Instituto Alberto Luiz Coimbra de Pós-Graduação e Pesquisa de Engenharia

DECOMPOSITION OF $(2K+1)$-REGULAR GRAPHS CONTAINING SPECIAL SPANNING $2K$-REGULAR CAYLEY GRAPHS INTO PATHS OF LENGTH $2K+1$

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Dissertação de Mestrado apresentada ao Programa de Pós-graduação em Engenharia de Sistemas e Computação, COPPE, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Mestre em Engenharia de Sistemas e Computação.

Orientador: Fábio Happ Botler

Rio de Janeiro
Fevereiro de 2020

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DISSERTAÇÃO SUBMETIDA AO CORPO DOCENTE DO INSTITUTO ALBERTO LUIZ COIMBRA DE PÓS-GRADUAÇÃO E PESQUISA DE ENGENHARIA DA UNIVERSIDADE FEDERAL DO RIO DE JANEIRO COMO PARTE DOS REQUISITOS NECESSÁRIOS PARA A OBTENÇÃO DO GRAU DE MESTRE EM CIÊNCIAS EM ENGENHARIA DE SISTEMAS E COMPUTAÇÃO.

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RIO DE JANEIRO, RJ – BRASIL
FEVEREIRO DE 2020

Hoffmann, Luiz Henrique Silva

Decomposition of $(2k + 1)$-regular graphs containing special spanning $2k$-regular Cayley Graphs into paths of length $2k + 1$/Luiz Henrique Silva Hoffmann. – Rio de Janeiro: UFRJ/COPPE, 2020.

VIII, 32 p.: il.; $29,7$cm.

Orientador: Fábio Happ Botler

Dissertação (mestrado) – UFRJ/COPPE/Programa de Engenharia de Sistemas e Computação, 2020.

Referências Bibliográficas: p. 31 – 32.

1. Decomposition. 2. Path. 3. Regular graph. 4. Cayley graph. I. Botler, Fábio Happ. II. Universidade Federal do Rio de Janeiro, COPPE, Programa de Engenharia de Sistemas e Computação. III. Título.

Resumo da Dissertação apresentada à COPPE/UFRJ como parte dos requisitos necessários para a obtenção do grau de Mestre em Ciências (M.Sc.)

DECOMPOSIÇÃO DE GRAFOS $(2K+1)$-REGULARES CONTENDO GRAFOS DE CAYLEY $2K$-REGULARES GERADORES ESPECIAIS EM CAMINHOS DE COMPRIMENTO $2K+1$

Luiz Henrique Silva Hoffmann

Fevereiro/2020

Orientador: Fábio Happ Botler

Programa: Engenharia de Sistemas e Computação

Uma P_ℓ-decomposição de um grafo G é um conjunto de caminhos aresta-disjuntas com ℓ arestas em G que cobre o conjunto de arestas de G. Favaron, Genest, e Kouider (2010) conjecturaram que todo grafo $(2k+1)$-regular que contém um emparelhamento perfeito admite uma P_{2k+1}-decomposição. Eles também verificaram essa conjectura para grafos 5-regulares sem ciclos de comprimento 4. Em 2015, Botler, Mota, e Wakabayashi estenderam esse resultado para grafos 5-regulares sem triângulos, e em 2017, Botler, Mota, Oshiro e Wakabayashi generalizaram esse resultado para grafos $(2k+1)$-regulares com cintura de tamanho pelo menos $2k$. Nessa dissertação, verificamos essa conjectura para grafos $(2k+1)$-regulares que contêm a k-ésima potência de ciclo; e para grafos 5-regulares que contêm um grafo de Cayley gerador 4-regular gerado por dois elementos comutativos.

Abstract of Dissertation presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Master of Science (M.Sc.)

DECOMPOSITION OF $(2K+1)$-REGULAR GRAPHS CONTAINING SPECIAL SPANNING $2K$-REGULAR CAYLEY GRAPHS INTO PATHS OF LENGTH $2K+1$

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February/2020

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A P_ℓ-decomposition of a graph G is a set of edge-disjoint paths with ℓ edges in G that cover the edge set of G. Favaron, Genest, and Kouider (2010) conjectured that every simple $(2k+1)$-regular graph that contains a perfect matching admits a P_{2k+1}-decomposition. They also verified this conjecture for 5-regular graphs without cycles of length 4. In 2015, Botler, Mota, and Wakabayashi extended this result to 5-regular graphs without triangles, and in 2017, Botler, Mota, Oshiro and Wakabayashi generalized this result to $(2k+1)$-regular graphs with girth at least $2k$. In this dissertation, we verify this conjecture for $(2k+1)$-regular graphs that contain the k-th power of the cycle; and for 5-regular graphs that contain spanning 4-regular Cayley graph generated by two commutative elements.

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Chapter 1

Introduction

All graphs in this dissertation are simple, i.e., have no loops nor multiple edges. A *decomposition* of a graph G is a set $\mathcal{D}$ of edge-disjoint subgraphs of G that cover its edge set. If every element of $\mathcal{D}$ is isomorphic to a fixed graph H, then we say that $\mathcal{D}$ is an *H-decomposition.*

The literature related to H-decompositions is quite vast and contains results for decompositions of $2k$-regular graphs into trees with k edges. For instance, Ringel [18] conjectured that the complete graph $K_{2\ell+1}$ admits a T-decomposition for any tree with ℓ edges. Moreover, Ringel's Conjecture holds for many classes of trees (see [6, 11]). Häggkvist [13] generalized Ringel's Conjecture for regular graphs as follows.

Conjecture 1 (Graham-Häggkvist, 1989). *Let T be a tree with ℓ edges. If G is a 2ℓ-regular graph, then G admits a T-decomposition.*

For more results on decompositions of regular graphs into trees see [7, 9, 14]. For the case $T = P_\ell$, Kouider and Lonc [16] improved Häggkvist's result proving that if G is a 2ℓ-regular graph with girth $g \geq (\ell + 3)/2$, then G admits a special P_ℓ-decomposition called *balanced P_ℓ-decomposition*, that is a path decomposition $\mathcal{D}$ where each vertex is the end vertex of exactly two paths of $\mathcal{D}$. These authors also stated the following strengthening of Conjecture 1 for paths.

Conjecture 2 (Kouider-Lonc, 1999). *Let ℓ be a positive integer. if G is a 2ℓ-regular graph, then G admits a balanced P_ℓ-decomposition.*

In this dissertation, we focus on the case in which H is the simple path with $2k + 1$ edges, which we denote by P_{2k+1} (see Figure 2.1). Note that this notation is not standard. In 1957, Kotzig [15] (see also [3]) proved the following theorem.

Theorem 1 (Kotzig, 1957). *A 3-regular graph G admits a P_3-decomposition if and only if G contains a perfect matching.*

Proof. First, suppose that G admits a P_3-decomposition $\mathcal{D}$. Since a 3-regular graph has $3n/2$ edges, we have $|\mathcal{D}| = n/2$. Note that every element of $\mathcal{D}$ has only one internal edge. Let G' be the graph consisting of the internal edges of every element of D. Let $P = a_0a_1a_2a_3$ be a path of $\mathcal{D}$. Since $d_P(a_1) = d_P(a_2) = 2$ and there are $n/2$ paths in $\mathcal{D}$, $d_{G'}(x) = 1$ for every $x \in V(G')$. Thus, $E(G')$ is a perfect matching of G.

Now, suppose that G has a perfect matching M. Note that, $G - M$ is a 2-regular graph, and hence admits an Eulerian orientation, i.e., an orientation of the edges of $G - M$ in which for each vertex v, precisely half of the edges incident to v are oriented towards v. For each $e = xy \in M$ let $P_e = a_0a_1a_2a_3$, where a_0a_1 and a_2a_3 are edges leaving, respectively, a_1 and a_2. Since $G - M$ admits an Eulerian orientation, there are no two in-edges of the same vertex. Thus, $a_0 \neq a_3$ and P_e is a path for every $e \in M$. $\square$

In 2010, Favaron, Genest, and Kouider [8] extended this result by proving that every 5-regular graph that contains a perfect matching and no cycles of length 4 admits a P_5-decomposition; and proposed the following conjecture to generalize Kotzig's result. They also present a 5-regular graph that admits a P_5-decomposition and does not contain a perfect matching (see Figure 1.1). This motivated them to pose the following conjecture.

