## GRACEFUL LABELING OF GRAPHS

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Programa: Engenharia de Sistemas e Computação

Em 1966, A. Rosa propôs uma nova coloração de grafos chamada coloração- $\beta$ em que os vértices são coloridos com números distintos entre 0 a $m$, onde $m$ é o número de arestas, tal que cada aresta é rotulada com o módulo da diferença das cores dos seus vértices extremos e cada um é único no grafo. Alguns anos depois, S. W. Golomb renomeou essa coloração de coloração graciosa, como é conhecida hoje em dia.

Esta definição permitiu que Rosa mostrasse que se toda árvore admitisse uma coloração graciosa, então uma conjectura de G. Ringel seria verdadeira. A partir disso, foi conjecturado que toda árvore fosse graciosa, a Conjectura das Árvores Graciosas.

Este trabalho apresenta alguns dos principais resultados em coloração graciosa de grafos e também apresenta esforços computacionais na direção da Conjectura das Árvores Graciosas. Inspirado por isso, nós também tomamos a abordagem computacional para estender a graciosidade de grafos cones generalizados.

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## GRACEFUL LABELING OF GRAPHS

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In 1966, A. Rosa introduced a new graph labeling called $\beta$-labeling in which the vertices are labeled with distinct numbers chosen from 0 to $m$, where $m$ is the number of edges, such that each edge is labeled with the absolute difference of the labels of its end vertices and it is unique in the graph. A few years later, S. W. Golomb renamed $\beta$-labeling as graceful labeling as it is known today.

This definition allowed Rosa to show that if every tree admits a graceful labeling, then a conjecture from G. Ringel would hold, from which it was conjectured that every tree is graceful, the Graceful Tree Conjecture.

This work presents some of the main results on graceful labeling of graphs and also presents computational efforts in the direction of the Graceful Tree Conjecture. Inspired by that, we also took the computational approach to extend the gracefulness of generalized cone graphs.

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## Chapter 1

## Introduction

Suppose we want to decompose a complete graph $G$ into trees, all of them isomorphic between themselves. In other words, we want to partition the edges of $G$ such that the subgraph induced by each set of edges of the partition is isomorphic to a given tree $T$. Ringel [20] conjectured that, for any tree $T$ with $n$ vertices, the complete graph $K_{2 n-1}$ can be decomposed into $2 n-1$ trees isomorphic to $T$.

Rosa [21] introduced graceful labeling in 1966, and, back then, he called it $\beta$ labeling. The term "graceful" was introduced by Golomb [12] in 1972. Rosa showed that if every tree is graceful, then Ringel's conjecture holds. Since then, researchers have been trying to prove Ringel's conjecture through the Graceful Tree Conjecture, which claims that every tree is graceful.

However, graceful graphs gained their own merit of study over the years. David S. Johnson, in his NP-completeness column of 1983 [16], includes the decision problem of graceful labeling as the "Open Problem of the Month". Moreover, there is the International Workshop on Graph Labeling in which graceful labeling is one of the main themes, and a complete survey on the subject from Gallian [11] that is constantly updated.

In Section 1.1, we give the definitions used throughout the text. In Chapter 2, we present the formal definition of graceful labeling of a graph and present the gracefulness of some graph classes as well as some general results about graceful labeling of graphs. In Chapter 3, we focus on results towards the Graceful Tree Conjecture, presenting different approaches to tackle the conjecture. Finally, in Chapter 4 we change our focus to generalized cone graphs, a graph class defined by the join of two graphs. We review known theoretical results and propose new computational results which establish the gracefulness of families of generalized cone graphs and suggest a conjecture regarding the non-graceful ones.

### 1.1 Definitions

In this section, we give most of the definitions and notation of graph theory used in this text. For any missing definition, see Bondy and Murty [7].

A graph $G$ is an ordered pair $(V, E)$ where $V$ is a set of elements called vertices and $E$ is a set of unordered pairs of distinct vertices from $V$ called edges. We say an edge $e$ connects two vertices $u$ and $v$, denoting as $e=u v$, and we say $u$ and $v$ are adjacent if they are connected by an edge. The set of adjacent vertices of a vertex $u$ is denoted as $N(u)$, and it is also called the set of neighbors of $u$. The degree of a vertex $u$ is $d(u)=|N(u)|$, the number of neighbors of $u$.

For a given graph $G$, when the vertex set and the edge set are not given explicitly, we refer to them as $V(G)$ and $E(G)$, and we use the letters $n$ and $m$ as the number of vertices and edges, respectively.

A subgraph $H$ of $G$ is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a graph $G=(V, E)$ and a subset $W \subseteq V$, the subgraph of $G$ induced by $W$, denoted as $G[W]$, is the graph $H=(W, F)$ such that, for all $u, v \in W$, if $u v \in E$, then $u v \in F$. We say $H$ is an induced subgraph of $G$. Equivalently, we can define subgraphs and induced subgraphs in terms of deletion of vertices and edges: $H$ is an induced subgraph of $G$ if it is obtained by deletion of vertices, and $H$ is a subgraph of $G$ if it is obtained by deletion of vertices and edges.

A walk in a graph is a finite sequence of vertices $W=\left(v_{0}, v_{1}, \ldots, v_{k}\right)$ such that $v_{i} v_{i+1}$ is an edge of the graph. If the walk $W$ does not go through an edge twice, we say $W$ is a trail, and if it does not go through a vertex twice, we say $W$ is a path. A path starting in $u$ and ending in $v$ is called a uv-path.

The length of a path is the number of its edges and the distance between two vertices $u$ and $v$ is the length of the shortest path between them and denoted as $\operatorname{dist}(u, v)$. If there is no path between $u$ and $v$, then $\operatorname{dist}(u, v)=\infty$.

A walk is said to be closed if the first and the last vertices are the same. A cycle is a closed trail in which all vertices, but the last, are distinct.

An Eulerian trail (or Eulerian path) of a graph is a trail that traverses each edge of the graph exactly once. Similarly, an Eulerian tour (or Eulerian cycle) is a cycle that traverses each edge exactly once. A graph is Eulerian if it admits an Eulerian cycle.

A graph $G$ is said to be connected if every pair of vertices is connected by a path. If there is exactly one path connecting each pair of vertices, we say $G$ is a tree. Equivalently, a tree is a connected graph with $n-1$ edges (see [7]).

A path graph $P_{n}$ is a connected graph on $n$ vertices such that each vertex has degree at most 2. A cycle graph $C_{n}$ is a connected graph on $n$ vertices such that every vertex has degree 2 .

A complete graph $K_{n}$ is a graph with $n$ vertices such that every vertex is adjacent to all the others. On the other hand, an independent set is a set of vertices of a graph in which no two vertices are adjacent. We denote $I_{n}$ for an independent set with $n$ vertices.

A bipartite graph $G=(V, E)$ is a graph such that there exists a partition $\mathcal{P}=(A, B)$ of $V$ such that every edge of $G$ connects a vertex in $A$ to one in $B$. Equivalently, $G$ is said to be bipartite if $A$ and $B$ are independent sets. The bipartite graph is also denoted as $G=(A, B, E)$.

The join of two graphs $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$ with disjoint vertex sets is the graph $G=(V, E)$ such that $V=V_{1} \cup V_{2}$ and $E=E_{1} \cup E_{2} \cup\left\{u v: u \in V_{1}, v \in\right.$ $\left.V_{2}\right\}$, that is, $G$ is obtained by connecting every vertex of $G_{1}$ to every vertex of $G_{2}$.

Finally, for a given graph $G=(V, E)$, a vertex labeling (or vertex coloring) of $G$ is a function $f: V \rightarrow \mathbb{N}$, and an edge labeling (or edge coloring) of $G$ is a function $g: E \rightarrow \mathbb{N}$. Intuitively, we are assigning labels (colors) to vertices and/or edges of the graph. Throughout this text, we have the codomains as a finite subset of $\mathbb{N}$, and we denote $[a, b]=\{a, a+1, \ldots, b\}$.

Many problems of graph theory consist in finding a vertex or an edge labeling for a graph satisfying certain properties. For example, a proper vertex coloring is a vertex coloring such that adjacent vertices have different colors, and a very well known problem is to find for a given graph $G$ the minimum $k$ such that there exists a proper vertex coloring $f$ of $G$ with $|\operatorname{Im}(f)|=k$. In our case, we are interested in graceful labeling.

## Chapter 2

## Graceful Labeling

A graceful labeling of a graph $G$ is a vertex labeling $f: V \rightarrow[0, m]$ such that $f$ is injective and the edge labeling $f_{\gamma}: E \rightarrow[1, m]$ defined by $f_{\gamma}(u v)=|f(u)-f(v)|$ is also injective. If a graph $G$ admits a graceful labeling, we say $G$ is a graceful graph.

Although it has been studied for 50 years, not many general results are known about graceful labeling. Most of the results are about asserting the gracefulness of a graph class since it suffices to show a graceful labeling for each graph in the class. On the other hand, results on non-gracefulness of a graph rely basically on a necessary condition only valid for Eulerian graphs (see Theorem 2.4) or on trying to label the graph gracefully until reaching a contradiction, which is not very effective in most of the cases.


Figure 2.1: Graceful labeling of $P_{3}$ and $K_{1,3}$.

To gain some intuition on how to label a graph gracefully, let us show how to label a path graph. So, take a path graph $P_{n}$ and let $V\left(P_{n}\right)=\left\{u_{0}, u_{1}, \ldots, u_{n-1}\right\}$ be the set of vertices such that $u_{k-1} u_{k} \in E\left(P_{n}\right)$ for $0<k<n$. Since $P_{n}$ has $m=n-1$ edges, we must label the vertices with numbers from 0 to $n-1$ so that every number in $[1, n-1]$ appears as an edge label. We start with edge label $n-1$ since there is only one way to get an absolute difference equal to $n-1$, which is having a vertex with label 0 adjacent to a vertex with label $n-1$. Thus, let us try labeling $u_{0}$ with 0 and $u_{1}$ with $n-1$. Next, let us try to get an edge label with value $n-2$. There are only two possible ways to get $n-2$ as an absolute difference: $n-2=|(n-2)-0|=|(n-1)-1|$. Since $u_{0}$ has no more unlabeled adjacent
vertices, we can only get the edge label $n-2$ by labeling $u_{2}$ with 1 . Going on with this strategy, our resulting labeling will be as follows:

$$
f\left(u_{k}\right)= \begin{cases}\frac{k}{2} & \text { if } k \text { is even } \\ n-\frac{k+1}{2} & \text { if } k \text { is odd }\end{cases}
$$

Now, to show that $f$ is indeed a graceful labeling of $P_{n}$, it suffices to show that the edge label 1 appears, which is expected to appear on the last edge $u_{n-2} u_{n-1}$. If $n$ is even, then $f\left(u_{n-1}\right)=\frac{n}{2}$ and $f\left(u_{n-2}\right)=\frac{n-2}{2}$. Hence, $f_{\gamma}\left(u_{n-1} u_{n-2}\right)=\frac{n}{2}-\frac{n-2}{2}=1$. If $n$ is odd, an analogous argument establishes the edge label 1 . Therefore, the following proposition holds.

Proposition 2.1. The path graph $P_{n}$ is graceful for all $n \geq 1$.
For a second example, we try to find a graceful labeling for the complete graph $K_{n}$. Since $K_{1}$ and $K_{2}$ are also path graphs, they are graceful. For $K_{3}$ and $K_{4}$, Figure 2.2 presents a graceful labeling for each one.


