H(U \setminus \{u\})$$

is not empty. This notion allows an equivalent definition of the Carathéodory number as the largest cardinality of a Carathéodory set. Carathéodory sets are also known as *irredundant sets* or *C-independent sets.*

There are two natural algorithmic problems associated with the Carathéodory number.

Carathéodory Number
Instance: A finite convexity space $(V, \mathcal{C})$ and an integer k.
Question: Is the Carathéodory number of $(V, \mathcal{C})$ at least k, or equivalently, is there a Carathéodory set of order at least k?

Local Carathéodory Number
Instance: A finite convexity space $(V, \mathcal{C})$, a subset U of V, an element u of $H(U)$, and an integer k.
Question: Is there a subset F of U with $u \in H(F)$ and $|F| \leq k$?

Whenever we consider the complexity of one of these problems for a specific convexity, we need to specify the way in which this convexity is encoded.

1.2 Graph convexities

We only consider finite, simple, and undirected graphs. Most established convexity spaces derived from such a graph G are defined using a set $\mathcal{P}$ of paths in G [10, 14, 21, 23]. Given G and $\mathcal{P}$, a subset U of the vertex set $V(G)$ of G is *convex* if for every path $P : u_0 \dots u_l$ in $\mathcal{P}$ whose endvertices u_0 and u_l belong to U also all internal vertices $u_1, \dots, u_{l-1}$ belong to U. If $\mathcal{C}$ denotes the collection of all such convex sets, then $(V(G), \mathcal{C})$ is a finite convexity space.

Natural and well studied choices for $\mathcal{P}$ include

- the set of all shortest paths leading to the *geodetic convexity* of G [3, 12, 19, 24],
- the set of all induced paths leading to the *monophonic convexity* of G [13, 16, 23],
- the set of all paths leading to the *all paths convexity* of G [8],
- the set of all triangle paths leading to the *triangle path convexity* of G [9, 11],
- the set of all induced paths of length at least 3 leading to the m^3*-convexity* of G [15], and
- the set of all paths of length two leading to the P_3*-convexity* of G [2, 6, 7].

The P_3-convexity was originally considered first for directed graphs [18, 25–27].

For some of the above convexities, the Carathéodory number is well understood. In the monophonic convexity [16] it is 2 for graphs with a non-complete component, and 1 for all other graphs. In the triangle path convexity [9] it is 2 for graphs with a component of order at least 3 and 1 for all other graphs. Similarly, it is very easy to see that the Carathéodory number in the all paths convexity is the same as in the triangle path convexity.

For certain graphs, some of the above convexities coincide. In distance-hereditary graphs [22], for instance, the geodetic and the monophonic convexity are the same, which implies that the Carathéodory number of the geodetic convexity of such graphs is as specified above.

In [2] we showed that CARATHÉODORY NUMBER is NP-complete in P_3-convexity even restricted to bipartite graphs. Here an instance of CARATHÉODORY NUMBER would be a graph G and an integer k, that is, the P_3-convexity of G is only encoded implicitly by G itself. Next to this complexity result, we established an upper bound on the Carathéodory number in P_3-convexity and developed an efficient algorithm for block graphs.

1.3 Our contribution

In the present paper we extend and complement earlier results. In Section 2 we consider r-interval convexities, which generalize the considered graph convexities. We prove the existence of Carathéodory sets of many sizes in such convexities and establish a best possible upper bound on their Carathéodory number. In Section 3 we prove that CARATHÉODORY NUMBER and LOCAL CARATHÉODORY NUMBER are NP-complete in geodetic convexity. Furthermore, we prove that the Carathéodory Number of the geodetic convexity of split graphs is at most 3. Finally, in Section 4 we prove that LOCAL CARATHÉODORY NUMBER is NP-complete in P_3-convexity.

2 Generalized Interval Convexities

For a set V and an integer k, let $\binom{V}{k}$ denote the set of all k-element subsets of V and let 2^V denote the set of all subsets of V.

A finite convexity space $(V, \mathcal{C})$ is an *interval convexity* [4] if there is a so-called *interval function* $I : \binom{V}{2} \to 2^V$ such that a subset C of V belongs to $\mathcal{C}$ if and only if $I(\{x, y\}) \subseteq C$ for every two distinct elements x and y of C. Clearly, all graph convexities from Subsection 1.2 are interval convexities. In fact, given a graph G and a set $\mathcal{P}$ of paths in G, then

$$I(\{x, y\}) \quad = \quad \{x, y\} \cup \{z \mid z \text{ lies on a path in } \mathcal{P} \text{ between } x \text{ and } y\} \tag{1}$$

has the desired properties.

More generally, we say that a finite convexity space $(V, \mathcal{C})$ is an *r-interval convexity* for some integer $r \geq 2$ if there is a so-called *r-interval function* $I : \binom{V}{r} \to 2^V$ such that a subset C of V belongs to $\mathcal{C}$ if and only if $I(U) \subseteq C$ for every set $U \in \binom{C}{r}$. Clearly, we may always assume that $U \subseteq I(U)$ for every $U \in \binom{C}{r}$.

Theorem 2.1. *If $(V, \mathcal{C})$ is a finite r-interval convexity for some $r \geq 2$ and C is a Carathéodory set with $|C| \geq r$, then $H(C)$ contains a Carathéodory set C' with $|C| - (r-1) \leq |C'| \leq |C| - 1$.*

Proof. Let I denote an r-interval function of $(V, \mathcal{C})$. By the definition of r-interval convexities, there is a sequence

$$C_0 \subseteq C_1 \subseteq \ldots \subseteq C_p$$

with

- $C = C_0$,
- $C_i = C_{i-1} \cup I(U_{i-1})$ for $i \in \{1, \ldots, p\}$ and some $U_{i-1} \in \binom{C_{i-1}}{r}$, and
- $C_p = H(C)$.

Let $v \in \partial H(C)$. Clearly, $v \in H(C_i)$ for every $i \in \{1, \ldots, p\}$. For $i \in \{0, 1, \ldots, p\}$, let S_i denote a subset of C_i of minimum order such that $v \in H(S_i)$. This choice implies that $v \in \partial H(S_i)$, that is, S_i is a Carathéodory set contained in $H(C)$. Furthermore, $S_0 = C$ and $|S_i| = 1$ if $v \in C_i$.

At this point it suffices to prove $|S_i| \geq |S_{i-1}| - (r-1)$ for every $i \in \{1, \ldots, p\}$. Therefore, let $i \in \{1, \ldots, p\}$. If $S_i \subseteq C_{i-1}$, then $|S_{i-1}| = |S_i|$. Hence we may assume that $S_i \not\subseteq C_{i-1}$. Since

$S_i \setminus C_{i-1} \subseteq I(U_{i-1})$, we have $H(S_i) \subseteq H((S_i \cap C_{i-1}) \cup U_{i-1})$. Since $(S_i \cap C_{i-1}) \cup U_{i-1} \subseteq C_{i-1}$, this implies

$$|S_{i-1}| \leq |(S_i \cap C_{i-1}) \cup U_{i-1}| \leq |S_i \cap C_{i-1}| + |U_{i-1}| \leq (|S_i| - 1) + r,$$

which completes the proof. □

Theorem 2.1 implies some algorithmic consequences for Carathéodory Number. If a finite convexity space $(V, \mathcal{C})$ is an r-interval convexity, then there is a Carathéodory set of order at least k if and only if there is a Carathéodory set of some order in $\{k, k+1, \ldots, k+r-2\}$. Therefore, in order to solve Carathéodory Number for a fixed value of k, one only has to consider subsets of V of order between k and $k+r-1$.

The following consequence of Theorem 2.1 is implicit in [16].

Corollary 2.2. *Every finite 2-interval convexity with Carathéodory number c has Carathéodory set of every order between 1 and c.*

The following result generalizes a bound in [2].

A *strictly r-ary tree* is a rooted tree in which every internal vertex has exactly r children.

Theorem 2.3. *If $(V, \mathcal{C})$ is a finite r-interval convexity for some $r \geq 2$ with Carathéodory number c, then*

$$c \leq \frac{(r-1)|V| + 1}{r}. \tag{2}$$

If $V \neq \emptyset$, then equality holds in (2) if and only if for every Carathéodory set S of order c, there is exactly one vertex in $\partial H(S)$, say v, and there is a strictly r-ary tree T with root v, vertex set V, and set of leaves S such that for every vertex u of T, the set of leaves of T that are either u or a descendant of u in T is the unique minimal subset S_u of S with $u \in H(S_u)$.

Proof. Let I denote an r-interval function of $(V, \mathcal{C})$. Let S be a Carathéodory set of order c. If $c < r$, then, since every set of order less than r is convex, it follows that $c = 1$ and (2) is satisfied. Hence we may assume $c \geq r$, which implies that no element of S belongs to $\partial H(S)$. This implies the existence of a sequence $u_1, \ldots, u_l$ of elements of $V \setminus S$ such that

- $u_i \in I(N^+(u_i))$ for some subset $N^+(u_i)$ of $S \cup \{u_j \mid 1 \leq j \leq i-1\}$ of order r for every i with $1 \leq i \leq l$ and
- $u_l \in \partial H(S)$.

We assume that the sequence $u_1, \ldots, u_l$ is chosen shortest possible, that is, l is minimum. Let $U = \{u_i \mid 1 \leq i \leq l\}$. Let the directed graph T have vertex set $V(T) = S \cup U$ and arc set

$$A(T) = \{(u, v) \mid u \in U \text{ and } v \in N^+(u)\}.$$

We consider the outdegree $d_T^+(u)$ and indegree $d_T^-(u)$ of vertices u of T. By definition, $d_T^+(u) = 0$ for every vertex u in S and $d_T^+(u) = |N^+(u)| = r$ for every vertex u in U. Since $u_l \in \partial H(S)$, we have $d_T^-(u) \geq 1$ for every vertex u in S. By the choice of the sequence, we have $d_T^-(u) \geq 1$ for every vertex u in $U \setminus \{u_l\}$. By definition, $d_T^-(u_l) = 0$. This implies

$$c + (l-1) \leq \sum_{u \in V(T)} d_T^-(u) = |A(T)| = \sum_{u \in V(T)} d_T^+(u) = rl \tag{3}$$

and hence $c \leq (r-1)l + 1$. Since $|V| \geq |V(T)| = c + l$, we obtain $c \leq (r-1)l + 1 \leq (r-1)(|V| - c) + 1$, which implies (2).

We proceed to the characterization of the cases leading to equality in (2) for $V \neq \emptyset$. If $c \leq 1$, then equality in (2) implies $c = 1$ and $|V| = 1$. Conversely, $|V| = 1$ implies $c = 1$. Therefore, if $c \leq 1$, then the desired equivalence holds and we may assume $c \geq 2$. First, we establish the "only if" part. Therefore, let (2) hold with equality. By the above reasoning, this implies that equality holds throughout (3) and $V = V(T)$. This implies that $d_T^-(u) = 1$ for every vertex u in $V(T) \setminus \{u_l\}$, which immediately implies that T is a strictly r-ary tree with root u_l, vertex set V, and set of leaves S. By the choice of the sequence, u_l is the only vertex of $\partial H(S)$ in U. Since $c \geq 2$, no vertex in S belongs to $\partial H(S)$. Hence $\partial H(S)$ contains exactly one element, the root u_l of T. For a vertex u of T, let S_u be the set of leaves of T that are either u or descendants of u in T. By the definition of the sequence, we obtain $u \in H(S_u)$. If there is some subset S'_u of S with $S_u \not\subseteq S'_u$ and $u \in H(S'_u)$, then $S' = (S \setminus S_u) \cup S'_u$ is a proper subset of S and $H(S')$ contains every vertex of T that is neither u nor an ancestor of u in T. Since u belongs to $H(S'_u)$, this implies that u and all ancestors of u in T belong to $H(S')$. Hence u_l belongs to $H(S')$, which contradicts $u_l \in \partial H(S)$. Hence for every vertex u of T, the set S_u is the unique minimal subset of S with $u \in H(S_u)$, which completes the proof of the "only if" part.

Now, we establish the "if" part. Let T be as in the statement. Since T is a strictly r-ary tree, the set S of leaves contains exactly $\frac{(r-1)|V|+1}{r}$ many vertices. Since S is the unique minimal subset S_v of S with $v \in H(S_v)$, the set S is a Carathéodory set. By (2), this implies equality in (2), which completes the proof. □

Let $(V, \mathcal{C})$ be a finite r-interval convexity for some $r \geq 2$ and let I be an r-interval function of $(V, \mathcal{C})$. For every subset U of V, let

$$\begin{aligned} I^0(U) &:= U, \\ I^1(U) &:= I(U) := U \cup \bigcup_{S \in \binom{U}{r}} I(S) \text{ and} \\ I^k(U) &:= I(I^{k-1}(U)) \text{ for every } k \geq 2. \end{aligned} \tag{4}$$

Proposition 2.4. *Let $(V, \mathcal{C})$, r, and I be as above.*

If $x \in I^k(U)$ for some subset U of V and $k \geq 0$, then $x \in H(F)$ for some subset F of U with $|F| \leq r^k$.

Proof. The proof is by induction on k. If $k = 0$, then $x \in I^0(U)$ implies that $x \in U$ and $F = \{x\}$ has the desired properties. Now let $k \geq 1$. By definition, $x \in I(S)$ for some $S \in \binom{I^{k-1}(U)}{r}$. By induction, for every element y of S, we have $y \in H(F_y)$ for some subset F_y of U with $|F_y| \leq r^{k-1}$. Now $x \in H(F)$ where $F = \bigcup\limits_{y \in S} F_y$ and $|F| \leq r^k$, which completes the proof. □

3 Geodetic Convexity

Theorem 3.1. Carathéodory Number *is NP-complete in geodetic convexity.*

Proof. Using shortest paths algorithms, the convex hull of a set of vertices of some graph can be determined efficiently. Therefore, it can be checked in polynomial time whether a given set of vertices is a Carathéodory set, which implies that Carathéodory Number in geodetic convexity is in NP. In order to complete the proof, we describe a polynomial reduction of the NP-complete problem $3\mathrm{SAT}_{\leq 5}$ (cf. [LO2] in [20]) to Carathéodory Number.

3SAT$_{\leq 5}$

Instance:	A set X of variables and a collection $\mathcal{C}$ of clauses over X such that every clause in $\mathcal{C}$ contains exactly 3 literals and for every variable x in X, there are at most 5 clauses in $\mathcal{C}$ that contain either x or $\bar{x}$.
Question:	Is $\mathcal{C}$ satisfiable?

Let $(X, \mathcal{C})$ be an instance of 3SAT$_{\leq 5}$ with $m = |\mathcal{C}| \geq 6$ clauses $C_1, \ldots, C_m$ in n boolean variables $x_1, \ldots, x_n$. We denote the three literals of each clause C_i of $\mathcal{C}$ by $\ell_{i,a}, \ell_{i,b}$, and $\ell_{i,c}$.

Starting with the empty graph, we construct a graph G as follows.

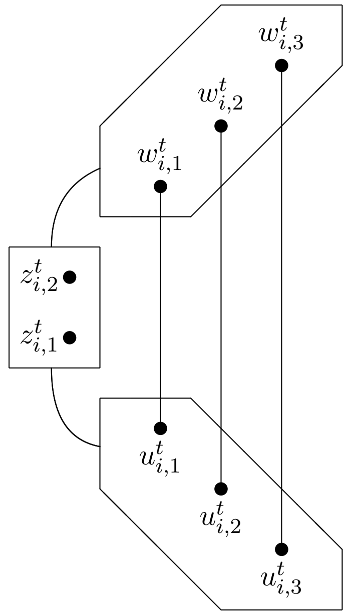

Figure 1: There are all possible 12 edges between $\{z^t_{i,1}, z^t_{i,2}\}$ and $\{u^t_{i,1}, u^t_{i,2}, u^t_{i,3}\} \cup \{w^t_{i,1}, w^t_{i,2}, w^t_{i,3}\}$.

- For the clause C_1, add to G a copy of the gadget in Figure 1 with $i = 1$ and $t = 3$ and denote its vertex set by $V_1 = V_1^3$.
- For the clause C_i with $i \in \{2, \ldots, m-1\}$, add to G a subgraph formed by the union of three disjoint copies of the gadget in Figure 1 and denote their vertex sets by V_i^1, V_i^2, V_i^3, respectively. Let $V_i = V_i^1 \cup V_i^2 \cup V_i^3$. Furthermore, add edges between the vertices in V_i to form the gadget in Figure 2.
- For the clause C_m, add to G a copy of the gadget in Figure 1 with $i = m$ and $t = 1$ and denote its vertex set by $V_m = V_m^1$.
- Denote

  by Z the set of all vertices $z^t_{i,j}$,

  by W the set of all vertices $w^t_{i,j}$, and

  by U the set of all vertices $u^t_{i,j}$.
- Add two vertices a and a' that are adjacent to all vertices of Z.
- For every $i \in \{1, \ldots, m-1\}$ and $j, j' \in \{1, 2, 3\}$, add the edge $w^3_{i,j} w^1_{i+1,j'}$.

- For every pair of literals $\ell_{i,j}, \ell_{p,q}$ of $\mathcal{C}$ that belong to different clauses and are not the negation of each other proceed as follows.
  - If $\{i,p\} \cap \{1,m\} = \emptyset$, then add the 9 edges
  $$u^1_{i,j}u^1_{p,q}, u^1_{i,j}u^2_{p,q}, u^1_{i,j}u^3_{p,q}, u^2_{i,j}u^1_{p,q}, u^2_{i,j}u^2_{p,q}, u^2_{i,j}u^3_{p,q}, u^3_{i,j}u^1_{p,q}, u^3_{i,j}u^2_{p,q}, u^3_{i,j}u^3_{p,q}.$$
  - If $\{i,p\} \cap \{1,m\} = \{1\}$, say $i = 1$, then add the 3 edges $u^3_{1,j}u^1_{p,q}, u^3_{1,j}u^2_{p,q}, u^3_{1,j}u^3_{p,q}$.
  - If $\{i,p\} \cap \{1,m\} = \{m\}$, say $i = m$, then add the 3 edges $u^1_{m,j}u^1_{p,q}, u^1_{m,j}u^2_{p,q}, u^1_{m,j}u^3_{p,q}$.
  - If $\{i,p\} = \{1,m\}$, say $i = 1$ and $p = m$, then add the edge $u^3_{1,j}u^1_{m,q}$.

This completes the construction of G.

Let $k = 3m - 4$.

It remains to prove that $\mathcal{C}$ is satisfiable if and only if G contains a Carathéodory set with at least k vertices.

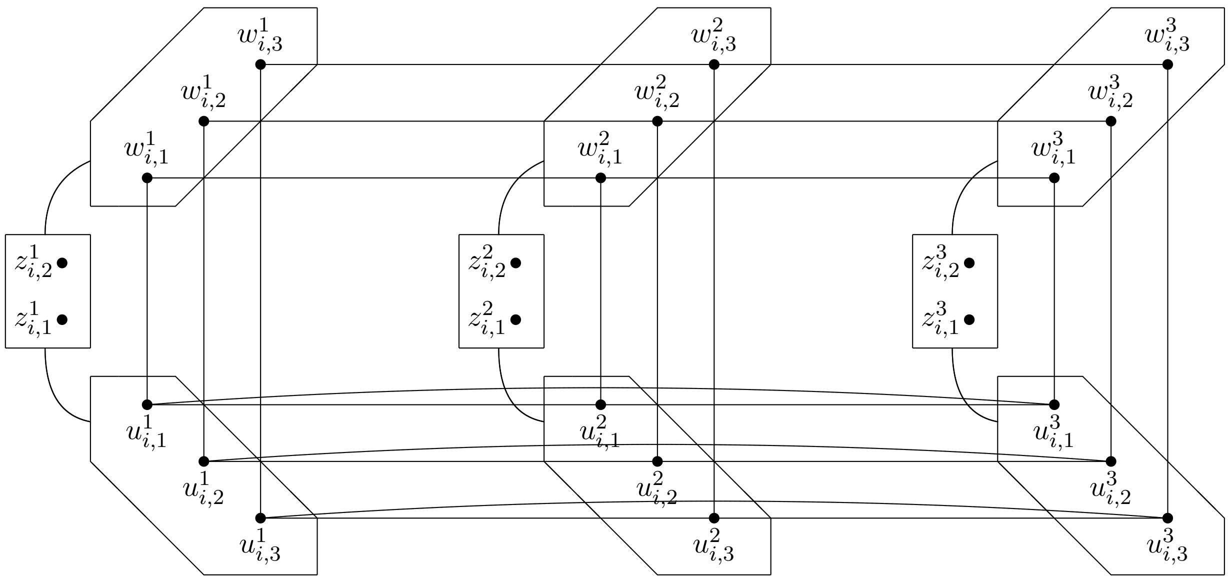

Figure 2: For $j \in \{1,2,3\}$, $w^1_{i,j}w^2_{i,j}w^3_{i,j}$ is an induced path and $\{u^1_{i,j}, u^2_{i,j}, u^3_{i,j}\}$ is a complete set.

We first prove the necessity, that is, we assume that $\mathcal{C}$ is satisfiable. Let $\mathcal{A}$ be a truth assignment satisfying $\mathcal{C}$. We define a set $S \subseteq V(G)$ of order k as follows. For each clause C_i, choose exactly one literal ℓ_{i,j_i} satisfied by $\mathcal{A}$ and add to S

- the vertex w^3_{i,j_i} for $i = 1$,
- the vertices $u^1_{i,j_i}, u^2_{i,j_i}, u^3_{i,j_i}$ for $i \in \{2,\ldots,m-1\}$, and
- the vertex u^1_{i,j_i} for $i = m$.

We will show that $w^1_{m,j_m} \in \partial H(S)$, which implies that S is a Carathéodory set of order k.

By construction, it follows easily that $w^1_{m,j_m} \in H(S)$.

Since $S \setminus \{w^3_{i,j_i}\}$ is a complete set, we have $w^1_{m,j_m} \notin H(S \setminus \{w^3_{1,j_1}\})$.

Now let $u^r_{p,j_p} \in S$ for some $p \in \{2,\ldots,m\}$. We will show that $H(S \setminus \{u^r_{p,j_p}\}) = (S \setminus \{u^r_{p,j_p}\}) \cup W'$ where W' contains exactly the vertices

- u^3_{1,j_1},
- w^s_{i,j_i} for $i \in \{2,\ldots,p-1\}$ and $s \in \{1,2,3\}$, and
- w^s_{p,j_p} for $s \in \{1,\ldots,r-1\}$.

Clearly, this implies $w^1_{m,j_m} \notin H(S \setminus \{u^r_{p,j_p}\})$.

By construction, it follows easily that $W' \subseteq H(S \setminus \{u^r_{p,j_p}\})$. Hence it suffices to prove that $S \setminus \{u^r_{p,j_p}\} \cup W'$ is convex. This follows easily from the construction using the facts that $S \setminus \{w^3_{1,j_1}\}$ is a complete set, the maximum distance between a vertex in $S \setminus \{u^r_{p,j_p}\}$ and a vertex in W' is at most 2, and the maximum distance between two vertices in W' is at most 3.

This shows that $w^1_{m,j_m} \in \partial H(S)$ and completes the proof of the necessity.

For the proof of the sufficiency, we establish the following claim.

Claim *Every Carathéodory set S of G with at least k vertices has the following properties.*

(i) $S \cap (Z \cup \{a,a'\}) = \emptyset$.

(ii) *$S \cap U$ is a complete set.*

(iii) *$S \cap W \cap V_i$ contains at most 2 vertices for $i \in \{1,\ldots,m\}$. Furthermore, if $S \cap W \cap V_i$ contains 2 vertices $w^t_{i,j}$ and $w^{t'}_{i,j'}$, then $j = j'$ and $t \neq t'$.*

(iv) *$|S \cap V^t_i| \leq 1$ for $i \in \{1,\ldots,m\}$ and $t \in \{1,2,3\}$.*

(v) *$|S \cap V^t_i| = 1$ for $i \in \{1,\ldots,m\}$ and $t \in \{1,2,3\}$. Furthermore, if U' denotes the set of neighbours in U of the vertices in $S \cap W$, then $(S \cap U) \cup U'$ is a complete set.*

Proof of the claim: Since S is a Carathéodory set, no proper subset S' of S can satisfy $V(G) = H(S')$.

(i) Since $H(\{z,z'\}) = V(G)$ for every two vertices z and z' in Z, the set S contains at most one vertex from Z. Hence $|S \cap (W \cup U)| \geq k - 3 = 3m - 7 > 0$. Since $H(\{a,x\}) = H(\{a',x\}) = V(G)$ for every vertex x in $W \cup U$, the S contains neither a nor a'. Hence $|S \cap (W \cup U)| \geq k - 1 = 3m - 5$. At this point we may assume to the contrary that $x, z^t_{i,1} \in S$ with $x \in V_p$ with $i \neq p$. If the distance between x and $z^t_{i,1}$ is more than 2, then $H(\{x, z^t_{i,1}\})$ contains a and a', which implies $H(\{x, z^t_{i,1}\}) = V(G)$. Hence, by construction, we may assume that the distance between x and $z^t_{i,1}$ is 2. This implies that $H(\{x, z^t_{i,1}\})$ contains either the three vertices $w^t_{i,1}$, $w^t_{i,2}$, and $w^t_{i,3}$ or two of the three vertices $u^t_{i,1}$, $u^t_{i,2}$, and $u^t_{i,3}$. In both cases, $H(\{x, z^t_{i,1}\})$ contains $z^t_{i,2}$, which implies $H(\{x, z^t_{i,1}\}) = V(G)$. This completes the proof of (i).