Conjecture 3 (Favaron–Genest–Kouider, 2010)**.** *If G is a $(2k+1)$-regular graph that contains a perfect matching, then G admits a P_{2k+1}-decomposition.*

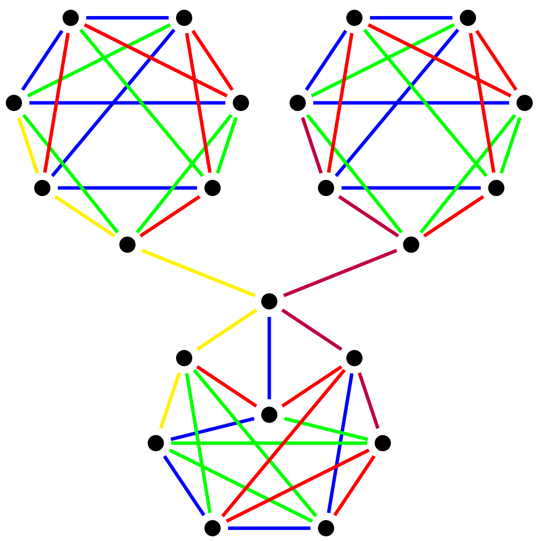


Figure 1.1: 5-regular graph that admits a P_5-decomposition, but has no perfect matching.

In 2015, Botler, Mota, and Wakabayashi [2] verified Conjecture 3 for triangle-free 5-regular graphs, and, more recently, Botler, Mota, Oshiro, and Wakabayashi [1] generalized this result for $(2k+1)$-regular graphs with girth at least $2k$.

It is clear that a 5-regular graph contains a perfect matching if and only if it contains a spanning 4-regular graph. In fact, by using a theorem of Petersen [17], one can prove that a $(2k+1)$-regular graph contains a perfect matching if and only if it contains a spanning $2k'$-regular graph for every $k' \leq k$.

Theorem 2 (Petersen, 1891)**.** *If G is a $2k$-regular graph, then G admits a decomposition into spanning 2-regular graphs.*

In this dissertation, we explore Conjecture 3 for $(2k + 1)$-regular graphs that contain special spanning $2k$-regular graphs as follows. Throughout the text, Γ denotes a finite group of order n; $+$ denotes the group operation of Γ; and 0 denotes the identity of Γ. As usual, for each $x \in \Gamma$, we denote by $-x$ the element $y \in \Gamma$ such that $x + y = 0$, and the operation $-$ denotes the default binary operation $-\colon (x, y) \mapsto x + (-y)$. Let $S \subseteq \Gamma$ be a set not containing the identity of Γ, and such that $-x \in S$ for every $x \in S$ (i.e., S is closed under taking inverses). The *Cayley graph* $X(\Gamma, S)$ is the graph H with $V(H) = \Gamma$, and $E(H) = \big\{\{x, y\}\colon y - x \in S\big\}$ (see [10]). [1] In this dissertation, we consider the general case in which S is not a set generating Γ, and hence $X(\Gamma, S)$ is not necessarily connected. We say that H is *simply commutative* if (i) $x+y = y+x$ for every $x, y \in S$, and (ii) $x+y \neq 0$ for every $x, y \in S$ with $y \neq -x$. Since $0 \notin S$, condition (ii) guarantees that H is a simple graph. It is not hard to check that, in such a graph, the neighborhood of a vertex $v \in \Gamma$ is $N(v) = S + v = \{x + v\colon x \in S\}$. Although the definition of Cayley graphs can be extended to multigraphs and directed graphs, in this dissertation we consider only non-directed simple graphs in order to tackle Conjecture 3. The reader may regard the graphs studied here as simple graphs that contain underlying graphs of directed Cayley graphs.

In this dissertation, we present two results regarding Conjecture 3. We verify it for $(2k + 1)$-regular graphs that contain the k-th power of a cycle (see Chapter 2); and for 5-regular graphs that contain spanning simply commutative 4-regular Cayley graphs (see Chapter 3).

We believe that, due to the underlying group structure, the techniques developed here can be extended for dealing with $(2k + 1)$-regular graphs that contain more general spanning Cayley graphs, and also $(2k+1)$-regular graphs that contain special spanning Schreier Graphs, which can give us significant insight with respect to the general case of Conjecture 3 (see Chapter 4).

---

[1] Although this definition is well-known, there is another definition which give us a colored directed graph as Cayley Graph, let X be a group and S a generating set of X, the Cayley Graph $\Gamma = \Gamma(X, S)$ is constructed as follows: (i) $X = V(\Gamma)$; (ii) each generator s of S is assigned a color c_s; (iii) for any $x \in X$ and $s \in S$, the vertices corresponding to the elements x and xs are joined by a directed edge of color c_s. Thus, the edge set $E(\Gamma)$ consists of pairs of the form (x, xs) in which $s \in S$ provides the color (see [4]).

Chapter 2

Regular graphs that contain spanning powers of cycles

Given positive integers k and n, the k-th power of the cycle on n vertices, which we denote by C_n^k, is the graph on the vertex set $\{0, \ldots, n-1\}$ and such that, for every vertex v, we have $x \in N(v)$ if and only if $x = v + r \pmod{n}$, where $r \in \{-k, \ldots, -1\} \cup \{1, \ldots, k\}$. Given a perfect matching M of a graph G we say that a P_ℓ-decomposition $\mathcal{D}$ of a graph G is *M-centered* if for every $P = a_0 a_1 \ldots a_i a_{i+1} \ldots a_{\ell-1} a_\ell \in \mathcal{D}$, we have $a_i a_{i+1} \in M$ for $i = (\ell - 1)/2$. The next results are examples of M-centered decomposition.

Proposition 1. *If G is a 5-regular graph that contains a spanning copy K of $K_{4,4}$, and $M = G - E(K)$, then G admits a M-centered P_5-decomposition.*

Proof. Let G, K, and M be as in the statement. Let (R, L) be the bipartition of K, where $R = \{r_1, r_2, r_3, r_4\}$ and $L = \{l_1, l_2, l_3, l_4\}$ be the partition classes of K. Since K is a complete bipartite graph, if $xy \in M$, then either $x, y \in R$ or $x, y \in L$. Thus, we may suppose, without loss of generality, that $M = \{r_1 r_2, r_3 r_4, l_1 l_2, l_3 l_4\}$. Therefore, $\mathcal{D} = \{l_1 r_1 l_3 l_4 r_2 l_2,\ l_3 r_3 l_1 l_2 r_4 l_4,\ r_1 l_2 r_3 r_4 l_1 r_2,\ r_3 l_4 r_1 r_2 l_3 r_4\}$ is an M-centered decomposition of G as desired (see Figure 2.1). $\square$

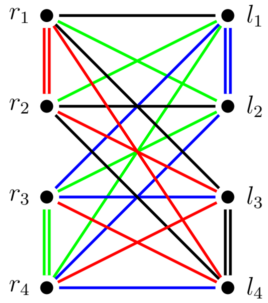


Figure 2.1: P_5-decomposition of a 5-regular graph that contains a spanning copy of a $K_{4,4}$.

Proposition 2. *Let $n, k \in \mathbb{N}$ with $k < n/2$. If G is a simple $(2k+1)$-regular graph on n vertices that contains a copy C of C_n^k, and $M = G - E(C)$, then G admits an M-centered P_{2k+1}-decomposition.*

Proof. Let G, C, and M be as in the statement, and let $V(C) = \{0, \dots, n-1\}$ as above. Since C is a $2k$-regular graph, M is a perfect matching of G. Given $i \in V(C)$, let Q_i be the path $v_0 v_1 \dots v_k$ in which $v_0 = i$; and, for $j = 1, \dots, k$, we have $v_j = v_{j-1} + j$ if j is odd; and $v_j = v_{j-1} - j$ if j is even. Note that for every $j = 1, \dots, k$, the path Q_i contains an edge xy such that $|x - y| = j$. Also, we have $V(Q_i) = \big\{i + r \pmod{n} \colon r \in \{-\lfloor k/2 \rfloor, -\lfloor k/2 \rfloor + 1, \dots, \lceil k/2 \rceil\}\big\}$. It is not hard to check that the set $\mathcal{Q} = \big\{Q_i \colon i \in V(C)\big\}$ is a P_k-decomposition of C.

Given an edge $e = ij \in M$, let $P_e = Q_i \cup \{ij\} \cup Q_j$. Since Q_i and Q_j have, respectively, i and j as end vertices, and $E(Q_i) \cap E(Q_j) = \emptyset$, the graph P_e is a trail of length $2k+1$. Thus, since $\mathcal{Q}$ is a P_k-decomposition of C, and M is a perfect matching of G, the set $\mathcal{D} = \{P_e \colon e \in M\}$ is a decomposition of G into trails of length $2k+1$. We claim that, in fact, $\mathcal{D}$ is a P_{2k+1}-decomposition of G. For that, we prove that if $ij \in M$, then $V(Q_i) \cap V(Q_j) = \emptyset$. Indeed, note that for every $e = ij \in M$, we have $|i - j| > k$, otherwise we have $ij \in E(C)$. Now, suppose that there is a vertex v in $V(Q_i) \cap V(Q_j)$. Then, there are r_1, r_2 with $-\lfloor k/2 \rfloor \leq r_1, r_2 \leq \lceil k/2 \rceil$, and such that $i + r_1 = v = j + r_2$. Suppose, without loss of generality, that $i > j$. Then, we have $r_2 - r_1 = i - j > k$, but $r_2 - r_1 \leq \lceil k/2 \rceil + \lfloor k/2 \rfloor = k$, a contradiction. □

Note that, from the proof of Proposition 2 we also obtain a construction for the Hamilton path decomposition of complete graphs of even order.

