TECHNICAL REPORT
Nonmonotonicity and Quasiconvexity on Equilibrium Problems

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Abstract

In this note, some results are introduced considering the assumptions of quasiconvexity and nonmonotonicity, finally an application and an idea to solve the quasiconvex equilibrium problem are presented considering these new results.

Keywords: Quasiconvexity; paramonotonicity; equilibrium problems.

Resumo

Nesta nota, alguns resultados são introduzidos considerando as suposições de quase-convexidade e não -monotonicidade, finalmente uma aplicação e uma idéia para resolver o problema de equilíbrio quase-convexa são apresentadas considerando estes novos resultados.

Palavras-chave: Quase-convexidade; paramonotonicidade; problemas de equilíbrio.

1 Introduction and Preliminaries

The equilibrium problems generalize minimization problems, variational inequalities problems, fixed point problems, linear complementarity problems, vector minimization problems and Nash equilibria problems with noncooperative games, see Blum and Oettli [8], Flores-Bázan [14], Iusem and Sosa [17]. Some recent work on equilibrium problems are Mansour et al. [1] and Cotrina and García [9].

Now, let K be a nonempty closed and convex subset of $\mathbb{R}^n$ and $f : K \times K \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for every $x \in K$. The equilibrium problem, shortly (EP) is, find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \ \forall y \in K. \tag{1.1}$$

On the other hand, in the literature there are two properties for f, which are called by

V. Strict paramonotonicity property (cutting property)

$$x \in S(f,K),\ y \in K,\ f(y,x) = 0 \Rightarrow y \in S(f,K).$$

M. Paramonotonicity property

$$x \in S(f,K),\ y \in K,\ f(y,x) = f(x,y) = 0 \Rightarrow y \in S(f,K).$$

where $S(f,K) = \{x \in K : f(x,v) \geq 0, \forall v \in K\}$ is the solution for the problem (1.1). The condition **V** is normaly used in Lemma 4.1 by Raupp and Sosa [23]. Note that **M** is obtained through **V** and considering f pseudomonotone, see Anh and Muu [3]. We can also see that **M** is important to guarantee convergence in the equilibrium problem, see Bello Cruz et al. [5].

Definition 1.1 *Let K be a nonempty convex set. A bifunction $f : K \times K \to I\!R$ is said to be*

(i) monotone on K if

$$f(x,y) + f(y,x) \leq 0,\ \forall x,y \in K;$$

(ii) pseudomonotone on K if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0,\ \forall x,y \in K;$$

(iii) quasimonotone on K if

$$f(x,y) > 0 \Rightarrow f(y,x) \leq 0,\ \forall x,y \in K.$$

We can easily verify that $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Definition 1.2 *A bifunction f will be called cyclically monotone if one has*

$$f(x_0,x_1) + f(x_1,x_2) + ... + f(x_m,x_0) \leq 0,$$

for any $m \in I\!N$ and any set of points $x_0, x_1, ..., x_m \in K$.

Definition 1.3 *A bifunction $f(w,\cdot) : K \subseteq I\!R^n \to I\!R$ is quasiconvex if for every $x,y \in K$ and $t \in [0,1]$ the following inequality holds*

$$f(w,(1-t)x + ty) \leq max\{f(w,x), f(w,y)\}.$$

Definition 1.4 *The function $f : K \times K \to I\!R$ is said to be properly quasimonotone if for all $x_1, ..., x_n \in K$ and all $\lambda_1, ..., \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$ it holds that*

$$\min_{1 \leq i \leq n} f\left(x_i, \sum_{j=1}^{n} \lambda_j x_j\right) \leq 0. \tag{1.2}$$

The Definition 1.4 is studied and widely used in Bianchi and Pini [6] and Farajzadeh and Zafarani [12], respectively. Now, consider the following conditions.

h1. $f(x,x)=0$ for each $x\in K$.

h2. $f(x,\cdot):K\to I\!R$ is lower semicontinuous for all $x\in K$.

Now consider the following variant of properly quasimonotone.

H2. For every $x_1,x_2,...,x_k\in K$ and $\lambda_1,...,\lambda_k\geq 0$ such that $\sum_{i=1}^{k}\lambda_i=1$, it holds that

$$\sum_{i=1}^{k}\lambda_i f(x_i,\sum_{j=1}^{k}\lambda_j x_j)\leq 0. \tag{1.3}$$

H3. $f:K\times K\to I\!R$ is positively homogeneous means that there exists $p>0$ such that

$$t^p f(x,y)=f(x,x+t(y-x)),$$

for every $t>0$.

H5. $f(w,\cdot)$ is quasiconvex.