Figure 2.2: Graceful labeling of $K_{3}$ and $K_{4}$.
Before analysing the general case, let us first introduce a property of graceful labelings. Given a graph with a graceful labeling, if we swap every vertex label $k$ with $m-k$, the resulting labeling is also graceful since the edge labels will not have changed: the end vertices of an edge with labels $a$ and $b$ become $m-a$ and $m-b$, and $|a-b|=|(m-a)-(m-b)|$. This is called the complementarity property.

Now, for $K_{n}$ with $n>4$, as before, we must have a vertex with label 0 adjacent to a vertex labeled $m$ to get the edge label $m$. But, in this case, every vertex is adjacent to every other vertex. Thus, we can label any vertex with 0 and any other one with $m$ without loss of generality. To get the edge label $m-1$, we have two options: $m-1=|(m-1)-0|=|m-1|$. However, the complementarity property allows us to choose either one without loss of generality. Choosing to label a vertex with 1 , we get edge labels 1 and $m-1$. Now we need to get the edge label $m-2=|(m-2)-0|=|(m-1)-1|=|m-2|$. We can not label a vertex with $m-1$ or 2 because it would create a duplicate edge label. Hence, our only option is to label a vertex with $m-2$, obtaining edge labels $2, m-3$ and $m-2$.

Since $m-3$ has already appeared on an edge, the next edge label we must obtain is $m-4=|(m-4)-0|=|(m-3)-1|=|(m-2)-2|=|(m-1)-3|=|m-4|$. Again, we only have one option without creating duplicate edge labels, which is to label a vertex with 4 , obtaining edge labels $3,4, m-6$ and $m-4$. At this point, we have labeled five vertices. However, for $K_{5}$, we would have $m-6=4$ as a duplicate edge label. For $n \geq 6$, the next edge label to get is $m-5$. But, all the five possible ways to get $m-5$ lead to a duplicate edge label. Therefore, there is no way to get label $m-5$ on an edge and the following proposition holds.

Proposition 2.2. The complete graph $K_{n}$ is graceful if, and only if, $n \leq 4$.
Given the initial intuition on how to gracefully label a graph, Section 2.1 presents some general results on graceful graphs and Section 2.2 shows the gracefulness of some graph classes.

### 2.1 General results

We start by showing a couple of results concerning necessary conditions to the existence of a graceful labeling of a graph. The first one is a straightforward condition given by Golomb [12].

Proposition 2.3. If $G=(V, E)$ is graceful, then there exists a partition $\mathcal{P}=(A, B)$ of $V$ such that the number of edges with one end in $A$ and the other in $B$ is $\left\lceil\frac{m}{2}\right\rceil$.

Proof. Let $G=(V, E)$ be a graph with a graceful labeling $f$ and consider the partition $\mathcal{P}=(A, B)$ of $V$ such that $A=\{u \in V: f(u) \equiv 0(\bmod 2)\}$. Since there are $\left\lceil\frac{m}{2}\right\rceil$ odd values between 1 and $m$, and an odd difference is only possible by subtracting an even value from an odd one, the number of edges connecting two vertices with different parities must be exactly $\left\lceil\frac{m}{2}\right\rceil$.

Although Proposition 2.3 gives a necessary condition to the existence of a graceful labeling for a graph, it has no practical use since it would be necessary to check all the $2^{n-1}$ possible partitions of $V$ to decide if a graph can admit a graceful labeling.

A more useful necessary condition was given by Rosa [21], but it only applies to Eulerian graphs. It is known as the parity condition.

Theorem 2.4. Let $G$ be an Eulerian graph. If $m \equiv 1,2(\bmod 4)$, then $G$ is not graceful.

Proof. Suppose $G=(V, E)$ is a graceful Eulerian graph. Let $f: V \rightarrow[0, m]$ be a graceful labeling of $G$ and $C=\left(u_{0}, u_{1}, \ldots, u_{m-1}, u_{m}=u_{0}\right)$ be an Eulerian cycle of
$G$. Taking the sum of the edge labels of $C$ modulo 2 , we have:

$$
\begin{align*}
\sum_{i=1}^{m} f_{\gamma}\left(u_{i-1} u_{i}\right) & =\sum_{i=1}^{m}\left|f\left(u_{i-1}\right)-f\left(u_{i}\right)\right|  \tag{2.1}\\
& \equiv \sum_{i=1}^{m} f\left(u_{i-1}\right)-f\left(u_{i}\right) \equiv 0 \quad(\bmod 2)
\end{align*}
$$

And, since $C$ is an Eulerian cycle, i.e., the cycle $C$ goes through each edge exactly once, and $f$ is a graceful labeling of $G$, we have:

$$
\begin{equation*}
\sum_{e \in E} f_{\gamma}(e)=\sum_{k=1}^{m} k=\frac{m(m+1)}{2} \stackrel{(2.1)}{\equiv} 0 \quad(\bmod 2) \tag{2.2}
\end{equation*}
$$

Thus, we must have $m \equiv 0,3(\bmod 4)$ in order to satisfy equation (2.2).
The parity condition, unlike Proposition 2.3, provides a simple way to test if an Eulerian graph can be graceful or not. And an interesting question arises: is there a graph class for which the parity condition is also a sufficient condition? As we will see, the parity condition does characterize at least one graph class.

In graph theory, it is natural to think of substructures that make a graph not satisfy a certain property, in this case being graceful. Such substructures can be subgraphs, induced subgraphs, or others, and they are called forbidden substructures for the graph class. Thus, one might think of finding forbidden substructures for the class of graceful graphs. However, Arumugam and Bagga [3] proved that every graph is an induced subgraph of a graceful graph.

Theorem 2.5. Every graph is an induced subgraph of a graceful graph.
Proof. Given a graph $G=(V, E)$, let us construct a graph $H$ from $G$ such that $H$ is graceful and $G$ is an induced subgraph of $H$. Consider a vertex labeling $f: V \rightarrow[0, k]$ injective for some $k \geq m$ such that the edge labeling $f_{\gamma}: E \rightarrow \mathbb{N}$ is also injective, and there exist $u, v \in V$ with $f(u)=0$ and $f(v)=k$. Let $\left\{x_{1}, x_{2}, \ldots, x_{r}\right\}$ be the set of missing edge labels. Without loss of generality, $x_{1}, x_{2}, \ldots, x_{s}$ are not vertex labels and $x_{s+1}, \ldots, x_{r}$ are vertex labels. For each $x_{i}, 1 \leq i \leq s$, add a vertex $w_{i}$ with label $x_{i}$ and add an edge connecting $w_{i}$ to $u$ so that $f_{\gamma}\left(u w_{i}\right)=x_{i}$. For each $x_{i}, s+1 \leq i \leq r$, add a vertex $w_{i}$ with label $k+x_{i}$ and connect $w_{i}$ to $u$ and $v$ so that $f_{\gamma}\left(u w_{i}\right)=k+x_{i}$ and $f_{\gamma}\left(v w_{i}\right)=x_{i}$. Note that the last step might have introduced new missing edge labels by creating vertex labels with values greater than $k$. However, these new missing edge labels are not vertex labels. So, for each new missing edge label $y$, add a new vertex $z_{y}$ with label $y$ and connect $z_{y}$ to $u$ so that $f_{\gamma}\left(u z_{y}\right)=y$. The resulting graph $H$ is graceful and it contains $G$ as an induced subgraph.


Figure 2.3: Constructing a graceful graph from $C_{5}$.

Theorem 2.5 says that a graph $G$ being non-graceful does not matter for graphs for which $G$ is an induced subgraph. It also says that we can always construct a graceful graph from any graph.

So far, we have characterized the gracefulness of two families of graphs: the path graphs and the complete graphs. The first one is a family of graceful graphs and the second one, for $n \geq 5$, is a family of non-graceful graphs. We have also shown that we can construct a graceful graph from any graph, graceful or not.

Next, we present an unpublished result of Erdős [12]. The following proof was given by Graham and Sloane[13].

Theorem 2.6. Almost all graphs are not graceful.
Proof. We show that for a fixed number $m$, almost all graphs with $n$ vertices and $m$ edges are not graceful as $n \rightarrow \infty$.

First, note that there are $\binom{n(n-1) / 2}{m}$ labeled graphs with $n$ vertices and $m$ edges. So, the number of unlabeled graphs is at least $\frac{1}{n!}(\underset{m}{n(n-1) / 2})$.

Let $f$ be a vertex labeling on $n$ vertices with distinct numbers from $[0, m]$. There are $\frac{(m+1)!}{(m-n+1)!} \leq(m+1)^{n}$ such labelings. Let us count how many graphs there are for which $f$ is a graceful labeling. Let $p_{i}$ be the number of pairs of vertices $\{u, v\}$ with $|f(u)-f(v)|=i$. Clearly, $\sum_{i=1}^{m} p_{i}=\binom{n}{2}$. If we construct a graph by taking one edge from each class counted by $p_{i}$, the resulting graph is graceful. Thus, there are $\prod_{i=1}^{m} p_{i}$ labeled graphs for which $f$ is a graceful labeling. Since this product is maximized when all $p_{i}$ 's are equal, $\prod_{i=1}^{m} p_{i} \leq\left(\frac{n(n-1)}{2 m}\right)^{m}$. Therefore, there are at most $(m+1)^{n}\left(\frac{n(n-1)}{2 m}\right)^{m}$ graceful labeled graphs, and this is also an upper bound for the number of graceful unlabeled graphs. Finally, we show that the ratio

$$
\rho=\frac{(m+1)^{n}\left(\frac{n(n-1)}{2 m}\right)^{m}}{\frac{1}{n!\binom{n-1) / 2}{m}}}
$$

goes to 0 as $n \rightarrow \infty$. Writing $m=\left(\frac{1}{2}-\mu\right)\binom{n}{2}$ with $\mu \in\left(-\frac{1}{2}, \frac{1}{2}\right)$, we have

$$
\rho=\frac{(m+1)^{n} n!}{\left(\frac{1}{2}-\mu\right)^{m}\left(\begin{array}{c}
n \\
2 \\
m
\end{array}\right)}<\frac{(m+1)^{n} n!\sqrt{8\binom{n}{2}\left(\frac{1}{2}-\mu\right)\left(\frac{1}{2}+\mu\right)}}{\left(\frac{1}{2}-\mu\right)^{m} 2^{\binom{n}{2} H_{2}(1 / 2-\mu)}}
$$

where $H_{2}(x)=-x \log _{2}(x)-(1-x) \log _{2}(1-x)($ cf. [18, p. 309]). Simplifying the denominator,

$$
\rho<\frac{(m+1)^{n} n!\sqrt{8\binom{n}{2}\left(\frac{1}{2}-\mu\right)\left(\frac{1}{2}+\mu\right)}}{2^{-\binom{n}{2}\left(\frac{1}{2}+\mu\right) \log _{2}\left(\frac{1}{2}+\mu\right)}}
$$

Taking the logarithm, it is easy to show that the right hand side of the inequality goes to $-\infty$ as $n \rightarrow \infty$. Therefore, $\rho \rightarrow 0$ as $n \rightarrow \infty$.

We finish this section by giving another construction of graceful graphs given by Acharya [1].

The full augmentation of a graceful graph $G=(V, E)$ is the addition of an isolated vertex to $G$ for each vertex label not used. Formally, $G_{f}=G \cup I_{m-n+1}$ is the full augmentation of $G$. Clearly, $G_{f}$ is also graceful and, in particular, graceful trees are already full augmented.

Theorem 2.7. If $G$ is a graceful graph and $G_{f}$ is its full augmentation, then $G_{f}+I_{q}$ is graceful for all $q \geq 1$.