(ii) First, we assume to the contrary that $S \cap U$ contains two vertices u and u' that are not adjacent. Recall that u and u' correspond to distinct literals. Since $m \geq 6$ and $\mathcal{C}$ is an instance of $3\mathrm{SAT}_{\leq 5}$, there is a clause C_i such that $u, u' \notin V_i$ and C_i contains at least two literals that are not the negations of either of the two literals associated with u and u'. Now $H(\{u,u'\})$ contains $z^t_{i,1}$ and $z^t_{i,2}$ for $t \in \{1,2,3\}$ and hence $H(\{u,u'\}) = V(G)$, which is a contradiction. Similarly, for every two vertices w and w' in $S \cap W \cap V_i$ that are not adjacent, we have $H(\{w,w'\}) = V(G)$. This completes the proof of (ii).

(iii) If $S \cap W \cap V_i$ contains 3 vertices, then either one lies on a shortest path between two others or there are two whose convex hull is $V(G)$. Hence $S \cap W \cap V_i$ contains at most 2 vertices. If $S \cap W \cap V_i$ contains $w^t_{i,j}$ and $w^{t'}_{i,j'}$ but either $j \neq j'$ or $t = t'$, then $H(\{w^t_{i,j}, w^{t'}_{i,j'}\}) = V(G)$. This completes the proof of (iii).

(iv) If $S \cap V_i^t$ contains at least 2 vertices, then, by (i), (ii), and (iii), $S \cap V_i^t$ contains exactly two vertices $w_{i,j}^t$ and $u_{i,j'}^t$ with $j, j' \in \{1,2,3\}$. If $j \neq j'$, then $H(\{w_{i,j}^t, u_{i,j'}^t\}) = V(G)$, which is a contradiction. Hence $j = j'$. Since $k = 3m - 4$ and $m \geq 6$, (i), (ii), and (iii) imply that $S \cap U$ contains some $u_{i'',j''}^{t''}$ with $i'' \neq i$, which is a neighbour of $u_{i,j'}^t$. Now $u_{i,j'}^t$ lies on a shortest path between $w_{i,j}^t$ and $u_{i'',j''}^{t''}$, which is a contradiction. This completes the proof of (iv).

(v) Since S has at least $3k-4$ vertices, (iv) implies that $|S \cap V_i^t| = 1$ for $i \in \{1, \ldots, m\}$ and $t \in \{1,2,3\}$. Furthermore, $S \cap V_i$ contains at least one element from U for $i \in \{2, \ldots, m-1\}$.

It remains to prove that $(S \cap U) \cup U'$ is a complete set. By (ii), $S \cap U$ is a complete set.

If $w_{i,j}^t$ and $u_{i',j'}^{t'}$ belong to S such that $u_{i,j}^t$ is not adjacent to $u_{i',j'}^{t'}$, then the distance between $w_{i,j}^t$ and $u_{i',j'}^{t'}$ is 3. This implies that $z_{i,1}^t$ and $z_{i,2}^t$ lie on shortest paths between $w_{i,j}^t$ and $u_{i',j'}^{t'}$, which implies the contradiction $H(\{w_{i,j}^t, u_{i',j'}^{t'}\})$.

Finally, if $w_{i,j}^t$ and $w_{i',j'}^{t'}$ belong to S such that $u_{i,j}^t$ is not adjacent to $u_{i',j'}^{t'}$, then, by the previous observations, S contains a vertex $u_{i'',j''}^{t''}$ such that $u = u_{i,j}^t$ lies on a shortest path between $w_{i,j}^t$ and $u_{i'',j''}^{t''}$ and $u' = u_{i',j'}^{t'}$ lies on a shortest path between $w_{i',j'}^{t'}$ and $u_{i'',j''}^{t''}$. Since $m \geq 6$ and $\mathcal{C}$ is an instance of 3SAT$_{\leq 5}$, there is a clause C_i such that $u, u' \notin V_i$ and C_i contains at least two literals that are not the negations of either of the two literals associated with u and u'. Now $H(\{u, u'\})$ contains $z_{i,1}^t$ and $z_{i,2}^t$ for $t \in \{1,2,3\}$. Hence $H(\{w_{i,j}^t, w_{i',j'}^{t'}, u_{i'',j''}^{t''}\}) = V(G)$, which is a contradiction.

This completes the proof of the claim. □

We are now ready to complete the proof of sufficiency. Let S be a Carathéodory set of G with at least k vertices. By (vi) of the claim, the set $(S \cap U) \cup U'$ corresponds to a set of literals that contains one literal from each clause such that no two of these literals are negations of each other. Hence $(S \cap U) \cup U'$ indicates a satisfying truth assignment for $\mathcal{C}$, which completes the proof. □

Theorem 3.2. Local Carathéodory Number *is NP-complete in geodetic convexity even restricted to bipartite graphs.*

Proof. Similarly as in the proof of Theorem 3.1, it follows easily that Local Carathéodory Number in geodetic convexity is in NP. In order to complete the proof, we describe a polynomial reduction of the problem Geodetic Hull Number restricted to instances (G, k) where G is a bipartite graph with a vertex of degree 1, to Local Carathéodory Number. Note that Geodetic Hull Number restricted to such instances is still NP-complete [1].

Geodetic Hull Number
Instance: A graph G and an integer k.
Question: Is there a set S of at most k vertices of G with $H(S) = V(G)$ in geodetic convexity?

Let (G, k) be such a restricted instance of Geodetic Hull Number. Let $V(G) = \{v_1, \ldots, v_n\}$ and let $d_G(v_1) = 1$. We describe the construction of an instance (G', U', u', k') of Local Carathéodory Number.

- Add to G n new vertices $w_1, \ldots, w_n$.
- Add all edges $w_i v_i$ for $i \in \{1, \ldots, n\}$ and $w_i w_{i+1}$ for $i \in \{1, \ldots, n-1\}$.
- Subdivide each edge $w_i w_{i+1}$ exactly $\mathrm{dist}_G(v_i, v_{i+1}) - 1$ times.

This completes the description of G'. Note that G' is bipartite.

Let $U' = (V(G) \setminus \{v_1\}) \cup \{w_1\}$, $u' = w_n$, and $k' = k$.

Since we are considering convex hulls in the geodetic convexities induced by the two graphs G and G', we will add an index indicating the corresponding graph. It remains to prove that there is a set S of at most k vertices of G with $H_G(S) = V(G)$ if and only if there is a subset F of U' of order at most k with $w_n \in H_{G'}(F)$.

We first prove the necessity. Let S be a set of at most k vertices of G with $H_G(S) = V(G)$. Since v_1 has degree 1, we have $v_1 \in S$. If $F = (S \setminus \{v_1\}) \cup \{w_1\}$, then $|F| = |S| \leq k$. Furthermore, for every shortest path $x_0 \dots x_l$ in G between $v_1 = x_0$ and $v_i = x_l$, the path $w_1 x_0 \dots x_l$ is a shortest path in G' between w_1 and $v_i = x_l$. This implies that $V(G) \subseteq H_{G'}(F)$ and, by construction, $w_n \in V(G') = H_{G'}(F)$.

Now we proceed to the proof of the sufficiency. Let F be a subset of U' of order at most k with $w_n \in H(F)$. Note that no vertex in $\{w_1, \dots, w_n\}$ lies on a shortest path in G' between two vertices in $V(G)$. Since $w_n \in H(F)$, this implies $w_1 \in F$.

Since the geodetic convexity is an interval convexity, we may consider its interval function I as in (1). Furthermore, we consider its iterated interval function I^k as in (4). For $i \in \{1, \dots, n\}$, let $k_i = \min\{k \mid w_i \in I^k(F)\}$ where $\min \emptyset = \infty$. Since $w_1 \in F$ and $w_n \in H(F)$, we have $k_1 = 0$ and $k_n < \infty$. Since w_n has degree 2 and w_i has degree 3 for $i \in \{2, \dots, n-1\}$, it follows easily by an inductive argument that $k_1 \leq k_2 \leq \dots \leq k_n$, that is, all k_i are finite, which implies $\{w_1, \dots, w_n\} \subseteq H_{G'}(F)$. By construction, if some vertex w_i lies on a shortest path in G' between some vertex w_s and some vertex v_t, then v_i also lies on a shortest path in G' between w_s and v_t. This implies $\{v_1, \dots, v_n\} \subseteq H_{G'}(F)$, that is $V(G') = H_{G'}(F)$. Furthermore, it implies that $S = (F \setminus \{w_1\}) \cup \{v_1\}$ is a set of at most k vertices of G with $V(G) = H_G(S)$, which completes the proof. □

A graph G is a *split graph* if its vertex set can be partitioned into a complete set and an independent set.

Theorem 3.3. *The Carathéodory number of the geodetic convexity of split graphs is at most* 3.

Proof. Let G be a split graph. Let $V(G) = C \cup I$ where C is a complete set and I is an independent set with $C \cap I = \emptyset$. Let U be a Carathéodory set of G of maximum cardinality. We will prove that U contains at most 3 elements. Let $v \in \partial H(U)$.

As before we consider the interval function I as in (1) and the iterated interval function I^k as in (4). Let k be minimal with $v \in I^k(U)$. If $k \leq 1$, then either v belongs to U or v lies on a shortest path between two vertices in U, which implies $|U| \leq 2$. Hence we may assume that $k \geq 2$.

Since no vertex in I lies on a shortest path between two other vertices, the set $H(U) \setminus U$ contains no element of I. This implies $v \in C$. Since $k \geq 2$, the vertex v lies on a shortest path between two vertices u_{k-1} and w_k in $I^{k-1}(U)$. Necessarily, one of the two vertices, say w_k, belongs to $U \cap I$. Furthermore, by the minimality of k, the other vertex u_{k-1} belongs to $I^{k-1}(U) \setminus I^{k-2}(U)$ and hence $u_{k-1} \in C$. Similarly, since $k \geq 2$, the vertex u_{k-1} lies on a shortest path between two vertices u_{k-2} and w_{k-1} in $I^{k-2}(U)$. As before, we may assume that $w_{k-1} \in U \cap I$. If $k \geq 3$, then this implies that u_{k-2} belongs to $I^{k-2}(U) \setminus I^{k-3}(U)$. Iterating this argument, we obtain vertices $u_0, u_1, \dots, u_{k-1}$ and $w_1, w_2, \dots, w_k$ with

- $u_0 \in U$, $u_1, \dots, u_{k-1} \in C$, $w_1, w_2, \dots, w_k \in U \cap I$, and
- $u_i \in I(\{u_{i-1}, w_i\})$ and $u_i \in I^i(U) \setminus I^{i-1}(U)$ for $i \in \{1, \dots, k-1\}$.

Since $v \in H(\{u_0, w_1, \dots, w_k\})$, we may assume that $k \geq 3$. By the minimality of k, the vertex u_i is a neighbour of w_k for $i \in \{1, \dots, k-2\}$, since otherwise $v \in I(\{u_i, w_k\})$ and hence $v \in I^{i+1}(U)$, which is a contradiction. Similarly, the vertex u_i is a neighbour of w_{k-1} for $i \in \{1, \dots, k-3\}$, since otherwise $u_{k-1} \in I(\{u_i, w_{k-1}\})$ and hence $v \in I^{i+2}(U)$, which is a contradiction.

If $k = 3$, then $u_1 \in I(\{w_1, w_3\})$, $u_2 \in I(\{u_1, w_2\})$, and $v \in I(\{u_2, w_3\})$, which implies $v \in H(\{w_1, w_2, w_3\})$ and hence $|U| = 3$.

If $k \geq 4$, then $u_{k-3} \in I(\{w_{k-1}, w_k\})$, $u_{k-2} \in I(\{u_{k-3}, w_{k-2}\})$, $u_{k-1} \in I(\{u_{k-2}, w_{k-1}\})$, and $v \in I(\{u_{k-1}, w_k\})$, which implies $v \in H(\{w_{k-2}, w_{k-1}, w_k\})$ and hence $|U| = 3$, which completes the proof. □

4 P_3-Convexity

Theorem 4.1. LOCAL CARATHÉODORY NUMBER *is NP-complete in* P_3*-convexity.*

Proof. Since the convex hull of a set of vertices of some graph can be computed efficiently in P_3-convexity, LOCAL CARATHÉODORY NUMBER in P_3-convexity is in NP. In order to complete the proof, we describe a polynomial reduction of the NP-complete problem P_3-HULL NUMBER [7] to LOCAL CARATHÉODORY NUMBER.

P_3-HULL NUMBER
Instance: A graph G and an integer k.
Question: Is there a set S of at most k vertices of G with $H(S) = V(G)$ in P_3-convexity?

Let (G, k) be an instance of P_3-HULL NUMBER. Possibly by adding isolated vertices to G and increasing k by the number of added vertices, we may assume that the order n of G is a power of 2, that is, $n = 2^p$ for some integer p. Let the graph G' arise from G by adding $n-1$ new vertices and $2(n-1)$ new edges such that G' contains a subgraph that is complete binary tree T of height p whose leaves are the vertices of G. If r denoted the root of T, then for some subset F of $V(G)$, it follows easily that the convex hull of F in G is $V(G)$ if and only if the convex hull of F in G' contains r. Considering the instance $(G', V(G), r, k)$ of LOCAL CARATHÉODORY NUMBER, this completes the proof. □

Acknowledgement The authors acknowledge support by a CAPES/DAAD Probral project "Cycles, Convexity, and Searching in Graphs". Vinícius Fernandes dos Santos acknowledges support by Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ).

References

[1] J. Araujo, V. Campos, F. Giroire, L. Sampaio, and R. Soares, *On the hull number of some graph classes*, Electronic Notes in Discrete Mathematics **38** (2011), 49-55.

[2] R.M. Barbosa, E.M.M. Coelho, M.C. Dourado, D. Rautenbach, J.L. Szwarcfiter, *On the Carathéodory number for the convexity of paths of order three*, to appear in SIAM J. Discrete Math., an extended abstract appeared in the Proceedings of *EuroComb '11*, Electronic Notes in Discrete Mathematics **38** (2011), 105-110.

[3] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, and C. Seara, *On geodetic sets formed by boundary vertices*, Discrete Math. **306** (2006), 188-198.

[4] J. Calder, *Some elementary properties of interval convexities*, J. London Math. Soc. **3** (1971), 422-428.

[5] C. Carathéodory, *Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen*, Rend. Circ. Mat. Palermo **32** (1911), 193-217.

[6] C.C. Centeno, S. Dantas, M.C. Dourado, D. Rautenbach, and J.L. Szwarcfiter, *Convex Partitions of Graphs induced by Paths of Order Three*, Discrete Mathematics and Theoretical Computer Science **12** (2010), 175-184.

[7] C.C. Centeno, M.C. Dourado, L.D. Penso, D. Rautenbach, and J.L. Szwarcfiter, *Irreversible Conversion of Graphs*, Theor. Comput. Sci. **412** (2011), 3693-3700.

[8] M. Changat, S. Klavžar, and H.M. Mulder, *The all-paths transit function of a graph*, Czech. Math. J. **51 (126)** (2001), 439-448.

[9] M. Changat and J. Mathew, *On triangle path convexity in graphs*, Discrete Math. **206** (1999), 91-95.

[10] M. Changat, H.M. Mulder, and G. Sierksma, *Convexities related to path properties on graphs*, Discrete Math. **290** (2005), 117-131.

[11] M. Changat, G.N. Prasanth, and J. Mathews, *Triangle path transit functions, betweenness and pseudo-modular graphs*, Discrete Math. **309** (2009), 1575-1583.

[12] M.C. Dourado, F. Protti, D. Rautenbach, and J.L. Szwarcfiter, *On the hull number of triangle-free graphs*, SIAM J. Discrete Math. **23** (2010), 2163-2172.

[13] M.C. Dourado, F. Protti, and J.L. Szwarcfiter, *Complexity results related to monophonic convexity*, Discrete Appl. Math. **158** (2010), 1269-1274.

[14] M.C. Dourado, D. Rautenbach, and P.M. Schäfer, *On Finite Convexity Spaces Induced by Sets of Paths in Graphs*, Discrete Math. **311** (2011), 616-619.

[15] F. Dragan, F. Nicolai, and A. Brandstädt, *Convexity and HHD-free graphs*, SIAM J. Discrete Math. **12** (1999), 119-135.

[16] P. Duchet, *Convex sets in graphs II: Minimal path convexity*, J. Combin. Theory, Ser. B **44** (1988), 307-316.

[17] J. Eckhoff, *Helly, Radon, and Carathéodory type theorems*, Handbook of Convex Geometry, A, B, Amsterdam: North-Holland, 1993, pp. 389-448.

[18] P. Erdős, E. Fried, A. Hajnal, and E.C. Milner, *Some remarks on simple tournaments*, Algebra Univers. **2** (1972), 238-245.

[19] M.G. Everett and S.B. Seidman, *The hull number of a graph*, Discrete Math. **57** (1985), 217-223.

[20] M.R. Garey and D.S. Johnson, *Computers and intractability A Guide to the Theory of NP-Completeness*, W.H. Freeman and Company, 1979.

[21] F. Harary and J. Nieminen, *Convexity in graphs*, J. Differential Geom. **16** (1981), 185-197.

[22] E. Howorka, *A characterization of distance-hereditary graphs*, Quart. J. Math. Oxford, Ser. 2 **28** (1977), 417-420.

[23] M. Farber and R.E. Jamison, *Convexity in graphs and hypergraphs*, SIAM J. Algebraic Discrete Methods **7** (1986), 433-444.

[24] M. Farber and R.E. Jamison, *On local convexity in graphs*, Discrete Math. **66** (1987), 231-247.

[25] J.W. Moon, *Embedding tournaments in simple tournaments*, Discrete Math. **2** (1972), 389-395.

[26] D.B. Parker, R.F. Westhoff, and M.J. Wolf, *On two-path convexity in multipartite tournaments*, European J. Combin. **29** (2008), 641-651.

[27] J.C. Varlet, *Convexity in tournaments*, Bull. Soc. R. Sci. Liège **45** (1976), 570-586.

[28] M.L.J. van de Vel, *Theory of Convex Structures*, North-Holland, Amsterdam, 1993.

Apêndice F

Algorithmic and Structural Aspects of the P_3-Radon Number

Este apêndice contém o artigo "*Algorithmic and Structural Aspects of the P_3-Radon Number*" aceito para publicação no periódico *Annals of Operations Research*.

Algorithmic and Structural Aspects of the P_3-Radon Number

Mitre C. Dourado[1]
Dieter Rautenbach[2]
Vinícius Fernandes dos Santos[3]
Philipp M. Schäfer[2]
Jayme L. Szwarcfiter[1,3,4]
Alexandre Toman[1]

[1] Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil, mitre@nce.ufrj.br, jayme@nce.ufrj.br, alexandre21campos@ppgi.ufrj.br

[2] Institut für Optimierung und Operations Research, Universität Ulm, Ulm, Germany, dieter.rautenbach@uni-ulm.de, philipp.schaefer@gmail.com

[3] PESC, COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil, vinicius.santos@gmail.com

[4] NCE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil

Abstract

The generalization of classical results about convex sets in $\mathbb{R}^n$ to abstract convexity spaces, defined by sets of paths in graphs, leads to many challenging structural and algorithmic problems. Here we study the Radon number for the P_3-convexity on graphs. P_3-convexity has been proposed in connection with rumour and disease spreading processes in networks and the Radon number allows generalizations of Radon's classical convexity result. We establish hardness results and describe efficient algorithms for trees.

Keywords: Graph convexity; Radon partition; Radon number
MSC 2010 classification:

1 Introduction

When does an individual within a network adopt an opinion or contract a disease? How does a rumour or a computer virus spread within a network? As a natural model for such processes [5] one can consider a set of vertices R in a graph G to represent the set of infected individuals and iteratively add further vertices u to R whenever sufficiently many neighbours of u belong to R, that is, someone adopts an opinion/contracts a disease if sufficiently many of his contacts did so.

In the simplest non-trivial case, vertices are added to R whenever at least two of their neighbours belong to R. The collection of all sets of vertices to which no further vertices will be added defines the so-called P_3-convexity on the graph G, that is, a set R of vertices of G is considered to be convex exactly if no vertex outside of R is the middle vertex of a path of order three starting and ending in R. Next to the geodetic convexity [8] defined by shortest paths, and the monophonic convexity [6] defined by induced paths in similar ways, this is one of the natural and well studied convexity spaces defined by paths in graphs. The P_3-convexity was first considered for directed graphs, more specifically for tournaments [7, 10, 11, 13].

Several of the classical convexity parameters have been considered for P_3-convexity. The geodetic number of P_3-convexity is the same as the well known 2-domination number [3]. It corresponds to the minimum number of infected individuals that will infect the entire network in one step. The hull number, which corresponds to the minimum number of infected individuals that will eventually infect the entire network, was investigated in [2, 5]. Also the Carathéodory number [1] was considered.

In the present paper we study the so-called Radon number of P_3-convexity. In 1921 Radon [12] proved that every set of $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect. His result naturally leads to the definition of the Radon number of a general convexity space $(X, \mathcal{C})$ [14] as the smallest integer k for which every set of k points in X can be partitioned into two sets whose convex hulls with respect to $\mathcal{C}$ intersect. A set of vertices R of some graph G that does not have a partition as in Radon's result with respect to P_3-convexity corresponds to a group of individuals with the property that no matter in which way two possible opinions are distributed among the members of the group and then propagated through the network according to P_3-convexity, no individual will ever get under conflicting influences.

Our contributions are as follows. First we introduce relevant notions and terminology in Section 2. In Section 3 we study the algorithmic problem to decide whether a given set of vertices of some graph allows a partition as in Radon's result. In Section 4 we study the algorithmic problem to determine the Radon number of the P_3-convexity of some graph. In both sections we prove hardness results and describe efficient algorithms for trees.

2 Preliminaries

We consider finite, simple, and undirected graphs. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set. For a vertex u of a graph G, let $N_G(u)$ and $d_G(u)$ denote the neighbourhood and degree of u in G. Furthermore, let $N_G[u]$ denote the closed neighbourhood $N_G(u) \cup \{u\}$ of u in G. For a set U of vertices of a graph G, let $G[U]$ denote the subgraph of G induced by U and let $G - U$ denote $G[V(G) \setminus U]$.

Let G be a graph and let R be a set of vertices of G. The set R is *convex in G* if no vertex in $V(G) \setminus R$ has two neighbours in R. The *convex hull $H_G(R)$ of R in G* is the intersection of all convex sets in G containing R. Equivalently, $H_G(R)$ is the smallest convex set in G containing R. A *Radon partition of R* is a partition of R into two disjoint sets R_1 and R_2 with $H_G(R_1) \cap H_G(R_2) \neq \emptyset$. The set R is an *anti-Radon set of G* if it has no Radon partition. The *Radon number $r(G)$ of G* is the minimum integer r such that every set of at least r vertices of G has a Radon partition. Equivalently, the Radon number of G is the maximum cardinality of an anti-Radon set of G plus one, i.e.

$$r(G) = \max\{|R| \mid R \text{ is an anti-Radon set of } G\} + 1.$$

Clearly, if R is an anti-Radon set of a graph G and H is a subgraph of G, then every subset of $R \cap V(H)$ is an anti-Radon set of H.

For a non-negative integer n, let $[n] = \{1, \ldots, n\}$.

3 Recognizing anti-Radon sets

In this section we consider the algorithmic problem to recognize anti-Radon sets in graphs.

ANTI-RADON SET RECOGNITION
Instance: A graph G and a set R of vertices of G.
Question: Does R have a Radon partition?

We prove that ANTI-RADON SET RECOGNITION is NP-complete for bipartite graphs. Furthermore, we give a charaterization of anti-Radon sets, which leads to an efficient algorithm solving ANTI-RADON SET RECOGNITION for trees.

while $\exists u \in V(G) \setminus R$ *with* $|N_G(u) \cap R| \geq 2$ **do**
  $R \leftarrow R \cup \{u\}$;
end
return R;

Algorithm 1: Procedure that determines the convex hull $H_G(R)$.

Theorem 1 ANTI-RADON SET RECOGNITION *is NP-complete even restricted to input graphs that are bipartite.*

Proof: Obviously, using the procedure in Algorithm 1 it is possible to determine the convex hull $H_G(R)$ in G of a set R of vertices of G in polynomial time. This implies that it can be checked in polynomial time whether a given partition of a set of vertices is a Radon partition and hence ANTI-RADON SET RECOGNITION is in NP.