Corollary 1. *The complete graph K_{2k+2} admits a P_{2k+1}-decomposition.*

A slight variation of the proof of Proposition 2 also provides the following result.

Corollary 2. *Let G be an ℓ-regular graph, ℓ odd, and let M be a perfect matching of G. If each component of $G \setminus M$ is a k-th power of a cycle, then G admits an M-centered P_ℓ-decomposition.*

Let G_1 and G_2 be disjoint graphs with perfect matchings M_1 and M_2, respectively. Let $a_1b_1, \dots, a_kb_k \in M_1$ and $x_1y_1, \dots, x_ky_k \in M_2$ be distinct edges, and let G be the graph obtained from the disjoint union $G_1 \cup G_2$ by removing $a_1b_1, \dots, a_kb_k, x_1y_1, \dots, x_ky_k$ and adding the edges $a_1x_1, b_1y_1, \dots, a_kx_k, b_ky_k$. We say that G is a *collage* of G_1 and G_2 *over edges* of M_1 and M_2, and denote by M_G the perfect matching $\big(M_1 \cup M_2 \cup \{a_1x_1, b_1y_1, \dots, a_kx_k, b_ky_k\}\big) \setminus \{a_1b_1, \dots, a_kb_k, x_1y_1, \dots, x_ky_k\}$. When M_1 and M_2 are clear from the context, we say simply that G is a *collage* of G_1 and G_2. Note that G is ℓ-regular if and only if G_1 and G_2 are ℓ-regular.

Let G be an ℓ-regular graph, where ℓ is an odd positive integer, and let M be a perfect matching of G. We say that G is *M-constructable* if either G admits an M-centered P_ℓ-decomposition, or G is the collage of an M_1-constructable graph and an M_2-constructable graph over edges of M_1 and M_2 and $M = M_G$. The next result is a useful tool in the proof of Theorem 3.

Lemma 1. *Every M-constructable ℓ-regular graph admits an M-centered P_ℓ-decomposition.*

Proof. Suppose, for a contradiction, that the statement does not hold, and let G be a counterexample with a minimum number of vertices. By the definition of M-constructable, G is the collage of an M_1-constructable graph G_1 and an M_2-constructable graph G_2 over edges of M_1 and M_2. By the minimality of G, the graph G_i admits an M_i-centered P_ℓ-decomposition $\mathcal{D}_i$, for $i = 1, 2$. Let a_i, b_i, x_i, y_i, for $i = 1, \ldots, k$ be such that G is the graph obtained from $G_1 \cup G_2$ by removing $a_1b_1, \ldots, a_kb_k, x_1y_1, \ldots, x_ky_k$ and adding $a_1x_1, b_1y_1, \ldots, a_kx_k, b_ky_k$ as above. For $i = 1, \ldots, k$, let $P_i \in \mathcal{D}_1$ and $Q_i \in \mathcal{D}_2$ be the paths containing the edges a_ib_i and x_iy_i, respectively. By the definition of M_1- and M_2-centered P_ℓ-decomposition, for $i = 1, \ldots, k$, we may write $P_i = P_{i,1}a_ib_iP_{i,2}$ and $Q_i = Q_{i,1}x_iy_iQ_{i,2}$, where $P_{i,1}, P_{i,2}, Q_{i,1}$ and $Q_{i,2}$ are paths of length $(\ell-1)/2$. Since G_1 and G_2 are disjoint, $V(P_{i,j}) \cap V(Q_{i,j}) = \emptyset$ for $i = 1, \ldots, k$ and $j = 1, 2$. Let $R_{i,1} = P_{i,1} \cup \{a_ix_i\} \cup Q_{i,1}$ and $R_{i,2} = P_{i,2} \cup \{b_iy_i\} \cup Q_{i,2}$, and note that $\mathcal{D} = \big(\mathcal{D}_1 \setminus \{P_1, \ldots, P_k\}\big) \cup \big(\mathcal{D}_2 \setminus \{Q_1, \ldots, Q_k\}\big) \cup \{R_{i,j} \colon i = 1, \ldots, k \text{ and } j = 1, 2\}$ is an M_G-centered P_ℓ-decomposition of G, a contradiction. $\square$

Corollary 3. *If G is a 5-regular graph that contains a $K_{4,4}$-factorization K and $M = G - E(K)$, then G admits an M-centered P_5-decomposition.*

Chapter 3

5-regular graphs that contain spanning Cayley graphs

In this chapter, we explore 5-regular graphs that contain spanning simply commutative 4-regular Cayley graphs. In [2], Botler et al. showed that every triangle-free 5-regular graph G that has a perfect matching admits a P_5-decomposition. For that, they applied the following strategy: i) to find an initial decomposition of G into paths and trails; and ii) to perform some exchange of edges on the elements of $\mathcal{D}$ preserving a special invariant, while minimizing the number of trails that are not paths. The proof of our main theorem follows this framework, but consists of three steps. First, from the structure of Cayley graphs, we find an initial decomposition into trails, not necessarily paths (see Proposition 3). Then, we show how to exchange edges to obtain a decomposition in which the *bad elements* (the trails that are not paths) are distributed in circular fashion (see Lemma 5). Finally, we show how to deal with these "cycles of bad elements" (see Theorem 3).

3.1 Preliminary remarks

In this section, we present four special types of trails used throughout this text, and the following useful lemma.

Lemma 2. *Let $k \in \mathbb{N}$. If G is a $(2k+1)$-regular graph, and $\mathcal{D}$ is a decomposition of G into trails of length $2k+1$, then each vertex of G is the end vertex of precisely one element of $\mathcal{D}$.*

Proof. Let k, G and $\mathcal{D}$ be as in the statement. Let $n = |V(G)|$. Given an element $T \in \mathcal{D}$, we denote by $o(T)$ the number of its vertices with odd degree in T, and given a vertex $v \in V(G)$, we denote by $\mathcal{D}(v)$ the number of trails in $\mathcal{D}$ having v as end vertex. Clearly, $\sum_{T\in\mathcal{D}} o(T) = \sum_{v\in V(G)} \mathcal{D}(v)$. Since T is a trail, we have $o(T) \leq 2$, for every T in $\mathcal{D}$. Also, since every element of $\mathcal{D}$ has $2k+1$ edges, we have

$|\mathcal{D}| = \frac{1}{2k+1}|E(G)| = \frac{1}{2k+1}\frac{1}{2}(2k+1)n = \frac{1}{2}n$. Thus, we have $\sum_{T\in\mathcal{D}} o(T) \leq 2|\mathcal{D}| = n$. Now, since $v \in V(G)$ has odd degree, v must have odd degree in at least one element of $\mathcal{D}$, and hence $\mathcal{D}(v) \geq 1$. Thus, we have $\sum_{v\in V(G)} \mathcal{D}(v) \geq n$, and hence $\sum_{v\in V(G)} = \sum_{T\in\mathcal{D}} o(T) = n$. This implies that $\mathcal{D}(v) = 1$ for every $v \in V(G)$, as desired. $\square$

Recall that Γ is a finite group of order n with operation $+$. Given two elements g, r of Γ, we say that $\{g, r\}$ is a *simple commutative generator* (SCG) if (i) $0 \notin \{g, r, 2g, 2r\}$; (ii) $g \notin \{r, -r\}$; and (iii) $g + r = r + g$. Let $S = \{g, -g, r, -r\}$, and consider the Cayley graph $C = X(\Gamma, S)$. By construction, C is a simply commutative Cayley graph (see Chapter 1). Conditions (i) and (ii) guarantee that C is a simple graph, while condition (iii) introduces the main restriction explored in this dissertation. In this case, we say that C is the graph *generated* by $\{g, r\}$, and that $\{g, r\}$ is the *generator* of C. Given an SCG $\{g, r\}$, we say that a simple 5-regular graph G with vertex set Γ is a *$\{g, r\}$-graph* if G contains a spanning Cayley graph C generated by $\{g, r\}$. We say that G is a *simply commutative generated graph* or, for short, *SCG-graph* if G is a $\{g, r\}$-graph for some SCG $\{g, r\}$. In this chapter, we verify Conjecture 3 for SCG-graphs.