Proof. Let $f: V\left(G_{f}\right) \rightarrow[0, m]$ be a graceful labeling of $G_{f}$ and $V\left(I_{q}\right)=\left\{v_{0}, v_{1}, \ldots\right.$, $\left.v_{q-1}\right\}$. Then, we can extend the labeling $f$ to $V\left(I_{q}\right)$ as follows: label $v_{i}$ with $f\left(v_{i}\right)=$ $m+(i+1)(m+1)$.

We have $\left|E\left(G_{f}+I_{q}\right)\right|=m+q(m+1)$ and, since we are extending $f$, we already have $\operatorname{Im}\left(\left.f\right|_{V\left(G_{f}\right)}\right)=[0, m]$ and $\operatorname{Im}\left(\left.f_{\gamma}\right|_{E\left(G_{f}\right)}\right)=[1, m]$. By definition of $f$, every vertex label is unique, and the set of labels on the edges incident with $v_{i}$ is exactly $[m+i(m+1), m+(i+1)(m+1)]$. Then, every label in $[1, m+q(m+1)]$ appears once on some edge. Therefore, the extension of $f$ is a graceful labeling of $G_{f}+I_{q}$.

### 2.2 Gracefulness of graph classes

In this section, we present the gracefulness of some graph classes. Most of the results asserting the gracefulness of a graph class are given by explicit graceful labelings. For the non-gracefulness of a graph class, there are only a few tools for that. Basically, we only have Proposition 2.3 and theorem 2.4. We can also prove by trying to label the graph and finding a contradiction. For instance, Rosa [21] showed Proposition 2.2 this way. Although the last method is not effective if done
by hand, if it is done computationally, it may result in something useful, as we will see later in following chapters.

It was already shown that the path graph $P_{n}$ is graceful and the complete graph $K_{n}$ is graceful if, and only if, $n \leq 4$. Next, we present the gracefulness cycle graphs, which was characterized by Rosa [21].

Proposition 2.8. The cycle graph $C_{n}$ is graceful if, and only if, $n \equiv 0,3(\bmod 4)$.
Proof. Cycle graphs are Eulerian graphs. Therefore, by the parity condition, if $n \equiv$ $1,2(\bmod 4)$, then $C_{n}$ is not graceful. Otherwise, let us call $V\left(C_{n}\right)=\left\{u_{0}, u_{1}, \ldots\right.$, $\left.u_{n-1}\right\}$ such that $u_{k} u_{k+1} \in E\left(C_{n}\right)$ for $0 \leq k \leq n-1$ and $u_{n}=u_{0}$.

If $n \equiv 0(\bmod 4)$, then label the vertices according to the following formula:

$$
f\left(u_{i}\right)= \begin{cases}\frac{i}{2} & \text { if } i=0,2,4, \ldots, n-2 \\ n-\frac{i-1}{2} & \text { if } i=1,3,5, \ldots, \frac{n}{2}-1 \\ n-\frac{i-1}{2}-1 & \text { if } i=\frac{n}{2}+1, \frac{n}{2}+3, \ldots, n-1\end{cases}
$$

If $n \equiv 3(\bmod 4)$, then label $V\left(C_{n}\right)$ as follows:

$$
f\left(u_{i}\right)= \begin{cases}\frac{i}{2} & \text { if } i=0,2,4, \ldots, n-1 \\ n-\frac{i-1}{2} & \text { if } i=1,3,5, \ldots, \frac{n+1}{2}-1 \\ n-\frac{i-1}{2}-1 & \text { if } i=\frac{n+1}{2}+1, \frac{n+1}{2}+3, \ldots, n-2\end{cases}
$$

Note that the parity condition characterizes the gracefulness of cycle graphs.
The wheel graph $W_{p}$ is the join of a cycle graph $C_{p}$ with a singleton graph, i.e., $W_{p}=C_{p}+K_{1}$. Frucht [10] showed that all wheels are graceful.

Proposition 2.9. The wheel graph $W_{p}$ is graceful for all $p \geq 3$.
Proof. Let $V\left(W_{p}\right)=\left\{u_{0}, u_{1}, \ldots, u_{p-1}, v\right\}$ be the set of vertices where $v$ is the vertex joined with the cycle and consider the following two cases.

1. If $p \equiv 0(\bmod 2)$, then the following formula gives a graceful labeling:

$$
\begin{aligned}
& f(v)=0 \\
& f\left(u_{i}\right)= \begin{cases}2 p & \text { if } i=0 \\
2 & \text { if } i=p-1 \\
i & \text { if } i=1,3,5, \ldots, p-3 \\
2 p-i-1 & \text { if } i=2,4,6, \ldots, p-2\end{cases}
\end{aligned}
$$

2. If $p \equiv 1(\bmod 2)$, then the following formula gives a graceful labeling:

$$
\begin{aligned}
& f(v)=0 \\
& f\left(u_{i}\right)= \begin{cases}2 p & \text { if } i=0 \\
2 & \text { if } i=1 \\
p+i & \text { if } i=2,4,6, \ldots, p-1 \\
p+1-i & \text { if } i=3,5,7, \ldots, p-2\end{cases}
\end{aligned}
$$

A caterpillar is a tree in which the removal of all leaves results in a path graph. It was proven by Rosa [21] that they are all graceful.

Proposition 2.10. All caterpillar trees are graceful.
Proof. Draw the caterpillar tree as a planar bipartite representation and label it as shown in Figure 2.4. It is easy to check that such drawing scheme is always possible.


Figure 2.4: Graceful labeling of caterpillar tree.

Note that a path graph $P_{n}$ is also a caterpillar tree and the labeling scheme given by Proposition 2.10, when applied to a path graph, yields the same labeling constructed before.

The complete bipartite graph $K_{p, q}$ is a bipartite graph $G=(A, B, E)$ such that $|A|=p,|B|=q$, and if $u \in A$ and $v \in B$, then $u v \in E$. In particular, the star graph is the complete bipartite graph $K_{1, q}$.

It was shown that for all positives values of $p$ and $q$, the complete bipartite graphs are graceful [12, 21].

Proposition 2.11. The complete bipartite graph $K_{p, q}$ is graceful for all $p, q \geq 1$.

Proof. Let $G=(A, B, E)$ be a bipartite graph with $a=|A|$ and $b=|B|$. Assign the vertices from $A$ with numbers $0,1, \ldots, a-1$, and assign the vertices from $B$ with numbers $a, 2 a, \ldots, b a$.

We can generalize the concept of bipartite graph to multipartite graph and, in a similar fashion, we have the complete multipartite graph. It was proven the following proposition regarding the gracefulness of complete multipartite graphs [5].

Proposition 2.12. The complete multipartite graphs $K_{p, q}, K_{1, p, q}, K_{2, p, q}$, and $K_{1,1, p, q}$ are graceful.

Proof. The graceful labelings are given in Figure 2.5.


Figure 2.5: Graceful labelings of $K_{p, q}, K_{1, p, q}, K_{2, p, q}$, and $K_{1,1, p, q}$.

Furthermore, Beutner [5] conjectured that these graphs are the only complete multipartite graphs which are graceful, and showed computationally that it is valid for all complete multipartite graphs up to 23 vertices.

## Chapter 3

## Trees

The Graceful Tree Conjecture remains unsolved to these days and there have been a few different approaches researchers have been trying to prove the conjecture. In this section, we present results on the gracefulness of trees and the different ways in which the conjecture has been tackled.

Conjecture 3.1 (Graceful Tree Conjecture). Every tree is graceful.
As shown in Chapter 2, paths and caterpillars are graceful. A first approach would be to extend the definition of caterpillars to new families of trees, i.e., look at the class of trees in which the removal of all leaves results in a caterpillar tree - the lobsters-, and so on. However, even the lobster trees have not been characterized yet. Bermond [4] conjectured in 1979 that all lobsters are graceful. This chapter presents others approaches which have shown to be more interesting.

### 3.1 Trees with limited diameter

The diameter of a tree $T$ is the maximum distance between two vertices, i.e., $\operatorname{diam}(T)=\max \{\operatorname{dist}(u, v): u, v \in V(T)\}$. Trees with small diameter have been proved to be graceful. We already showed that trees with diameter 1 (only $K_{2}$ ), diameter 2 (star graphs), and diameter 3 (a subclass of caterpillar trees) are graceful since they are all also caterpillar trees. For greater diameters, Zhao [28] proved in 1989 that all trees with diameter 4 are graceful, Hrnčiar and Haviar [15] proved in 2001 that all trees with diameter 5 are graceful, and Superdock [23, 24] proved more recently that some subclasses of trees with diameter 6 are graceful.

We show in this section that all trees with diameter 4 are graceful. The proof presented here was given by Hrnčiar and Haviar [15] since it is simpler than the original proof of Zhao [28].

Lemma 3.1. Let $T$ be a tree with a graceful labeling $f$ and let $u \in V(T)$ the vertex with $f(u)=0$. If $T^{\prime}$ is the tree obtained from $T$ by adding a new vertex $v$ only adjacent to $u$, then $T^{\prime}$ is graceful.

Proof. If $m$ is the number of edges of $T$, then the vertex labeling $f^{\prime}$ such that $\left.f^{\prime}\right|_{V(T)}=f$ and $f^{\prime}(v)=m+1$ is a graceful labeling of $T^{\prime}$.

Corollary 3.1.1. If $w \in V(T)$ has label $m$, then adding a new vertex only adjacent to $w$ also results in a graceful tree.

Proof. Just consider the complementary graceful labeling of $f$.
Corollary 3.1.2. If $u \in V(T)$ has label 0 (or $m$ ) and $H$ is a caterpillar tree, then adding an edge between $u$ and $a$ vertex of $H$ with maximum eccentricity also results in a graceful tree.

Proof. Apply iteratively Lemma 3.1 giving preference to adding leaves first whenever it is possible. Also note that the corollary is valid for any graceful graph $G$ as long as $u \in V(G)$ has label 0 (or $m$ ).

Lemma 3.1 allows us to obtain new graceful graphs from smaller ones by adding a vertex. Then, it is reasonable to ask if this could be used to prove the Graceful Tree Conjecture, i.e., somehow show that for any tree, there is a finite sequence of graceful trees starting from a single vertex such that each tree is the previous one in the sequence plus a vertex, and the last tree of the sequence is the target tree itself.

One sufficient condition to the existence of such sequence is if every tree admits a graceful labeling in which the label 0 can be assigned to any vertex. In the general context, such graphs are called 0 -rotatable graceful graphs. However, it is not true that every tree is 0 -rotatable graceful [26].

Let $T$ be a tree and $u v \in E(T)$. We denote by $T_{u, v}$ the subtree of $T$ containing $v$ after the removal of the edge $u v$. Precisely, if $S=\{w \in V(T): v$ is on the $u w$-path $\}$, then $T_{u, v}=T[S]$.

Lemma 3.2. Let $T$ be a tree with a graceful labeling $f$ and let $u \in V(T)$ be a vertex adjacent to $u_{1}$ and $u_{2}$. Consider $T^{\prime}=T-\left(V\left(T_{u, u_{1}}\right) \cup V\left(T_{u, u_{2}}\right)\right)$ and let $v \in V\left(T^{\prime}\right)$, $v \neq u$.
(a) If $u_{1} \neq u_{2}$ and $f\left(u_{1}\right)+f\left(u_{2}\right)=f(u)+f(v)$, then the tree obtained by a disjoint union of $T^{\prime}, T_{u, u_{1}}$ and $T_{u, u_{2}}$, and connecting $v$ to $u_{1}$ and $u_{2}$ is graceful with the same graceful labeling $f$.
(b) If $u_{1}=u_{2}$ and $2 f\left(u_{1}\right)=f(u)+f(v)$, then the tree obtained by a disjoint union of $T^{\prime}$ and $T_{u, u_{1}}$, and connecting $v$ to $u_{1}$ is graceful with the same graceful labeling $f$.