In order to prove NP-completeness, we describe a polynomial reduction of 3SAT [4] to ANTI-RADON SET RECOGNITION. Therefore, let $\mathcal{C}$ be an instance of 3SAT with m clauses $C_1, \ldots, C_m$ in n boolean variables $x_1, \ldots, x_n$. We will construct an instance (G, R) of ANTI-RADON SET RECOGNITION such that $\mathcal{C}$ has a satisfying truth assignment if and only if R has a Radon partition and the encoding length of (G, R) is polynomially bounded in the encoding length of $\mathcal{C}$.

By duplicating clauses and adding further variables, we may assume that $m = 2^{k_m}$ and $n = 2^{k_n}$ for positive integers k_m and k_n such that $k_m + k_n$ is odd. Note that this increases the encoding length of $\mathcal{C}$ only a constant factor. We begin the construction of (G, R).

- For every literal x in $\{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$, we create a literal gadget as in Figure 1 consisting of eight vertices
$$x, u_1(x), u_2(x), u_3(x), u_4(x), v_1(x), v_2(x), w(x)$$
and ten edges
$$\begin{aligned} &xu_1(x), xu_2(x), xu_3(x), xu_4(x),\\ &u_1(x)v_1(x), u_2(x)v_1(x),\\ &u_3(x)v_2(x), u_4(x)v_2(x),\\ &v_1(x)w(x), v_2(x)w(x). \end{aligned}$$

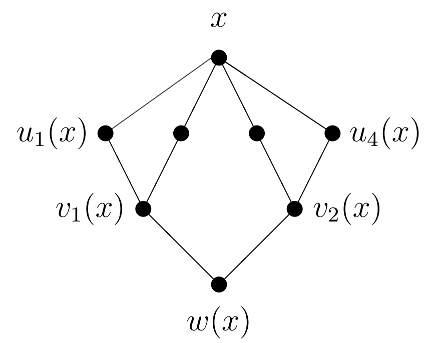

Figure 1: A literal gadget.

- For every i in $\{1, \ldots, \frac{n}{2}\}$, we create four vertices $y(x_{2i-1})$, $y(x_{2i})$, $y'(x_{2i-1})$, and $y'(x_{2i})$ and fourteen edges
$$\begin{aligned} &y'(x_{2i-1})v_1(x_{2i-1}), y'(x_{2i-1})v_1(\bar{x}_{2i-1}), y'(x_{2i-1})v_1(x_{2i}), y'(x_{2i-1})v_1(\bar{x}_{2i}),\\ &y'(x_{2i})v_2(x_{2i-1}), y'(x_{2i})v_2(\bar{x}_{2i-1}), y'(x_{2i})v_2(x_{2i}), y'(x_{2i})v_2(\bar{x}_{2i}),\\ &y(x_{2i-1})w(x_{2i-1}), y(x_{2i-1})w(\bar{x}_{2i-1}), y(x_{2i-1})y'(x_{2i-1}),\\ &y(x_{2i})w(x_{2i}), y(x_{2i})w(\bar{x}_{2i}), y(x_{2i})y'(x_{2i}) \end{aligned}$$
as illustrated in Figure 2.

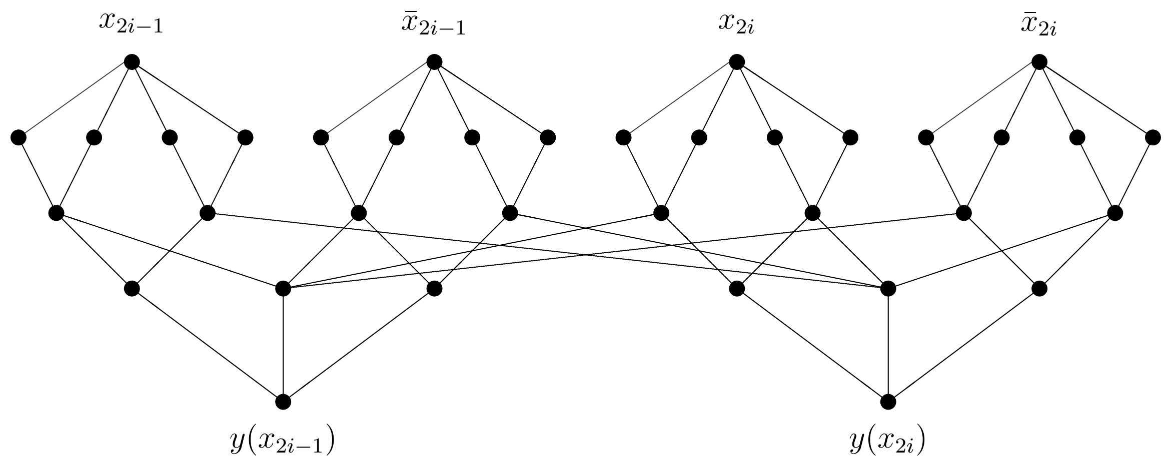

Figure 2: A gadget for the variable pair x_{2i-1} and x_{2i}.

- For every clause C with $C \in \{C_1, \ldots, C_m\}$, we create a vertex C.
- For every literal $x \in \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$ and every clause $C \in \{C_1, \ldots, C_m\}$ such that x occurs in C, we create four vertices $l_1(C,x)$, $l_2(C,x)$, $l_3(C,x)$, and $l_4(C,x)$ and eight edges

$$\begin{aligned}&l_1(C,x)l_3(C,x), l_1(C,x)l_4(C,x), l_2(C,x)l_3(C,x), l_2(C,x)l_4(C,x),\\ &l_3(C,x)C, l_4(C,x)C,\\ &xl_1(C,x), xl_2(C,x)\end{aligned}$$

  as illustrated in Figure 3.
- We create a vertex p and edges between p and all vertices in

$$\bigcup_{C \in \{C_1,\ldots,C_m\}} \quad \bigcup_{x : x \text{ is a literal in } C} \quad \{l_1(C,x), l_2(C,x)\}$$

  as illustrated in Figure 3.
- We create a vertex q and edges between q and all vertices in

$$\bigcup_{i=1}^{n}\bigcup_{j=1}^{4}\{u_j(x_i), u_j(\bar{x}_i)\}$$

  as illustrated in Figure 3.
- We create the inner vertices and all edges of a strictly binary tree T with root z such that $y(x_1), \ldots, y(x_n)$ are the leaves of T as illustrated in Figure 3.
- We create the inner vertices and all edges of a strictly binary tree T' with root z' such that $C_1, \ldots, C_m$ are the leaves of T' as illustrated in Figure 3.

Note that the graph constructed so far is bipartite.

- We identify z with z'.

Note that T has height k_n and T' has height k_m. Since $k_n + k_m$ is odd, identifying z with z' results in the bipartite graph G, which completes the construction of G.

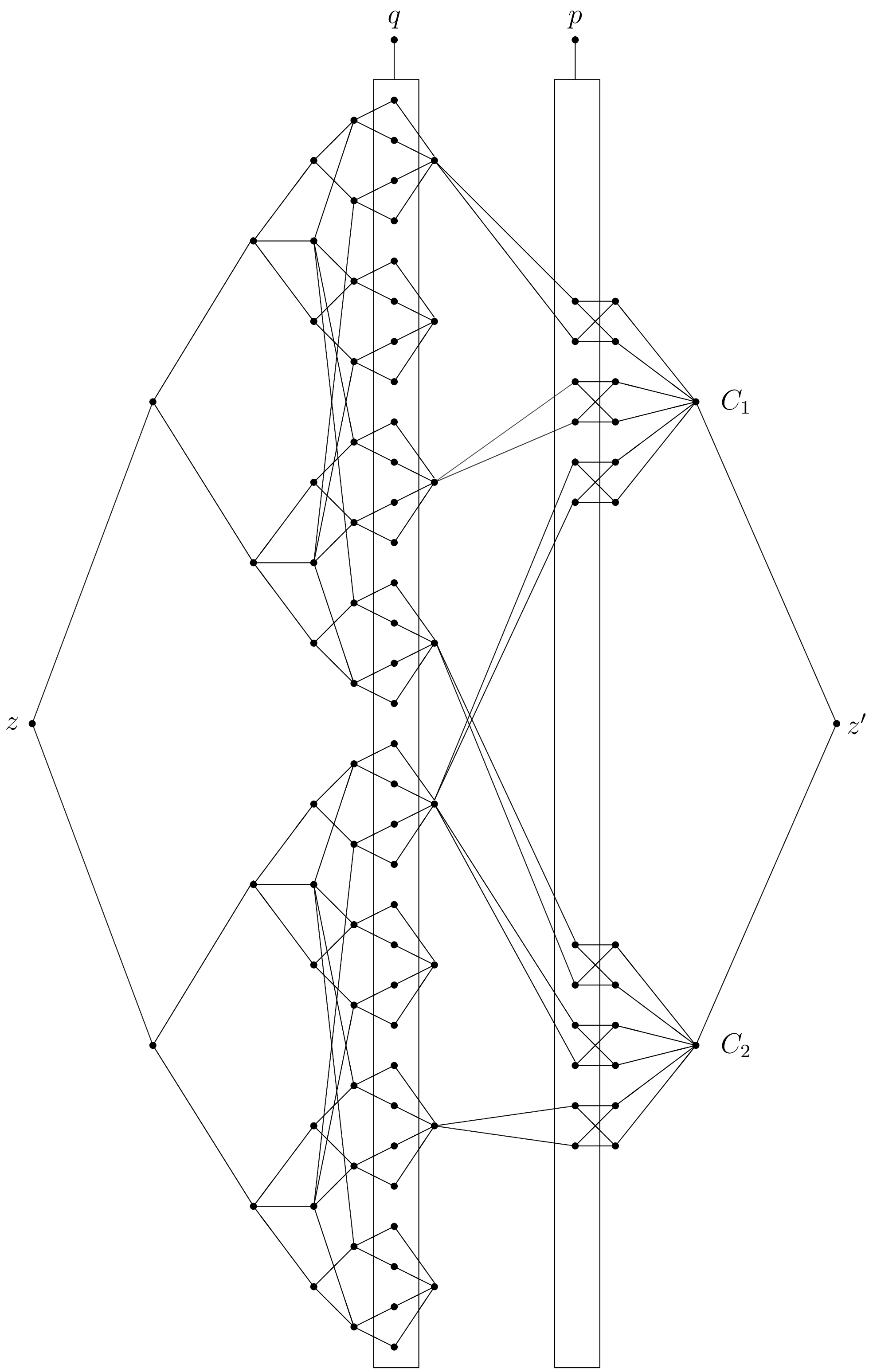

Figure 3: An example of the complete construction for the two clauses $C_1 = \{x_1, x_2, x_3\}$ and $C_2 = \{\bar{x}_2, x_3, x_4\}$ over the four boolean variables x_1, x_2, x_3, and x_4. The vertices z and z' have to be identified.

Let

$$R = \{p, q\} \cup \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}.$$

Clearly, the encoding length of (G, R) is polynomially bounded in n and m.

If $\mathcal{C}$ has a satisfying truth assignment A, then let R_1 consist of p and all literal vertices $x \in \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$ that are `true` in A. Let $R_2 = R \setminus R_1$.

If x is a `true` literal in some clause C, then $l_1(C, x)$ and $l_2(C, x)$ are adjacent to x and p in R_1. Since every clause in $\mathcal{C}$ contains a `true` literal, this easily implies $z' \in H_G(R_1)$.

If $x \in \{x_i, \bar{x}_i\}$ is `false`, then $u_1(x)$ to $u_4(x)$ are adjacent to x and q in R_2. Since for every variable x_i, one literal $x \in \{x_i, \bar{x}_i\}$ is `false`, this easily implies that $z \in H_G(R_2)$.

Since $z = z'$, we obtain that $H_G(R_1)$ and $H_G(R_2)$ intersect, that is, $R = R_1 \cup R_2$ is a Radon partition of R.

Now let $R = R_1 \cup R_2$ be a Radon partition of R with $p \in R_1$.

Claim 1 *$q \in R_2$.*

Proof of Claim 1: For contradiction, we assume that $q \in R_1$. This implies that R_2 is a subset of $\{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$. Since no vertex has two neighbours in R_2, we have $H_G(R_2) = R_2$. Hence $H_G(R_1)$ must contain a literal vertex, say x, from R_2. Let

$$\begin{aligned} U(x) &= \{x, u_1(x), u_2(x), u_3(x), u_4(x), v_1(x), v_2(x), w(x)\} \cup \\ & \bigcup_{C \in \{C_1, \ldots, C_m\}: x \text{ is a literal in } C} \{l_1(C, x), l_2(C, x), l_3(C, x), l_4(C, x)\}. \end{aligned}$$

Since no vertex in $U(x)$ has two neighbours in $V(G) \setminus U(x)$ and $R_1 \subseteq V(G) \setminus U(x)$, we obtain that $H_G(R_1) \subseteq V(G) \setminus U(x)$. Since $x \in U(x)$, we obtain the contradiction that $x \notin H_G(R_1)$, which completes the proof of the claim. $\Box$

Let P and Q denote the vertex sets of the two components of

$$G - (\{z\} \cup \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\})$$

that contain p and q, respectively. Since no vertex in P has two neighbours in $V(G) \setminus P$ and $R_2 \subseteq V(G) \setminus P$, we obtain that $H_G(R_2)$ contains no vertex from P. Similarly, since no vertex in Q has two neighbours in $V(G) \setminus Q$ and $R_1 \subseteq V(G) \setminus Q$, we obtain that $H_G(R_1)$ contains no vertex from Q. Furthermore, considering the set $U(x)$ as in the proof of Claim 1, we obtain that $H_G(R_1)$ contains no vertex from R_2 and $H_G(R_2)$ contains no vertex from R_1. Since $R = R_1 \cup R_2$ is a Radon partition, this implies that $z = z'$ is the unique vertex in $H_G(R_1) \cap H_G(R_2)$. In view of the strictly binary trees T and T', this implies that all vertices of T belong to $H_G(R_2)$ and all vertices of T' belong to $H_G(R_1)$. This implies that for every variable x_i with $i \in \{1, \ldots, n\}$, the set R_2 contains either x_i or $\bar{x}_i$, and for every clause C_j with $j \in \{1, \ldots, m\}$, the set R_1 contains one of the literals in C_j. Altogether, setting all literals x in R_1 to `true` — and consequently their negations to `false` — yields a satisfying partial truth assignment that can be extended to all variables. Therefore, $\mathcal{C}$ has a satisfying truth assignment, which completes the proof. $\Box$

The Algorithm 1 considered in the proof of Theorem 1 leads to some useful observations. If $R = R_1 \cup R_2$ is a Radon partition of a set R of vertices of a graph G, then forming the two intersecting convex hulls $H_G(R_1)$ and $H_G(R_2)$ by iteratively adding single vertices to either R_1 or R_2, there is a first vertex that belongs to both sets. We call such a vertex a *Radon witness vertex for R*. Note that Radon witness vertices are not unique. The following lemma makes this observation more precise.

Lemma 2 *Let G be a graph and let R be a set of vertices of G.*

R is an anti-Radon set of G if and only if there are no two sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ of vertices of G such that

(i) $x_1, \ldots, x_a$ *are distinct,*

(ii) $y_1, \ldots, y_b$ *are distinct,*

(iii) $x_a = y_b$ *and* $\{x_1, \ldots, x_a\} \cap \{y_1, \ldots, y_b\} = \{x_a\}$,

(iv) either $x_i \in R$ *or* x_i *has two neighbours in* $\{x_1, \ldots, x_{i-1}\}$ *for* $i \in [a]$, *and*

(v) either $y_j \in R \setminus \{x_a\}$ *or* y_j *has two neighbours in* $\{y_1, \ldots, y_{j-1}\}$ *for* $j \in [b]$.

Proof: First, we assume that R is not an anti-Radon set of G. This implies the existence of a partition $R = R_1 \cup R_2$ of R such that $H_G(R_1) \cap H_G(R_2) \neq \emptyset$. Let $R_1 = \{x_1, \ldots, x_{a'}\}$ and $R_2 = \{y_1, \ldots, y_{b'}\}$. As noted above, forming the convex hull of R_1 and R_2 by iteratively adding single vertices to either R_1 or R_2, there is a first vertex, say z, that belongs to both sets. This implies that there are two sequences $x_1, \ldots, x_a$ with $a \geq a'$ and $y_1, \ldots, y_b$ with $b \geq b'$ such that $z = x_a = y_b$ that satisfy

- (i), (ii), (iii),
- either $x_i \in R_1$ or x_i has two neighbours in $\{x_1, \ldots, x_{i-1}\}$ for $i \in [a]$, and
- either $y_j \in R_2$ or y_j has two neighbours in $\{y_1, \ldots, y_{j-1}\}$ for $j \in [b]$.

Clearly, $x_1, \ldots, x_a$ satisfies (iv). Since R_1 and R_2 are disjoint, we may assume that $z \notin R_2$, which implies that $y_1, \ldots, y_b$ satisfies (v).

Conversely, if the two sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ of vertices of G satisfy (i) to (v), then $R_1 = R \cap \{x_1, \ldots, x_a\}$ and $R_2 = R \cap \{y_1, \ldots, y_{b-1}\}$ are disjoint subsets of R with $x_a \in H_G(R_1) \cap H_G(R_2)$. Hence R is not an anti-Radon set, which completes the proof. $\Box$

We call a pair of sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ as in Lemma 2 a *Radon witness sequences for R*. Note that a vertex z is a Radon witness vertex for R if and only if there are Radon witness sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ for R with $z = x_a = y_b$. If the vertex x_a belongs to R, then we may assume that $a = 1$, that is, the sequence $x_1, \ldots, x_a$ contains one element only. Altogether, by Lemma 2, a set R of vertices of a graph G is not an anti-Radon set of G if and only if there are Radon witness sequences for R if and only if there is a Radon witness vertex for R.

These notions are helpful to efficiently solve Anti-Radon Set Recognition for trees.

Theorem 3 *Let T be a tree and let R be a set of vertices of T.*

R is an anti-Radon set of T if and only if there is no vertex z of T such that

(i) either $z \notin R$ *and four neighbours of* z *in* T *are in* $H_{T-z}(R)$

(ii) or $z \in R$ and two neighbours of z in T are in $H_{T-z}(R \setminus \{z\})$.

Proof: First we assume that R is no anti-Radon set. By Lemma 2, there are Radon witness sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ for R. Let $z = x_a = y_b$ be the corresponding Radon witness vertex. Let $R_1 = R \cap \{x_1, \ldots, x_a\}$ and $R_2 = R \cap \{y_1, \ldots, y_{b-1}\}$.

If $z \notin R$, then the conditions in Lemma 2 imply that there are four distinct neighbours u_1, v_1, u_2, and v_2 of z in T such that $u_1, v_1 \in H_{T-z}(R_1)$ and $u_2, v_2 \in H_{T-z}(R_2)$, which implies that all four vertices belong to $H_{T-z}(R)$, that is, (i) holds. If $z \in R$, then, as noted above, we may assume $a = 1$. Now the conditions for $y_1, \ldots, y_b$ in Lemma 2 imply that there are two neighbours u_2 and v_2 of z in T such that $u_2, v_2 \in H_{T-z}(R_2)$, which implies that these two vertices belong to $H_{T-z}(R \setminus \{z\})$, that is, (ii) holds.

Conversely, if (i) holds, then let u_1, v_1, u_2, and v_2 be four distinct neighbours of z in T such that $u_1, v_1, u_2, v_2 \in H_{T-z}(R)$. For $i \in [2]$, let R_i denote the set of vertices in R that belong to the same component of $T - z$ as either u_i or v_i. Note that R_1 and R_2 are disjoint, because T is a tree. Clearly, $u_1, v_1 \in H_{T-z}(R_1)$ and $u_2, v_2 \in H_{T-z}(R_2)$, which implies that R is not an anti-Radon set. Finally, if (ii) holds, then a similar construction implies that R is not an anti-Radon set, which completes the proof. $\square$

Since the convex hull of a set of vertices in a graph can be determined in polynomial time using Algorithm 1, Theorem 3 leads to an efficient algorithm solving Anti-Radon Set Recognition for trees. At the end of the next section we explain how to obtain a linear time algorithm.

4 Maximizing anti-Radon sets

In this section we consider the algorithmic problem to determine the Radon number and anti-Radon sets of maximum cardinality for a given graph. We establish the NP-hardness of the following decision problem.

Maximum Anti-Radon Set
Instance: A graph G and an integer k.
Question: Does G have an anti-Radon set of size k?

Furthermore, we develop an efficient algorithm for trees.

A natural certificate for a "Yes"-instance of Maximum Anti-Radon Set would certainly be an anti-Radon set of G of size k. Since Anti-Radon Set Recognition is NP-complete, such a certificate can most probably not be checked efficiently and we do not know whether Maximum Anti-Radon Set lies in NP.

A graph G is a *split graph* if its vertex set admits a partition $V(G) = C \cup I$ into a clique C and an independent set I.

Theorem 4 Maximum Anti-Radon Set *is NP-hard even restricted to input graphs that are split graphs.*

Proof: For the reduction, we use the Set Packing problem, which is known to be NP-complete [9]. The latter problem has as input a family $\mathcal{S} = \{S_1, \ldots, S_n\}$ of non-empty sets and an integer l. The question is whether $\mathcal{S}$ contains l mutually disjoint sets. Given $\mathcal{S}$ and l, we construct an instance (G, k) of Maximum Anti-Radon Set. The elements of the ground set $S_1 \cup \ldots \cup S_n$ of $\mathcal{S}$ are all vertices of the graph G. Besides, G contains a pair of new distinguished vertices w_i and z_i for each set S_i in $\mathcal{S}$. The edges of G are as follows. The set $C = (S_1 \cup \{z_1\}) \cup \ldots \cup (S_n \cup \{z_n\})$ forms a clique of G. In addition, for each of the distinguished vertices w_i, we add an edge $w_i z_i$ and edges $w_i v$ for each $v \in S_i$. Finally, we define $k = l$. This completes the construction of (G, k). Observe that $I = \{w_1, \ldots, w_n\}$ is an independent set of G. Therefore, G is a split graph with partition $V(G) = C \cup I$. Without loss of generality, we may assume $k = l \geq 4$. We prove that $\mathcal{S}$ contains l mutually disjoint sets if and only if G has an anti-Radon set of size k.

Suppose that $\mathcal{S}$ contains l mutually disjoint sets, say $S_1, \ldots, S_l$. Since $S_i \cap S_j = \emptyset$ for distinct $i, j \in [l]$, we obtain $N_G[w_i] \cap N_G[w_j] = \emptyset$ for distinct $i, j \in [k]$. This implies that $H_G(J) = J$ for every subset J of $\{w_1, \ldots, w_k\}$ and hence $\{w_1, \ldots, w_k\}$ is an anti-Radon set of G of size k.

Conversely, suppose that G has an anti-Radon set R of size $k \geq 4$. If R contains two vertices v_1 and v_2 such that $v_1 \in C$, then, since every vertex in I has at least two neighbours in C and every vertex in I has a common neighbour with every vertex in C, we obtain $V(G) = H_G(\{v_1, v_2\})$. This implies that $R = R_1 \cup R_2$ with $R_1 = \{v_1, v_2\}$ and $R_2 = R \setminus R_1$ is a Radon partition of R, which is a contradiction. Hence R is a subset of $I = \{w_1, \ldots, w_n\}$. If R contains two vertices, say w_i and w_j, from I such that $S_i \cap S_j$ contains a vertex, say v, then again $V(G) = H_G(\{w_i, w_j\})$, which is a contradiction. Hence the sets S_i for $i \in [n]$ with $w_i \in R$ are l mutually disjoint sets in $\mathcal{S}$, which completes the proof. $\Box$

We develop a reduction principle relating anti-Radon sets of a connected graph G that has a vertex u such that all edges of G incident with u are bridges of G, to anti-Radon sets of the components of $G - u$. The main application of this principle is an efficient algorithm computing the Radon number and largest anti-Radon sets of trees.