H6. The bifunction $-f$ is triangular, i.e.,

$$f(u,w)+f(w,v)\leq f(u,v),\ \ \forall u,v,w\in K.$$

Remark 1.1 *The conditions* **h1**, **h2** *and* **H2** *are used in Iusem et al. [18]. The condition* **H3** *is used in Bianchi and Pini [6], Hu and Fang [16], a similar definition of* **H3** *can be seen in Polyak [22], pp. 84. Condition* **H5** *is studied by Bianchi and Pini [7], Cruz Neto et al. [10] and Flores-Bazán [13].*

2 Main Results

Lemma 2.1 *Let $K\neq\emptyset$, convex, closed subset of $I\!R^n$ and let $f:K\times K\to I\!R$ a bifunction, f satisfying the following assumptions* **h2**, **H3** *and* **H6**. *Then, if*

$$f(\bar{x},y)=0\Rightarrow f(y,\bar{x})\leq 0.$$

Proof. We take any y and $\bar{x}\in K$, and $\bar{x}\neq y$ with $f(\bar{x},y)=0$, for each $t\in\,]0,1[$, define $z_t=\bar{x}+t(y-\bar{x})\in K$, then by **H3**, there is a $p>0$ and **H6**, we obtain

$$f(\bar{x},y)+f(y,z_t)\leq t^p f(\bar{x},y),$$

by $f(\bar{x},y)=0$

$$f(y,z_t)\leq 0,$$

now, use **h2** with $t\to 0$, so we get the desired $f(y,\bar{x})\leq 0$. ■

Remark 2.1 *The first proposed lemma, can be applied in Anh and Muu [3] (proof of claim 3).*

Corollary 2.1 *Let $K \neq \emptyset$, convex, closed subset of $I\!R^n$ and let $f : K \times K \to I\!R$ a bifunction, f satisfying the following assumptions* **h1**, **h2**, **H5** *and* **H6**. *Then, if*

$$f(\bar{x}, y) = 0 \Rightarrow f(y, \bar{x}) \leq 0.$$

Proof. We take any y and $\bar{x} \in K$, and $\bar{x} \neq y$ with $f(\bar{x}, y) = 0$, for each $t \in]0, 1[$, define $z_t = \bar{x} + t(y - \bar{x}) \in K$, then by **H6** and **H5** we obtain

$$f(\bar{x}, y) + f(y, z_t) \leq f(\bar{x}, z_t) \leq max\{f(\bar{x}, y), f(\bar{x}, \bar{x})\},$$

the following is similar to Lemma 2.1. ■

Remark 2.2 *Note that the condition* **h1** *and* **H6***, imply monotonicity, when $u = v$. Note also, that from the condition* **H6***, we can obtain the cyclic monotonicity of f, see Giuli [15].*

Corollary 2.2 *Let $K \neq \emptyset$, convex, closed subset of $I\!R^n$ and let $f : K \times K \to I\!R$ a bifunction, f satisfying the following assumptions* **h2**, **H2** *and* **H3**. *Then, if*

$$f(\bar{x}, y) = 0 \Rightarrow f(y, \bar{x}) \leq 0.$$

Proof. We take any y and $\bar{x} \in K$, and $\bar{x} \neq y$ with $f(\bar{x}, y) = 0$, for each $t \in]0, 1[$, define $z_t = \bar{x} + t(y - \bar{x}) \in K$, then by **H2** we obtain

$$(1 - t)f(\bar{x}, z_t) + tf(y, z_t) \leq 0,$$

then, now by **H3**, we obtain

$$(1 - t)t^p f(\bar{x}, y) + tf(y, z_t) \leq 0,$$

the following is similar to Lemma 2.1. ■

Remark 2.3 *We note that the condition* **H2** *is proposed in Zhou and Chen [25] and in Bianchi and Pini [6], the condition* **H2** *is used to guarantee the solution existence of the dual equilibrium problem.*

3 Application

In this section will be apply the lemma 2.1, corollary 2.1 and corollary 2.2.

3.1 Paramonotonicity Property

Consider the following assumptions.

T1. If x^*, $\bar{x} \in K$, satisfy $f(\bar{x}, x^*) = f(x^*, \bar{x}) = 0$ then $x^* \in S(f, K) \Rightarrow \bar{x} \in S(f, K)$ (paramonotonicity property).

L1. $S(f, K) \neq \emptyset$.