Proof. It suffices to show that the edge labels of $u u_{1}$ and $u u_{2}$ are the same as of $v u_{1}$ and $v u_{2}$.
(a) $\left|f\left(u_{1}\right)-f(u)\right|=\left|f(u)+f(v)-f\left(u_{2}\right)-f(u)\right|=\left|f(v)-f\left(u_{2}\right)\right|$
$\left|f\left(u_{2}\right)-f(u)\right|=\left|f(u)+f(v)-f\left(u_{1}\right)-f(u)\right|=\left|f(v)-f\left(u_{1}\right)\right|$
(b) $\left|f\left(u_{1}\right)-f(u)\right|=\left|\frac{f(u)+f(v)}{2}-f(u)\right|=\left|\frac{f(v)-f(u)}{2}\right|$
$\left|f\left(u_{1}\right)-f(v)\right|=\left|\frac{f(u)+f(v)}{2}-f(v)\right|=\left|\frac{f(u)-f(v)}{2}\right|$


Figure 3.1: Transfer of subtrees from $u$ to $v$.

This operation is called a transfer and we mostly do transfers of leaves from one vertex to another. For the remaining of this section, for a graceful tree, we no longer distinguish the vertex label from the vertex itself since in a tree every number from [ $0, n-1]$ must appear as a vertex label.

As an example, take the star graph $K_{1, m}$. We can transfer some leaves, which is connected to vertex 0 , to the vertex $m$ (see Figure 3.2). For an example, we can transfer $k$ and $m-k$ from 0 to $m$ since $k+(m-k)=0+m$. As said before, the subtree being transferred is usually a leaf and we denote a sequence of transfers of leaves adjacent to $u$ to $v$ as $u \rightarrow v$. Although the notation is not precise, the context will make clear how many and which leaves are being transferred.


Figure 3.2: Transfer of leaves from $m$ to 0 ( $m \rightarrow 0$ transfer).

Proposition 3.3. All trees with diameter 4 are graceful.
Proof. Consider the following types of transfers.
A $u \rightarrow v$ transfer is of type 1 if the leaves being transferred are $k, k+1, \ldots, k+s$. This type of transfer can be realized if $u+v=k+(k+s)$. We use this type of transfer when we want to leave an odd number of vertices connected to $u$.

A $u \rightarrow v$ transfer is of type 2 if the leaves being transferred are $k, k+1, \ldots, k+s$ and $l, l+1, \ldots, l+s$ with $k+s<l$. This type of transfer can be realized if $u+v=k+(l+s)$. We use this type of transfer when we want to leave an even number of vertices connected to $u$.

By Lemma 3.1, it is sufficient to show that every tree $T$ of diameter 4 with central vertex (which is unique in $T$ ) of odd degree has a graceful labeling with the central vertex having the maximum label. This is true because, in a tree of diameter 4, any subtree rooted at one of the children of central vertex is a caterpillar tree.

Let $w$ be the central vertex of $T, x$ be the number of vertices adjacent to $w$ with even degree, and $y$ be the number of vertices adjacent to $w$ with odd degree greater than 1. Let $d(w)=2 k+1$ and consider the tree of Figure 3.2 b . We can obtain $T$ from that tree by the following sequence of transfers: $0 \rightarrow m-1 \rightarrow 1 \rightarrow m-2 \rightarrow$ $2 \rightarrow m-3 \rightarrow \cdots$, where the first $x$ transfers (or $x-1$ if $y=0$ ) are of type 1 and the next $y-1$ transfers (if $y>1$ ) are of type 2 .

In order to verify that this sequence works, let us analyse the first transfer. Suppose $\left\{u_{1}, \ldots, u_{x}\right\}$ is the set of vertices adjacent to $w$ with even degree. Starting with the tree on Figure 3.2b, the central vertex $w$ is the one with label $m$. The first transfer is $0 \rightarrow m-1$. Then, $u_{1}$ is the vertex 0 and we want to leave $d\left(u_{1}\right)-1$ vertices attached to it. Initially, we have the vertices $k+1, k+2, \ldots, m-k-2, m-k-1$ adjacent to 0 . Since $0+(m-1)=(k+1)+(m-k-2)$, it is possible to leave $d\left(u_{1}\right)-1$ vertices by doing a type 1 transfer of a continuous sequence of vertices to $m-1$. Going on with an analogous analysis, it can be seen that this sequence works.

Proposition 3.4. All trees with diameter 5 are graceful.
The proof of Proposition 3.4 also uses the transfers operations used in the proof of Proposition 3.3. However, since it is divided in several cases and it does not add much to the discussion, we omit it.

### 3.2 All trees up to 35 vertices are graceful

Given that the Graceful Tree Conjecture has remained open for a long time, it is valid to question if it can be false. For that, it would suffice to come up with a tree
that does not admit a graceful labeling. In order to show that a tree does not admit such labeling, one must verify an exponential number of possible ways to label it. Thus, a computational approach is more suited for the task.

Fang [9] took this approach and proved in 2010 that all trees up to 35 vertices are graceful. Fang's result replaces previous ones in this direction: Aldred and McKay [2] established in 1998 that all trees up to 27 vertices are graceful, and Horton [14] verified in 2003 that all trees with at most 29 vertices are graceful.

Proposition 3.5. All trees up to 35 vertices are graceful.
For the verification of Proposition 3.5, Fang used the algorithm described by Wright et al. [27] to enumerate all trees which has amortized constant time complexity to generate each of them.

For each tree, the algorithm to find a graceful labeling is divided in two parts. First, it tries to find a graceful labeling using a backtracking search with a fixed maximum number of iterations. If it does not find one, then it tries to find a graceful labeling through a combinatorial optimization approach, and it uses a hill-climbing tabu search combined with ideas from simulated annealing.

The backtracking search tries to construct a graceful labeling $f$ for the tree with $f(r)=0$ where $r$ is the root, which is the center vertex in a central tree or one of the centers in a bicentral tree. Then, at each iteration, it tries to create a new edge label $k$ by labeling a not yet labeled vertex $u$ adjacent to an already labeled vertex $v$ such that $|f(u)-f(v)|=k$. In order to avoid branching the decision tree, the search goes from edge label $n-1$ to 1 . As noted before, the higher the value, the less the number of possible ways to get that value as an absolute difference.

As usual of backtracking search algorithms, the decision tree can grow exponentially as $n$ increases. Then, Fang added a threshold to the number of backtracks, preventing searching for very long time. This threshold was chosen empirically and set to $(n-19) * 11000-1000$. Algorithm 3.1 is a pseudocode for the backtracking search algorithm.

If the backtracking search does not return a graceful labeling, a combinatorial optimization approach is taken. Solving a decision problem by this approach requires formulating an evaluation function such that the answer is "yes" if, and only if, the function reaches a certain extreme value. For deciding if a tree admits a graceful labeling, the following function is taken:

$$
h(f)=\sum_{k \in[1, n-1] \backslash \operatorname{Im}\left(f_{\gamma}\right)} k
$$

where $f$ is an injective vertex labeling of the tree.
Given a vertex labeling $f$, the evaluation function $h$ is summing the edge labels

```
Algorithm 3.1: Backtracking search
    Function Search(k):
        if k=0 then
            return true
        if iterations exceed threshold then
            return false
        for every vertex v}\mathrm{ without label with its parent v}\mathrm{ v labeled do
            if label f(v')+k is valid and not yet used then
                    label v}\mathrm{ with }f(\mp@subsup{v}{}{\prime})+
                    if Search(k-1) then
                    return true
                    unlabel v
                if label f(v')-k is valid and not yet used then
                    label v}\mathrm{ with }f(\mp@subsup{v}{}{\prime})+
                    if Search(k-1) then
                    return true
                    unlabel v
        return false
```

that did not appeared on any of the edges of the tree. Hence, $f$ is a graceful labeling of the tree if, and only if, $h(f)=0$. Thus, since $h$ is always non-negative, we are interested in minimizing $h$.

Since the graph is a tree, an injective vertex labeling is also a permutation of $[0, n-1]$ on its vertices. Then, the domain of exploration of $h$ is all permutations of $[0, n-1]$. The local search uses the hill-climbing method: at each iteration, it selects a number of random pairs of vertices, swaps their labels and picks the best one that improves the current solution.

As it is known, the hill-climbing method purely can get stuck in a local minimum. To avoid this problem, two strategies are adopted. The first one is the use of tabu search which forbids certain moves if they were made very recently, unless it results in a graceful labeling. The second strategy is based on an idea from the simulated annealing technique in which it is allowed to worsen the solution with a certain probability. Algorithm 3.2 is a pseudocode of these ideas.

This hybrid algorithm combining backtracking search and combinatorial optimization approach allowed the verification of the gracefulness of all trees up to 35 vertices. It is worth mentioning that the task was accomplished with the help of a community of volunteers in which the task was divided and distributed between them. Details of the performance of the algorithm can be found in the Fang's paper [9].

```
Algorithm 3.2: Local search using metaheuristics
    let \(f\) be the vertex labeling corresponding to the identity permutation
    \(v \leftarrow h(f)\)
    while \(v \neq 0\) do
        randomly choose \(2 n\) pairs of vertices
        foreach pair of vertices \((x, y)\) chosen do
            swap the values of \(f(x)\) and \(f(y)\)
            evaluate \(h\) for the modified labeling
            swap back \(f(x)\) with \(f(y)\)
        choose the pair \((x, y)\) that minimizes \(h\)
        let \(f^{\prime}\) be the labeling obtained by swapping \(f(x)\) with \(f(y)\)
        \(v^{\prime} \leftarrow h\left(f^{\prime}\right)\)
        if \(f(x), f(y)\) was not swapped in the last \(\lfloor n / 3\rfloor\) iterations then
            if \(v>v^{\prime}\) then
                swap \(f(x)\) with \(f(y)\), and update \(v\)
            else
                with probability \(p, \operatorname{swap} f(x)\) with \(f(y)\), and update \(v\)
        else if \(v^{\prime}=0\) then
            swap \(f(x)\) with \(f(y)\), and update \(v\)
    return \(f\)
```


### 3.3 Relaxed versions

Relaxed versions of graceful labeling have been studied for as long as graceful labeling itself. Rosa himself introduced together with graceful labeling some variants of it, both stronger and weaker versions of graceful labeling.

Usually, one only consider relaxed versions when the graph is not graceful. However, here we consider a couple of relaxed versions of graceful labeling for trees, and, with the purpose of getting closer to the Graceful Tree Conjecture, the goal has been in trying to improve bounds for these labelings.

Probably, the following relaxed graceful labelings are the most intuitive ones.

1. Edge-relaxed: $f_{\gamma}$ can be non-injective.
2. Vertex-relaxed: $f$ can be non-injective ( $f_{\gamma}$ must still be injective).
3. Range-relaxed: $f: V(G) \rightarrow[0, k]$ for some $k \geq m$.

Bounds have been established for all these three versions. Rosa and Širáň [22] showed that every tree has a edge-relaxed graceful labeling with at least $5 \mathrm{~m} / 7$ different edge labels. Van Bussel [25] showed two results, one concerning vertexrelaxed graceful labeling of trees and the other concerning the range-relaxed graceful labeling of trees, which we present next.

Theorem 3.6. Every tree $T$ has a range-relaxed graceful labeling with vertex labels in the range $[0,2 m-\operatorname{diam}(T)]$.