Let G be a graph and let u be a vertex of G. Throughout this section, let $G^{u \leftarrow x}$ denote the graph that arises by adding to G a new vertex x and a new edge ux. Let

$$\begin{aligned}
\mathcal{R}_+(G, u) &= \{R \mid R \text{ is an anti-Radon set of } G \text{ and } u \in H_G(R)\},\\
\mathcal{R}_-(G, u) &= \{R \mid R \text{ is an anti-Radon set of } G \text{ and } u \notin H_G(R)\},\\
\mathcal{R}'_+(G, u) &= \{R \mid R \subseteq V(G),\ \{x\} \cup R \text{ is an anti-Radon set of } G^{u \leftarrow x}, \text{ and } u \in H_G(R)\}, \text{ and}\\
\mathcal{R}'_-(G, u) &= \{R \mid R \subseteq V(G), \{x\} \cup R \text{ is an anti-Radon set of } G^{u \leftarrow x}, \text{ and } u \notin H_G(R)\}.
\end{aligned}$$

Furthermore, for a set S, let

$$\begin{aligned}
r_+(G, S, u) &= \max\{|R| \mid R \in \mathcal{R}_+(G, u) \text{ and } R \subseteq S\},\\
r_-(G, S, u) &= \max\{|R| \mid R \in \mathcal{R}_-(G, u) \text{ and } R \subseteq S\},\\
r'_+(G, S, u) &= \max\{|R| \mid R \in \mathcal{R}'_+(G, u) \text{ and } R \subseteq S\}, \text{ and}\\
r'_-(G, S, u) &= \max\{|R| \mid R \in \mathcal{R}'_-(G, u) \text{ and } R \subseteq S\}.
\end{aligned}$$

These definitions immediately imply

$$r(G) = \max\{r_+(G, V(G), u), r_-(G, V(G), u)\} + 1. \tag{1}$$

Now let G be a connected graph and let u be a vertex of G such that all edges of G incident with u are bridges of G. Let $G_1, \ldots, G_k$ denote the components of $G - u$ and let u_i denote the unique neighbour of u in $V(G_i)$ for $i \in [k]$. Let R be a set of vertices of G and let $R_i = R \cap V(G_i)$ for $i \in [k]$. Let S be a set.

Lemma 5 *Let G, u, R, G_i, u_i, R_i for $i \in [k]$, and S be as above.*

R belongs to $\mathcal{R}_+(G,u)$ and $R \subseteq S$ if and only if

- *$u \in R$ only if $u \in S$,*
- *$R_i \subseteq S$ for $i \in [k]$,*
- *and one of the following cases occurs.*

  (i) *$u \in R$ and $R_i \in \mathcal{R}'_-(G_i, u_i)$ for $i \in [k]$.*

  (ii) *$u \in R$ and there is some index $i_1 \in [k]$ such that $R_{i_1} \in \mathcal{R}'_+(G_{i_1}, u_{i_1})$ and $R_i \in \mathcal{R}'_-(G_i, u_i)$ for $i \in [k] \setminus \{i_1\}$.*

  (iii) *$u \notin R$ and there are two distinct indices $i_1, i_2 \in [k]$ such that $R_{i_j} \in \mathcal{R}_+(G_{i_j}, u_{i_j})$ for $j \in [2]$ and $R_i \in \mathcal{R}'_-(G_i, u_i)$ for $i \in [k] \setminus \{i_1, i_2\}$.*

  (iv) *$u \notin R$ and there are three distinct indices $i_1, i_2, i_3 \in [k]$ such that $R_{i_j} \in \mathcal{R}'_+(G_{i_j}, u_{i_j})$ for $j \in [3]$ and $R_i \in \mathcal{R}'_-(G_i, u_i)$ for $i \in [k] \setminus \{i_1, i_2, i_3\}$.*

Proof: $R \subseteq S$ is obviously equivalent to

- $u \in R$ only if $u \in S$
- and $R_i \subseteq S$ for $i \in [k]$.

Therefore, it remains to prove that $R \in \mathcal{R}_+(G,u)$ if and only if one of the cases (i) to (iv) occurs.

For $i \in [k]$, let $R'_i = R_i \cup \{u\}$ and let G'_i be the subgraph of G induced by $\{u\} \cup V(G_i)$. Note that $G_i^{u_i \leftarrow x}$ is isomorphic to G'_i.

First, we prove the necessity and assume that R belongs to $\mathcal{R}_+(G,u)$. Let l denote the number of indices i in $[k]$ with $u_i \in H_{G_i}(R_i)$. Since R is an anti-Radon set, we have that either $u \in R$ and $l \leq 1$ or $u \notin R$ and $l \leq 3$. Furthermore, if $u \notin R$, then $u \in H_G(R)$ implies that $l \geq 2$. Accordingly, we consider four different cases.

Case 1 *$u \in R$ and $l = 0$.*

Since R is an anti-Radon set of G, the subset R'_i of R is an anti-Radon set of G'_i for $i \in [k]$. Since $l = 0$, we have $u_i \notin H_{G_i}(R_i)$ for $i \in [k]$. Together this implies $R_i \in \mathcal{R}'_-(G_i, u_i)$ for $i \in [k]$, that is, (i) holds.

Case 2 *$u \in R$ and $l = 1$.*

Since $l = 1$, there is a unique index $i_1 \in [k]$ with $u_{i_1} \in H_{G_{i_1}}(R_{i_1})$. Since R is an anti-Radon set of G, the subset R'_i of R is an anti-Radon set of G'_i for $i \in [k]$. By the definition of l, this implies that $R_{i_1} \in \mathcal{R}'_+(G_{i_1}, u_{i_1})$ and $R_i \in \mathcal{R}'_-(G_i, u_i)$ for $i \in [k] \setminus \{i_1\}$, that is, (ii) holds.

Case 3 *$u \notin R$ and $l = 2$.*

Since $l = 2$, there are exactly two distinct indices $i_1, i_2 \in [k]$ with $u_{i_j} \in H_{G_{i_j}}(R_{i_j})$ for $j \in [2]$. Since R_i is an anti-Radon set of G_i, this implies $R_{i_j} \in \mathcal{R}_+(G_{i_j}, u_{i_j})$ for $j \in [2]$. Since $u \in H_G(R_{i_1} \cup R_{i_2})$, the set R'_i is an anti-Radon set of G'_i for $i \in [k] \setminus \{j_1, j_2\}$. Together, this implies that (iii) holds.

Case 4 *$u \notin R$ and $l = 3$.*

Since $l = 3$, there are exactly three distinct indices $i_1, i_2, i_3 \in [k]$ with $u_{i_j} \in H_{G_{i_j}}(R_{i_j})$ for $j \in [3]$. Since $u \in H_G(R_{i_1} \cup R_{i_2}) \cap H_G(R_{i_1} \cup R_{i_3}) \cap H_G(R_{i_2} \cup R_{i_3})$, the set R'_i is an anti-Radon set of G'_i for $i \in [k]$, which implies that (iv) holds.

This completes the proof of the necessity.

Now, we prove the sufficiency and assume that one of the Cases (i) to (iv) occurs. It is obvious that $u \in H_G(R)$ in each of these cases. Hence, for contradiction, we assume that R is not an anti-Radon set. By Lemma 2, there is a pair $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ of Radon witness sequences for R. Let $z = x_a = y_b$. If $z = u$, then

- either $u \in R$ and $u_i \in H_{G_i}(R_i)$ for at least two indices $i \in [k]$
- or $u \not\in R$ and $u_i \in H_{G_i}(R_i)$ for at least four indices $i \in [k]$.

Clearly, this does not occur in any of the Cases (i) to (iv). Hence z belongs to G_{i^*} for some $i^* \in [k]$. Since R_{i^*} is an anti-Radon set of G_{i^*} in each of the Cases (i) to (iv), deleting from the two Radon witness sequences for R all vertices that do not belong to R_{i^*} results in two subsequences that are not Radon witness sequences for R_{i^*}. This implies that u belongs to exactly one of the two witness sequences and that u precedes u_{i^*} in that sequence. Therefore, $u \in H_G(R \setminus R_{i^*})$ and R'_{i^*} is no anti-Radon set of G'_{i^*}. We obtain that $R_{i^*} \not\in \mathcal{R}'_+(G_{i^*}, u_{i^*}) \cup \mathcal{R}'_-(G_{i^*}, u_{i^*})$. This necessarily implies that Case (iii) occurs with $i^* \in \{i_1, i_2\}$. Nevertheless, $u \not\in H_G(R \setminus R_{i^*})$ in this case, which is a contradiction and completes the proof. $\Box$

The proof of the following lemma is very similar to the proof of Lemma 5 and we leave it to the reader.

Lemma 6 *Let G, u, R, G_i, u_i, R_i for $i \in [k]$, and S be as above.*
R belongs to $\mathcal{R}_-(G, u)$ and $R \subseteq S$ if and only if

- *$u \not\in R$,*
- *$R_i \subseteq S$ for $i \in [k]$,*
- *and one of the following cases occurs.*
  - *(i) $R_i \in \mathcal{R}_-(G_i, u_i)$ for $i \in [k]$.*
  - *(ii) There is some index $i_1 \in [k]$ such that $R_{i_1} \in \mathcal{R}_+(G_{i_1}, u_{i_1})$ and $R_i \in \mathcal{R}_-(G_i, u_i)$ for $i \in [k] \setminus \{i_1\}$.*

Since the definition of $\mathcal{R}'_+(G, u)$ involves conditions for two distinct sets $\{x\} \cup R$ and R, the proof of the following lemma contains some subtleties.

Lemma 7 *Let G, u, R, G_i, u_i, R_i for $i \in [k]$, and S be as above.*
R belongs to $\mathcal{R}'_+(G, u)$ and $R \subseteq S$ if and only if

- *$u \in R$ only if $u \in S$,*
- *$R_i \subseteq S$ for $i \in [k]$,*
- *and one of the following cases occurs.*

(i) $u \in R$ *and* $R_i \in \mathcal{R}'_-(G_i, u_i)$ *for* $i \in [k]$.

(ii) $u \notin R$ *and there are two distinct indices* $i_1, i_2 \in [k]$ *such that* $R_{i_j} \in \mathcal{R}'_+(G_{i_j}, u_{i_j})$ *for* $j \in [2]$ *and* $R_i \in \mathcal{R}'_-(G_i, u_i)$ *for* $i \in [k] \setminus \{i_1, i_2\}$.

Proof: $R \subseteq S$ is obviously equivalent to

- $u \in R$ only if $u \in S$
- and $R_i \subseteq S$ for $i \in [k]$.

Therefore, it remains to prove that $R \in \mathcal{R}'_+(G, u)$ if and only if one of the cases (i) and (ii) occurs.

For $i \in [k]$, let $R'_i = R_i \cup \{u\}$ and let G'_i be the subgraph of G induced by $\{u\} \cup V(G_i)$.

First, we prove the necessity and assume that R belongs to $\mathcal{R}'_+(G, u)$. Let l denote the number of indices i in $[k]$ with $u_i \in H_{G_i}(R_i)$. Since $\{x\} \cup R$ is an anti-Radon set of $G^{u \leftarrow x}$, we have that either $u \in R$ and $l = 0$ or $u \notin R$ and $l \leq 2$. Furthermore, if $u \notin R$, then $u \in H_G(R)$ implies that $l \geq 2$, that is, $l = 2$ in this case. Accordingly, we consider two different cases.

Case 1 $u \in R$ *and* $l = 0$.

It follows as in the proof of Lemma 5 that (i) holds.

Case 2 $u \notin R$ *and* $l = 2$.

Since $l = 2$, there are exactly two distinct indices $i_1, i_2 \in [k]$ with $u_{i_j} \in H_{G_{i_j}}(R_{i_j})$ for $j \in [2]$. Since $u \in H_{G^{u \leftarrow x}}(R_{i_1} \cup R_{i_2}) \cap H_{G^{u \leftarrow x}}(R_{i_1} \cup \{x\}) \cap H_{G^{u \leftarrow x}}(R_{i_2} \cup \{x\})$, the set R'_i is an anti-Radon set of G'_i for $i \in [k]$, which implies that (ii) holds.

This completes the proof of the necessity. The proof of the sufficiency is straightforward and left to the reader. $\square$

The proof of the following lemma is very similar to the proof of Lemma 7 and we leave it to the reader.

Lemma 8 *Let* G, u, R, G_i, u_i, R_i *for* $i \in [k]$, *and* S *be as above.*

R belongs to $\mathcal{R}'_-(G, u)$ and $R \subseteq S$ if and only if

- $u \notin R$,
- $R_i \subseteq S$ *for* $i \in [k]$,
- *and one of the following cases occurs.*

(i) $R_i \in \mathcal{R}_-(G_i, u_i)$ *for* $i \in [k]$.

(ii) There is some index $i_1 \in [k]$ *such that* $R_{i_1} \in \mathcal{R}_+(G_{i_1}, u_{i_1})$ *and* $R_i \in \mathcal{R}'_-(G_i, u_i)$ *for* $i \in [k] \setminus \{i_1\}$.

The Lemmas 5 to 8 immediately imply the following recurrence formulas.

Corollary 9 *Let* G, u, R, G_i, u_i, R_i *for* $i \in [k]$, *and* S *be as above.*

(i) If $u \in S$, then $r_+(G,S,u)$ equals the maximum of the following four expressions.

$$1 + \sum_{i\in[k]} r'_-(G_i,S,u_i),$$

$$\max_{i_1\in[k]}\left(1 + r'_+(G_{i_1},S,u_{i_1}) + \sum_{i\in[k]\setminus\{i_1\}} r'_-(G_i,S,u_i)\right),$$

$$\max_{i_1,i_2\in[k]}\left(r_+(G_{i_1},S,u_{i_1}) + r_+(G_{i_2},S,u_{i_2}) + \sum_{i\in[k]\setminus\{i_1,i_2\}} r'_-(G_i,S,u_i)\right), \text{ and}$$

$$\max_{i_1,i_2,i_3\in[k]}\left(r'_+(G_{i_1},S,u_{i_1}) + r'_+(G_{i_2},S,u_{i_2}) + r'_+(G_{i_3},S,u_{i_3}) + \sum_{i\in[k]\setminus\{i_1,i_2,i_3\}} r'_-(G_i,S,u_i)\right)$$

where the individual maxima are taken over distinct indices i_j.

(ii) If $u \notin S$, then $r_+(G,S,u)$ equals the maximum of the following two expressions.

$$\max_{i_1,i_2\in[k]}\left(r_+(G_{i_1},S,u_{i_1}) + r_+(G_{i_2},S,u_{i_2}) + \sum_{i\in[k]\setminus\{i_1,i_2\}} r'_-(G_i,S,u_i)\right) \text{ and}$$

$$\max_{i_1,i_2,i_3\in[k]}\left(r'_+(G_{i_1},S,u_{i_1}) + r'_+(G_{i_2},S,u_{i_2}) + r'_+(G_{i_3},S,u_{i_3}) + \sum_{i\in[k]\setminus\{i_1,i_2,i_3\}} r'_-(G_i,S,u_i)\right)$$

where the individual maxima are taken over distinct indices i_j.

(iii) $r_-(G,S,u)$ equals

$$\max\left\{\sum_{i\in[k]} r_-(G_i,S,u_i),\quad \max_{i_1\in[k]}\left(r_+(G_{i_1},S,u_{i_1}) + \sum_{i\in[k]\setminus\{i_1\}} r_-(G_i,S,u_i)\right)\right\}.$$

(iv) If $u \in S$, then $r'_+(G,S,u)$ equals the maximum of the following two expressions.

$$1 + \sum_{i\in[k]} r'_-(G_i,S,u_i), \text{ and}$$

$$\max_{i_1,i_2\in[k]}\left(r'_+(G_{i_1},S,u_{i_1}) + r'_+(G_{i_2},S,u_{i_2}) + \sum_{i\in[k]\setminus\{i_1,i_2\}} r'_-(G_i,S,u_i)\right)$$

where the maximum is taken over distinct indices i_1 and i_2.

(v) If $u \notin S$, then $r'_+(G,S,u)$ equals

$$\max_{i_1,i_2\in[k]}\left(r'_+(G_{i_1},S,u_{i_1}) + r'_+(G_{i_2},S,u_{i_2}) + \sum_{i\in[k]\setminus\{i_1,i_2\}} r'_-(G_i,S,u_i)\right)$$

where the maximum is taken over distinct indices i_1 and i_2.

(vi) $r'_-(G, S, u)$ *equals*

$$\max\left\{\sum_{i\in[k]} r_-(G_i, S, u_i),\ \ \max_{i_1\in[k]}\left(r_+(G_{i_1}, S, u_{i_1}) + \sum_{i\in[k]\setminus\{i_1\}} r'_-(G_i, S, u_i)\right)\right\}.$$

If G is a graph with exactly one vertex u and S is a set, then $u \in S$ implies

$$r_+(G, S, u) = r'_+(G, S, u) = 1 \text{ and } r_-(G, S, u) = r'_-(G, S, u) = 0$$

and $u \notin S$ implies

$$r_+(G, S, u) = r'_+(G, S, u) = r_-(G, S, u) = r'_-(G, S, u) = 0.$$

These initial values together with the recurrences from Corollary 9 yield an efficient algorithm solving the following problem on trees.

Maximum Anti-Radon Subset

Instance:	A graph G and a set S of vertices of G.
Task:	Determine an anti-Radon set R of G of maximum cardinality that is a subset of S.

Note that Maximum Anti-Radon Subset can be considered a common generalization of Anti-Radon Set Recognition and Maximum Anti-Radon Set. In fact, a set S of vertices of some graph G is an anti-Radon set of G exactly if Maximum Anti-Radon Subset on the instance (G, S) would have $R = S$ as unique solution. Furthermore, if R is the solution of Maximum Anti-Radon Subset on the instance $(G, S) = (G, V(G))$, then the Radon number of G equals $|R| + 1$.

Theorem 10 *There is a linear time algorithm solving* Maximum Anti-Radon Subset *on trees.*

Proof: Given a tree T and a set S of vertices of T, we select an arbitrary vertex r of T and consider the rooted version of T with r as root. Using the initial values and the recurrences, we process the vertices of T in an order of non-increasing depth and determine for every vertex u of T, the values

$$r_+\left(T_{\leq u}, S, u\right), r'_+\left(T_{\leq u}, S, u\right), r_-\left(T_{\leq u}, S, u\right), \text{ and } r'_-\left(T_{\leq u}, S, u\right)$$

where $T_{\leq u}$ denotes the subtree of T rooted in u that contains u and all descendants of u in T. In view of the recurrences, these four values can be determined in $O(d_T(u))$ time for each vertex u using the corresponding values of the children of u in T. Altogether, this results in $O\left(\sum\limits_{u\in V(T)} d_T(u)\right)$ time, which is linear in the order of the tree T. By definition, the maximum cardinality of an anti-Radon set R of T that is a subset of S equals $\max\{r_+(T, S, r), r_-(T, S, r)\}$. Furthermore, keeping track of the maximizers in the individual recurrences, it is possible to determine a corresponding anti-Radon set R in the same time. $\Box$

Working on the proof of Theorem 10, we noticed the following remarkable property of trees without vertices of degree 2.

Theorem 11 *If T is a tree with no vertex of degree 2, then some anti-Radon set of T of maximum cardinality contains only endvertices of T.*

Before we prove this result, we need two lemmas.

Lemma 12 *If T is a tree, u_0 is an endvertex of T, u_1 is the neighbour of u_0 in T, and R^- is a subset of $V(T) \setminus \{u_0\}$ that is an anti-Radon set of T with $u_1 \not\in H_T(R^-)$, then $R^- \cup \{u_0\}$ is an anti-Radon set of T.*

Proof: Let $R = R^- \cup \{u_0\}$. For contradiction, we assume that R is not an anti-Radon set of T. By Lemma 2, there are Radon witness sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ for R. Since u_0 is an endvertex, the vertex u_0 is no Radon witness vertex for R, that is, $x_a \neq u_0$. Let $P : u_0u_1 \ldots u_l$ be the path in T between u_0 and $u_l = x_a$. Since R^- is an anti-Radon set of T, the vertex u_0 belongs to exactly one of the Radon witness sequences. Let R_1 be the set of elements of R that belong to the Radon witness sequence that does not contain u_0. Let $R_2 = R \setminus R_1$. Clearly, $u_0 \in R_2$ and $u_l \in H_T(R_1) \cap H_T(R_2)$. Let $R_2^- = R_2 \setminus \{u_0\}$. Since R^- is an anti-Radon set of T, we have $u_l \not\in H_T(R_2^-)$. This implies that for $i \in [l-1]$, the set $H_T(R_2^-)$ contains exactly one neighbour of u_i, say v_i, that does not lie on P. Since $u_1 \in H_T(\{u_l, v_1, \ldots, v_{l-1}\})$, this implies the contradiction $u_1 \in H_T(R_1 \cup R_2^-) = H_T(R^-)$. □

Lemma 13 *If T is a tree with no vertex of degree 2, u_0 is an endvertex of T, u_1 is the neighbour of u_0 in T, and R is an anti-Radon set of T with $u_0 \in R$ and $u_1 \not\in H_T(R \setminus \{u_0\})$, then there is a path $P : u_0 \ldots u_l$ in T with $u_l \not\in H_T(R)$ such that $R' = (R \setminus \{u_0\}) \cup \{u_l\}$ is an anti-Radon set of T with $u_{l-1} \not\in H_T(R' \setminus \{u_l\})$.*

Proof: Let $R^- = R \setminus \{u_0\}$.

Claim 1 *There is a path $P : u_0u_1 \ldots u_l$ and vertices $v_i \not\in V(P)$ for $i \in [l-1]$ such that $u_i \in H_T(R) \setminus H_T(R^-)$ for $i \in [l-1]$, v_i is a unique neighbour of u_i in $H_T(R^-)$ for $i \in [l-1]$, and $u_l \not\in H_T(R)$.*

Proof of Claim 1: Let the path $P : u_0 \ldots u_l$ and the vertices $v_i \not\in V(P)$ for $i \in [l-1]$ be such that $u_i \in H_T(R) \setminus H_T(R^-)$ and v_i is a unique neighbour of u_i in $H_T(R^-)$ for $i \in [l-1]$. Note that $P : u_0u_1$ trivially satisfies these properties.

If $u_l \in H_T(R^-)$, then $u_{l-1} \in H_T(\{v_{l-1}, u_l\}) \subseteq H_T(R^-)$, which is a contradiction. Hence $u_l \not\in H_T(R^-)$. If $u_l \not\in H_T(R)$, then the path P and the vertices v_i for $i \in [l-1]$ have the desired properties as stated in the claim. Hence we may assume that $u_l \in H_T(R) \setminus H_T(R^-)$. This implies that exactly one neighbour of u_l distinct from u_{l-1}, say v_l, belongs to $H_T(R^-)$. Now $d_T(u_l) \geq 2$. Since T has no vertex of degree 2, the vertex u_l has a neighbour distinct from u_{l-1} and v_l, say u_{l+1}. Now the path $P' : u_0u_1 \ldots u_{l+1}$ and the vertices $v_i \not\in V(P')$ for $i \in [l]$ satisfy the properties stated at the beginning of the proof of this claim.

Since T is a finite tree, this easily implies the claim. □

Let $R' = (R \setminus \{u_0\}) \cup \{u_l\}$. Note that $R' \setminus \{u_l\} = R^-$ and hence $u_{l-1} \not\in H_T(R' \setminus \{u_l\})$. It remains to prove that R' is an anti-Radon set of T. For contradiction, we assume that R' is not an anti-Radon set and that z is a Radon witness vertex for R'. Since $u_l \not\in H_T(R)$ and $u_{l-1} \not\in H_T(R^-)$, no neighbour of u_l belongs to $H_T(R^-)$. By Lemma 12, this implies that the vertex z does not belong to the component of $T - u_{l-1}$ that contains u_l. Since for $i \in [l-1]$, we have that $u_i \notin R'$ and that $H_{T-u_i}(R')$ contains exactly the two neighbours u_{i+1} and v_i of u_i, it follows that every remaining vertex that would be a Radon witness vertex for R' would also be a Radon witness vertex for R, which is a contradiction and completes the proof. $\square$

We are now in a position to prove Theorem 11.

Proof of Theorem 11: Let R be an anti-Radon set of T of maximum cardinality such that the number of vertices in R of degree at least 3 is as small as possible. For contradiction, we assume that $u_0 \in R$ is such that $d_T(u_0) \geq 3$. Let $F_1, \ldots, F_k$ denote the components K of $T - u_0$ with the property that the neighbour of u_0 in $V(K)$ does not belong to $H_T(R \setminus \{u_0\})$. Let $|V(F_1)| \geq \ldots \geq |V(F_k)|$. Since R is an anti-Radon set of T and $d_T(u_0) \geq 3$, we have $k \geq 2$. We assume that R and u_0 are chosen subject to the above conditions such that $|V(F_1)|$ is as large as possible. Applying Lemma 13 to the tree $T' = T[\{u_0\} \cup V(F_2)]$ and the vertex u_0 yields a path $P : u_0 \ldots u_l$ in T'. The properties of P given in Lemma 13 easily imply that the set $R' = (R \setminus \{u_0\}) \cup \{u_l\}$ is an anti-Radon set of T with $|R'| = |R|$ and $u_{l-1} \notin H_T(R' \setminus \{u_l\})$. If u_l is an endvertex of T, then R' has less vertices of degree at least 3 than R, which contradicts the choice of R. Hence, we may assume that $d_T(u_l) \geq 3$. Now the component K' of $T - u_l$ that contains u_{l-1} has strictly more vertices than F_1 and the neighbour u_{l-1} of u_l in $V(K')$ does not belong to $H_T(R' \setminus \{u_l\})$. This contradicts the choice of R and u_0, which completes the proof. $\square$

It seems possible that for all trees T that have only vertices of degree 3 and 1, all anti-Radon sets of maximum cardinality consist only of endvertices, that is, for such trees one might be able to replace "some" with "all" in Theorem 11. The example in Figure 4 shows that this is impossible in general.