A 2-factor in a graph G is a spanning 2-regular subgraph of G. Let $\{g, r\}$ be an SCG. If C is the graph generated by $\{g, r\}$, and $x \in \{g, r\}$, then we denote by F_x the 2-factor of C with edge set $E(F_x) = \{v + x \colon v \in \Gamma\}$. If G is a $\{g, r\}$-graph, then we denote by $M_{g,r}$ the perfect matching $G \setminus E(F_g \cup F_r)$, and the triple $\{M_{g,r}, F_g, F_r\}$ is called the *base factorization* of G. Although G is a simple graph, for ease of notation, we refer to an edge $uv \in F_x$, with $x \in \{g, r\}$, as a *green* (resp. *red*) *out-edge* of u and *in-edge* of v if $v = u + x$ and $x = g$ (resp. $x = r$). In the figures throughout the text, the edges in F_g, F_r, $M_{g,r}$ are illustrated, respectively, in green, red, and double black pattern, while edges without specific affiliation are illustrated in gray. Moreover, if such an edge has a specific direction (i.e., in-edge or out-edge), it is illustrated accordingly. Note that each vertex u has precisely one edge of each type (green in-edge, green out-edge, red in-edge, red out-edge), and is incident to precisely one edge of $M_{g,r}$. Note that the group structure overcomes Theorem 2 by giving a decomposition of C into 2-factors in terms of the elements g and r. The next definition presents the main elements of the decompositions in our proofs.

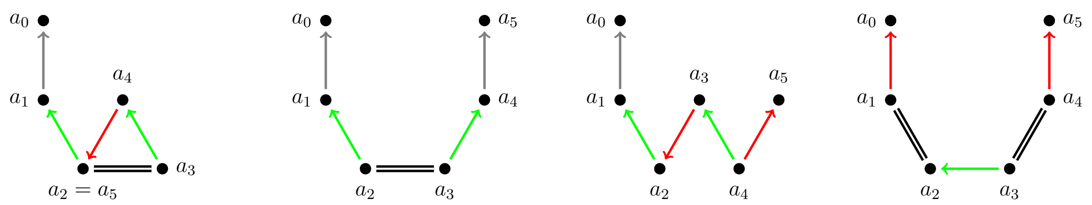
Definition 1. *We say that a trail T in a $\{g, r\}$-graph is of type A, B, C, or D if T can be written as $a_0a_1a_2a_3a_4a_5$ as follows.*

type A: a_0, a_1, a_2, a_3, a_4 are distinct vertices and $a_2 = a_5$ and the following hold: $a_2a_3 \in M_{g,r}$, $a_2a_1, a_3a_4 \in F_g$, $a_5 = a_4a_5$, and $a_1a_0 \in F_g \cup F_r \cup M_{g,r}$, i.e., a_1a_0 is an out-edge of a_1, or $a_1a_0 \in M_{g,r}$ (see Figure 3.1a). In this case, we say that a_3 is the primary connection vertex *of T, a_2 is the* secondary connection vertex *of* T*;* a_1 *is the* auxiliary vertex *of* T*; and* a_4 *is the* tricky vertex *of* T*. We denote these vertices, respectively, by* $\mathrm{cv}_1(T)$*,* $\mathrm{cv}_2(T)$*,* $\mathrm{aux}(T)$*, and* $\mathrm{tr}(T)$*;*

type B: $a_0, a_1, a_2, a_3, a_4, a_5$ *are distinct vertices and the following hold:* $a_2a_3 \in M_{g,r}$ $a_2a_1, a_3a_4 \in F_g$*,* $a_1a_0, a_4a_5 \in F_g \cup F_r \cup M_{g,r}$ *(see Figure 3.1b);*

type C: $a_0, a_1, a_2, a_3, a_4, a_5$ *are distinct vertices and the following hold:* $a_2a_1, a_4a_3 \in F_g$*,* $a_3a_2, a_4a_5 \in F_r$*,* $a_1a_0 \in F_g \cup F_r \cup M_{g,r}$*, and, moreover, we have* $a_2a_4 \in E(G)$ *and* $a_2a_4 \in M_{g,r}$ *(see Figure 3.1c);*

type D: $a_0, a_1, a_2, a_3, a_4, a_5$ *are distinct vertices and the following hold:* $a_1a_0, a_4a_5 \in F_r$*,* $a_1a_2, a_3a_4 \in M_{g,r}$*, and* $a_3a_2 \in F_g$ *(see Figure 3.1d).*



(a) Type A. (b) Type B. (c) Type C. (d) Type D.

Figure 3.1: The main types of trails.

We remark that elements of type A are not paths, while elements of type B, C, and D are paths. Moreover, the connection vertices of an element T are always incident to an edge of $M_{g,r}$ in T, and hence, no vertex of a $\{g, r\}$-graph is a connection vertex of two edge-disjoint elements of type A in a graph.

Given a trail (not necessarily a path) $T = a_0a_1a_2a_3a_4a_5$ in a decomposition $\mathcal{D}$ of a $\{g, r\}$-graph G, we say that the edge a_1a_0 (resp. a_4a_5) is a *hanging edge* at a_1 (resp. a_4) if $a_1a_0 \in M_{g,r} \cup F_g \cup F_r$ (resp. $a_4a_5 \in M_{g,r} \cup F_g \cup F_r$), i.e., the hanging edges of T are the end edges of T that are in $M_{g,r}$ or that are in-edges of its end vertices. By Definition 1, all end edges of elements of type A, B, C, or D are hanging edges. Note that if T is an element of type A where $a_5 = a_2$, then a_1a_0, a_2a_3 and a_4a_2 are hanging edges of T at, respectively, a_1, a_3, and a_4. Given a trail decomposition $\mathcal{D}$ of a graph G and a vertex $u \in V(G)$, we denote by $\mathrm{hang}_{\mathcal{D}}(u)$ the number of edges of G that are hanging edges at u.

3.2 Complete decompositions

The proof of the main theorem of this dissertation relies on the following definition, which consist of a set of properties that hold for decompositions of the given graph, and that are invariant due to a series of operations performed throughout the proof.

Definition 2. *A decomposition $\mathcal{D}$ of a $\{g, r\}$-graph G into trails of length 5 is* complete *if the following hold.*

(a) $\operatorname{hang}_{\mathcal{D}}(u) > 0$ *for every vertex u that is a connection vertex of an element of $\mathcal{D}$;*

(b) Every element in $\mathcal{D}$ is of type A, B, C or D;

(c) If T can be written as $a_0a_1a_2a_3a_4a_5$ and $a_0a_1 \in M_{g,r}$, then $a_0 = a_1 + g + r$.

The first step of our proof is given by the next proposition, which presents an initial decomposition for the graphs studied.

Proposition 3. *If G is an SCG-graph, then G admits a complete decomposition.*

Proof. Let $\{F_g, F_r, M_{g,r}\}$ be the base factorization of G. For each $e = xy \in M_{g,r}$, let $P_e = a_0a_1a_2a_3a_4a_5$, where $a_1a_0, a_4a_5 \in F_r$, $a_2a_1, a_3a_4 \in F_g$, $a_2 = x$, and $a_3 = y$. We claim that $\mathcal{D} = \{P_e \colon e \in M_{g,r}\}$ is complete. Clearly, P_e is an element of type A or B, for every $e \in M_{g,r}$, and hence $\mathcal{D}$ satisfies Definition 2(b). Moreover, note that a_0a_1 (resp. a_4a_5) is a hanging edge at a_1 (resp. a_4). Thus, given $z \in V(G)$, let $e' = xy \in M_{g,r}$ be such that $x = z - g$, then $P_{e'}$ contains a hanging edge at z, and hence there is a hanging edge at every vertex of G. This proves Definition 2(a). Finally, if $P \in \mathcal{D}$ can be written as $a_0a_1a_2a_3a_4a_5$ where $a_0a_1 \in M_{g,r}$, then P must be of type A, and we have $a_1a_2, a_3a_4 \in F_g$, $a_2a_3, a_4a_5 \in F_r$ and $a_0 = a_3 = a_2 + r = a_1 + g + r$, and hence $\mathcal{D}$ satisfies Definition 2(c). $\square$

The next lemma is used often in our proof and presents a consequence of the exchange of hanging edges at primary connection vertices.

Lemma 3. *If $T = a_0a_1a_2a_3a_4a_5$ is an element of type A in a decomposition of a $\{g, r\}$-graph G into trails of length 5, where $a_5 = a_2$ and $a_2a_3 \in M_{g,r}$, and $u \in V(G)$ is such that a_3u is a hanging edge at $a_3 = \operatorname{cv}_1(T)$, then $T' = a_0a_1a_2a_4a_3u$ is of type C.*

Proof. Let T, u, and T' be as in the statement. Since a_3a_4 is a green out-edge of a_3 and a_2a_3 is an edge of $M_{g,r}$ incident to a_3, we conclude that a_3u is a red out-edge of a_3, and hence $u = a_3 + r$. Now, since G is simple, we have $u \notin \{a_2, a_3, a_4\}$; if $u = a_1$, then we have $a_3 + r = u = a_1 = a_3 + g + r + g$, which implies $2g = 0$, a contradiction to the definition of SCG. Finally, by Lemma 2 we have $u \neq a_0$. Thus, T' is a path. Since $a_3u \in F_r$, T' is of type C. $\square$

Figure 3.2: Exchange of edges between elements of type A and a hanging edge in the proof of Lemma 3.