Proof. Let $T=(V, E)$ be a tree and $v_{0} \in V$, and consider the tree $T$ rooted at $v_{0}$. Also consider that the longest path from $v_{0}$ is at the leftmost in a planar representation of $T$. Let the length of this path be $\ell$, the vertices of the path be $v_{0}, v_{1}, \ldots, v_{\ell}$, and $h_{i}$ be the number of vertices at level $i$. The following construction provides a vertex labeling for $T$ in the range $[0,2 m-\ell]$.

1. Label $v_{0}$ temporarily with $\alpha$ and $v_{1}$ with $\alpha+1$. After labeling all vertices, we shift all labels by a constant so that the smallest value is 0 .
2. For $i>1$, label $v_{i}$ as follows:

$$
f\left(v_{i}\right)= \begin{cases}f\left(v_{i-2}\right)-h_{i-2}-h_{i-1}+1=\alpha-\sum_{j=0}^{i-1} h_{j}+\frac{i}{2} & \text { if } i \text { is even } \\ f\left(v_{i-2}\right)+h_{i-2}+h_{i-1}-1=\alpha+\sum_{j=0}^{i-1} h_{j}-\frac{i-1}{2} & \text { if } i \text { is odd }\end{cases}
$$

3. At each level $i$, consider the order in which the vertices are represented in the place. Label the $k$-th vertex $u_{i, k}$ at level $i, k \in\left[0, h_{i}-1\right]$, as follows:

$$
f\left(u_{i, k}\right)= \begin{cases}f\left(v_{i}\right)-k & \text { if } i \text { is even } \\ f\left(v_{i}\right)+k & \text { if } i \text { is odd }\end{cases}
$$



Figure 3.3: Labeling of Theorem 3.6 at level $i$ even.
It is clear that all vertex labels are distinct: as we go from top to bottom, left to right, on even levels they are decreasing, and on odd levels they are increasing. Moreover, all edge labels are distinct. Indeed, the edge labels are increasing as we go from top to bottom, left to right.

Consider two edges $u_{i} u_{i+1}$ and $w_{i} w_{i+1}$ where $u_{i}$ and $w_{i}$ are at the same level $i$. If $i$ is even, then $f\left(u_{i}\right)>f\left(w_{i}\right)$ and $f\left(u_{i+1}\right)<f\left(w_{i+1}\right)$. Then,

$$
f_{\gamma}\left(u_{i} u_{i+1}\right)=f\left(u_{i+1}\right)-f\left(u_{i}\right)<f\left(w_{i+1}\right)-f\left(w_{i}\right)=f_{\gamma}\left(w_{i} w_{i+1}\right)
$$

Analogously, the same holds if $i$ is odd.
Now, let $u_{i}$ and $u_{i+1}$ be the rightmost vertices at levels $i$ and $i+1$, respectively, and consider the edge $u_{i} u_{i+1}$, which is not necessarily an edge of the tree. By what we just showed, it has the largest edge label from level $i$ to $i+1$. So, it suffices to show that $f_{\gamma}\left(u_{i} u_{i+1}\right)<f_{\gamma}\left(v_{i+1} v_{i+2}\right)$, since $v_{i+1} v_{i+2}$ has the smallest edge label from level $i+1$ to $i+2$. Assuming $i$ even, we have

$$
\begin{aligned}
f_{\gamma}\left(u_{i} u_{i+1}\right) & =f\left(u_{i+1}\right)-f\left(u_{i}\right) \\
& =f\left(v_{i+1}\right)+h_{i+1}-1-\left(f\left(v_{i}\right)-h_{i}+1\right) \\
& <f\left(v_{i+1}\right)-\left(f\left(v_{i}\right)-h_{i}-h_{i+1}+1\right) \\
& =f_{\gamma}\left(v_{i+1} v_{i+2}\right)
\end{aligned}
$$

Again, the same holds if $i$ is odd by an analogous proof.
Finally, we check that the labels are inside the range. Let $f_{\text {max }}$ and $f_{\text {min }}$ be the maximum and the minimum vertex labels, respectively. If $\ell$ is even, then the largest vertex label is at level $\ell-1$ and the smallest one is at level $\ell$.

$$
\begin{aligned}
f_{\max } & =f\left(v_{\ell-1}\right)+h_{\ell-1}-1 \\
& =\alpha+\sum_{j=0}^{\ell-2} h_{j}-\frac{\ell-2}{2}+h_{\ell-1}-1 \\
& =\alpha+m+1-h_{\ell}-\frac{\ell}{2} \\
f_{\min } & =f\left(v_{\ell}\right)-h_{\ell}+1 \\
& =\alpha-\sum_{j=0}^{\ell-1} h_{j}+\frac{\ell}{2}-h_{\ell}+1 \\
& =\alpha-m+\frac{\ell}{2} \\
f_{\max }-f_{\min } & =\alpha+m+1-h_{\ell}-\frac{\ell}{2}-\left(\alpha-m+\frac{\ell}{2}\right) \\
& =2 m-l-h_{\ell}+1 \\
& \leq 2 m-l
\end{aligned}
$$

Thus, if we choose our root as one of the end vertices of a longest path in the tree, we obtain a range-relaxed graceful labeling in the range $[0,2 m-\operatorname{diam}(T)]$.

Theorem 3.7. Every tree $T$ has a vertex-relaxed graceful labeling with more than $\frac{n}{2}$ distinct vertex labels.

Instead of proving Theorem 3.7 directly, Van Bussel proved a stronger result. Before that, we must define the following labeling. We say a vertex labeling $f$ is locally bipartite if there is a bipartition of $V(G)=A \cup B$ such that

1. $\forall u \in A \forall v \in N(u): f(u)<f(v)$
2. $\forall v \in B \forall u \in N(v): f(u)<f(v)$

Note that if a graph $G$ admits such labeling, then $G$ must be bipartite.
Theorem 3.8. Let $T=(V, E)$ be a tree with a bipartition of $V=A \cup B$, and let $v \in A$ be an arbitrary vertex. Then, there exists a vertex-relaxed graceful labeling $f$ of $T$ satisfying the following properties:

1. $f$ is locally bipartite;
2. $f(v)=0$;
3. $f(x) \neq f(y)$ for all $x, y \in B$.

Proof. We prove by induction on $n$. For $n=1$ and $n=2$, it is clear that such labeling exists. Suppose $n>2$, and let $v$ be an arbitrary vertex of $T$. We divide in two cases.

Case 1. Assume $d(v) \geq 2$. Since $v$ has at least two adjacent vertices, we can split $v$ into two vertices $v_{1}$ and $v_{2}$ and obtain two trees $T_{1}$ and $T_{2}$ strictly smaller than $T$ such that $T$ is the union of $T_{1}$ and $T_{2}$ by identifying $v_{1}$ with $v_{2}$. By induction


Figure 3.4: Splitting the tree at the vertex $v$.
hypothesis, $T_{1}$ has a bipartition $V\left(T_{1}\right)=A_{1} \cup B_{1}$ with $v_{1} \in A_{1}$ and a vertex-relaxed graceful labeling $f_{1}$ satisfying those properties with respect to $v_{1}$. Similarly, we have $A_{2}, B_{2}$, and $f_{2}$ for $T_{2}$. Thus, if $m_{1}=\left|E\left(T_{1}\right)\right|$, the labeling $f$ of $T$ defined by

$$
f(u)= \begin{cases}f_{1}(u) & \text { if } u \in V\left(T_{1}\right) \\ f_{2}(u) & \text { if } u \in A_{2} \\ f_{2}(u)+m_{1} & \text { if } u \in B_{2}\end{cases}
$$

is the required labeling.

1. $f(v)=f\left(v_{1}\right)=f\left(v_{2}\right)=0$
2. Since we are adding a constant to all vertex labels in $B_{2}, f_{2}$ remains locally bipartite. Hence, $f$ is also locally bipartite.
3. The edge labels in $T_{1}$ remains the same in $T$ and those in $T_{2}$ are shifted by $m_{1}$, generating edge labels $\left\{m_{1}+1, \ldots, m_{1}+m_{2}\right\}$. Hence, $f$ is a vertex-relaxed graceful labeling of $T$.
4. All vertex labels in $B_{1}$ and $B_{2}$ are distinct. Thus,

$$
\min \left\{f_{2}\left(B_{2}\right)\right\}+m_{1} \geq m_{1}+1>m_{1}=\max \left\{f_{1}\left(B_{1}\right)\right\}
$$

and we have that all vertex labels in $B_{1} \cup B_{2}$ are distinct.
Case 2. Assume $d(v)=1$. Let $w$ be the adjacent vertex of $v$. Since $n \geq 3$, we have $d(w) \geq 2$. Let $r_{1}, r_{2}, \ldots, r_{k}$, where $k=d(w)-1$, be the vertices adjacent to $w$ except for $v$. Let the trees of $T-w$ rooted at $r_{i}$ be $T_{i}$ with $m_{i}$ edges, bipartition $\left(A_{i}, B_{i}\right)$, and $r_{i} \in A_{i}$.

Since $T_{i}$ is smaller than $T$ by at least 2 vertices, by induction hypothesis, $T_{i}$ has a vertex-relaxed graceful labeling $f_{i}$ satisfying those properties with respect to $r_{i}$. The labeling $f$ of $T$ defined below is as required.

$$
\begin{aligned}
f(v) & =0 \\
f(w) & =m \\
f(u) & = \begin{cases}f_{i}(u)+i & \text { if } u \in A_{i} \\
f_{i}(u)+\sum_{j=1}^{i-1} m_{j}+i & \text { if } u \in B_{i}\end{cases}
\end{aligned}
$$

For the verification, let us denote $M_{i}=\sum_{j=1}^{i-1} m_{j}$.

1. For each tree $T_{i}, f$ adds the constant $i$ to all vertices and also adds $M_{i}$ to the vertices in $B_{i}$, which means that $f$ is locally bipartite in $T_{i}$. And, since $w$ gets the largest vertex label possible, we have that $f$ is locally bipartite in $T$.
2. For each tree $T_{i}, f$ shifts all edge labels by $M_{i}$. Together with edges incident with $w$, which has labels $\{m, m-1, \ldots, m-k\}$, we have that each edge label in $[1, m]$ appears in some edge. Hence, $f$ is a vertex-relaxed graceful labeling.
3. It is clear that in each $B_{i}$, all vertices have different labels. And, since $\max \left\{f\left(B_{i}\right)\right\}=M_{i}+m_{i}+i<M_{i+1}+i+1=\min \left\{f\left(B_{i}\right)\right\}$, we have that all labels in $\bigcup_{i=1}^{k} B_{i}$ are distinct, and the maximum of these labels is $M_{k}+m_{k}+k=$ $m-1$. Hence, all the vertex labels in $B=\{w\} \cup \bigcup_{i=1}^{k} B_{i}$ are distinct.

Therefore, every tree $T$ admits a vertex-relaxed graceful labeling satisfying those properties. If we take $A$ as the smallest set of the bipartition of $T$, we have $|B| \geq \frac{n}{2}$. And, since the vertex label 0 can not appear in $B$, we have at least $\frac{n}{2}+1$ distinct vertex labels, as required in Theorem 3.7.

Although it is clear that every graph admits an edge-relaxed and a range-relaxed graceful labelings, not all graphs have a vertex-relaxed graceful labeling [25]. Furthermore, it is still unknown a connected non-graceful graph that has a vertexrelaxed graceful labeling.

## Chapter 4

## Generalized Cone Graphs

In Chapter 2, we presented the gracefulness of some graph classes and how to construct bigger graceful graphs from smaller ones. In this chapter, we generalize the wheel graphs, also known as cone graphs, and study its gracefulness. This graph class was first studied by Bhat-Nayak and Selvam [6] in 2003 and not much progress has been made since then.