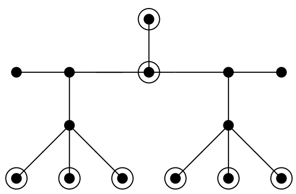

Figure 4: An anti-Radon set of maximum cardinality that contains an internal vertex.

Acknowledgement This work has been done in the context of the CAPES-DAAD PROBRAL project "Cycles, Convexity, and Searching in Graphs". The first and fifth authors have been supported by CNPq and FAPERJ. The third author has been supported by FAPERJ. The sixth author has been supported by CAPES.

References

[1] R.M. Barbosa, E.M.M. Coelho, M.C. Dourado, D. Rautenbach, and J.L. Szwarcfiter, On the Carathéodory number for the convexity of paths of order three, to appear in *Electronic Notes in Discrete Mathematics*.

[2] C.C. Centeno, M.C. Dourado, L.D. Penso, D. Rautenbach, and J.L. Szwarcfiter, Irreversible conversion of graphs, *Theor. Comput. Sci.* **412** (2011), 3693-3700.

[3] M. Chellali, O. Favaron, A. Hansberg, and L. Volkmann, k-Domination and k-Independence in Graphs: A Survey, *Graphs Combin.*, DOI: 10.1007/s00373-011-1040-3.

[4] S.A. Cook, The complexity of theorem-proving procedures, *ACM, Proc. 3rd ann. ACM Sympos. Theory Computing*, Shaker Heights, Ohio 1971, 151-158 (1971).

[5] P.A. Dreyer and F.S. Roberts, Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion, *Discrete Appl. Math.* **157** (2009), 1615-1627.

[6] P. Duchet, Convex sets in graphs. II: Minimal path convexity, *J. Comb. Theory, Ser. B* **44** (1988), 307-316.

[7] P. Erdős, E. Fried, A. Hajnal, and E.C. Milner, Some remarks on simple tournaments, *Algebra Univers.* **2** (1972), 238-245.

[8] M. Farber and R.E. Jamison, Convexity in graphs and hypergraphs, *SIAM J. Algebraic Discrete Methods* **7** (1986), 433-444.

[9] R.M. Karp, Reducibility among combinatorial problems, in Complexity of Computer Computations 1972, Plenum Press, New York, pp. 85-103.

[10] J.W. Moon, Embedding tournaments in simple tournaments, *Discrete Math.* **2** (1972), 389-395.

[11] D.B. Parker, R.F. Westhoff, and M.J. Wolf, On two-path convexity in multipartite tournaments, *European J. Combin.* **29** (2008), 641-651.

[12] J. Radon, Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten, *Math. Ann.* **83** (1921), 113-115.

[13] J.C. Varlet, Convexity in tournaments, *Bull. Soc. R. Sci. Liège* **45** (1976), 570-586.

[14] M.L.J. van de Vel, Theory of Convex Structures, North-Holland, Amsterdam, 1993.

Apêndice G

An upper bound on the P_3-Radon number

Este apêndice contém o artigo "*An upper bound on the P_3-Radon number*", publicado no periódico *Discrete Mathematics*.

Discrete Mathematics 312 (2012) 2433–2437

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

An upper bound on the P_3-Radon number

Mitre C. Dourado [a], Dieter Rautenbach [b,*], Vinícius Fernandes dos Santos [c], Philipp M. Schäfer [b], Jayme L. Szwarcfiter [a,c,d], Alexandre Toman [a]

[a] Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil
[b] Institut für Optimierung und Operations Research, Universität Ulm, Ulm, Germany
[c] PESC, COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil
[d] NCE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil

ARTICLE INFO

Article history:
Received 4 October 2011
Received in revised form 26 April 2012
Accepted 2 May 2012
Available online 26 May 2012

Keywords:
Graph convexity
Radon partition
Radon number

ABSTRACT

The generalization of classical results about convex sets in $\mathbb{R}^n$ to abstract convexity spaces, defined by sets of paths in graphs, leads to many challenging structural and algorithmic problems. Here we study the Radon number for the P_3-convexity on graphs.

A set R of vertices of a graph G is P_3-convex if no vertex in $V(G) \setminus R$ has two neighbours in R. The P_3-convex hull of a set of vertices is the smallest P_3-convex set containing it. The P_3-Radon number $r(G)$ of a graph G is the smallest integer r such that every set R of r vertices of G has a partition $R = R_1 \cup R_2$ such that the P_3-convex hulls of R_1 and R_2 intersect. We prove that $r(G) \leq \frac{2}{3}(n(G)+1)+1$ for every connected graph G and characterize all extremal graphs.

1. Introduction

In 1921 Radon [20] proved that every set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect. His result naturally leads to the definition of the Radon number of a general convexity space $(X, \mathcal{C})$ [22] as the smallest integer k for which every set of k points in X can be partitioned into two sets whose convex hulls with respect to $\mathcal{C}$ intersect.

In the present paper we study the Radon number of the P_3-convexity of finite graphs, where some set U of vertices of a finite graph G is considered to be convex exactly if no vertex of G that does not belong to U has two neighbours in U, that is, no vertex outside of U is the middle vertex of a path of order 3 starting and ending in U. Next to the geodetic convexity [16] defined by shortest paths, and the monophonic convexity [11] defined by induced paths in similar ways, this is one of the natural and well studied convexity spaces defined by paths in graphs. See [4,5,9] for further examples and insights concerning such convexity spaces. The P_3-convexity was first considered for directed graphs, and more specifically for tournaments [15,18,23,19].

Several of the classical convexity parameters have been considered for abstract convexity spaces [13], such as P_3-convexity. The geodetic number of P_3-convexity is the same as the well known 2-domination number [6]. Also the hull number [3,10] and the Carathéodory number [2,7] have been studied. In [8] we investigated the algorithmic aspects of the Radon number of P_3-convexity proving that:

- it is NP-complete to decide whether a given set R of vertices of a bipartite graph G has a partition as in Radon's result and
- it is NP-hard to decide for a given split graph G and integer k whether the Radon number of G is at least k.

* Corresponding author.
E-mail addresses: mitre@nce.ufrj.br (M.C. Dourado), dieter.rautenbach@uni-ulm.de, dieter.rautenbach@gmx.de (D. Rautenbach), vinicius.santos@gmail.com (V. Fernandes dos Santos), philipp.schaefer@gmail.com (P.M. Schäfer), jayme@nce.ufrj.br (J.L. Szwarcfiter), alexandre21campos@ppgi.ufrj.br (A. Toman).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2012.05.002

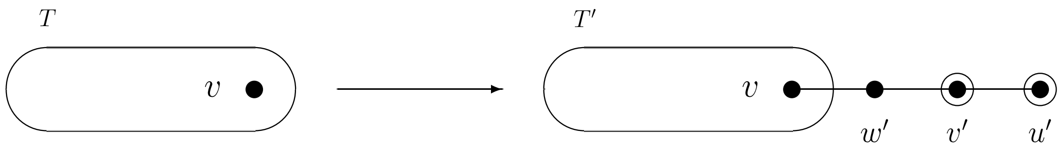

Fig. 1. A type 1 extension.

These strong hardness results motivate the search for good bounds on the Radon number of P_3-convexity. After introducing relevant notions and terminology in Section 2, we prove an upper bound on the P_3-Radon number of connected graphs and characterize all extremal graphs in Section 3. In Section 4 we conclude with some open problems.

2. Preliminaries

We consider finite, simple, and undirected graphs. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set. For a vertex u of a graph G, let $N_G(u)$ and $d_G(u)$ denote the neighbourhood and degree of u in G. Furthermore, let $N_G[u]$ denote the closed neighbourhood $N_G(u) \cup \{u\}$ of u in G. For a set U of vertices of a graph G, let $G[U]$ denote the subgraph of G induced by U and let $G - U$ denote $G[V(G) \setminus U]$.

Let G be a graph and let R be a set of vertices of G. The set R is *convex*[1] *in* G if no vertex in $V(G) \setminus R$ has two neighbours in R. The *convex hull* $H_G(R)$ *of* R *in* G is the intersection of all convex sets in G containing R. Equivalently, $H_G(R)$ is the smallest set containing R that is convex in G. A *Radon partition of* R is a partition of R into two disjoint sets R_1 and R_2 with $H_G(R_1) \cap H_G(R_2) \neq \emptyset$. The set R is an *anti-Radon set of* G if it has no Radon partition. The *Radon number* $r(G)$ *of* G is the minimum integer r such that every set of at least r vertices of G has a Radon partition. Equivalently, the Radon number of G is the maximum cardinality of an anti-Radon set of G plus 1, i.e.

$$r(G) = \max\{|R| \mid R \text{ is an anti-Radon set of } G\} + 1.$$

Clearly, if R is an anti-Radon set of a graph G and H is a subgraph of G, then every subset of $R \cap V(H)$ is an anti-Radon set of H.

For a non-negative integer n, let $[n] = \{1, \ldots, n\}$.

The convex hull of a set R of vertices can be formed by iteratively adding vertices to R that have two neighbours in R. If $R = R_1 \cup R_2$ is a Radon partition of R, then forming the convex hulls of R_1 and R_2 in this way, iteratively adding individual vertices one at a time to R_1 or R_2, there is a first vertex that belongs to both sets. This observation immediately implies that R has a Radon partition exactly if there are two sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ such that:

- all x_i are distinct,
- all y_j are distinct,
- $x_a = y_b$ is the only common element of the two sequences,
- for every $i \in [a]$, either $x_i \in R$ or $|N_G(x_i) \cap \{x_1, \ldots, x_{i-1}\}| \geq 2$, and
- for every $j \in [b]$, either $y_j \in R \setminus \{y_b\}$ or $|N_G(y_j) \cap \{y_1, \ldots, y_{j-1}\}| \geq 2$.

Note that the last two conditions are not symmetric for the two sequences, that is, y_b is necessarily required to have two neighbours among $y_1, \ldots, y_{b-1}$. We call the two sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ *Radon witness sequences for* R and the vertex $x_a = y_b$ a *Radon witness vertex for* R.

3. The bound

In this section we prove our upper bound on the Radon number of connected graphs and characterize all extremal graphs.

We define a set $\mathcal{T}$ of trees using the following two extension operations.

- If T is a tree, v is a vertex of T, and T' arises from T by adding three new vertices u', v', and w' and three new edges $u'v'$, $v'w'$, and $w'v$, then T' is said to arise from T by a *type* 1 *extension*. See Fig. 1.
- If T is a tree, v is an endvertex of T, w is the neighbour of v in T, w is of degree 2 in T, and T'' arises from T by adding three new vertices u, u', and v' and three new edges uv, $u'v'$, and $v'w$, then T'' is said to arise from T by a *type 2 extension*. See Fig. 2.

Let $\mathcal{T}$ be defined recursively as the set of trees that consists of K_2 and every tree T that arises from a smaller tree in $\mathcal{T}$ by a type 1 extension or a type 2 extension.

Our main result is the following.

Theorem 1. *If G is a connected graph, then $r(G) \leq \frac{2}{3}(n(G) + 1) + 1$ with equality if and only if G belongs to $\mathcal{T}$.*

Before we prove this result, we need some lemmas.

Lemma 2. *If T is a tree and R is an anti-Radon set of T, then $|R| \leq \frac{2}{3}(n(T) + 1)$. Furthermore, if $|R| = \frac{2}{3}(n(T) + 1)$, then $T \in \mathcal{T}$.*

[1] For the sake of brevity and since we only consider one type of convexity, we omit the "P_3-" from now on.

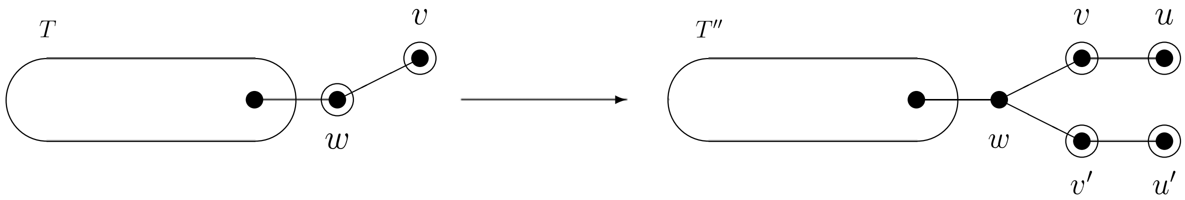

Fig. 2. A type 2 extension.

Proof. We use induction on $n(T)$. For $n(T) \leq 2$, the desired statement is obvious. Therefore, we assume that $n(T) \geq 3$. For a contradiction, we assume that either $|R| > \frac{2}{3}(n(T)+1)$ or $|R| = \frac{2}{3}(n(T)+1)$ but $T \notin \mathcal{T}$. $\square$

Claim 1. *R contains all endvertices of T.*

Proof of Claim 1. For a contradiction, we assume that R does not contain the endvertex u of T. Clearly, R is an anti-Radon set of the tree $T - \{u\}$. Hence, by induction, $|R| \leq \frac{2}{3}((n(T)-1)+1) < \frac{2}{3}(n(T)+1)$, which is a contradiction. $\square$

Claim 2. *No vertex of T is adjacent to two endvertices.*

Proof of Claim 2. For a contradiction, we assume that the vertex u is adjacent to the two endvertices v and w in T. Claim 1 implies $v, w \in R$. Since R is an anti-Radon set of T and $u \in H_T(\{v, w\})$, the vertex u does not belong to R and $R' = (R \backslash \{v, w\}) \cup \{u\}$ is an anti-Radon set of the tree $T - \{v, w\}$. Hence, by induction, $|R| \leq |R'| + 1 \leq \frac{2}{3}((n(T)-2)+1)+1 < \frac{2}{3}(n(T)+1)$, which is a contradiction. $\square$

Let $P : u_1 u_2 \dots u_l$ be a longest path in T, that is, u_1 is an endvertex of T. Claims 1 and 2 imply $u_1 \in R$ and $d_T(u_2) = 2$. Since the desired statement is obvious for stars, we may assume that $l \geq 4$. Hence $d_T(u_3) \geq 2$.

Claim 3. *$u_2 \in R$ and $u_3 \notin R$.*

Proof of Claim 3. For a contradiction, we assume that $u_2 \notin R$. The set $R \setminus \{u_1\}$ is an anti-Radon set of the tree $T - \{u_1, u_2\}$. Hence, by induction, $|R| \leq \frac{2}{3}((n(T)-2)+1)+1 < \frac{2}{3}(n(T)+1)$, which is a contradiction. Hence $u_2 \in R$. Since $u_2 \in H_T(\{u_1, u_3\})$ and R is an anti-Radon set of T, we obtain $u_3 \notin R$. $\square$

If $d_T(u_3) = 2$, then T arises from the tree $F = T - \{u_1, u_2, u_3\}$ by a type 1 extension and $S = R \setminus \{u_1, u_2\}$ is an anti-Radon set of F. By induction, $|R| \leq |S| + 2 \leq \frac{2}{3}(n(F)+1)+2 = \frac{2}{3}(n(T)+1)$. Hence either $|R| < \frac{2}{3}(n(T)+1)$ or $|R| = \frac{2}{3}(n(T)+1)$ and, by induction and the definition of $\mathcal{T}$, $T \in \mathcal{T}$, which is a contradiction. Therefore, we may assume that $d_T(u_3) \geq 3$.

Recall that P is a longest path in T, which implies that every path in T between u_3 and an endvertex of T that does not contain u_4 has length either 1 or 2. If $d_T(u_3) \geq 4$, then, by the choice of P, symmetry, and Claims 1–3, we obtain the existence of two neighbours of u_3 distinct from u_2, say v and w, that belong to R. Now $u_1, u_2, v, w \in R$ and $u_2 \in H_T(\{u_1, v, w\})$, which is a contradiction. Hence $d_T(u_3) = 3$. By Claim 2, it suffices to consider the following two cases.

Case 1. *u_3 is adjacent to an endvertex u_2'.*

By Claim 1, we have $u_2' \in R$. Let $S = (R \setminus \{u_1, u_2'\}) \cup \{u_3\}$ and $F = T - \{u_1, u_2'\}$. For a contradiction, we assume that S is not an anti-Radon set of F. Since R is an anti-Radon set of T and $u_3 \in H_T(\{u_2, u_2'\})$, the set $S \setminus \{u_2\}$ is an anti-Radon set of F. This implies that u_3 is the only Radon witness vertex for S in F. Now $u_3 \in H_F(S \setminus \{u_3\}) \subseteq H_T(R \setminus \{u_1, u_2'\})$. This implies that $u_3 \in H_T(R \setminus \{u_1, u_2\})$ and hence $u_2 \in H_T(R \setminus \{u_2\})$, which is a contradiction. Hence S is an anti-Radon set of F and, by induction, $|R| \leq |S| + 1 \leq \frac{2}{3}(n(F)+1)+1 < \frac{2}{3}(n(T)+1)$, which is a contradiction.

Case 2. *u_3 is adjacent to a vertex u_2' of degree 2 that is distinct from u_2, and u_2' is adjacent to an endvertex u_1'.*

By Claims 1 and 3, we have $u_1', u_2' \in R$. Like in Case 1, it follows that $S = (R \setminus \{u_1, u_1', u_2'\}) \cup \{u_3\}$ is an anti-Radon set of $F = T - \{u_1, u_1', u_2'\}$. Hence, by induction, $|R| \leq |S| + 2 \leq \frac{2}{3}(n(F)+1)+2 = \frac{2}{3}(n(T)+1)$. Since T arises from F by a type 2 extension, we obtain that either $|R| < \frac{2}{3}(n(T)+1)$ or $|R| = \frac{2}{3}(n(T)+1)$ and, by induction and the definition of $\mathcal{T}$, $T \in \mathcal{T}$, which is a contradiction and completes the proof. $\square$

To each tree T in $\mathcal{T}$, we assign a set $R(T)$ of vertices of T as follows. We denote the vertices as in the definition of the extensions.

- Let $R(K_2) = V(K_2)$.
- If T' arises from $T \in \mathcal{T}$ by a type 1 extension, then let $R(T') = R(T) \cup \{u', v'\}$. See Fig. 1.
- If T'' arises from $T \in \mathcal{T}$ by a type 2 extension, then let $R(T') = (R(T) \setminus \{w\}) \cup \{u, u', v'\}$. See Fig. 2.

Lemma 3. *If T belongs to $\mathcal{T}$, then the following statements hold.*

(i) *$R(T)$ is an anti-Radon set of T of maximum cardinality, $|R(T)| = \frac{2}{3}(n(T)+1)$, and $H_T(R(T)) = V(T)$.*
(ii) *$R(T)$ is the unique anti-Radon set of T of maximum cardinality.*

Proof. Since (i) can easily be proved by induction on the order using Lemma 2 and arguments similar to those in the proof of Lemma 2, we give details only for (ii). We use induction on the order. Clearly, (ii) holds for K_2. Now let T be a tree in $\mathcal{T}$ for which (ii) holds. In view of the definition of $\mathcal{T}$, we need to consider the two trees arising from T by a type 1 extension or a type 2 extension.

Let T' arise from T by a type 1 extension and denote the vertices as in the definition of type 1 extensions. Let R' be an anti-Radon set of T' of maximum cardinality. By (i), we have $|R'| = \frac{2}{3}(n(T') + 1)$. Now, arguments similar to those in the proofs of Claims 1 and 3 in the proof of Lemma 2 imply that $u', v' \in R'$ and $w' \notin R'$. Since $R = R' \setminus \{u', v'\}$ is an anti-Radon set of T and $|R| = \frac{2}{3}(n(T) + 1)$, we obtain, by induction, that $R = R(T)$. Hence $R' = R(T')$.

Next, let T'' arise from T by a type 2 extension and denote the vertices as in the definition of type 2 extensions. Let R'' be an anti-Radon set of T'' of maximum cardinality. By (i), we have $|R''| = \frac{2}{3}(n(T'') + 1)$. Again, the same arguments as in the proof of Claims 1 and 3 in the proof of Lemma 2 imply that $u, u', v, v' \in R''$ and $w \notin R''$. Since $R = (R'' \setminus \{u, u', v'\}) \cup \{w\}$ is an anti-Radon set of T and $|R| = \frac{2}{3}(n(T) + 1)$, we obtain, by induction, that $R = R(T)$. Hence $R'' = R(T'')$. $\square$

Lemma 4. *If G arises by adding a new edge xy to a tree F in $\mathcal{T}$, then $r(G) < r(F)$.*

Proof. Let R be an anti-Radon set of G of maximum cardinality. For a contradiction, we assume that $|R| \geq r(F) - 1$. Since R is also an anti-Radon set of F, Lemmas 2 and 3 imply that $R = R(F)$. Since K_2 is complete, F is not K_2. Therefore, in view of the definition of $\mathcal{T}$, we need to consider the following two cases.

Case 1. *$F = T'$ and T' arises from some tree T in $\mathcal{T}$ by a type* 1 *extension.*

We denote the vertices as in the definition of type 1 extensions. Note that $R = R(T') = R(T) \cup \{u', v'\}$ and G arises by adding xy to T'. If $x, y \in V(T)$, then, by induction, $R(T)$ is not an anti-Radon set of $T + xy$, which implies the contradiction that R is not an anti-Radon set of G. If $x, y \in \{u', v', w'\}$, then $xy = u'w'$. By Lemma 3(i), we have $v \in H_T(R(T))$. Hence $u' \in H_G(R(T) \cup \{v'\})$, which implies the contradiction that R is not an anti-Radon set of G. Hence, we may assume that $x \in V(T)$ and $y \in \{u', v', w'\}$. If $y = w'$ or $y = v'$, then Lemma 3(i) implies that $v' \in H_G(R(T) \cup \{u'\})$, which implies the contradiction that R is not an anti-Radon set of G. Hence $y = u'$. Now Lemma 3(i) implies that $u' \in H_G(R(T) \cup \{v'\})$, which implies the contradiction that R is not an anti-Radon set of G.

Case 2. *$F = T''$ and T'' arises from some tree T in $\mathcal{T}$ by a type* 2 *extension.*

We denote the vertices as in the definition of type 2 extensions. Note that $R = R(T'') = (R(T) \setminus \{w\}) \cup \{u, u', v'\}$ and G arises by adding xy to T''.

If $x, y \in V(T) \setminus \{v, w\}$, then, by induction, $R(T)$ is not an anti-Radon set of $T + xy$. If there is some Radon partition of $R(T)$ in the graph $T + xy$ into two sets R_1 and R_2 with $v, w \in R_1$, then $(R_1 \setminus \{w\}) \cup \{u, u', v'\}$ and R_2 define a Radon partition of R in G, which is a contradiction. Hence, for every Radon partition of $R(T)$ in the graph $T + xy$ into two sets R_1 and R_2, the two vertices v and w do not belong to the same set. This implies that w is a Radon witness vertex for $R(T)$ in $T + xy$. Hence, $w \in H_{T+xy}(R(T) \setminus \{w\})$, which implies that a neighbour of w different from v belongs to $H_{T+xy}(R(T) \setminus \{v, w\})$. This implies that $v \in H_G(R \setminus \{v\})$, which is a contradiction.

If $x, y \in \{u, u', v, v', w\}$, then $\{u, u', v, v'\}$ is not an anti-Radon set of G, which implies the contradiction that R is not an anti-Radon set of G. Hence, we may assume that $x \in V(T) \setminus \{v, w\}$ and $y \in \{u, u', v, v', w\}$. By symmetry, it suffices to consider $y \in \{u, v, w\}$.

If $y = u$, then Lemma 3(i) implies that $u \in H_G(R \setminus \{u\})$, which implies the contradiction that R is not an anti-Radon set of G. Hence we may assume that $y \in \{v, w\}$.

Regardless of whether y equals v or w, we consider $T + xw$. By induction, $R(T)$ is not an anti-Radon set of $T + xw$. If there is some Radon partition of $R(T)$ in the graph $T + xw$ into two sets R_1 and R_2 with $v, w \in R_1$, then, regardless of whether y equals v or w, $(R_1 \setminus \{w\}) \cup \{u, u', v'\}$ and R_2 define a Radon partition of R in G, which is a contradiction. Hence, for every Radon partition of $R(T)$ in the graph $T + xw$ into two sets R_1 and R_2, the two vertices v and w do not belong to the same set. This implies that w is a Radon witness vertex for $R(T)$ in $T + xw$. Hence, $w \in H_{T+xw}(R(T) \setminus \{w\})$. Since $R(T)$ is an anti-Radon set of T, this implies that $x \in H_T(R(T) \setminus \{v, w\})$. Now, regardless of whether y equals v or w, we have $v \in H_G(R \setminus \{v\})$, which is a contradiction and completes the proof. $\square$

We are now in a position to prove the main result of this section.