Lemma 3.1 *Let $K \neq \emptyset$, convex, closed subset of $I\!R^n$ and let $f : K \times K \to I\!R$ a bifunction. Now assume that* **h2**, **H3**, **H6**, **T1** *and* **L1** *hold. Then, given $\bar{x} \in K$ and $x^* \in S(f,K)$, if $f(\bar{x},x^*) = 0$ then $\bar{x}$ is a solution of $S(f,K)$.*

Proof. Let $\bar{x} \in K$ and by **L1**, $x^* \in S(f,K)$ with $f(\bar{x},x^*) = 0$, then by the lemma 2.1, we obtain $f(x^*,\bar{x}) \leq 0$ and as $x^* \in S(f,K)$, we obtain $f(x^*,\bar{x}) \geq 0$. In this way we get that $f(\bar{x},x^*) = f(x^*,\bar{x}) = 0$, then by **T1** we get what we want. ■

Corollary 3.1 *Let $K \neq \emptyset$, convex, closed subset of $I\!R^n$ and let $f : K \times K \to I\!R$ a bifunction. Now assume that* **h1**, **h2**, **H5**, **H6**, **T1** *and* **L1** *hold. Then, given $\bar{x} \in K$ and $x^* \in S(f,K)$, if $f(\bar{x},x^*) = 0$ then $\bar{x}$ is a solution of $S(f,K)$.*

Proof. Let $\bar{x} \in K$ and by **L1**, $x^* \in S(f,K)$ with $f(\bar{x},x^*) = 0$, then by the Corollary 2.1, we obtain $f(x^*,\bar{x}) \leq 0$ and as $x^* \in S(f,K)$, we obtain $f(x^*,\bar{x}) \geq 0$. In this way we get that $f(\bar{x},x^*) = f(x^*,\bar{x}) = 0$, then by **T1** we get what we want. ■

4 Future Works

In this section, we propose some suggestions for future work, where we could apply the Section 2 of this work in Section 5 of Mallma et al. [20].

4.1 "An inexact proximal method with proximal distances for quasimonotone equilibrium problems". Mallma, Papa and Oliveira. 2017.

Now, we consider the notation, the assumptions and results obtained in Mallma et al. [20]. In this section, we will use the results obtained in the Section 2 of this paper.
We are interested in Section 4 of Mallma et al. [20], which is called "Convergence Results", in that section, the authors use the following particular solution set

$$S^*(f,\bar{C}) = \{x \in S(f,\bar{C}) : f(x,w) > 0, \forall\, w \in C\}, \tag{4.4}$$

this set will be important, to introduce the quasimonotonicity in the convergence of the proposed algorithm. Now, we will introduce our proposed lemmas in the following proposition of [20], for this reason, we consider the following particular set

$$S^\bullet(f,\bar{C}) = \{x \in S(f,\bar{C}) : f(x,w) = 0, \forall\, w \in C\}, \tag{4.5}$$

and we consider the following assumption.

Assumption H5' $S^\bullet(f,\bar{C}) \neq \emptyset$.

Proposition 4.1 *Under the assumptions by Mallma et al. [20]:* **H2**, **H3** *and* **H4**, $(d,H) \in \mathcal{F}(\bar{C})$ *and considering the new conditions* **h1**, **h2**, **H5**, **H6** *and* **H5'** *(of this note), we obtain*

$$H(x^*,x^k) \leq H(x^*,x^{k-1}) + \frac{1}{\lambda_k}\left\langle e^k, x^k - x^* \right\rangle, \tag{4.6}$$

for all $x^* \in S^\bullet(f,\bar{C})$.

Proof. Given $x^* \in S^{\bullet}(f, \bar{C})$ by **H5'**, then $f(x^*, w) = 0$, $\forall w \in C$ and as $x^k \in C$, we obtain $f(x^*, x^k) = 0$, by Corollary 2.1 (from this note) we have $f(x^k, x^*) \leq 0$. What follows from the proof is equal to Proposition 4.1 by Mallma et al. [20]. ∎

Remark 4.1 *The definitions of proximal distance and induced proximal distance it was introduced by Auslender and Teboulle [4], and for example used in Papa et al. [21] to solve the quasiconvex minimization problem, this proximal distance is also studied in Alvarez et al. [2].*

Now, the idea is to get an inexact proximal method with proximal distances for quasiconvexity equilibrium problems, then it will be a later work to future analyze the implications of the Proposition 4.1 in the paper of Mallma et al. [20].

On the other hand, one last future work would be to apply the Lemma 2.1 in the Theorem 6.1, case **(a)**, by Khatibzadeh et al. [19], also apply it in the Section 4 of the paper of Iusem et al. [18] and also in Santos and Scheimberg [24]. Another natural idea would be to extend the results in the work of Cruz Neto et al. [11], in order to generate an algorithm to solve the quasiconvex equilibrium problem on Hadamard manifolds.