A generalized cone graph is the join of a cycle graph $C_{p}$ and an independent set $I_{q}$, where $p \geq 3$ and $q \geq 0$. For instance, for $q=0$ and $q=1$, we simply have the cycle graphs and the wheel graphs, respectively.

Throughout this chapter, we denote the vertices of the generalized cone graphs as $V\left(C_{p}+I_{q}\right)=\left\{u_{0}, u_{1}, \ldots, u_{p-1}, v_{0}, v_{1}, \ldots, v_{q-1}\right\}$ where $u_{k} \in V\left(C_{p}\right), u_{k} u_{k+1} \in E\left(C_{p}\right)$ for $0 \leq k<p$ and $u_{p}=u_{0}$, and $v_{k} \in V\left(I_{q}\right)$. Also, from now on, we simply call generalized cone graphs as cone graphs.

The first result we show is concerning the non-graceful cone graphs. As we said in Chapter 2, the only useful theoretical tool for proving the non-existence of graceful labeling for a given graph is the parity condition, which only applies to Eulerian graphs. Thus, applying the parity condition to Eulerian cone graphs, the following holds.

Proposition 4.1. The cone graph $C_{p}+I_{q}$ is not graceful for $p \equiv 2(\bmod 4)$ and $q \equiv 0(\bmod 2)$.

Proof. For $p \equiv 2(\bmod 4)$ and $q \equiv 0(\bmod 2)$, the cone $C_{p}+I_{q}$ is Eulerian since the degree of every vertex is even (cf. [7]), and it has $m=p(q+1)$ edges. Writing $p=4 s+2$ and $q=2 t$, we have $m=(4 s+2)(2 t+1) \equiv 2(\bmod 4)$. Hence, by the parity condition, $C_{p}+I_{q}$ is not graceful.

### 4.1 Graceful cones

For $q=0$ and $q=1$, we have the cycle graphs and the wheel graphs, respectively, and their gracefulness is already characterized in Chapter 2. For $q=2$, we have the double cones, and it is still an open problem to characterize them. By Proposition 4.1, the double cone $C_{p}+I_{2}$ is not graceful for $p \equiv 2(\bmod 4)$, and so far they are the only non-graceful double cones $[6,11,19]$.


Figure 4.1: Graceful labeling of $C_{4}+I_{2}$.

For the general case, Bhat-Nayak and Selvam [6] proved the following theorem.
Proposition 4.2. The cone graph $C_{p}+I_{q}$ is graceful for $p \equiv 0,3(\bmod 12)$ and $q \geq 1$.

For the proof of Proposition 4.2, Bhat-Nayak and Selvam introduced a new graph labeling and showed a more general result similar to Theorem 2.7.

A vertex labeling $f$ of a graph $G$ with $n$ vertices is said to be a special labeling if it satisfies the following conditions:

1. For every $i \in[1, n]$, there exists a vertex $u_{i} \in V(G)$ such that $f\left(u_{i}\right)$ is either $2 i-1$ or $2 i$.
2. $\operatorname{Im}\left(f_{\gamma}\right)=[1,2 n] \backslash \operatorname{Im}(f)$.
3. If $f(x)$ and $f_{\gamma}(x y)$ are odd, then $f(x)<f(y)$.

Note that conditions 1 and 2 imply that the number of vertices must be the same as the number of edges, i.e., $n=m$.

Theorem 4.3. If a graph $G$ has a special labeling, then the graph $G+I_{q}$ is graceful for all $q \geq 1$.

Proof. Let $G$ be a graph on $p$ vertices and $f$ be a special labeling of $G$. Define the vertex labeling $g$ for $G+I_{q}$ as follows, where $V(G)=\left\{u_{1}, \ldots, u_{p}\right\}$ and $V\left(I_{q}\right)=$ $\left\{v_{1}, \ldots, v_{q}\right\}$ :

$$
\begin{aligned}
& g\left(v_{j}\right)=j-1 \\
& g\left(u_{i}\right)= \begin{cases}i(q+1) & \text { if } f\left(u_{i}\right)=2 i \\
i(q+1)-1 & \text { if } f\left(u_{i}\right)=2 i-1\end{cases}
\end{aligned}
$$

We claim $g$ is a graceful labeling of $G+I_{q}$. As noted before, since $G$ has a special labeling, $G$ has $p$ edges. Thus, the number of edges of $G+I_{q}$ is $p+p q$. Clearly, $g: V\left(G+I_{q}\right) \rightarrow[0, p(q+1)]$ and it is injective. So, we have to prove that $g_{\gamma}$ is onto $[1, p(q+1)]$. For that, we show that for each $i \in[1, p]$ and $j \in[1, q+1]$, there is an edge $e$ with $g_{\gamma}(e)=(i-1)(q+1)+j$.

Consider a pair $(i, j)$. Since $f$ is a special labeling of $G$, by condition 1 , there is a vertex $u_{i} \in V(G)$ with $f\left(u_{i}\right)=2 i-1$ or $f\left(u_{i}\right)=2 i$.

Case 1. $f\left(u_{i}\right)=2 i-1$ and $1 \leq j \leq q$.
We have $g\left(u_{i}\right)=i(q+1)-1$ and $g\left(v_{q-j+1}\right)=q-j$. Since $q-j<i(q+1)-1$, the edge label on $u_{i} v_{q-j+1}$ is $i(q+1)-1-(q-j)=(i-1)(q+1)+j$.

Case 2. $f\left(u_{i}\right)=2 i-1$ and $j=q+1$.
By condition 2, there is an edge $e=x y \in E(G)$ with $f_{\gamma}(x y)=2 i$. Hence, $f(x)$ and $f(y)$ have the same parity. Suppose $f(x)=2 a+r$ and $f(y)=2 b+r$, where $r \in\{0,1\}$ is the parity. Then, $f_{\gamma}(x y)=2 i=|(2 a+r)-(2 b+r)|=2|a-b|$, and $i=|a-b|$. Therefore, $g_{\gamma}(x y)=|(a(q+1)-r)-(b(q+1)-r)|=$ $(q+1)|a-b|=i(q+1)=(i-1)(q+1)+(q+1)$

Case 3. $f\left(u_{i}\right)=2 i$ and $2 \leq j \leq q+1$.
We have $g\left(u_{i}\right)=i(q+1)$ and $g\left(v_{q-j+2}\right)=q-j+1$. Since $q-j+1<i(q+1)$, the edge label on $u_{i} v_{q-j+2}$ is $i(q+1)-(q-j+1)=(i-1)(q+1)+j$.

Case 4. $f\left(u_{i}\right)=2 i$ and $j=1$.
By condition 2, there is an edge $e=x y \in E(G)$ with $f_{\gamma}(x y)=2 i-1$. Now, $f(x)$ and $f(y)$ have different parities. Without loss of generality, suppose $f(x)$ odd and let $f(x)=2 a-1$ and $f(y)=2 b$. By condition 3, we have $f(x)<f(y)$ which implies $g(x)<g(y)$. Thus, $f_{\gamma}(x y)=2 i-1=2 b-(2 a-1)$ implies $i-1=b-a$. Finally, $g_{\gamma}(x y)=b(q+1)-(a(q+1)-1)=(b-a)(q+1)-1=$ $(i-1)(q+1)-1$.

Thus, we have proved that $\operatorname{Im}\left(g_{\gamma}\right)=[1, p(q+1)]$ and therefore $g$ is a graceful labeling of $G+I_{q}$.

We do not present here the complete proof of Proposition 4.2. Here, we only show a partial result which says that $C_{24 k}+I_{q}$ is graceful. For that, Bhat-Nayak and Selvam proved the following lemmas.

Lemma 4.4. For $k \geq 2, P_{4 k-3}$ has a vertex labeling $f$ such that $\operatorname{Im}(f)=[k+2,2 k] \cup$ $[2 k+3,3 k+1] \cup[5 k+1,7 k-1], \operatorname{Im}\left(f_{\gamma}\right)=[2 k+1,6 k-4]$, and the end vertices receive the labels $5 k+1$ and $7 k-1$.

Proof. Let $P_{4 k-3}=u_{1} u_{2} \cdots u_{4 k-3}$ and define the vertex labeling $f$ as follows:

$$
\begin{aligned}
f\left(u_{2 i-1}\right) & =5 k+i & & \text { for } 1 \leq i \leq 2 k-1 \\
f\left(u_{2 i}\right) & =k+2 & & \text { for } i=1 \\
& =3 k+3-i & & \text { for } 2 \leq i \leq k \\
& =3 k+1-i & & \text { for } k+1 \leq i \leq 2 k-2
\end{aligned}
$$

Now, it is easy to verify directly that $\operatorname{Im}\left(f_{\gamma}\right)=[2 k+1,6 k-4]$.
Remark 4.1. For $k=1$, consider the single vertex of $P_{1}$ labeled with 6.
Lemma 4.5. For $k \geq 1, P_{8 k-1}$ has a vertex labeling $f$ such that $\operatorname{Im}(f)=[1, k] \cup$ $[k+2,8 k], \operatorname{Im}\left(f_{\gamma}\right)=[1,8 k-2]$, and the end vertices receive the labels $2 k+1$ and $8 k$.

Proof. Let $P_{8 k-1}=u_{1} u_{2} \cdots u_{8 k-1}$ and define the vertex labeling $f$ as follows:

$$
\begin{aligned}
f\left(u_{1}\right) & =2 k+1 & & \\
f\left(u_{2 i+1}\right) & =4 k+1+i & & \text { for } 1 \leq i \leq k \\
f\left(u_{2 i}\right) & =4 k+2-i & & \text { for } 1 \leq i \leq k \\
f\left(u_{8 k+1-2 i}\right) & =8 k+1-i & & \text { for } 1 \leq i \leq k+2 \\
f\left(u_{8 k-2}\right) & =2 k+2 & & \\
f\left(u_{8 k-2-2 i}\right) & =i & & \text { for } 1 \leq i \leq k
\end{aligned}
$$

Thus, we labeled the vertices $u_{1}, \ldots, u_{2 k+1}, u_{6 k-3}, \ldots, u_{8 k-1}$ with labels in $[1, k] \cup$ $[2 k+1,2 k+2] \cup[3 k+2,5 k+1] \cup[7 k-1,8 k]$, and obtained edge labels in $[1,2 k] \cup$ [ $6 k-3,8 k-2$ ]. For the remaining subpath $u_{2 k+1} u_{2 k+2} \cdots u_{6 k-3}$, label it as given by Lemma 4.4 to obtain the desired labeling.

Lemma 4.6. For $k \geq 1, P_{8 k-1}$ has a vertex labeling $g$ such that $\operatorname{Im}(g)=\{16 k+$ $2,16 k+4, \ldots, 18 k\} \cup\{18 k+4,18 k+6, \ldots, 32 k\}, \operatorname{Im}\left(f_{\gamma}\right)=\{2,4, \ldots, 16 k-4\}$, and the end vertices receive the labels $20 k+2$ and $32 k$.

Proof. Let $f$ be the vertex labeling obtained from Lemma 4.5. Then, defining $g$ as $g(u)=2 f(u)+16 k$ gives the required labeling.

Lemma 4.7. For $k \geq 1, P_{16 k+3}$ has a vertex labeling $f$ such that $\operatorname{Im}(f)=\{1,3, \ldots$, $16 k-1,18 m+2,20 k+2,32 k, 32 k+2, \ldots, 48 k\}, \operatorname{Im}\left(f_{\gamma}\right)=\{16 k-2,16 k, 16 k+1$, $16 k+3, \ldots, 48 k-1\}$, and the end vertices receive the labels $20 k+2$ and $32 k$.