Proof of Theorem 1. Let R be an anti-Radon set of G of maximum cardinality and let F be a spanning tree of G. Since R is an anti-Radon set of F, Lemmas 2 and 4 imply that $|R| + 1 = r(G) \leq r(F) \leq \frac{2}{3}(n(G) + 1) + 1$ with equality only if $F \in \mathcal{T}$ and $G = F$, that is, $G \in \mathcal{T}$. Furthermore, by Lemma 3, every graph G in $\mathcal{T}$ satisfies $r(G) = \frac{2}{3}(n(G) + 1) + 1$, which completes the proof. $\square$

Note that if the graph G has the connected components $G_1, \ldots, G_l$, then $r(G) - 1 = (r(G_1) - 1) + \cdots + (r(G_l) - 1)$.

4. Conclusion

The Radon number of the geodetic convexity of finite graphs has received an especially large amount of attention. This is probably due to Eckhoff's conjecture [14] related to Tverberg's generalization [21] of Radon's result [20]. Jamison [17] proved this conjecture for the geodetic convexity of trees, and Bandelt and Pesch [1] relate the Radon number for Helly graphs in geodetic convexity to the clique number of these graphs. The special role of the geodetic convexity in this context was justified by Duchet who actually announced [12] that the partition conjecture would hold in general if it holds for the geodetic convexity of finite graphs. It is an open problem whether Eckhoff's conjecture holds for P_3-convexity. The precise statement would be that for every $m > 2$, every set R of vertices of some graph G with $|R| \geq (m-1)(r(G)-1)+1$ has a partition $R = R_1 \cup \ldots \cup R_m$ into m sets such that $H_G(R_1) \cap \ldots \cap H_G(R_m) \neq \emptyset$. For further details, please refer to [14].

We close with another open problem concerning a bound on the Radon number. For a graph G, let $\tilde{\alpha}(G)$ denote the maximum order of a set U of vertices of G such that every vertex of G has at most one neighbour in U. Clearly, $H_G(U) = U$ for each such set, which immediately implies that U is an anti-Radon set and thus $r(G) \geq \tilde{\alpha}(G) + 1$. We conjecture the existence of upper bounds on $r(G)$ in terms of $\tilde{\alpha}(G)$ for general graphs G or at least for trees G.

Acknowledgements

This work has been done in the context of the CAPES-DAAD PROBRAL project "Cycles, Convexity, and Searching in Graphs". The first and fifth authors were supported by CNPq and FAPERJ. The third author was supported by FAPERJ. The sixth author was supported by CAPES.

References

[1] H.-J. Bandelt, E. Pesch, A Radon theorem for Helly graphs, Arch. Math. 52 (1989) 95–98.
[2] R.M. Barbosa, E.M.M. Coelho, M.C. Dourado, D. Rautenbach, J.L. Szwarcfiter, On the Carathéodory number for the convexity of paths of order three, SIAM J. Discrete Math. (in press).
[3] C.C. Centeno, M.C. Dourado, L.D. Penso, D. Rautenbach, J.L. Szwarcfiter, Irreversible conversion of graphs, Theor. Comput. Sci. 412 (2011) 3693–3700.
[4] M. Changat, S. Klavžar, H.M. Mulder, The all-paths transit function of a graph, Czech. Math. J. 51 (126) (2001) 439–448.
[5] M. Changat, J. Mathew, On triangle path convexity in graphs, Discrete Math. 206 (1999) 91–95.
[6] M. Chellali, O. Favaron, A. Hansberg, L. Volkmann, k-domination and k-independence in graphs: a survey, Graphs Combin. 28 (2012) 1–55. http://dx.doi.org/10.1007/s00373-011-1040-3.
[7] M.C. Dourado, D. Rautenbach, V. Fernandes dos Santos, P.M. Schäfer, J.L. Szwarcfiter, On the Carathéodory number of interval and graph convexities, manuscript 2012.
[8] M.C. Dourado, D. Rautenbach, V. Fernandes dos Santos, P.M. Schäfer, J.L. Szwarcfiter, A. Toman, Algorithmic and structural aspects of the p_3-radon number, Manuscript 2011.
[9] M.C. Dourado, D. Rautenbach, P.M. Schäfer, On finite convexity spaces induced by sets of paths in graphs, Discrete Math. 311 (2011) 616–619.
[10] P.A. Dreyer, F.S. Roberts, Irreversible k-threshold processes: graph-theoretical threshold models of the spread of disease and of opinion, Discrete Appl. Math. 157 (2009) 1615–1627.
[11] P. Duchet, Convex sets in graphs. II: minimal path convexity, J. Comb. Theory, Ser. B 44 (1988) 307–316.
[12] P. Duchet, Discrete convexity: retractions, morphisms and the partition problem, in: R. Balakrishnan, et al. (Eds.), Graph Connections. Proceedings of the Conference, Cochin, India January 28–31, 1998, Allied Publishers Limited, New Delhi, 1999, pp. 10–18.
[13] J. Eckhoff, Helly, Radon, and Carathéodory type theorems, in: Handbook of Convex Geometry, A, B, North-Holland, Amsterdam, 1993, pp. 389–448.
[14] J. Eckhoff, The partition conjecture, Discrete Math. 221 (2000) 61–78.
[15] P. Erdős, E. Fried, A. Hajnal, E.C. Milner, Some remarks on simple tournaments, Algebra Univers. 2 (1972) 238–245.
[16] M. Farber, R.E. Jamison, Convexity in graphs and hypergraphs, SIAM J. Algebraic Discrete Methods 7 (1986) 433–444.
[17] R.E. Jamison, Partition numbers for trees and ordered sets, Pac. J. Math. 96 (1981) 115–140.
[18] J.W. Moon, Embedding tournaments in simple tournaments, Discrete Math. 2 (1972) 389–395.
[19] D.B. Parker, R.F. Westhoff, M.J. Wolf, On two-path convexity in multipartite tournaments, European J. Combin. 29 (2008) 641–651.
[20] J. Radon, Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten, Math. Ann. 83 (1921) 113–115.
[21] H. Tverberg, A generalization of Radon's theorem, J. Lond. Math. Soc. 41 (1966) 123–128.
[22] M.L.J. van de Vel, Theory of Convex Structures, North-Holland, Amsterdam, 1993.
[23] J.C. Varlet, Convexity in tournaments, Bull. Soc. R. Sci. Liège 45 (1976) 570–586.

Apêndice H

Characterization and recognition of Radon-independent sets in split graphs

Este apêndice contém o artigo "*Characterization and recognition of Radon-independent sets in split graphs*", publicado no periódico *Information Processing Letters.*

Information Processing Letters 112 (2012) 948–952

Contents lists available at SciVerse ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Characterization and recognition of Radon-independent sets in split graphs

Mitre C. Dourado [a], Dieter Rautenbach [b,*], Vinícius Fernandes dos Santos [c], Jayme L. Szwarcfiter [a,c]

[a] *Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil*
[b] *Institut für Optimierung und Operations Research, Universität Ulm, Ulm, Germany*
[c] *PESC, COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil*

ARTICLE INFO

Article history:
Received 12 January 2012
Received in revised form 24 April 2012
Accepted 22 August 2012
Available online 3 September 2012
Communicated by B. Doerr

Keywords:
Combinatorial problems
Graph algorithms
Radon number
Radon partition
Split graph
Graph convexity

ABSTRACT

Let R be a set of vertices of a split graph G. We characterize when R allows a partition into two disjoint set R_1 and R_2 such that the convex hulls of R_1 and R_2 with respect to the P_3-convexity of G intersect. Furthermore, we describe a linear time algorithm that decides the existence of such a partition. Our results are related to the so-called Radon number of the P_3-convexity of G and complement earlier results in this area.

1. Introduction

Inspired by Radon's well-known theorem [12], the Radon number of a convexity [14] is the smallest integer r with the property that every set with at least r elements allows a partition into two disjoint sets whose convex hulls intersect. With this definition, Radon's theorem states that the Radon number of $\mathbb{R}^d$ with respect to the usual convexity notion equals $d+2$. In the present paper we consider the convexity induced on the vertex set of a finite, simple, and undirected graph G by paths of order 3. More precisely, a set C of vertices of G is *convex* provided C contains every vertex adjacent to two vertices in C. Each set S of vertices of G is contained in a smallest convex set conv(S), the *convex hull of* S. The convex sets are obviously closed under intersections and form a convexity in the sense of van de Vel [14]. This convexity is usually referred to as the P_3*-convexity* of G and was first studied for directed graphs [6,10,11,13]. Next to the geodetic convexity [7] defined by shortest paths, and the monophonic convexity [5] defined by induced paths in similar ways, this is one of the natural and well studied convexities defined by paths in graphs [1,2]. In a series of papers [3,4,9] we investigate algorithmic as well as structural aspects of the Radon number of P_3-convexity. In order to motivate and phrase our present contribution, we need some more terminology.

Let G be a graph. For a set R of vertices of G, a partition of R into two disjoint sets R_1 and R_2 is a *Radon partition* of R, if conv(R_1) and conv(R_2) intersect. If R admits a Radon partition, then R is called *Radon-dependent*, otherwise it is called *Radon-independent*. As stated above, the *Radon number* $r(G)$ *of* G is the smallest integer r with the property that every set with at least r vertices is Radon-dependent. Equivalently, the Radon number of G

* Corresponding author.
E-mail addresses: mitre@nce.ufrj.br (M.C. Dourado), dieter.rautenbach@uni-ulm.de (D. Rautenbach), vinicius.santos@gmail.com (V.F. dos Santos), jayme@nce.ufrj.br (J.L. Szwarcfiter).

0020-0190/$ – see front matter
http://dx.doi.org/10.1016/j.ipl.2012.08.013

is exactly 1 more than the largest order of a Radon-independent set in G.

As it turned out in [3], the Radon number is a parameter that is algorithmically very hard to compute. More specifically, we obtained the following two complexity results concerning bipartite graphs and split graphs. Recall that a graph is a split graph, if its vertex set allows a partition into a clique and an independent set.

Theorem A. *(See Dourado et al. [3].) It is NP-complete to decide whether a given set of vertices of a given bipartite graph is Radon-dependent with respect to P_3-convexity.*

Theorem B. *(See Dourado et al. [3].) It is NP-hard to determine the Radon number of a given split graph with respect to P_3-convexity.*

By Theorem A, the recognition of Radon-independent sets is hard. As a consequence, it is unknown whether the problem of deciding for a given graph G and a given integer r, if the Radon number of G is at least r, even belongs to NP. In conjunction Theorems A and B motivate the question whether recognizing Radon-independent sets is at least tractable for split graphs even though the Radon number remains hard for these graphs. Our contribution in the present paper is that this is indeed the case. More precisely, in Section 3 we provide a structural characterization of Radon-independent sets in split graphs using vertex-partitioned, not necessarily induced subgraphs. Furthermore, we describe a linear time algorithm for their recognition. In order to obtain these results, we will first study (inclusionwise) minimal Radon-dependent sets in Section 2.

2. Minimal Radon-dependent sets

Throughout this section let G be a split graph without isolated vertices. Let C and I partition the vertex set of G such that C is a clique and I is an independent set. Note that a split graph has no isolated vertices if and only if it its connected.

Let R be a set of vertices of G. Let $R_C = R \cap C$ and $R_I = R \cap I$.

The neighborhood and the degree of a vertex u in G are denoted $N(u)$ and $d(u)$, respectively.

Lemma 1. *$C \subseteq \mathrm{conv}(R)$ if and only if*

(1.1) *$|R_C| \geqslant 2$, or*
(1.2) *$|R_C| = 1$ and, if $|C| > 1$, then $N(w) \neq R_C$ for some $w \in R_I$, or*
(1.3) *$|R_C| = 0$ and $N(w_1) \cap N(w_2) \neq \emptyset$ for distinct vertices $w_1, w_2 \in R_I$. In addition, if $|C| > 1$, then $N(z) \neq N(w_1) \cap N(w_2)$, for some $z \in R_I$.*

Proof. Let $C \subseteq \mathrm{conv}(R)$. If $|C| = 1$, then clearly, either $|R_C| = 1$ or R_I contains distinct vertices with a common neighbor, that is, we obtain (1.2) and (1.3), respectively. Now assume $|C| > 1$. If $|R_C| = 2$, then (1.1). If $|R_C| = 1$, then C must contain an additional vertex belonging to $\mathrm{conv}(R)$. This implies that some $w \in R_I$ has a neighbor in C other than the vertex contained in R_C, meaning that (1.2) holds. Finally, let $R_C = \emptyset$. We know that we need some vertex $u_1 \in C$ to lie in $N(w_1) \cap N(w_2)$ for some distinct vertices $w_1, w_2 \in R_I$. Furthermore, we need a vertex $u_2 \in C$ with $u_2 \neq u_1$, to belong to the neighborhood of some $z \in R_I$. Hence (1.3) holds.

Conversely, if one of (1.1), (1.2), or (1.3) holds, then $\mathrm{conv}(R)$ contains at least one vertex of C and, if $|C| > 1$, then $\mathrm{conv}(R)$ contains at least two vertices of C. Since C is a clique, this implies $C \subseteq \mathrm{conv}(R)$, which completes the proof. □

In the following *minimal* always means inclusionwise minimal. If $\mathrm{conv}(R)$ contains at most one vertex from C, then we say that R is *weak*.

Lemma 2. *If R is a minimal Radon-dependent set of G and $R = R_1 \cup R_2$ is a Radon partition of R, then the following statements hold.*

(2.1) *At least one of the two sets R_1 and R_2 is weak.*
(2.2) *$\mathrm{conv}(R_1) \cap \mathrm{conv}(R_2)$ contains exactly one vertex.*

Proof. For contradiction, we assume that (2.1) does not hold, which implies that $C \subseteq \mathrm{conv}(R_1), \mathrm{conv}(R_2)$ and $|C| > 1$. We consider the three cases from Lemma 1 for R_2. If (1.1) or (1.2) hold for R_2 and $v \in R_2 \cap C$, then $v \in \mathrm{conv}(R_1) \cap \mathrm{conv}(\{v\})$ and $R_1 \cup \{v\}$ is a Radon-dependent set properly contained in R, which is a contradiction. If (1.3) holds for R_2 and z coincides with one of the w_i, then $z \in \mathrm{conv}(R_1) \cap \mathrm{conv}(\{z\})$ and $R_1 \cup \{z\}$ is a Radon-dependent set properly contained in R, which is a contradiction. Finally, if (1.3) holds for R_2, z is distinct from both w_i, and $v \in N(w_1) \cap N(w_2)$, then $v \in \mathrm{conv}(R_1) \cap \mathrm{conv}(\{w_1, w_2\})$ and $R_1 \cup \{w_1, w_2\}$ is a Radon-dependent set properly contained in R, which is a contradiction and completes the proof of (2.1).

For contradiction, we assume that (2.2) does not hold, that is, $\mathrm{conv}(R_1) \cap \mathrm{conv}(R_2)$ contains two distinct vertices. If $\mathrm{conv}(R_1) \cap \mathrm{conv}(R_2)$ contains a vertex from R, say $u \in R_2$, then $u \in \mathrm{conv}(R_1) \cap \mathrm{conv}(\{u\})$ and $R_1 \cup \{u\}$ is a Radon-dependent set properly contained in R, which is a contradiction. Hence $\mathrm{conv}(R_1) \cap \mathrm{conv}(R_2)$ contains no vertex from R. By (2.1), we may assume that R_2 is weak. This implies that $\mathrm{conv}(R_1) \cap \mathrm{conv}(R_2)$ does not contain two vertices from C. Hence there is some $u \in \mathrm{conv}(R_1) \cap \mathrm{conv}(R_2)$ that belongs to $I \setminus R$. Since $N(u) \subseteq C$, this implies that $\mathrm{conv}(R_2)$ contains two vertices from C, which implies the contradiction that R_2 is not weak and completes the proof of (2.2). □

The following is the main result of this section. We use the convention that vertices in R_C will be denoted v_i and vertices in R_I will be denoted w_j. Furthermore, distinct indices imply distinct vertices, that is, $R_C = \{v_1, \ldots, v_{|R_C|}\}$ and $R_I = \{w_1, \ldots, w_{|R_I|}\}$.

Theorem 3. *R is a minimal Radon-dependent set of G if and only if*

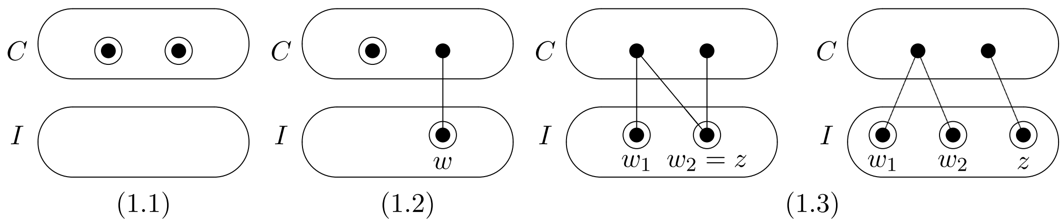

Fig. 1. The three cases from Lemma 1. Note that in (1.3) the vertex z might coincide with one of the w_i as illustrated.

(3.1) $|R| = 3$ *and*
  (a) $|R_C| \geqslant 2$, *or*
  (b) $|R_C| = 1$ *and* $v_1 \in N(w_1) \cap N(w_2)$, *or*
  (c) $|R_C| = 1$, $N(w_1) \cap N(w_2) \neq \emptyset$, *and* $d(w_1) > 1$, *or*
  (d) $|R_C| = 1$, $N(w_1) \neq R_C$, *and* $d(w_2) > 1$, *or*
  (e) $R_C = \emptyset$, $N(w_1) \cap N(w_2) \neq \emptyset$, $d(w_1) > 1$, *and* $d(w_3) > 1$, *or*

(3.2) $|R| = 4$, *no subset of R satisfies* (3.1), *and*
  (a) $|R_C| = 1$ *and* $N(w_1) \cap N(w_2) \neq \emptyset$, *or*
  (b) $R_C = \emptyset$ *and* $N(w_1) \cap N(w_2) \cap N(w_3) \cap N(w_4) \neq \emptyset$, *or*
  (c) $R_C = \emptyset$, $N(w_1) \cap N(w_2) \neq \emptyset$, $N(w_3) \cap N(w_4) \neq \emptyset$, *and* $d(w_1) > 1$, *or*
  (d) $R_C = \emptyset, d(w_1) = d(w_2) = 1, \emptyset \neq N(w_1) \cap N(w_2) \neq N(w_3)$, *and* $d(w_4) > 1$, *or*

(3.3) $|R| = 5$, *no subset of R satisfies* (3.1) *or* (3.2), *and*
  (a) $R_C = \emptyset$, $N(w_1) \cap N(w_2) \neq \emptyset$, *and* $N(w_3) \cap N(w_4) \neq \emptyset$.

Proof. Let R be a minimal Radon-dependent set with a Radon partition into R_1 and R_2. By Lemma 2, we may assume that R_2 is weak and that $\mathrm{conv}(R_1) \cap \mathrm{conv}(R_2)$ contains exactly one vertex v. We show that one of the above conditions necessarily holds and consider the following alternatives.

Case 1. $v \in R_C$.

First, we assume that $v \in R_2$. If R_1 is weak, then $\mathrm{conv}(R_1) \cap C = \{v\}$, which implies the existence of two neighbors of v in $R_1 \cap I$ and (3.1)(b) holds. Now, let R_1 not be weak, which implies that $C \subseteq \mathrm{conv}(R_1)$ and $|C| > 1$. We consider the three alternatives of Lemma 1 for R_1. If (1.1) or (1.2) hold for R_1, then R_1 contains an element of C distinct from v and the minimality of R easily implies that (3.1)(a) holds. If (1.3) holds for R_1, then there are vertices $u_1, u_2 \in C \setminus R_1$ and $w_1, w_2, z \in R_1 \cap I$, with $u_1 \neq u_2$, $w_1 \neq w_2$, $u_1 \in N(w_1) \cap N(w_2)$, and $u_2 \in N(z)$. If $z \in \{w_1, w_2\}$, then (3.1)(c) holds, otherwise (3.2)(a) holds.

Next, we assume that $v \in R_1$. Since R_2 is weak, we obtain, as above, the existence of two neighbors of v in $R_2 \cap I$ and (3.1)(b) holds.

Case 2. $v \in R_I$.

First, we assume that $v \in R_2$. Since $v \in \mathrm{conv}(R_1) \setminus R_1$, v must have two distinct neighbors $u_1, u_2 \in \mathrm{conv}(R_1) \cap C$, implying that $C \subseteq \mathrm{conv}(R_1)$. Again we consider the three alternatives of Lemma 1 for R_1. If (1.1) holds, then $|R_C| = 2$ and (3.1)(a) holds. If (1.2) holds, then (3.1)(d) holds. Finally, if (1.3) holds, then we obtain that (3.1)(e) or (3.2)(d) holds, depending on whether (1.3) involves two or three vertices in $R_1 \cap I$, respectively.

Next, we assume that $v \in R_1$. Since $v \in \mathrm{conv}(R_2) \setminus R_2$, v must have two distinct neighbors $u_1, u_2 \in \mathrm{conv}(R_2) \cap C$, which contradicts the fact that R_2 is weak. So, this case does not occur.

Case 3. $v \in C \setminus R$.

Since R_2 is weak, v is the only vertex of $\mathrm{conv}(R_2)$ in C. The minimality of R implies that R_2 consists of two distinct vertices $w_1, w_2 \in I$ satisfying $v \in N(w_1) \cap N(w_2)$. If R_1 is weak, then, by symmetry, R_1 consists of two distinct vertices $w_3, w_4 \in I$ satisfying $v \in N(w_3) \cap N(w_4)$ and (3.2)(b) holds. Hence we assume now that R_1 is not weak and examine the alternatives of Lemma 1 for R_1. If (1.1) holds, then w_1 together with two vertices in $R_1 \cap C$ would be a Radon-dependent set contradicting the minimality of R. If (1.2) holds, then (3.2)(a) holds. Finally, if (1.3) holds, then we obtain either (3.2)(c) or (3.3)(a), depending on whether (1.3) involves two or three vertices in $R_1 \cap I$, respectively.

Case 4. $v \in I \setminus R$.

In this case, v must have two neighbors in $\mathrm{conv}(R_2) \cap C$ contradicting the fact that R_2 is weak. So, this case does not occur.

This completes the proof of the necessity. For the sufficiency, we assume that one of the conditions (3.1)(a) to (3.3)(a) holds and exhibit a Radon partition of R in each case in the following table.

In all cases, it is easy to verify that the removal of any vertex from R leads to a Radon-independent set. Note that some conditions are implicit in the phrases "... no subset of R satisfies ..." in the statements of (3.2) and (3.3). Hence R is indeed a minimal Radon-dependent set, which completes the proof. □

Theorem 3 has an immediate corollary.

Corollary 4. *The cardinality of a minimal Radon-dependent set of a split graph without isolated vertices is between* 3 *and* 5.

3. Radon-independent sets

Theorem 3 yields a list of forbidden configurations characterizing Radon-independent sets of split graphs. In order to reduce the length of this list, it is convenient to use as forbidden configurations vertex-partitioned, not necessarily induced subgraphs. Note that if u is an isolated vertex of a graph G with vertex set V, then some set $R \subseteq V \setminus \{v\}$ is a Radon-independent set of $G - v$ if and only if $R \cup \{v\}$ is a Radon-independent set of G. Therefore, it suffices to consider graphs without isolated vertices.

Table 1
Radon partitions for Theorem 3. Recall that vertices in R_C are denoted v_i and vertices in R_I are denoted w_j.