The following lemma is also used often in our proof and shows how two elements of type A may be connected.

Lemma 4. *If T_1 and T_2 are two edge-disjoint elements of type A in a $\{g, r\}$-graph G such that* $\mathrm{tr}(T_2) = \mathrm{cv}_1(T_1)$, *then* $\mathrm{aux}(T_2) = \mathrm{cv}_2(T_1)$.

Proof. Let $T_1 = a_0a_1a_2a_3a_4a_5$ and $T_2 = b_0b_1b_2b_3b_4b_5$, where $a_5 = a_2$ and $b_5 = b_2$ and $a_2a_3, b_2b_3 \in M_{g,r}$. If $\mathrm{cv}_1(T_1) = \mathrm{tr}(T_2)$, then $a_3 = b_4$. Since $b_1 = b_4 + r + g$ and $a_2 = a_3+g+r$. Thus $\mathrm{aux}(T_2) = b_1 = b_4+r+g = a_3+r+g = a_3+g+r = a_2 = \mathrm{cv}_2(T_1)$, as desired (see Figure 3.3). $\square$

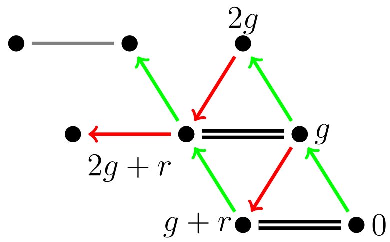


Figure 3.3: Identities given by Lemma 4 when $b_3 = 0$.

We say that an element T of type A in a complete decomposition $\mathcal{D}$ is *free* if $\mathrm{tr}(T) \neq \mathrm{cv}_i(T')$ for every $T' \in \mathcal{D}$ and $i \in \{1, 2\}$. An *A-chain* is a sequence $T_0, T_1, \ldots, T_{s-1}$ of elements of type A such that for each $j \in \{0, \ldots, s-1\}$, we have $\mathrm{tr}(T_j) = \mathrm{cv}_i(T_{j-1})$, for some $i \in \{1, 2\}$ (subtraction on the indices are taken modulo s). Note that A-chains do not consider the auxiliary vertex when allowing two elements to be consecutive. Thus, elements, say T_1 and T_2, of type A that are not consecutive in an A-chain, or that are in different A-chains, may still share a vertex u for which $\mathrm{cv}_i(T_1) = u = \mathrm{aux}(T_2)$.

Given a decomposition $\mathcal{D}$ of a graph G into trails of length 5, denote by $\tau(\mathcal{D})$ the number of elements that are not paths. By exchanging edges between the elements of a decomposition given by Proposition 3, we can show that a complete decomposition that minimizes $\tau(\mathcal{D})$ has no free element, and hence its elements of type A are partitioned into A-chains.

Lemma 5. *Every $\{g, r\}$-graph for which $2g + 2r \neq 0$ admits a complete decomposition in which the elements of type A are partitioned into A-chains.*

Proof. Let G be a $\{g,r\}$-graph for which $2g+2r \neq 0$ and put $M = M_{g,r}$. By Proposition 3, G admits a complete decomposition. Let $\mathcal{D}$ be a complete decomposition of G that minimizes $\tau(\mathcal{D})$. In what follows, we prove that $\mathcal{D}$ contains no free element of type A.

For that, in each step, we exchange edges between some elements of $\mathcal{D}$ and obtain a complete decomposition $\mathcal{D}'$ into trails of length five such that $\tau(\mathcal{D}') < \tau(\mathcal{D})$, which is a contradiction to the minimality of $\mathcal{D}$. To check that $\mathcal{D}'$ is a complete decomposition, we observe the three following items: (i) The only vertex that has a hanging edge of $\mathcal{D}$ but does not have a hanging edge of $\mathcal{D}'$ is the tricky vertex of a free element of type A, which by the definition of free element, is not a connection vertex of any element of $\mathcal{D}'$, and hence Definition 2(a) holds for $\mathcal{D}'$; (ii) every element of $\mathcal{D}'$ that is not an element of $\mathcal{D}$, i.e., the elements involved in the exchange of edges, is of type A, B, C, or D, and hence 2(b) holds for $\mathcal{D}'$; (iii) every edge $uv \in M$ in the elements considered that can be viewed as an end edge in $\mathcal{D}'$ (some elements may be expressed by more than two trails) either can be viewed as end edges in $\mathcal{D}$ or is obtained from an element of type A and satisfies $u = v + g + r$, and hence Definition 2(c) holds for $\mathcal{D}'$.

Claim 1. *No element of type B or C has a hanging edge at the primary connection vertex of a free element of type A.*

Proof. Let $T_1 \in \mathcal{D}$ be a free element of type A, and let $T_2 \in \mathcal{D}$ be an element of type B or C that contains a hanging edge at $\mathrm{cv}_1(T_1)$. Let $T_1 = a_0a_1a_2a_3a_4a_5$, where $a_5 = a_2$ and $a_2a_3 \in M$, and $T_2 = b_0b_1b_2b_3b_4b_5$. We divide the proof depending on whether T_2 is of type B or C.

$\mathbf{T_2}$ is of type B. Suppose, for a contradiction, that $b_4 = \mathrm{cv}_1(T_1) = a_3$. Put $T_1' = a_0a_1a_2a_4a_3b_5$, $T_2' = b_0b_1b_2b_3b_4a_2$ (see Figure 3.4), and let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2\}\big) \cup \{T_1', T_2'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. By Lemma 3, T_1' is an element of Type C. In what follows, we prove that T_2' is a path. For that, we prove that $a_2 \notin \{b_0, b_1, b_2, b_3, b_4\}$. Indeed, since G has no loops or multiple edges, $a_2 \notin \{b_3, b_4\}$. Since M is a matching, $a_2 \neq b_2$. If $a_2 = b_1$, then $b_2 = b_1 - g = a_2 - g = a_3 + g + r - g = b_5$, and hence T_2 is of type A, a contradiction. Finally, by Lemma 2 $a_2 \neq b_0$. Thus, T_2' is a path. Moreover, T_2' is an element of type B.

In what follows, we prove that $\mathcal{D}'$ is a complete decomposition. Note that $\mathrm{hang}_{\mathcal{D}'}(v) \geq \mathrm{hang}_{\mathcal{D}}(v)$ for every $v \in V(G) \setminus \{a_4\}$. Since a_4 is not a connection vertex of $\mathcal{D}'$, and $\mathrm{hang}_{\mathcal{D}}(v) > 0$ for every connection vertex v of $\mathcal{D}$, $\mathrm{hang}_{\mathcal{D}'}(v) > 0$ for every connection vertex v of $\mathcal{D}'$. Thus Definition 2(a) holds for $\mathcal{D}'$. Also, since T_1' is of type C and T_2' is of type B, Definition 2(b) holds for $\mathcal{D}'$. Moreover, note that $a_2a_3 \in M$ is an end edge of T_2'. Since T_1 is of type A, we have $a_2 = a_3 + g + r$. Thus, Definition 2(c) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is a complete decomposition such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 1 < \tau(\mathcal{D})$, a contradiction to the minimality of $\mathcal{D}$.

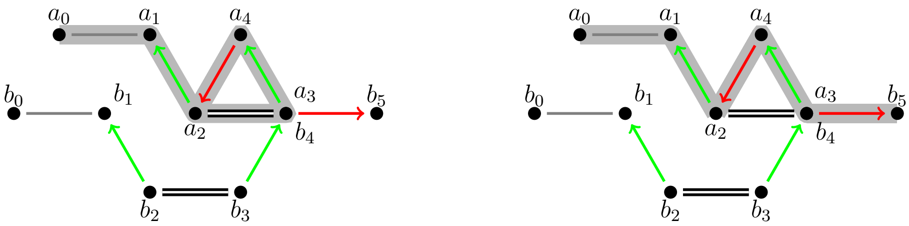


Figure 3.4: Exchange of edges between elements of type A and B in the proof of Claim 1.