Proof. Let $P_{16 k+3}=u_{1} u_{2} \cdots u_{16 k+3}$ and define the vertex labeling $f$ as follows:

$$
\begin{aligned}
f\left(u_{2 i-1}\right) & =20 k+2 & & \text { for } i=1 \\
& =48 k+4-2 i & & \text { for } 2 \leq i \leq 8 k+2 \\
f\left(u_{2 i}\right) & =2 i-1 & & \text { for } 1 \leq i \leq 7 k \\
& =18 k+2 & & \text { for } i=7 k+1 \\
& =2 i-3 & & \text { for } 7 k+2 \leq i \leq 8 k+1
\end{aligned}
$$

Now, it is easy to verify that $\operatorname{Im}\left(f_{\gamma}\right)$ is as required.
Proposition 4.8. The cone graph $C_{24 k}+I_{q}$ is graceful for all $k \geq 1$.
Proof. Consider $P_{8 k-1}$ and $P_{16 k+3}$ labeled as given by Lemmas 4.6 and 4.7 respectively. By joining the paths by identifying the end vertices with the same label, we get a $C_{24 k}$ with a vertex labeling $f$ such that $\operatorname{Im}(f)=\{1,3, \ldots, 16 k-1,16 k+$ $2,16 k+4, \ldots, 48 k\}$ and $\operatorname{Im}\left(f_{\gamma}\right)=\{2,4, \ldots, 16 k, 16 k+1,16 k+3, \ldots, 48 k-1\}$. Furthermore, the largest odd vertex label is less than the smallest even vertex label. Therefore, $f$ satisfies all three conditions of being a special labeling for $C_{24 k}$.

Therefore, by Theorem 4.3, $C_{24 k}+I_{q}$ is graceful.
For the proof of Proposition 4.2, Bhat-Nayak and Selvam proved not only Proposition 4.8, but also that $C_{p}+I_{q}$ is graceful for $p \equiv 3,12,15(\bmod 12)$, each of them following the same strategy as shown before: prove the existence of a specific vertex labeling of some specific paths and then join their end vertices to form a cycle graph.

Besides Proposition 4.2, Bhat-Nayak and Selvam also proved the following proposition.

Proposition 4.9. The cone graph $C_{p}+I_{q}$ is graceful for $p=7,11,19$ and $q \geq 1$.
Proof. The following vertex labelings are special labelings for their respective cycle.
$C_{7}: 1,14,5,7,10,4,12$.
$C_{11}: 1,22,5,18,7,15,9,12,14,4,20$.
$C_{19}: 1,36,3,34,5,32,7,30,12,26,16,22,20,24,13,28,9,17,38$.
Brundage [8] also worked on this problem and showed the following result.
Proposition 4.10. The cone graphs $C_{5}+I_{q}$ and $C_{8}+I_{q}$ are graceful for all $q \geq 1$.
Proof. Brundage gives a graceful labeling $f: V \rightarrow[0, m]$ for each case.
For $C_{5}+I_{q}$, label the vertices of $C_{5}$ with $0, m, m-3,3, m-1$ consecutively along the cycle, where $m=5(q+1)$ is the total number of edges. Now, label the vertices of $I_{q}$ as follows:

$$
f\left(v_{k}\right)= \begin{cases}2 & \text { if } k=0 \\ 5 k+3 & \text { if } k=1,2, \ldots, q-1\end{cases}
$$

Thus, for $0<k<q$, as $3<5 k+3<m-3$, the incident edges of $v_{k}$ have labels $5 k+3, m-(5 k+3),(m-3)-(5 k+3), 5 k,(m-1)-(5 k+3)$, which are all distinct since they have different residues modulo 5 :

$$
\begin{aligned}
5 k+3 & \equiv 3 & & (\bmod 5) \\
m-(5 k+3) & \equiv 2 & & (\bmod 5) \\
(m-3)-(5 k+3) & \equiv 4 & & (\bmod 5) \\
5 k & \equiv 0 & & (\bmod 5) \\
(m-1)-(5 k+3) & \equiv 1 & & (\bmod 5)
\end{aligned}
$$

It is now easy to see that the labels in the edges incident with $v_{k}, 0<k<q$, cover the whole interval [ $4, m-7$ ]. Along with the labels of the edges in $C_{5}$ ( $m, 3, m-$ $6, m-4, m-1)$ and those incident with $v_{0}(2, m-2, m-5,1, m-3)$, all the labels in $[1, m]$ appear exactly once. Thus, $f$ is a graceful labeling of $C_{5}+I_{q}$.

For $C_{8}+I_{q}$, label the vertices of $C_{8}$ with $0, m, 2,3, m-2,1, m-3, m-1$ along the cycle, where $m=8(q+1)$, and label each $v_{k}$ in $I_{q}$ with $4 k+6$. The proof that this is indeed a graceful labeling is analogous to the previous case.

Remark 4.2. Note that the graceful labeling for some families of cone graphs is often not unique. For instance, a graceful labeling for $C_{8}+I_{q}$ distinct from the one given by Brundage goes as follows. Label $C_{8}$ with $0, m, \frac{m}{2}, \frac{3 m}{4}+1, \frac{m}{2}+1, \frac{3 m}{4}, \frac{m}{4}-1, m-1$, and label $I_{q}$ with $2 k+2$ for $0 \leq k<q$, where $m=8(q+1)$.

Brundage [8] organized the gracefulness of cone graphs in a table (see Table 4.1) and made a conjecture characterizing this class.

Conjecture 4.1 (Brundage, 1994). The generalized cone graph $C_{p}+I_{q}$ is graceful if, and only if, the parity condition holds.

| $p$ | 3, 4 | 5 | 6 | 7, 8 | 9 | 10 | 11, 12 | 13 | 14 | comments |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | Y | N | N | Y | N | N | Y | N | N | Y iff $p \equiv 0,3(\bmod 4)$ |
| 1 | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y $\forall p$ |
| 2 | Y | Y | N | Y | Y | N | Y | ? | N | ?, $\mathrm{N} \forall p=6+4 k$ |
| 3 | Y | Y | Y | Y | Y | ? | Y | ? | ? | ? |
| 4 | Y | Y | N | Y | Y | N | Y | ? | N | ?, $\mathrm{N} \forall p=6+4 k$ |
| 5 | Y | Y | ? | Y | ? | ? | Y | ? | ? | ? |
| 6 | Y | Y | N | Y | ? | N | Y | ? | N | ?, $\mathrm{N} \forall p=6+4 k$ |
| 7 | Y | Y | ? | Y | ? | ? | Y | ? | ? | ? |
| 8 | Y | Y | N | Y | ? | N | Y | ? | N | ?, $\mathrm{N} \forall p=6+4 k$ |
| 9 | Y | Y | ? | Y | ? | ? | Y | ? | ? | ? |
| comments | Y | $\begin{gathered} \mathrm{Y} \\ \forall q \geq 1 \end{gathered}$ | $?$ <br> $\mathrm{N} \forall q$ even | Y | ? | ? <br> $\mathrm{N} \forall q$ even | Y | ? | $?$ <br> $\mathrm{N} \forall q$ even | ? <br> $\mathrm{N} \forall p=6+4 k, q$ even Y $\forall p \equiv 0,3(\bmod 12)$ |

Table 4.1: Gracefulness of $C_{p}+I_{q}$ (updated as of 2014).

### 4.2 Computational results

Questioning the validity of Conjecture 4.1, we started looking for counterexamples, i.e., find a cone graph for which the parity condition does not hold and it is not graceful. For this task, a backtracking search algorithm similar to the Fang's algorithm presented in Chapter 3 was implemented.

The strategy is the same as in Fang's algorithm: it tries to create a new edge label at each iteration by labeling a not yet labeled vertex. For reducing the search tree, some optimizations were made due to the inherent symmetries of cone graphs. The following observations eliminate most of search through equivalent labelings given by the symmetries of the graph.

Force $f\left(u_{0}\right)=0$ without loss of generality. Since the edge labeling function $f_{\gamma}$ is a bijection, i.e., in a graceful labeling, every possible edge label from 1 to $m$ must appear as a label of some edge, and an edge label is obtained as the absolute value of the difference of the labels of its incident vertices, it follows that the vertices labeled 0 and $m$ must be adjacent in the graph. Otherwise, no edge would be assigned label $m$. Furthermore, since all edges are incident with at least one vertex of the cycle, one of the vertices of the cycle must be labeled 0 or $m$. By symmetry, let $u_{0}$ be that vertex. Now, the complementarity property allows us to assume without loss of generality that $f\left(u_{0}\right)=0$.

Just two candidate recipients for vertex label $m$. Assuming $f\left(u_{0}\right)=0$, the vertex label $m$ must be assigned to a vertex that is adjacent to $u_{0}$, i.e., to either $u_{1}, u_{p-1}$ or $v_{k}$ for some $k \in[0, q-1]$. Again owing to the symmetries in both the cycle and the independent set, we can narrow down our options, without loss of generality, to only two among those vertices, say $u_{1}$ and $v_{0}$.

Constrained recipients for edge label $m-1$. If, in the previous step, we chose vertex $u_{1}$ to receive label $m$, then, because we had already assigned label 0 to vertex $u_{0}$, the edge label $m-1$ can only appear on an edge that is incident with either $u_{1}$ (a neighbor of $u_{1}$ would receive label 1) or $u_{0}$ (a neighbor of $u_{0}$ would receive label $m-1$ ). Owing to the symmetries (rotation, reflection) of the cycle and the complementarity property, these two cases are actually equivalent. We can therefore consider, without loss of generality, that the edge labeled $m-1$ will be incident with $u_{1}$. We must now pick a neighbor of vertex $u_{1}$ to assign label 1 . Since vertex $u_{0}$ is already labeled with 0 , the possible neighbors are $u_{2}$ or $v_{k}$. However, by the symmetry of the independent set, we can consider $v_{0}$ as the sole candidate to receive label 1, and our search is limited to just two cases. If, on the other hand, we chose vertex $v_{0}$ to receive label $m$, then we must either assign label $m-1$ to a neighbor
of $u_{0}$ (namely $u_{1}$ or $v_{1}$ without loss of generality), or assign label 1 to a neighbor of $v_{0}$ (namely $u_{k}$, where we can impose $1 \leq k \leq\left\lfloor\frac{p}{2}\right\rfloor$ owing to the reflection symmetry of the cycle).

Establish an order of labeling in $I_{q}$. Since all vertices in the independent set are indistinguishable between themselves (both from the standpoint of some vertex in the independent set, since there are no edges between any of them, and from the standpoint of some vertex in the cycle, since each vertex in the cycle is adjacent to all vertices in the independent set), we may assume an order in which the vertices of $I_{q}$ are labeled. This prevents looking for labelings that are identical up to a permutation of the vertices in $I_{q}$.