Condition	R	R_1	R_2	Notes
(3.1)(a)	$\{v_1, v_2, v_3\}$	$\{v_1, v_2\}$	$\{v_3\}$	$\lvert R_C\rvert = 3$
(3.1)(a)	$\{v_1, v_2, w_1\}$	$\{v_1, w_1\}$	$\{v_2\}$	$N(w_1) \neq \{v_1\}$
(3.1)(b)	$\{v_1, w_1, w_2\}$	$\{w_1, w_2\}$	$\{v_1\}$	
(3.1)(c)	$\{v_1, w_1, w_2\}$	$\{w_1, w_2\}$	$\{v_1\}$	
(3.1)(d)	$\{v_1, w_1, w_2\}$	$\{v_1, w_1\}$	$\{w_2\}$	
(3.1)(e)	$\{w_1, w_2, w_3\}$	$\{w_1, w_2\}$	$\{w_3\}$	
(3.2)(a)	$\{v_1, w_1, w_2, w_3\}$	$\{w_1, w_2, w_3\}$	$\{v_1\}$	$N(w_3) \neq N(w_1) \cap N(w_2)$
(3.2)(a)	$\{v_1, w_1, w_2, w_3\}$	$\{v_1, w_3\}$	$\{w_1, w_2\}$	$N(w_3) = N(w_1) \cap N(w_2)$
(3.2)(b)	$\{w_1, w_2, w_3, w_4\}$	$\{w_1, w_2\}$	$\{w_3, w_4\}$	
(3.2)(c)	$\{w_1, w_2, w_3, w_4\}$	$\{w_1, w_2\}$	$\{w_3, w_4\}$	
(3.2)(d)	$\{w_1, w_2, w_3, w_4\}$	$\{w_1, w_2, w_3\}$	$\{w_4\}$	
(3.3)(a)	$\{w_1, w_2, w_3, w_4, w_5\}$	$\{w_1, w_2, w_5\}$	$\{w_3, w_4\}$	$N(w_5) \neq N(w_1) \cap N(w_2)$

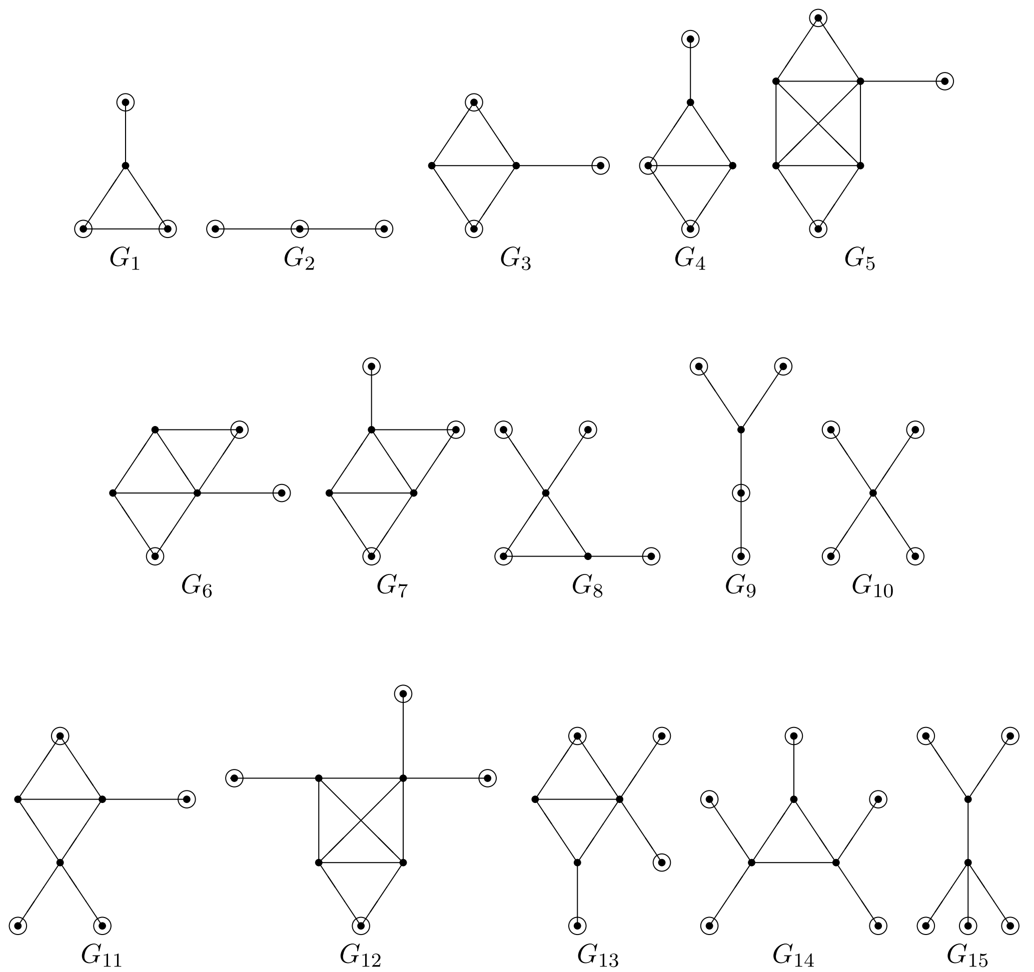

Fig. 2. Forbidden subgraphs for Theorem 5. The encircled vertices belong to R while the non-encircled vertices may or may not belong to R.

Theorem 5. *Let G be a split graph without isolated vertices. A set R of vertices of G is a Radon-independent set of G if and only if G does not contain one of the vertex-partitioned graphs in Fig. 2 as a subgraph such that the encircled vertices belong to R.*

Proof. If G contains some graph G_i from Fig. 2 as a subgraph such that the encircled vertices belong to R, then we say that G_i is a *special subgraph of G*. Since for each of the graphs in Fig. 2, the set of encircled vertices obviously forms a Radon-dependent set, all these graphs are forbidden as special subgraphs. This implies the necessity. For the sufficiency, we prove that if R is a Radon-dependent set, then G contains one of the graphs in Fig. 2 as a special subgraph. Indeed, if R is a Radon-dependent set, then R contains a minimal Radon-dependent set. We may actually assume that R itself is minimal and consider the alternatives from Theorem 3. The subsequent observations follow immediately from the possibilities that certain vertices do or do not coincide. We provide details in only for (3.1)(e), which gives a typical example of the necessary arguments and leave the further details to the reader.

If (3.1)(a) holds, then G contains G_1 or G_2 as a special subgraph.

If (3.1)(b) holds, then G contains G_2 as a special subgraph.

If (3.1)(c) holds, then G contains G_1 or G_2 or G_3 as a special subgraph.

If (3.1)(d) holds, then G contains G_1 or G_3 or G_4 or G_7 as a special subgraph.

If (3.1)(e) holds, then let u_1 denote a common neighbor of w_1 and w_2, let u_2 denote a neighbor of w_1 distinct from u_1, and let u_3 and u_4 be neighbors of w_3. If $u_1 \notin \{u_3, u_4\}$ and $u_2 \notin \{u_3, u_4\}$, then G contains G_5 as a special subgraph. If $u_1 \in \{u_3, u_4\}$ and $u_2 \notin \{u_3, u_4\}$, then G contains G_6 as a special subgraph. If $u_1 \notin \{u_3, u_4\}$ and $u_2 \in \{u_3, u_4\}$, then G contains G_7 as a special subgraph. If $u_1 \in \{u_3, u_4\}$ and $u_2 \in \{u_3, u_4\}$, then G contains G_3 as a special subgraph. Altogether, in this case G contains G_3 or G_5 or G_6 or G_7 as a special subgraph.

If (3.2)(a) holds, then G contains G_8 or G_9 or G_{10} as a special subgraph.

If (3.2)(b) holds, then G contains G_{10} as a special subgraph.

If (3.2)(c) holds, then G contains G_8 or G_{10} or G_{11} as a special subgraph.

If (3.2)(d) holds, then G contains G_8 or G_{11} or G_{12} or G_{13} as a special subgraph.

If (3.3)(a) holds, then G contains G_{14} or G_{15} as a special subgraph. □

The full list of forbidden vertex-partitioned induced subgraphs can be obtained by considering all possibilities of adding edges to the graphs in Fig. 2, while preserving them split graphs.

We conclude this paper with a description of a linear time algorithm that decides for a given split graph G and a given set R of vertices of G, whether R is a Radon-independent set of G. As noted above, we can first eliminate all isolated vertices from G and R without changing the outcome. Hence we may assume that G is connected. It is well known that we can determine a partition of the vertex set of G into a clique C and an independent set I in linear time [8]. Clearly, R is a Radon-independent set of G if and only if it does not contain a minimal Radon-dependent set as a subset. Therefore, in order to complete the description of the algorithm, we consider the alternatives (3.1)(a) to (3.3)(a) from Theorem 3 one by one in the given order and argue that for each alternative, we can decide in linear time whether R contains a subset that satisfies the given conditions. In many cases like for instance (3.1)(a), (3.1)(b), (3.2)(a), and (3.2)(b) this is very easy. In other cases one has to argue carefully in order to obtain a linear running time. Since the arguments are very similar we give details only for the case (3.2)(c), which is the most difficult one: Given G, C, I, and $R \subseteq I$, we need to decide whether there are four distinct vertices w_1, w_2, w_3, and w_4 in $R_I = R \cap I = R$ such that $N(w_1) \cap N(w_2) \neq \emptyset$, $N(w_3) \cap N(w_4) \neq \emptyset$, and $d(w_1) > 1$. We first determine the set D of vertices in C that have at least two neighbors in R_I. Since we consider the alternatives from Theorem 3 in order, we may assume that (3.2)(b) does not apply, that is, every vertex in C has at most 3 neighbors in R_I. If no vertex in D has a neighbor in R_I of degree > 1, then (3.2)(c) does not apply. Hence we assume that this is not the case and choose an arbitrary vertex v from D that has a neighbor in R_I of degree > 1. Let $N = N(v) \cap R_I$. Let $D^- = D \setminus \{v\}$. Recall that N has 2 or 3 elements and that it contains a vertex of degree > 1. If some vertex in D^- has two neighbors in $R_I \setminus N$, then (3.2)(c) applies. Hence, we assume that every vertex in D^- has at most one neighbor in $R_I \setminus N$. First, we assume that N has 3 elements. If some vertex in D^- has a neighbor in $R_I \setminus N$, then (3.2)(c) applies. If no vertex in D^- has a neighbor in $R_I \setminus N$, then (3.2)(c) does not apply. Next, we assume that N has 2 elements, say $N = \{w_1, w_2\}$. We partition D^- into sets $D_0 = \{u \in D^- \mid N(u) \cap R_I \subseteq N\}$ and $D_w = \{u \in D^- \mid (N(u) \cap R_I) \setminus N = \{w\}\}$ for $w \in R_I \setminus N$. If there are distinct vertices w_3 and w_4 from $R_I \setminus N$ such that D_{w_3} contains a neighbor of w_1 and D_{w_4} contains a neighbor of w_2, then (3.2)(c) applies. If there are no such vertices, then there is a vertex w in N with $N(u) \cap N \subseteq \{w\}$ for every vertex $u \in D^-$ and (3.2)(c) does not apply. The above observations easily lead to a linear time algorithm. Altogether, we obtain the following final result contrasting with Theorem A.

Corollary 6. *It can be decided in linear time whether a given set of vertices of a given split graph is a Radon-dependent set.*

References

[1] R.M. Barbosa, E.M.M. Coelho, M.C. Dourado, D. Rautenbach, J.L. Szwarcfiter, On the Carathéodory number for the convexity of paths of order three, SIAM J. Discrete Math. 26 (2012) 929–939.

[2] C.C. Centeno, M.C. Dourado, L.D. Penso, D. Rautenbach, J.L. Szwarcfiter, Irreversible conversion of graphs, Theor. Comput. Sci. 412 (2011) 3693–3700.

[3] M.C. Dourado, D. Rautenbach, V. Fernandes dos Santos, P.M. Schäfer, J.L. Szwarcfiter, A. Toman, Algorithmic and structural aspects of the P_3-Radon number, manuscript, 2011.

[4] M.C. Dourado, D. Rautenbach, V. Fernandes dos Santos, P.M. Schäfer, J.L. Szwarcfiter, A. Toman, An upper bound on the P_3-Radon number, Discrete Math. 312 (2012) 2433–2437.

[5] P. Duchet, Convex sets in graphs. II: Minimal path convexity, J. Comb. Theory, Ser. B 44 (1988) 307–316.

[6] P. Erdős, E. Fried, A. Hajnal, E.C. Milner, Some remarks on simple tournaments, Algebra Univers. 2 (1972) 238–245.

[7] M. Farber, R.E. Jamison, Convexity in graphs and hypergraphs, SIAM J. Algebraic Discrete Methods 7 (1986) 433–444.

[8] P. Heggernes, D. Kratsch, Linear-time certifying recognition algorithms and forbidden induced subgraphs, Nordic J. Comput. 14 (2007) 87–108.

[9] M.A. Henning, D. Rautenbach, P.M. Schäfer, Open packing, total domination, and the P_3-Radon number, 2011, submitted for publication.

[10] J.W. Moon, Embedding tournaments in simple tournaments, Discrete Math. 2 (1972) 389–395.

[11] D.B. Parker, R.F. Westhoff, M.J. Wolf, On two-path convexity in multipartite tournaments, European J. Combin. 29 (2008) 641–651.

[12] J. Radon, Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten, Math. Ann. 83 (1921) 113–115.

[13] J.C. Varlet, Convexity in tournaments, Bull. Soc. R. Sci. Liège 45 (1976) 570–586.

[14] M.L.J. van de Vel, Theory of Convex Structures, North-Holland, Amsterdam, 1993.

Apêndice I

On the Radon Number for P_3-Convexity

Este apêndice contém o artigo "*On the Radon Number for P_3-Convexity*", apresentado no *Latin American Symposium on Theoretical Informatics, 2012* cujos anais foram publicados no *Lecture Notes in Computer Science.*

On the Radon Number for P_3-Convexity

Mitre C. Dourado[1], Dieter Rautenbach[2], Vinícius Fernandes dos Santos[3], Philipp M. Schäfer[2], Jayme L. Szwarcfiter[1,3,4], and Alexandre Toman[1]

[1] Instituto de Matematica,
Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil
{mitre,jayme}@nce.ufrj.br, alexandre21campos@ppgi.ufrj.br
[2] Institut für Optimierung und Operations Research,
Universität Ulm, Ulm, Germany
dieter.rautenbach@uni-ulm.de, philipp.schaefer@gmail.com
[3] PESC, COPPE, Universidade Federal do Rio de Janeiro,
Rio de Janeiro, Brazil
vinicius.santos@gmail.com
[4] NCE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil

Abstract. The generalization of classical results about convex sets in $\mathbb{R}^n$ to abstract convexity spaces, defined by sets of paths in graphs, leads to many challenging structural and algorithmic problems. Here we study the Radon number for the P_3-convexity on graphs. P_3-convexity has been proposed in connection with rumour and disease spreading processes in networks and the Radon number allows generalizations of Radon's classical convexity result. We establish hardness results, describe efficient algorithms for trees, and prove a best-possible bound on the Radon number of connected graphs.

Keywords: Graph convexity, Radon partition, Radon number.

Area: Algorithms, combinatorics and graph theory, complexity theory.

1 Introduction

When does an individual within a network adopt an opinion or contract a disease? How does a rumour or a computer virus spread within a network? As a natural model for such processes [7] one can consider a set of vertices R in a graph G to represent the set of infected individuals and iteratively add further vertices u to R whenever sufficiently many neighbours of u belong to R, that is, someone adopts an opinion/contracts a disease if sufficiently many of his contacts did so.

In the simplest non-trivial case, vertices are added to R whenever at least two of their neighbours belong to R. The collection of all sets of vertices to which no further vertices will be added defines the so-called P_3-convexity on the graph G, that is, a set R of vertices of G is considered to be convex exactly if no vertex outside of R is the middle vertex of a path of order three starting and ending in R. Next to the geodetic convexity [12] defined by shortest paths, and

D. Fernández-Baca (Ed.): LATIN 2012, LNCS 7256, pp. 267–278, 2012.

the monophonic convexity [8] defined by induced paths in similar ways, this is one of the natural and well studied convexity spaces defined by paths in graphs. The P_3-convexity was first considered for directed graphs, more specifically for tournaments [11,15,19] and multipartite tournaments [16].

Several of the classical convexity parameters have been considered for P_3-convexity. The geodetic number of P_3-convexity is the same as the well known 2-domination number [5]. It corresponds to the minimum number of infected individuals that will infect the entire network in one step. The hull number, which corresponds to the minimum number of infected individuals that will eventually infect the entire network, was investigated in [4,7]. Also the Carathéodory number [2] was considered.

In the present paper we study the so-called Radon number of P_3-convexity. In 1921 Radon [17] proved that every set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect. His result naturally leads to the definition of the Radon number of a general convexity space $(X, \mathcal{C})$ [20] as the smallest integer k for which every set of k points in X can be partitioned into two sets whose convex hulls with respect to $\mathcal{C}$ intersect. A set of vertices R of some graph G that does not have a partition as in Radon's result with respect to P_3-convexity corresponds to a group of individuals with the property that no matter in which way two possible opinions are distributed among the members of the group and then propagated through the network according to P_3-convexity, no individual will ever get under conflicting influences.

Our contributions are as follows. First we introduce relevant notions and terminology in Section 2. In Section 3 we study the algorithmic problem to decide whether a given set of vertices of some graph allows a partition as in Radon's result. In Section 4 we study the algorithmic problem to determine the Radon number of the P_3-convexity of some graph. In both sections we prove hardness results and describe efficient algorithms for trees. In Section 5 we prove an upper bound on the P_3-Radon number of connected graphs and characterize all extremal graphs. Finally, in Section 6 we conclude with some open problems.

2 Preliminaries

We consider finite, simple, and undirected graphs and use standard notation.

Let G be a graph and let R be a set of vertices of G. The set R is *convex in G* if no vertex in $V(G) \setminus R$ has two neighbours in R. The *convex hull $H_G(R)$ of R in G* is the intersection of all convex sets in G containing R. Equivalently, $H_G(R)$ is the smallest set containing R that is convex in G. A *Radon partition of R* is a partition of R into two disjoint sets R_1 and R_2 with $H_G(R_1) \cap H_G(R_2) \neq \emptyset$. The set R is an *anti-Radon set of G* if it has no Radon partition. The *Radon number $r(G)$ of G* is the minimum integer r such that every set of at least r vertices of G has a Radon partition. Equivalently, the Radon number of G is the maximum cardinality of an anti-Radon set of G plus one, i.e.

$$r(G) = \max\{|R| \mid R \text{ is an anti-Radon set of } G\} + 1.$$

Clearly, if R is an anti-Radon set of a graph G and H is a subgraph of G, then every subset of $R \cap V(H)$ is an anti-Radon set of H.

For a non-negative integer n, let $[n] = \{1, \ldots, n\}$.

3 Recognizing Anti-Radon Sets

In this section we consider the algorithmic problem to recognize anti-Radon sets in graphs.

ANTI-RADON SET RECOGNITION
Instance: A graph G and a set R of vertices of G.
Question: Does R have a Radon partition?

We prove that ANTI-RADON SET RECOGNITION is NP-complete for bipartite graphs. Furthermore, we give a characterization of anti-Radon sets, which leads to an efficient algorithm solving ANTI-RADON SET RECOGNITION for trees.

Theorem 1. ANTI-RADON SET RECOGNITION *is NP-complete even when restricted to input graphs that are bipartite.*

Algorithm 1. Procedure that determines the convex hull $H_G(R)$

while $\exists u \in V(G) \setminus R$ *with* $|N_G(u) \cap R| \geq 2$ **do**
| $R \leftarrow R \cup \{u\}$;
end
return R;

The Algorithm 1 leads to some useful observations. If $R_1 \cup R_2$ is a Radon partition of a set R of vertices of a graph G, then forming the two intersecting convex hulls $H_G(R_1)$ and $H_G(R_2)$ by iteratively adding single vertices to either R_1 or R_2, there is a first vertex that belongs to both sets. We call such a vertex a *Radon witness vertex for R*. Note that Radon witness vertices are not unique. The following lemma makes this observation more precise.

Lemma 1. *Let G be a graph and let R be a set of vertices of G. R is an anti-Radon set of G if and only if there are no two sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ of vertices of G such that*

(i) $x_1, \ldots, x_a$ *are distinct,*
(ii) $y_1, \ldots, y_b$ *are distinct,*
(iii) $x_a = y_b$ *and* $\{x_1, \ldots, x_a\} \cap \{y_1, \ldots, y_b\} = \{x_a\}$,
(iv) either $x_i \in R$ *or* x_i *has two neighbours in* $\{x_1, \ldots, x_{i-1}\}$ *for* $i \in [a]$, *and*
(v) either $y_j \in R \setminus \{x_a\}$ *or* y_j *has two neighbours in* $\{y_1, \ldots, y_{j-1}\}$ *for* $j \in [b]$.

Proof: First, we assume that R is not an anti-Radon set of G. This implies the existence of a partition $R_1 \cup R_2$ of R such that $H_G(R_1) \cap H_G(R_2) \neq \emptyset$. Let $R_1 = \{x_1, \ldots, x_{a'}\}$ and $R_2 = \{y_1, \ldots, y_{b'}\}$. As noted above, forming the convex hull of R_1 and R_2 by iteratively adding single vertices to either R_1 or R_2, there is a first vertex, say z, that belongs to both sets. This implies that there are two sequences $x_1, \ldots, x_a$ with $a \geq a'$ and $y_1, \ldots, y_b$ with $b \geq b'$ such that $z = x_a = y_b$ that satisfy

- (i), (ii), (iii),
- either $x_i \in R_1$ or x_i has two neighbours in $\{x_1, \ldots, x_{i-1}\}$ for $i \in [a]$, and
- either $y_j \in R_2$ or y_j has two neighbours in $\{y_1, \ldots, y_{j-1}\}$ for $j \in [b]$.

Clearly, $x_1, \ldots, x_a$ satisfies (iv). Since R_1 and R_2 are disjoint, we may assume that $z \notin R_2$, which implies that $y_1, \ldots, y_b$ satisfies (v).

Conversely, if the two sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ of vertices of G satisfy (i) to (v), then $R_1 = R \cap \{x_1, \ldots, x_a\}$ and $R_2 = R \cap \{y_1, \ldots, y_{b-1}\}$ are disjoint subsets of R with $x_a \in H_G(R_1) \cap H_G(R_2)$. Hence R is not an anti-Radon set, which completes the proof. □

We call a pair of sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ as in Lemma 1 a *Radon witness sequences for R*. Note that a vertex z is a Radon witness vertex for R if and only if there are Radon witness sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ for R with $z = x_a = y_b$. If the vertex x_a belongs to R, then we may assume that $a = 1$, that is, the sequence $x_1, \ldots, x_a$ contains one element only. Altogether, by Lemma 1, a set R of vertices of a graph G is not an anti-Radon set of G if and only if there are Radon witness sequences for R if and only if there is a Radon witness vertex for R.

These notions are helpful to efficiently solve Anti-Radon Set Recognition for trees.

Theorem 2. *Let T be a tree and let R be a set of vertices of T. R is an anti-Radon set of T if and only if there is no vertex z of T such that*

(i) either $z \notin R$ and four neighbours of z in T are in $H_{T-z}(R)$
(ii) or $z \in R$ and two neighbours of z in T are in $H_{T-z}(R \setminus \{z\})$.

Proof: First we assume that R is no anti-Radon set. By Lemma 1, there are Radon witness sequences $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$ for R. Let $z = x_a = y_b$ be the corresponding Radon witness vertex. Let $R_1 = R \cap \{x_1, \ldots, x_a\}$ and $R_2 = R \cap \{y_1, \ldots, y_{b-1}\}$.

If $z \notin R$, then the conditions in Lemma 1 imply that there are four distinct neighbours u_1, v_1, u_2, and v_2 of z in T such that $u_1, v_1 \in H_{T-z}(R_1)$ and $u_2, v_2 \in H_{T-z}(R_2)$, which implies that all four vertices belong to $H_{T-z}(R)$, that is, (i) holds. If $z \in R$, then, as noted above, we may assume $a = 1$. Now the conditions for $y_1, \ldots, y_b$ in Lemma 1 imply that there are two neighbours u_2 and v_2 of z in T such that $u_2, v_2 \in H_{T-z}(R_2)$, which implies that these two vertices belong to $H_{T-z}(R \setminus \{z\})$, that is, (ii) holds.

Conversely, if (i) holds, then let u_1, v_1, u_2, and v_2 be four distinct neighbours of z in T such that $u_1, v_1, u_2, v_2 \in H_{T-z}(R)$. For $i \in [2]$, let R_i denote the set of vertices in R that belong to the same component of $T - z$ as either u_i or v_i. Note that R_1 and R_2 are disjoint, because T is a tree. Clearly, $u_1, v_1 \in H_{T-z}(R_1)$ and $u_2, v_2 \in H_{T-z}(R_2)$, which implies that R is not an anti-Radon set. Finally, if

(ii) holds, then a similar construction implies that R is not an anti-Radon set, which completes the proof. □

Since the convex hull of a set of vertices in a graph can be determined in polynomial time using Algorithm 1, Theorem 2 leads to an efficient algorithm solving Anti-Radon Set Recognition for trees. At the end of the next section we explain how to obtain a linear time algorithm.

4 Maximizing Anti-Radon Sets

In this section we consider the algorithmic problem to determine the Radon number and anti-Radon sets of maximum cardinality for a given graph. We establish the NP-hardness of the following decision problem.

Maximum Anti-Radon Set
Instance: A graph G and an integer k.
Question: Does G have an anti-Radon set of size k?

Furthermore, we develop an efficient algorithm for trees.

A natural certificate for a "Yes"-instance of Maximum Anti-Radon Set would certainly be an anti-Radon set of G of size k. Since Anti-Radon Set Recognition is NP-complete, such a certificate can most probably not be checked efficiently and we do not know whether Maximum Anti-Radon Set lies in NP.

A graph G is a *split graph* if its vertex set admits a partition $V(G) = C \cup I$ into a clique C and an independent set I.