$\mathbf{T_2}$ is of type C. We may assume $b_3b_2 \in F_r$. In this case we have $b_4b_3 \in F_g$. Since T_2 contains a hanging edge at $\mathrm{cv}_1(T_1)$, we have $a_3 = \mathrm{cv}_1(T_1) \in \{b_1, b_4\}$. If $b_4 = a_3$, then there are two green out-edges at a_3, namely a_3a_4, b_4a_3, a contradiction. Thus, we may assume that $a_3 = b_1$. Put $T_1' = a_0a_1a_2a_4a_3b_0$, $T_2' = a_2b_1b_2b_4b_3b_5$ (see Figure 3.5), and let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2\}\big) \cup \{T_1', T_2'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. By Lemma 3, T_1' is an element of Type C. In what follows we prove that T_2' is a path. For that, we prove that $a_2 \notin \{b_0, b_1, b_2, b_3, b_4\}$. Indeed, since G is simple, $a_2 \notin \{b_1, b_2\}$. If $a_2 = b_4$, then a_2a_1 and $b4b_3$ are two green out-edges at a_2, a contradiction. By Lemma 2, $a_2 \neq b_5$. Finally, $a_3 + g + r = a_2$ and $b_1 = b_3 + r + g$, if $a_2 = b_3$, then we have $a_3 + g + r = a_2 = b_3 = b_1 - g - r = a_3 - g - r$, which implies $2g + 2r = 0$, a contradiction. Thus T_2' is a path. Moreover, T_2' is an element of type C.

Now, we prove that $\mathcal{D}'$ is a complete decomposition. Analogously to the case above $\mathrm{hang}_{\mathcal{D}'}(v) \geq \mathrm{hang}_{\mathcal{D}}(v)$ for every $v \in V(G) \setminus \{a_4\}$. Since a_4 is not a connection vertex of $\mathcal{D}'$, and $\mathrm{hang}_{\mathcal{D}}(v) > 0$ for every connection vertex v of $\mathcal{D}$, $\mathrm{hang}_{\mathcal{D}'}(v) > 0$ for every connection vertex v of $\mathcal{D}'$. Thus Definition 2(a) holds for $\mathcal{D}'$. Also, since T_1' and T_2' are both elements of type C, Definition 2(b) holds. Moreover, note that a_2a_3 is an end edge of T_2', and, by definition of element of type A, Definition 2(c) holds. Thus, $\mathcal{D}'$ is a complete decomposition such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 1 < \tau(\mathcal{D})$, a contradiction to the minimality of $\mathcal{D}$. □

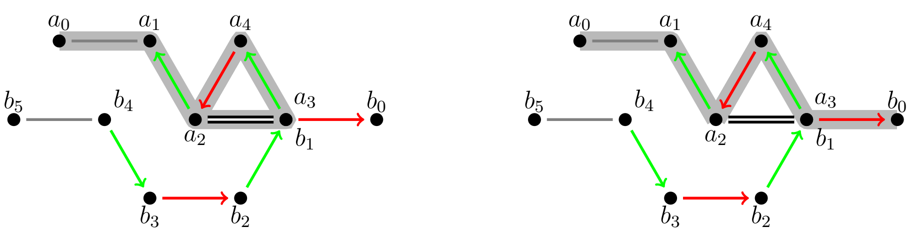


Figure 3.5: Exchange of edges between elements of type A and C in the proof of Claim 1.

Claim 2. *Let T_1 and T_2 be two elements of type A in $\mathcal{D}$. If T_1 is free and T_2 contains a hanging edge on $\mathrm{cv}_1(T_1)$, then no element of type A, B, or C in $\mathcal{D}\backslash\{T_1, T_2\}$ contains a hanging edge at $\mathrm{cv}_2(T_2)$.*

Proof. Let $T_1 = a_0a_1a_2a_3a_4a_5$ and $T_2 = b_0b_1b_2b_3b_4b_5$ be two elements of $\mathcal{D}$, where $a_5 = a_2$ and $b_5 = b_2$ and $a_2a_3, b_2b_3 \in M$. First, we prove that $\mathrm{cv}_1(T_1) = \mathrm{tr}(T_2)$ and $\mathrm{cv}_2(T_1) = \mathrm{aux}(T_2)$. Suppose, for contradiction, that $\mathrm{cv}_1(T_1) \neq \mathrm{tr}(T_2)$. Therefore, $b_1 = \mathrm{cv}_1(T_1) = a_3$. Now, put $T_1' = a_0a_1a_2a_4a_3b_0$, $T_2' = a_2b_1b_2b_3b_4b_2$ (see Figure 3.6) and let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2\}\big) \cup \{T_1', T_2'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. By Lemma 3, T_1' is an element of type C. We claim that T_2' is an element of type A. For that we prove that $a_2 \notin \{b_1, b_2, b_3, b_4\}$. Again, since G is a simple graph, we have $a_2 \notin \{b_1, b_2\}$. Since every vertex is incident to one edge of M, we have $a_2 \neq b_3$, and if $a_2 = b_4$, then we have $a_3 + g + r = a_2 = b_4 = a_3 - g - r$, which implies $2g + 2r = 0$, a contradiction. Now, we prove that $\mathcal{D}'$ is a complete decomposition. Analogously to the cases above, we have $\mathrm{hang}_{\mathcal{D}'}(v) \geq \mathrm{hang}_{\mathcal{D}}(v) \geq 0$ for every $v \in V(G) \setminus \{a_4\}$. Therefore, Definition 2(a) holds for $\mathcal{D}'$. Moreover, $a_3b_0 \notin M$ and if $a_0a_1 \in M$, then, by Definition 2(a), we have $a_0 = a_1 + g + r$, and hence Definition 2(c) holds for $\mathcal{D}'$. Thus, $\mathcal{D}'$ is a complete decomposition such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 1 < \tau(\mathcal{D})$, a contradiction to the minimality of $\mathcal{D}$. Finally, by Lemma 4, we have $\mathrm{cv}_2(T_1) = \mathrm{aux}(T_2)$.

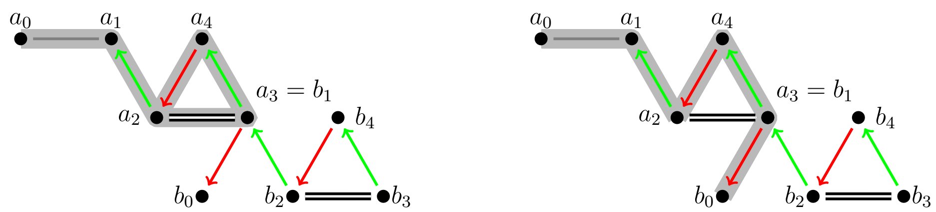


Figure 3.6: Exchange of edges between two elements of type A in the proof of Claim 2.

Now, let $T_3 \in \mathcal{D} \setminus \{T_1, T_2\}$ be an element of type A, B, or C, and suppose, for a contradiction, that T_3 contains a hanging edge at $\mathrm{cv}_2(T_2)$. Since $\mathrm{cv}_1(T_1) = \mathrm{tr}(T_2)$ and $\mathrm{cv}_2(T_1) = \mathrm{aux}(T_2)$, we have $a_3 = b_4$, $a_5 = a_2 = b_1$ and $b_5 = b_2$. In what follows, we divide the proof according to the type of T_3.

$\mathbf{T_3}$ is of type A. Let $T_3 = c_0c_1c_2c_3c_4c_5$, where $c_2 = c_5$ and $c_2c_3 \in M$. Since each vertex is incident to precisely one edge of M we have $c_3 \neq b_2 = \mathrm{cv}_2(T_2)$. Therefore we have $\mathrm{cv}_2(T_2) \in \{c_1, c_4\}$. Suppose that $\mathrm{cv}_2(T_2) = c_4$. Thus, we have $b_5 = b_2 = c_4$. Put $T_1' = a_0a_1a_2a_4a_3b_5$, $T_2' = b_0b_1b_4b_3b_2c_2$, $T_3' = b_1c_4c_3c_2c_1c_0$ (see Figure 3.7), and let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2, T_3\}\big) \cup \{T_1', T_2', T_3'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. In what follows, we prove that T_1', T_2' and T_3' are all paths. By Lemma 3, T_1' is an element of type C. Since G is simple, we have $c_2 \notin \{b_1, b_2, b_3, b_4\}$

and $b_1 \notin \{c_2, c_3, c_4\}$. By Lemma 2, $b_2 \neq a_0$, $c_2 \neq b_0$, $b_1 \neq c_0$. Therefore, T_2' is an element of type D. If $b_1 = c_1$, then b_2b_1 and c_2c_1 are two green in-edges at c_1, a contradiction. Therefore, T_3' is an element of type B.

We claim that $\mathcal{D}'$ is a complete decomposition. Analogously to the cases above, we have $\mathrm{hang}_{\mathcal{D}'}(v) \geq \mathrm{hang}_{\mathcal{D}}(v) \geq 0$ for every $v \in V(G) \setminus \{a_4\}$. Therefore, Definition 2(a) holds for $\mathcal{D}'$. Also, since T_1', T_2' and T_3' are of type C, D and B, respectively, Definition 2(b) holds for $\mathcal{D}'$. Moreover, none of the edges in $E(T_1) \cup E(T_2) \cup E(T_3)$ that may be in M and is not an end edge of T_1, T_2, or T_3, namely a_0a_1 and c_0c_1, became an end edge of T_1', T_2', or T_3'. Thus Definition 2(c) holds for $\mathcal{D}'$. Thus, $\mathcal{D}'$ is a complete decomposition such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 1 < \tau(\mathcal{D})$, a contradiction to the minimality of $\mathcal{D}$.