Putting together these ideas, Algorithm 4.1 shows a pseudocode for the backtracking search algorithm to find a graceful labeling for a cone graph.

```
Algorithm 4.1: Backtracking search for generalized cone graphs
    Function Search(upper):
        if upper \(=0\) then
            return true
        \(l b l \leftarrow\) largest edge label \(\leq\) upper not present yet
        foreach pair \((k, k c)\) with \(|k-k c|=l b l\) do
            if both \(k\) and \(k c\) are not vertex labels yet then
                foreach edge \(u v\) with both ends unlabeled do
                    label \(u\) with \(k\) and \(v\) with \(k c\)
                    if Check () and Search \((l b l-1)\) then return true
                    label \(u\) with \(k c\) and \(v\) with \(k\)
                    if Check() and Search (lbl-1) then return true
                    unlabel \(u\) and \(v\)
        else
                let \(k\) be the unused vertex label and \(u\) be the vertex with label \(k c\)
                if \(u \in V\left(C_{p}\right)\) then
                    foreach \(v \in N(u) \cap V\left(C_{p}\right), v\) unlabeled do
                    label \(v\) with \(k\)
                    if Check () and Search \((l b l-1)\) then return true
                    unlabel \(v\)
                        if there are unlabeled vertex in \(I_{q}\) then
                    \(v \leftarrow\) next unlabeled vertex from \(I_{q}\)
                    label \(v\) with \(k\)
                            if Check () and Search (lbl-1) then return true
                    unlabel \(v\)
                else
                    foreach \(v \in V\left(C_{p}\right), v\) unlabeled do
                    label \(v\) with \(k\)
                    if Check () and Search \((l b l-1)\) then return true
                    unlabel \(v\)
```

The function Check in the pseudocode checks if the current labeling is valid, i.e., it checks if there are no repeated edge or vertex labels. Unlike Fang's backtracking search algorithm, this check is necessary here because labeling a vertex can create more than just one edge label. So, a verification is necessary every time we label a new vertex before continuing the search.

Running the search for a graceful labeling for $C_{6}+I_{5}$, the smallest cone graph which was still unknown to be graceful or not, the algorithm returned no possible graceful labeling, refuting, therefore, Brundage's conjecture. Moreover, the algorithm did not find a graceful labeling for $C_{6}+I_{q}$ with $5 \leq q \leq 35$. Notice that we are only interested in odd values of $q$ since, for even values, the parity condition already settles that $C_{6}+I_{q}$ is not graceful.

Searching for more non-graceful cone graphs, it makes sense to look for cone graphs $C_{p}+I_{q}$ with $p \equiv 2(\bmod 4)$ as they are the only ones that, together with an even $q$, are not graceful by the parity condition. Then, the next subclass to search for non-graceful cones is $C_{10}+I_{q}$. We found that $C_{10}+I_{3}$ and $C_{10}+I_{5}$ are graceful. However, the algorithm returned no graceful labeling for $C_{10}+I_{q}$ with $7 \leq q \leq 25$. A similar result was gotten with $p=14$ : the cones $C_{14}+I_{3}$ and $C_{14}+I_{5}$ are graceful, but the cones $C_{14}+I_{7}$ and $C_{14}+I_{9}$ are not. The following propositions summarize these results.

Proposition 4.11. The cone graphs $C_{10}+I_{q}$ and $C_{14}+I_{q}$ are graceful for $q=3,5$.
Proof. We have the following labelings where the first $p$ labels are from the cycle and the last $q$ are from the independent set.

$$
\begin{aligned}
& C_{10}+I_{3}: 0,40,25,3,33,13,6,29,10,21 ; 37,38,39 . \\
& C_{10}+I_{5}: 0,27,1,57,14,13,2,16,3,15 ; 23,32,51,55,60 . \\
& C_{14}+I_{3}: 0,56,6,1,28,5,2,30,34,3,33,11,22,55 ; 40,47,54 . \\
& C_{14}+I_{5}: 0,84,33,17,82,34,47,54,64,68,69,32,49,83 ; 2,5,8,11,14 .
\end{aligned}
$$

Proposition 4.12. The cone graphs $C_{6}+I_{q}, 5 \leq q \leq 35, C_{10}+I_{q}, 7 \leq q \leq 25$, $C_{14}+I_{7}$, and $C_{14}+I_{9}$ are not graceful.

Proof. Proven computationally.
Proposition 4.12 not only disproves Conjecture 4.1, but also gives a stronger feeling about how the non-gracefulness of generalized cone graphs behaves, from which we conjecture the following.

Conjecture 4.2. For every $p \equiv 2(\bmod 4)$, there exists a $q_{p}>1$ such that the cone graph $C_{p}+I_{q}$ is not graceful for all $q \geq q_{p}$.

One might think of trying to prove it computationally, implementing an algorithm to do something similar to the proof of Proposition 2.2, exhausting all possibilities for all values of $q$ greater than a threshold. However, as it was noted,
the running times of the algorithm to establish the non-gracefulness were growing exponentially, which indicates that it is not possible to prove it in this way.

Besides the non-graceful cone graphs, we also searched for new families of cones which are graceful. We have seen two approaches to tackle this class: fixing the size of the independent size or fixing the size of the cycle. By taking the last one, we started to find graceful labelings for $C_{9}+I_{q}$, the smallest family of this kind which was still open, and tried to find a pattern in the labelings while increasing the size of the independent size. As seen in Proposition 4.10, a simple rule could be possible, and indeed we found a scheme of labeling, not only for $C_{9}+I_{q}$, but also for $C_{13}+I_{q}$.

Proposition 4.13. The cone graphs $C_{9}+I_{q}$ and $C_{13}+I_{q}$ are graceful for all $q \geq 1$.
Proof. For $C_{9}+I_{q}$, label the vertices of $C_{9}$ with $0, m, 5, m-7,3, m-8, m-3,4, m-2$ along the cycle, where $m=9(q+1)$ is the number of edges, and label $I_{q}$ as follows:

$$
f\left(v_{k}\right)= \begin{cases}1 & \text { if } k=0 \\ 9 k+4 & \text { if } k=1,2, \ldots, q-1\end{cases}
$$

For $C_{13}+I_{q}$, label the vertices of $C_{13}$ as $0, m, m-8,6, m-9,10, m-6,7, m-$ $4, m-7,5, m-1, m-3$ along the cycle, where $m=13(q+1)$ is the number of edges, and label $I_{q}$ as follows:

$$
f\left(v_{k}\right)= \begin{cases}1 & \text { if } k=0 \\ 13 k+4 & \text { if } k=1,2, \ldots, q-1\end{cases}
$$

As for the verification, since it is analogous to the proof of Proposition 4.10, it is omitted.

On the other hand, finding a pattern after having fixed the size of the independent set (and allowing the size of the cycle to grow freely) seems to be much harder. For instance, it seems that, for $p>5$ and $p \equiv 1(\bmod 4)$, the cone graph $C_{p}+I_{q}$ has a graceful labeling $f$ such that $f\left(v_{0}\right)=1$ and $f\left(v_{k}\right)=p k+4$ for $1 \leq k<q$, as it can be seen in Proposition 4.13; we have also verified it for several cones with $p=17$ and $p=21$. However, no pattern has been found for the cycles. Another example is the family of cone graphs $C_{p}+I_{q}$ with $p \equiv 0(\bmod 4)$ : each of them seems to have a graceful labeling with $f\left(v_{k}\right)=\frac{p}{4}(k+1)$ for $0 \leq k<q$. That is known to be true for $p=4[6]$, and now $p=8$ (see Remark 4.2); we have also verified that several cones with $p=12,16,20$ admit such labeling.

Table 4.2 summarizes the current state of the gracefulness of generalized cone graphs for small values and gives a comment for the state of each row and column.

|  | 3,4 | 5 | 6 | 7, 8 | 9 | 10 | 11, 12 | 13 | 14 | comments |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | Y | N | N | Y | N | N | Y | N | N | Y iff $p \equiv 0,3(\bmod 4)$ |
| 1 | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y $\forall p$ |
| 2 | Y | Y | N | Y | Y | N | Y | Y | N | ?, $\mathrm{N} \forall p=6+4 k$ |
| 3 | Y | Y | Y | Y | Y | Y | Y | Y | Y | ? |
| 4 | Y | Y | N | Y | Y | N | Y | Y | N | ?, $\mathrm{N} \forall p=6+4 k$ |
| 5 | Y | Y | N | Y | Y | Y | Y | Y | Y | ? |
| 6 | Y | Y | N | Y | Y | N | Y | Y | N | ?, $\mathrm{N} \forall p=6+4 k$ |
| 7 | Y | Y | N | Y | Y | N | Y | Y | N | ? |
| 8 | Y | Y | N | Y | Y | N | Y | Y | N | ?, $\mathrm{N} \forall p=6+4 k$ |
| 9 | Y | Y | N | Y | Y | N | Y | Y | N | ? |
| 10 | Y | Y | N | Y | Y | N | Y | Y | N | ?, $\mathrm{N} \forall p=6+4 k$ |
| 11 | Y | Y | N | Y | Y | N | Y | Y | ? | ? |
| comments | Y | $\stackrel{\mathrm{Y}}{\forall q \geq 1}$ | $?$ $\mathrm{N} \forall q$ even | Y | $\begin{gathered} \mathrm{Y} \\ \forall q \geq 1 \end{gathered}$ | ? $\mathrm{N} \forall q$ even | Y | $\begin{gathered} \mathrm{Y} \\ \forall q \geq 1 \end{gathered}$ | ? $\mathrm{N} \forall q$ even | $\begin{gathered} \text { ?, } \\ \mathrm{N} \forall p=6+4 k, q \text { even } \\ \mathrm{Y} \forall p \equiv 0,3(\bmod 12) \end{gathered}$ |

Table 4.2: Gracefulness of $C_{p}+I_{q}$ (shaded entries are new results).

## Chapter 5

## Conclusion

The graceful labeling of graphs has been a topic of research for 50 years and it still has many properties to be found. Although its primary interest was the graceful labeling of trees in order to solve Ringel's conjecture, graceful labeling of graphs gained over the years its own beauty and interest.

This work gives a brief overview of the subject, presenting not only theoretical results from the literature, but also some computational results. Furthermore, we give some contributions to this problem.

In Chapter 2, the problem is presented, as well as the gracefulness of some rather simple graph classes like cycles and wheels. We also show necessary conditions to the existence of a graceful labeling for a graph, and two methods of constructing graceful graphs. In particular, one of them shows that any graph is an induced subgraph of some graceful graph.

In Chapter 3, we focus on graceful labeling of trees, more specifically, on different ways to approach the Graceful Tree Conjecture. The first one tackle the trees by limiting the diameter by introducing the transfer operation to modify a tree keeping it graceful. The second one reinforces the conjecture by showing computationally that all trees up to 35 vertices are graceful. Finally, we present some relaxed version of graceful labeling in which the better the bound, the closer to the conjecture we are.

In Chapter 4, we move our focus to generalized cone graphs. Their gracefulness was first tackled by Bhat-Nayak and Selvam, although some particular cases were already known. Later, Brundage also worked on this graph class and made a conjecture characterizing the gracefulness of cone graphs.

We tackled the gracefulness of cone graphs computationally and were able to disprove Brundage's conjecture. We also establish the gracefulness of new families of cone graphs and make a new conjecture regarding the non-graceful cone graphs.

For future work on the subject, we could consider looking for a way to prove Conjecture 4.2, or even characterize the gracefulness of generalized cone graphs. As
we showed in Chapter 4, it seems that $C_{p}+I_{q}$ is graceful for $p \equiv 0,1,3(\bmod 4)$ and $q \geq 1$. For $p \equiv 2(\bmod 4)$, our conjecture says there is a $q_{p}>1$ such that the cone graph is not graceful for all $q \geq q_{p}$. If, moreover, we could find out the parameter $q_{p}$ for each $p \equiv 2(\bmod 4)$, we would have a characterization of the gracefulness of generalized cone graphs.

Another class of interest is the class of trees, being the main open class on this topic. It is already settled that many classes of trees are graceful, but also there are many classes, even simple ones like lobsters, that are still open. Finally, another approach to the problem is to relax the conditions of graceful labelings and find nearly graceful labelings. This approach by approximating the labeling is also a topic of research for both trees and graphs in general.

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