Theorem 3. Maximum Anti-Radon Set *is NP-hard even when restricted to input graphs that are split graphs.*

Proof: For the reduction, we use the Set Packing problem, which is known to be NP-complete [14]. The latter problem has as input a family $\mathcal{S} = \{S_1, \ldots, S_n\}$ of non-empty sets and an integer l. The question is whether $\mathcal{S}$ contains l mutually disjoint sets. Given $\mathcal{S}$ and l, we construct an instance (G, k) of Maximum Anti-Radon Set. The elements of the ground set $S_1 \cup \ldots \cup S_n$ of $\mathcal{S}$ are all vertices of the graph G. Besides, G contains a pair of new distinguished vertices w_i and z_i for each set S_i in $\mathcal{S}$. The edges of G are as follows. The set $C = (S_1 \cup \{z_1\}) \cup \ldots \cup (S_n \cup \{z_n\})$ forms a clique of G. In addition, for each of the distinguished vertices w_i, we add an edge $w_i z_i$ and edges $w_i v$ for each $v \in S_i$. Finally, we define $k = l$. This completes the construction of (G, k). Observe that $I = \{w_1, \ldots, w_n\}$ is an independent set of G. Therefore, G is a split graph with partition $V(G) = C \cup I$. Without loss of generality, we may assume $k = l \geq 4$. We prove that $\mathcal{S}$ contains l mutually disjoint sets if and only if G has an anti-Radon set of size k.

Suppose that $\mathcal{S}$ contains l mutually disjoint sets, say $S_1, \ldots, S_l$. Since $S_i \cap S_j = \emptyset$ for distinct $i, j \in [l]$, we obtain $N_G[w_i] \cap N_G[w_j] = \emptyset$ for distinct $i, j \in [k]$. This implies that $H_G(J) = J$ for every subset J of $\{w_1, \ldots, w_k\}$ and hence $\{w_1, \ldots, w_k\}$ is an anti-Radon set of G of size k.

Conversely, suppose that G has an anti-Radon set R of size $k \geq 4$. If R contains two vertices v_1 and v_2 such that $v_1 \in C$, then, since every vertex in I has at least two neighbours in C and every vertex in I has a common neighbour with every vertex in C, we obtain $V(G) = H_G(\{v_1, v_2\})$. This implies that $R_1 \cup R_2$ with $R_1 = \{v_1, v_2\}$ and $R_2 = R \setminus R_1$ is a Radon partition of R, which is a contradiction. Hence R is a subset of $I = \{w_1, \ldots, w_n\}$. If R contains two vertices, say w_i and w_j, from I such that $S_i \cap S_j$ contains a vertex, say v, then again $V(G) = H_G(\{w_i, w_j\})$, which is a contradiction. Hence the sets S_i for $i \in [n]$ with $w_i \in R$ are l mutually disjoint sets in $\mathcal{S}$, which completes the proof. $\square$

We develop a reduction principle relating anti-Radon sets of a connected graph G that has a vertex u such that all edges of G incident with u are bridges of G, to anti-Radon sets of the components of $G - u$. The main application of this principle is an efficient algorithm computing the Radon number and largest anti-Radon sets of trees.

Let G be a graph and let u be a vertex of G. Throughout this section, let $G^{u \leftarrow x}$ denote the graph that arises by adding to G a new vertex x and a new edge ux. Let

$$\begin{aligned}
\mathcal{R}_+(G,u) &= \{R \mid R \text{ is an anti-Radon set of } G \text{ and } u \in H_G(R)\},\\
\mathcal{R}_-(G,u) &= \{R \mid R \text{ is an anti-Radon set of } G \text{ and } u \notin H_G(R)\},\\
\mathcal{R}'_+(G,u) &= \{R \mid R \subseteq V(G),\ \{x\} \cup R \text{ is an anti-Radon set of } G^{u \leftarrow x},\\
&\qquad \text{and } u \in H_G(R)\}, \text{ and}\\
\mathcal{R}'_-(G,u) &= \{R \mid R \subseteq V(G), \{x\} \cup R \text{ is an anti-Radon set of } G^{u \leftarrow x},\\
&\qquad \text{and } u \notin H_G(R)\}.
\end{aligned}$$

Furthermore, let

$$\begin{aligned}
r_+(G,u) &= \max\{|R| \mid R \in \mathcal{R}_+(G,u)\},\\
r_-(G,u) &= \max\{|R| \mid R \in \mathcal{R}_-(G,u)\},\\
r'_+(G,u) &= \max\{|R| \mid R \in \mathcal{R}'_+(G,u)\}, \text{ and}\\
r'_-(G,u) &= \max\{|R| \mid R \in \mathcal{R}'_-(G,u)\}.
\end{aligned}$$

These definitions immediately imply

$$r(G) = \max\{r_+(G,u), r_-(G,u)\} + 1. \tag{1}$$

Now let G be a connected graph and let u be a vertex of G such that all edges of G incident with u are bridges of G. Let $G_1, \ldots, G_k$ denote the components of $G - u$ and let u_i denote the unique neighbour of u in $V(G_i)$ for $i \in [k]$. Let R be a set of vertices of G and let $R_i = R \cap V(G_i)$ for $i \in [k]$.

Lemma 2. *Let G, u, R, G_i, u_i, and R_i for $i \in [k]$ be as above.*
R belongs to $\mathcal{R}_+(G,u)$ if and only if one of the following cases occurs.

(i) $u \in R$ and $R_i \in \mathcal{R}'_-(G_i, u_i)$ for $i \in [k]$.

(ii) $u \in R$ *and there is some index* $i_1 \in [k]$ *such that* $R_{i_1} \in \mathcal{R}'_+(G_{i_1}, u_{i_1})$ *and* $R_i \in \mathcal{R}'_-(G_i, u_i)$ *for* $i \in [k] \setminus \{i_1\}$.

(iii) $u \notin R$ *and there are two distinct indices* $i_1, i_2 \in [k]$ *such that* $R_{i_j} \in \mathcal{R}_+(G_{i_j}, u_{i_j})$ *for* $j \in [2]$ *and* $R_i \in \mathcal{R}'_-(G_i, u_i)$ *for* $i \in [k] \setminus \{i_1, i_2\}$.

(iv) $u \notin R$ *and there are three distinct indices* $i_1, i_2, i_3 \in [k]$ *such that* $R_{i_j} \in \mathcal{R}'_+(G_{i_j}, u_{i_j})$ *for* $j \in [3]$ *and* $R_i \in \mathcal{R}'_-(G_i, u_i)$ *for* $i \in [k] \setminus \{i_1, i_2, i_3\}$.

Similar lemmas hold for $\mathcal{R}_-(G, u)$, $\mathcal{R}'_+(G, u)$, and $\mathcal{R}'_-(G, u)$, which immediately imply the following recurrence formulas.

Corollary 1. *Let* G, u, R, G_i, u_i, *and* R_i *for* $i \in [k]$ *be as above.*

(i) $r_+(G, u)$ *equals the maximum of the following four expressions*

$$1 + \sum_{i \in [k]} r'_-(G_i, u_i),$$

$$\max_{i_1 \in [k]} \left(1 + r'_+(G_{i_1}, u_{i_1}) + \sum_{i \in [k] \setminus \{i_1\}} r'_-(G_i, u_i) \right),$$

$$\max_{i_1, i_2 \in [k]} \left(r_+(G_{i_1}, u_{i_1}) + r_+(G_{i_2}, u_{i_2}) + \sum_{i \in [k] \setminus \{i_1, i_2\}} r'_-(G_i, u_i) \right),$$

$$\max_{i_1, i_2, i_3 \in [k]} \left(r'_+(G_{i_1}, u_{i_1}) + r'_+(G_{i_2}, u_{i_2}) + r'_+(G_{i_3}, u_{i_3}) + \sum_{i \in [k] \setminus \{i_1, i_2, i_3\}} r'_-(G_i, u_i) \right)$$

where the individual maxima are taken over distinct indices i_j.

(ii) $r_-(G, u)$ *equals*

$$\max \left\{ \sum_{i \in [k]} r_-(G_i, u_i), \quad \max_{i_1 \in [k]} \left(r_+(G_{i_1}, u_{i_1}) + \sum_{i \in [k] \setminus \{i_1\}} r_-(G_i, u_i) \right) \right\}.$$

(iii) $r'_+(G, u)$ *equals*

$$\max \left\{ 1 + \sum_{i \in [k]} r'_-(G_i, u_i), \quad \max_{i_1, i_2 \in [k]} \left(r'_+(G_{i_1}, u_{i_1}) + r'_+(G_{i_2}, u_{i_2}) + \sum_{i \in [k] \setminus \{i_1, i_2\}} r'_-(G_i, u_i) \right) \right\}$$

where the maximum is taken over distinct indices i_1 *and* i_2.

(iv) $r'_-(G, u)$ *equals*

$$\max \left\{ \sum_{i \in [k]} r_-(G_i, u_i), \quad \max_{i_1 \in [k]} \left(r_+(G_{i_1}, u_{i_1}) + \sum_{i \in [k] \setminus \{i_1\}} r'_-(G_i, u_i) \right) \right\}.$$

If G is a graph with exactly one vertex u, then

$$r_+(G,u) = r'_+(G,u) = 1 \text{ and } r_-(G,u) = r'_-(G,u) = 0.$$

These initial values together with (1) and the recurrences from Corollary 1 yield an efficient algorithm computing the Radon number of trees by dynamic programming. Keeping track of the corresponding maximizers in the individual recurrences, it is possible to determine largest anti-Radon sets of trees efficiently. Since for a vertex u in a tree T, the evaluation of the recurrences can be done in $O(d_T(u))$ time, the overall running time of this algorithm is $O\left(\sum\limits_{u \in V(T)} d_T(u)\right)$, which is $O(n)$ for trees.

It is not difficult to generalize the sketched algorithm in such a way that it solves the following more general problem on trees.

Maximum Anti-Radon Subset
Instance: A graph G and a set S of vertices of G.
Task: Determine a largest anti-Radon set R of G that is a subset of S.

In fact, the requirement $R \subseteq S$ just eliminates some of the terms from the recurrences whenever $u \notin S$. The above observations immediately imply the following.

Theorem 4. *There is a linear time algorithm solving* Maximum Anti-Radon Subset *on trees.*

Clearly, Maximum Anti-Radon Subset generalizes both problems considered above. Choosing $S = V(G)$, a largest anti-Radon subset of S is just a largest anti-Radon set of G. Furthermore, a given set S is an anti-Radon set exactly if the largest anti-Radon subset of S is S itself.

5 An Upper Bound on the Radon Number

In this section we prove an upper bound on the Radon number of connected graphs and characterize all extremal graphs.

We define a set $\mathcal{T}$ of trees using the following two extension operations.

- If T is a tree, v is a vertex of T, and T' arises from T by adding three new vertices u', v', and w' and three new edges $u'v'$, $v'w'$, and $w'v$, then T' is said to arise from T by a *type 1 extension.*
- If T is a tree, v is a endvertex of T, w is the neighbour of v in T, w is of degree 2 in T, and T'' arises from T by adding three new vertices u, u', and v' and three new edges uv, $u'v'$, and $v'w$, then T'' is said to arise from T by a *type 2 extension.*

Let $\mathcal{T}$ be defined recursively as the set of trees that consists of K_2 and every tree T that arises from a smaller tree in $\mathcal{T}$ by a type 1 extension or a type 2 extension.

Our main result is the following.

Theorem 5. *If G is a connected graph, then $r(G) \leq \frac{2}{3}(n(G)+1)+1$ with equality if and only if G belongs to $\mathcal{T}$.*

Before we prove this result, we need some lemmas.

Lemma 3. *If T is a tree and R is an anti-Radon set of T, then $|R| \leq \frac{2}{3}(n(T)+1)$. Furthermore, if $|R| = \frac{2}{3}(n(T)+1)$, then $T \in \mathcal{T}$.*

Proof: We use induction on $n(T)$. For $n(T) \leq 2$, the desired statement is obvious. Therefore, we assume $n(T) \geq 3$. For contradiction, we assume that either $|R| > \frac{2}{3}(n(T)+1)$ or $|R| = \frac{2}{3}(n(T)+1)$ but $T \not\in \mathcal{T}$.

Claim 1. *R contains all endvertices of T.*

Proof of Claim 1: For contradiction, we assume that R does not contain the endvertex u of T. Clearly, R is an anti-Radon set of the tree $T - \{u\}$. Hence, by induction, $|R| \leq \frac{2}{3}((n(T)-1)+1) < \frac{2}{3}(n(T)+1)$, which is a contradiction. □

Claim 2. *No vertex of T is adjacent to two endvertices.*

Proof of Claim 2: For contradiction, we assume that the vertex u is adjacent to the two endvertices v and w in T. Claim 1 implies $v, w \in R$. Since R is an anti-Radon set of T and $u \in H_T(\{v,w\})$, the vertex u does not belong to R and $R' = (R \setminus \{v,w\}) \cup \{u\}$ is an anti-Radon set of the tree $T - \{v,w\}$. Hence, by induction, $|R| \leq |R'| + 1 \leq \frac{2}{3}((n(T)-2)+1)+1 < \frac{2}{3}(n(T)+1)$, which is a contradiction. □

Let $P : u_1u_2 \ldots u_l$ be a longest path in T, that is, u_1 is an endvertex of T. Claims 1 and 2 imply $u_1 \in R$ and $d_G(u_2) = 2$. Since the desired statement is obvious for stars, we may assume that $l \geq 4$. Hence $d_T(u_3) \geq 2$.

Claim 3. *$u_2 \in R$ and $u_3 \not\in R$.*

Proof of Claim 3: For contradiction, we assume $u_2 \not\in R$. The set $R \setminus \{u_1\}$ is an anti-Radon set of the tree $T - \{u_1, u_2\}$. Hence, by induction, $|R| \leq \frac{2}{3}((n(T)-2)+1)+1 < \frac{2}{3}(n(T)+1)$, which is a contradiction. Hence $u_2 \in R$. Since $u_2 \in H_T(\{u_1,u_3\})$ and R is an anti-Radon set of T, we obtain $u_3 \not\in R$. □

If $d_T(u_3) = 2$, then T arises from the tree $F = T - \{u_1,u_2,u_3\}$ by a type 1 extension and $S = R \setminus \{u_1,u_2\}$ is an anti-Radon set of F. By induction, $|R| \leq |S| + 2 \leq \frac{2}{3}(n(F)+1)+2 = \frac{2}{3}(n(T)+1)$. Hence either $|R| < \frac{2}{3}(n(T)+1)$ or $|R| = \frac{2}{3}(n(T)+1)$ and, by induction and the definition of $\mathcal{T}$, $T \in \mathcal{T}$, which is a contradiction. Therefore, we may assume that $d_T(u_3) \geq 3$.

Recall that P is a longest path in T, which implies that every path in T between u_3 and an endvertex of T that does not contain u_4 has length either 1 or 2. If $d_T(u_3) \geq 4$, then, by the choice of P, symmetry, and Claims 1, 2, and 3, we obtain the existence of two neigbours of u_3 distinct from u_2, say v and

w, that belong to R. Now $u_1, u_2, v, w \in R$ and $u_2 \in H_T(\{u_1, v, w\})$, which is a contradiction. Hence $d_T(u_3) = 3$. By Claim 2, it suffices to consider the following two cases.

Case 1. *u_3 is adjacent to an endvertex u_2'.*

By Claim 1, we have $u_2' \in R$. Let $S = (R \setminus \{u_1, u_2'\}) \cup \{u_3\}$ and $F = T - \{u_1, u_2'\}$. For contradiction, we assume that S is no anti-Radon set of F. Since R is an anti-Radon set of T and $u_3 \in H_T(\{u_2, u_2'\})$, the set $S \setminus \{u_2\}$ is an anti-Radon set of F. This implies that u_3 is the only Radon witness vertex for S in F. Now

- firstly replacing u_2 with u_1 and u_2',
- and secondly replacing u_3 with u_2

in two Radon witness sequences for S, results in two Radon witness sequences for R with u_2 as Radon witness vertex, which is a contradiction. Hence S is an anti-Radon set of F and, by induction, $|R| \leq |S|+1 \leq \frac{2}{3}(n(F)+1)+1 < \frac{2}{3}(n(T)+1)$, which is a contradiction.

Case 2. *u_3 is adjacent to a vertex u_2' of degree 2 that is distinct from u_2, and u_2' is adjacent to an endvertex u_1'.*

By Claims 1 and 3, we have $u_1', u_2' \in R$. Similarly as in Case 1, it follows that $S = (R \setminus \{u_1, u_1', u_2'\}) \cup \{u_3\}$ is an anti-Radon set of $F = T - \{u_1, u_1', u_2'\}$. Hence, by induction, $|R| \leq |S| + 2 \leq \frac{2}{3}(n(F) + 1) + 2 = \frac{2}{3}(n(T) + 1)$. Since T arises from F by a type 2 extension, we obtain that either $|R| < \frac{2}{3}(n(T) + 1)$ or $|R| = \frac{2}{3}(n(T) + 1)$ and, by induction and the definition of $\mathcal{T}$, $T \in \mathcal{T}$, which is a contradiction and completes the proof. □

To each tree T in $\mathcal{T}$, we assign a set $R(T)$ of vertices of T as follows. We denote the vertices as in the definition of the extensions.

- Let $R(K_2) = V(K_2)$.
- If T' arises from $T \in \mathcal{T}$ by a type 1 extension, then let $R(T') = R(T) \cup \{u', v'\}$.
- If T'' arises from $T \in \mathcal{T}$ by a type 2 extension, then let $R(T') = (R(T) \setminus \{w\}) \cup \{u, u', v'\}$.

Lemma 4. *If T belongs to $\mathcal{T}$, then the following statements hold.*

(i) $R(T)$ is an anti-Radon set of T of maximum cardinality, $|R(T)| = \frac{2}{3}(n(T)+1)$, and $H_T(R(T)) = V(T)$.
(ii) $R(T)$ is the unique anti-Radon set of T of maximum cardinality.

Lemma 5. *If G arises by adding a new edge xy to a tree F in $\mathcal{T}$, then $r(G) < r(F)$.*

We are now in a position to prove the main result of this section.

Proof of Theorem 5: Let R be an anti-Radon set of G of maximum cardinality and let F be a spanning tree of G. Since $H_F(S) \subseteq H_G(S)$ for every set S of vertices of G, the set R is also an anti-Radon set of F and Lemmas 3 and 5 imply that $|R|+1 = r(G) \leq r(F) \leq \frac{2}{3}(n(G)+1)+1$ with equality only if $F \in \mathcal{T}$ and $G = F$, that is, $G \in \mathcal{T}$. Furthermore, by Lemma 4, every graph G in $\mathcal{T}$ satisfies $r(G) = \frac{2}{3}(n(G)+1)+1$, which completes the proof. $\square$

Note that if the graph G has the connected components $G_1, \ldots, G_l$, then $r(G) - 1 = (r(G_1)-1) + \ldots + (r(G_2)-1)$.

6 Conclusion

Especially the Radon number of the geodetic convexity of finite graphs has received a lot of attention. This is probably due to Eckhoff's conjecture [10] related to Tverberg's generalization [18] of Radon's result [17]. Jamison [13] proved this conjecture for the geodetic convexity of trees and Bandelt and Pesch [1] relate the Radon number for Helly graphs in geodetic convexity to the clique number of these graphs. The special role of the geodetic convexity in this context was justified by Duchet who actually announced [9] that the partition conjecture would hold in general if it holds for the geodetic convexity of finite graphs. It is an open problem whether Eckhoff's conjecture holds for P_3-convexity. The precise statement would be that for every $m > 2$, every set R of vertices of some graph G with $|R| \geq (m-1)(r(G)-1)+1$ has a partition $R_1 \cup \ldots \cup R_m$ into m sets such that $H_G(R_1) \cap \ldots \cap H_G(R_m) \neq \emptyset$. For further details, please refer to [10]. Very recently, the disproof of Eckhoff's conjecture in general was announced [3]. Nevertheless, it still remains open for P_3-convexity.

Acknowledgement. This work has been done in the context of the CAPES-DAAD PROBRAL project "Cycles, Convexity, and Searching in Graphs". The first and fifth authors have been supported by CNPq and FAPERJ. The third author has been supported by FAPERJ. The sixth author has been supported by CAPES.

References

1. Bandelt, H.-J., Pesch, E.: A Radon theorem for Helly graphs. Arch. Math. 52, 95–98 (1989)
2. Barbosa, R.M., Coelho, E.M.M., Dourado, M.C., Rautenbach, D., Szwarcfiter, J.L.: On the Carathéodory number for the convexity of paths of order three. Electronic Notes in Discrete Mathematics 38, 105–110 (2011)
3. Bukh, B.: Radon partitions in convexity spaces, arXiv:1009.2384
4. Centeno, C.C., Dourado, M.C., Penso, L.D., Rautenbach, D., Szwarcfiter, J.L.: Irreversible conversion of graphs. Theor. Comput. Sci. 412, 3693–3700 (2011)

5. Chellali, M., Favaron, O., Hansberg, A., Volkmann, L.: k-Domination and k-Independence in Graphs: A Survey. Graphs Combin., doi:10.1007/s00373-011-1040-3
6. Cook, S.A.: The complexity of theorem-proving procedures. In: ACM, Proc. 3rd Ann. ACM Sympos. Theory Computing, Shaker Heights, Ohio, pp. 151–158 (1971)
7. Dreyer, P.A., Roberts, F.S.: Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion. Discrete Appl. Math. 157, 1615–1627 (2009)
8. Duchet, P.: Convex sets in graphs. II: Minimal path convexity. J. Comb. Theory, Ser. B 44, 307–316 (1988)
9. Duchet, P.: Discrete convexity: Retractions, morphisms and the partition problem. In: Balakrishnan, R., et al. (eds.) Proceedings of the Conference Graph Connections, Cochin, India, January 28-31, 1998, pp. 10–18. Allied Publishers Limited, New Delhi (1999)
10. Eckhoff, J.: The partition conjecture. Discrete Math. 221, 61–78 (2000)
11. Erdős, P., Fried, E., Hajnal, A., Milner, E.C.: Some remarks on simple tournaments. Algebra Univers. 2, 238–245 (1972)
12. Farber, M., Jamison, R.E.: Convexity in graphs and hypergraphs. SIAM J. Algebraic Discrete Methods 7, 433–444 (1986)
13. Jamison, R.E.: Partition numbers for trees and ordered sets. Pac. J. Math. 96, 115–140 (1981)
14. Karp, R.M.: Reducibility among combinatorial problems. In: Complexity of Computer Computations 1972, pp. 85–103. Plenum Press, New York (1972)
15. Moon, J.W.: Embedding tournaments in simple tournaments. Discrete Math. 2, 389–395 (1972)
16. Parker, D.B., Westhoff, R.F., Wolf, M.J.: On two-path convexity in multipartite tournaments. European J. Combin. 29, 641–651 (2008)
17. Radon, J.: Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. Math. Ann. 83, 113–115 (1921)
18. Tverberg, H.: A generalization of Radon's theorem. J. Lond. Math. Soc. 41, 123–128 (1966)
19. Varlet, J.C.: Convexity in tournaments. Bull. Soc. R. Sci. Liège 45, 570–586 (1976)
20. van de Vel, M.L.J.: Theory of Convex Structures. North-Holland, Amsterdam (1993)

Apêndice J

On Clique Graphs of Chordal Comparability Graphs

Este apêndice contém o resumo do trabalho"*On Clique Graphs of Chordal Comparability Graphs*", apresentado no *Latin American Workshop on Cliques in Graphs, 2012*, cuja versão completa encontra-se em fase de elaboração.

On Clique Graphs of Chordal Comparability Graphs

Michel Habib[1] Denis Julien[1] Ross M. McConnell[2]
Vinícius Fernandes dos Santos[3,*] Jayme L. Szwarcfiter[3,4,†]

[1] LIAFA, CNRS and Université Paris Diderot, France
[2] Computer Science Department, Colorado State University, USA
[3] COPPE, Universidade Federal do Rio de Janeiro, Brasil
[4] IM and NCE, Universidade Federal do Rio de Janeiro, Brasil

The *clique graph* $K(G)$ of a graph G is the intersection of the maximal cliques of G. A well known characterization of clique graphs is that by Roberts and Spencer (1971). In addition there are characterizations for clique graphs of several graph classes. In this work, we add a new class to this list, by describing the clique graphs of chordal comparability graphs. It is based on a new characterization of chordal comparability graphs, in terms of their maximal cliques. We recall that clique graphs of chordal graphs have been already characterized, e.g. [1], [2], [3]. As for comparability graphs, only partial characterizations of subclasses, such as cographs [4], are known. The problem of characterizing the clique graphs of general comparability graphs remains open.

References

[1] C. Bornstein and J. L. Szwarcfiter, Clique graphs of chordal and path graphs, *SIAM J. on Discrete Mathematics* 7 (1994), 331-336.

[2] A. Brandstadt, V. D. Chepoi, F. F. Dragan and V. I. Voloshin, Dually chordal graphs, *SIAM J. on Discrete Mathematics* 11 (1998), 436-455

[3] M. Gutierrez, Tree-clique graphs, in J. L. Szwarcfiter, ed., *Workshop Internacional de Combinatória*, 7-26, Universdade Federal do Rio de Janeiro, Rio de Janeiro, 1996

[4] F. Larrión, C. P. Mello, A. Morgana, V. Neumann-Lara and M. Pizaña, The clique operator on cographs and serial graphs. *Discrete Mathematics* 282 (2004), 183-191

*Supported by FAPERJ
†Partially Suppoted by CNPq, CAPES and FAPERJ