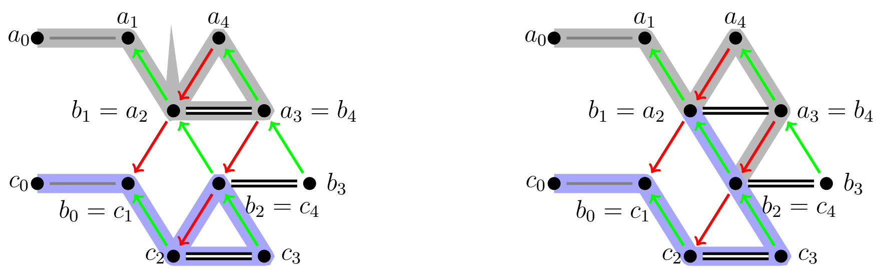


Figure 3.7: Exchange of edges between three elements of type A in the proof of Claim 2.

Thus, we may assume $\mathrm{cv}_2(T_2) = c_1$. This implies that $b_5 = b_2 = c_1$, and hence we have $b_3 = b_5 - r - g = c_1 - g - r = c_4$. Put $T_1' = a_0a_1a_2a_4a_3b_2$, $T_2' = b_0b_1b_4b_3b_2c_0$, $T_3' = b_1c_1c_2c_3c_4c_2$ (see Figure 3.8) and $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2, T_3\}\big) \cup \{T_1', T_2', T_3'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. Again, by Lemma 3, T_1' is an element of type C. We claim that T_2', T_3' are, respectively, of type D and A. Since G is simple, $c_0 \notin \{b_1, b_2, b_3, b_4\}$ and $b_1 \notin \{c_1, c_2, c_4\}$. By Lemma 2 we have $c_0 \neq b_0$. Therefore, T_2' is of type D. Finally, if $b_1 = c_3$, then a_2a_1 and c_3c_4 are two green out-edges at c_3, a contradiction. Therefore, T_3' is an element of type A. Analogously to the case above, $\mathcal{D}'$ is a complete decomposition such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 2 < \tau(\mathcal{D})$, a contradiction to the minimality of $\mathcal{D}$.

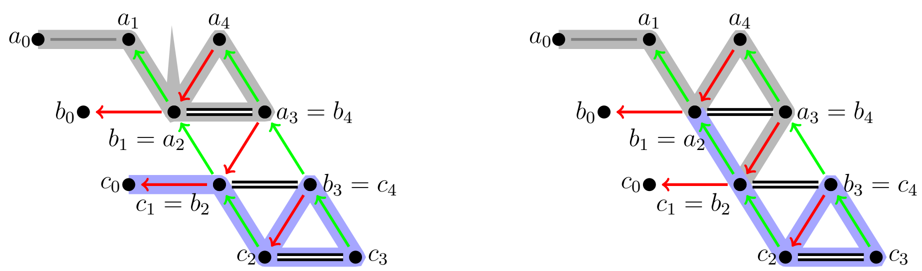


Figure 3.8: Exchange of edges between three elements of type A in the proof of Claim 2.

$\mathbf{T_3}$ is of type B. Let $T_3 = c_0c_1c_2c_3c_4c_5$ be an element of type B. Since T_3 contains a hanging edge on $\mathrm{cv}_2(T_2) = b_2$, we have $b_2 \in \{c_1, c_4\}$. By symmetry we may assume $b_2 = c_1$. Thus, put $T_1' = a_0a_1a_2a_4a_3b_2$, $T_2' = b_0b_1b_4b_3b_2c_0$, $T_3' = b_1c_1c_2c_3c_4c_5$ (see Figure 3.9) and let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2, T_3\}\big) \cup \{T_1', T_2', T_3'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. Again, by Lemma 3, T_1' is an element of type C. We prove that T_2' and T_3' are all paths. Since G is simple, we have $c_0 \notin \{b_1, b_2, b_3, b_4\}$ and $b_1 \notin \{c_1, c_2\}$. By Lemma 2, we have $c_0 \neq b_0$ and $b_1 \neq c_5$. Therefore, T_2' is an element of type D. Since $c_4 = c_3 + g$ and $b_1 = c_1 + g$, if $c_4 = b_1$, then $c_3 = c_1$, a contradiction. If $b_1 = c_3$, then a_2a_1 and c_3c_4 are two green out-edges of b_1, a contradiction, and if $b_1 = c_4$, then c_1b_1 and c_3c_4 are two green in-edges of b_1, again a contradiction. Therefore, T_3' is an element of type B. Analogously to the case above, $\mathcal{D}'$ is a complete decomposition such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 2 < \tau(\mathcal{D})$, a contradiction to the minimality of $\mathcal{D}$.

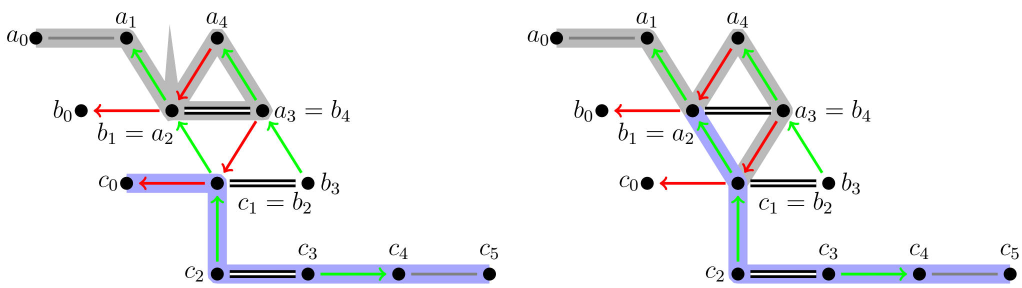


Figure 3.9: Exchange of edges between two elements of type A and another element of type B in the proof of Claim 2.

$\mathbf{T_3}$ is of type C. Let $T_3 = c_0c_1c_2c_3c_4c_5$ be an element of type C, where $c_3c_2 \in F_r$. This implies that $c_4c_3 \in F_g$. Since T_3 contains a hanging edge on $\mathrm{cv}_2(T_2) = b_2$, we have $b_2 \in \{c_1, c_4\}$. If $b_2 = c_4$, then c_4c_3 and b_2b_1 are two green out-edges of b_2, a contradiction. Thus, we may assume $b_2 = c_1$. Put $T_1' = a_0a_1a_2a_4a_3b_2$, $T_2' = b_0b_1b_4b_3b_2c_0$, $T_3' = b_1c_1c_2c_3c_4c_5$ (see Figure 3.10) and let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2, T_3\}\big) \cup \{T_1', T_2', T_3'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. Again, by Lemma 3, T_1' is an element of type C. We prove that T_2' and T_3' are all paths. Since G is simple, $c_0 \notin \{b_1, b_2, b_3, b_4\}$ and $b_1 \notin \{c_1, c_2\}$. By Lemma 2, we have $c_0 \neq b_0$ and $b_1 \neq c_5$. Therefore, T_2' is an element of type D. Analogously to the case above, if $b_1 = c_3$, then b_2b_1 and c_4c_3 are two green in-edges of b_1, a contradiction, and if $b_1 = c_4$, then a_2a_1 and c_4c_3 are two green out-edges of b_1, again a contradiction. Therefore, T_3' is an element of type C. Once more, analogously to the cases above, $\mathcal{D}'$ is a complete decomposition such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 2 < \tau(\mathcal{D})$, a contradiction to the minimality of $\mathcal{D}$.

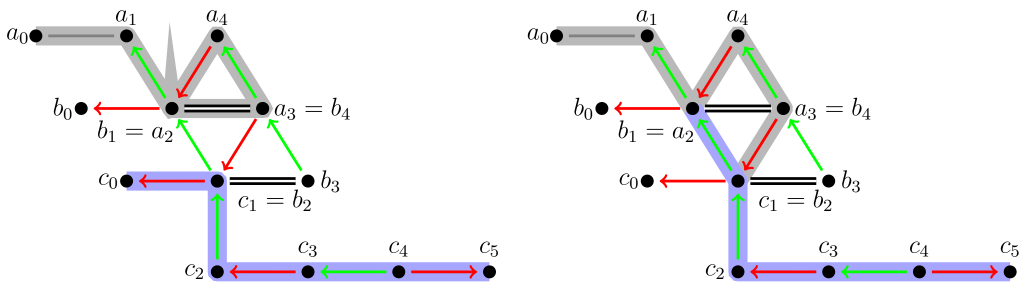


Figure 3.10: Exchange of edges between two elements of type A and another element of type C in the proof of Claim 2.