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References

[1] Ait Mansour M, Elakri RA and Laghdir M. "Equilibrium and Quasi-Equilibrium Problems under φ-Quasimonotonicity and φ-Quasiconvexity", Minimmax Theory and its Applications, vol. 2, pp. 175-229, 2017.

[2] Alvarez F, Correa R and Marechal M. "Regular self-proximal distances are Bregman", Journal of Convex Analysis, vol. 24, pp. 135-148, 2017.

[3] Anh PN and Muu LD, "A hybrid subgradient algorithm for nonexpansive mappings and equilibrium problems", Optim Lett., vol. 8, pp. 727-738, 2014.

[4] Auslender A and Teboulle M. "Interior gradient and proximal methods for convex and conic optimization", SIAM Journal of Optimization, vol. 16, pp. 697-725, 2006.

[5] Bello Cruz JY, Santos PSM and Scheimberg S, "A two-phase algorithm for a variational inequality formulation of equilibrium problems", Journal of Optimization Theory and Applications, vol. 159, pp. 562-575, 2013.

[6] Bianchi M and Pini R, "A note on equilibrium problems with properly quasimonotone bifunctions", Journal of Global Optimization, vol. 20, pp. 67-76, 2001.

[7] Bianchi M and Pini R, "Coercivity conditions for equilibrium problems", Journal of Optimization Theory and Applications, vol. 124, pp. 79-92, 2005.

[8] Blum E and Oettli W, “From optimization and variational inequalities to equilibrium problems”, Math. Student, vol. 63, pp. 123-145, 1994.

[9] Cotrina J and García Y, “Equilibrium problems: Existence results and applications”, Set-Valued and Variational Analysis, vol. 26, pp. 159-177, 2018.

[10] Cruz Neto JX, Lopes JO and Soares PA, “A minimization algorithm for equilibrium problems with polyhedral constraints”, Optimization, vol. 65, pp. 1061-1068, 2016.

[11] Cruz Neto JX, Santos PSM and Soares Jr PA, “An extragradient method for equilibrium problems on Hadamard manifolds”, Optimization Letters, vol. 10, pp. 1327-1336, 2016.

[12] Farajzadeh AP and Zafarani J , “Equilibrium problems and variational inequalities in topological vector spaces”, Optimization, vol. 59, pp. 485-499, 2010.

[13] Flores-Bazán F, “Existence theorems for generalized noncoercive equilibrium problems: the quasi-convex case”, SIAM J. Optim., vol. 11, pp. 675-690, 2000.

[14] Flores-Bazán F, “Existence theory for finite dimensional pseudomnonotone equilibrium problems”, Acta. Appl. Math., vol. 77, pp. 249-297, 2003.

[15] Giuli M, “Cyclically monotone equilibrium problems and Ekeland’s principle”, Decisions in Economics and Finance, vol. 40, pp. 231-242, 2017.

[16] Hu R and Fang YP, “Strict feasibility and solvability for vector equilibrium problems in reflexive Banach spaces”, Optim Lett., vol. 5, pp. 505-514, 2011.

[17] Iusem AN and Sosa W, “New existence results for equilibrium problems”, Nonlinear Analysis, vol. 52, pp. 621-635, 2003.

[18] Iusem AN, Kassay G and Sosa W, “On certain conditions for the existence of solutions of equilibrium problems”, Math. Program, vol. 11, pp. 259-273, 2009.

[19] Khatibzadeh H, Mohebbi V and Ranjbar S, “Convergence analysis of the proximal point algorithm for pseudo-monotone equilibrium problems”, Optimization Methods Software, vol. 30, pp. 1146-1163, 2015.

[20] Mallma Ramirez L, Papa Quiroz EA and Oliveira PR, “An inexact proximal method with proximal distances for quasimonotone equilibrium problems”, Journal of the Operations Research Society of China, vol. 5, pp. 545-561, 2017.

[21] Papa Quiroz EA, Mallma Ramirez L and Oliveira PR. “An inexact proximal method for quasiconvex minimization”, European Journal of Operational Research, vol. 246, pp. 721-729, 2015.

[22] Polyak BT, “Introduction to Optimization”, Optimization Software, New York. 1987.

[23] Raupp FMP and Sosa W, “An analytic center cutting plane algorithm for finding equilibrium point”, RAIRO Operations Research, vol. 40, pp. 37-52, 2006.

[24] Santos PSM and Scheimberg S, “Inexact subgradient algorithm for equilibrium problems”, Comput. Appl. Math., vol. 30, pp. 91-107, 2011.

[25] Zhou JX and Chen G, “Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities”, Journal of Mathematical Analysis and Applications, vol. 132, pp. 213-225, 1988.