$\square$

Claim 3. *There is no free element of type A.*

Proof. Suppose, for a contradiction, that $\mathcal{D}$ contains a free element, say T_1, of type A. By Definition 2(a), there is a hanging edge e_2 on $\mathrm{cv}_1(T_1)$. Let T_2 be the element of $\mathcal{D}$ that contains e_2. By Claim 1, T_2 is not of type B or C, and since M is a matching, T_2 is not of type D. Thus, T_2 is of type A. By Definition 2(a), there is a hanging edge e_3 on $\mathrm{cv}_2(T_2)$. Let T_3 be the element of $\mathcal{D}$ that contains e_3. By Claim 2, T_3 is of type D, which implies that there are two edges of M incident to $\mathrm{cv}_2(T_2)$, a contradiction. $\square$

Now, consider the auxiliary directed graph $D_{\mathcal{D}}$ in which $V(D_{\mathcal{D}}) = \mathcal{D}$ and $(T_1, T_2) \in A(D_{\mathcal{D}})$ if and only if $\mathrm{tr}(T_2) = \mathrm{cv}_i(T_1)$ for some $i \in \{1, 2\}$ (we ignore possible multiple edges). It is clear that the elements of type A in $\mathcal{D}$ are partitioned into A-chains if and only if $D_{\mathcal{D}}$ consists of vertex-disjoint directed cycles and isolated vertices. Note that if a vertex u is a (primary or secondary) connection vertex of an element $T \in \mathcal{D}$, then T is an element of type A and u is incident to an edge in $M \cap E(T)$. Therefore, every vertex of G is a connection vertex of at most one element of $\mathcal{D}$, and hence, by Claim 3, every vertex of $D_{\mathcal{D}}$ has in-degree precisely 1.

Note also that given two elements T_1 and T_2 we have $\mathrm{tr}(T_1) \neq \mathrm{tr}(T_2)$, otherwise there would be a vertex with two green in edges. This implies that every vertex of $D_{\mathcal{D}}$ has out-degree at most 2. Now, if T_1 and T_2 are two elements of type A in $\mathcal{D}$ such that $\mathrm{cv}_1(T_1) = \mathrm{tr}(T_2) = u$, by Lemma 4, we have $\mathrm{aux}(T_2) = \mathrm{cv}_2(T_1)$, which means that $E(T_1) \cup E(T_2)$ contains the five edges in $E(G)$ incident to u, and hence, no other element of $\mathcal{D}$ contains edges incident to u. This implies that every vertex of $D_{\mathcal{D}}$ has out-degree at most 1, and hence $D_{\mathcal{D}}$ consists of vertex-disjoint directed cycles and isolated vertices as desired. $\square$

3.3 Admissible decompositions

In this section, we present a new decomposition invariant, which we call *admissible decompositions*, and conclude our proof. For that, we introduce an important object, the exceptional pair. Let G be a $\{g,r\}$-graph, and let $\mathcal{D}$ be a decomposition of G into trails of length 5. We say that a pair (T_1,T_2) of elements of $\mathcal{D}$ is an *exceptional pair* if T_1 and T_2 are elements of type A and C, respectively, and can be written as $T_1 = a_0a_1a_2a_3a_4a_5$ and $T_2 = b_0b_1b_2b_3b_4b_5$ such that $a_2a_3 \in M_{g,r}$, $a_2 = a_5 = b_3$, and $a_2a_1, a_3a_4, b_2b_1, b_4b_3 \in F_g$, $a_4a_5, b_3b_2, b_4b_5 \in F_r$, $a_1a_0, b_1b_0 \in M_{g,r} \cup F_g \cup F_r$ (see Figure 3.11). Note that since G is a simple graph, we have $b_4 \neq a_3$. Also, if $2g+2r \neq 0$, then we have $b_1 \neq a_3$. This yields the following remark.

Remark 1. *If G is a $\{g,r\}$-graph for which $2g+2r \neq 0$ and (T_1,T_2) is an exceptional pair, then T_2 does not contain a hanging edge at* $\mathrm{cv}(T_1)$*.*

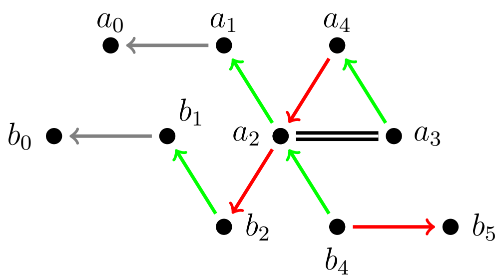


Figure 3.11: An exceptional pair.

An *open chain* is a sequence $T_0, T_1, \ldots, T_{s-1}$ such that the following hold. (i) T_0 is a free element of type A; (ii) T_j is an element of type A and $\mathrm{tr}(T_j) = \mathrm{cv}_i(T_{j-1})$, $j \in \{0,\ldots,s-2\}$ and some $i \in \{1,2\}$; and (iii) T_{s-1} is an element of type C such that (T_{s-2}, T_{s-1}) is an exceptional pair. The next step of our proof requires the following variation of Definition 2.

Definition 3. *We say that a decomposition $\mathcal{D}$ of a $\{g,r\}$-graph G into trails of length* 5 *is* admissible *if the following hold.*

(a) There is a hanging edge at every connection vertex, except possibly for at most one secondary connection vertex $\mathrm{cv}_2(T')$*, and, in this case, there is an open chain $S = T_0, \ldots, T_{s-2}, T_{s-1}$ in $\mathcal{D}$, such that $T_{s-2} = T'$;*

(b) Every element in $\mathcal{D}$ is either a path or an element of type A;

(c) If $T \in \mathcal{D}$ can be written as $a_0a_1a_2a_3a_4a_5$ and $a_0a_1 \in M$, then $a_0 = a_1 + g + r$;

(d) The elements of type A in $\mathcal{D}$ are partitioned into A-chains and at most one open chain.

It is not hard to check that the decomposition given by Lemma 5 is an admissible decomposition. Therefore, every $\{g,r\}$-graph such that $2g+2r \neq 0$ admits an admissible decomposition. By performing some more exchanging of edges between the elements of same A-chain of an admissible decomposition, we can show that an admissible decomposition that minimizes its number of elements of type A is in fact a P_5-decomposition.

Theorem 3. *Every $\{g,r\}$-graph for which $2g+2r \neq 0$ admits a P_5-decomposition.*

Proof. Let g and r be as in the statement, let G be a $\{g,r\}$-graph for which $2g+2r \neq 0$, and put $M = M_{g,r}$. By Lemma 5, G admits an admissible decomposition. Let $\mathcal{D}$ be an admissible decomposition of G that minimizes $\tau(\mathcal{D})$. In what follows, we prove that $\tau(\mathcal{D}) = 0$. Suppose, for a contradiction, that $\tau(\mathcal{D}) > 0$. We divide that A-chains into three types, according to the connections between its elements. Given $i \in \{1,2\}$, we say that an A-chain $S = T_0, T_1, \ldots, T_{s-1}$ is of type i if $\operatorname{tr}(T_j) = \operatorname{cv}_i(T_{j-1})$ for every $j \in \{0,\ldots,s-1\}$; and we say that S is a *mixed* A-chain if S is not of type 1 or 2.

Analogously to the proof of Lemma 5, in each step, we exchange edges between some elements of $\mathcal{D}$ and obtain an admissible decomposition $\mathcal{D}'$ into trails of length five such that $\tau(\mathcal{D}') < \tau(\mathcal{D})$, which is a contradiction to the minimality of $\mathcal{D}$. To check that $\mathcal{D}'$ is an admissible decomposition, we observe the four following items: (i) The only vertex that has a hanging edge of $\mathcal{D}$ but does not have a hanging edge of $\mathcal{D}'$ is the secondary connection vertex of an element T_1 of type A, and in this case there is an element T_2 of type C such that (T_1, T_2) is an exceptional pair, and hence Definition 3(a) holds for $\mathcal{D}'$; (ii) every element of $\mathcal{D}'$ that is not an element of $\mathcal{D}$, i.e., the elements involved in the exchange of edges, is a path or an element of type A, and hence 3(b) holds for $\mathcal{D}'$; (iii) every edge $uv \in M$ in the elements considered that can be viewed as an end edge in $\mathcal{D}'$ (some elements may be expressed by more than two trails) either can be viewed as end edges in $\mathcal{D}$ or is obtained from an element of type A and satisfies $u = v + g + r$, and hence Definition 3(c) holds for $\mathcal{D}'$; (iv) either an open chain is reduced by one element, an A-chain is converted into an open chain, or all the elements of an A-chain are replaced by paths of length 5.

Claim 4. *Every A-chain in $\mathcal{D}$ is mixed.*

Proof. Suppose, for a contradiction, that there is an A-chain $S = T_0, T_1, \ldots, T_{s-1}$ of type 1 or 2. Let $T_j = a_{0,j}a_{1,j}a_{2,j}a_{3,j}a_{4,j}a_{5,j}$, where $a_{5,j} = a_{2,j}$, $a_{2,j}a_{3,j} \in M$, $a_{1,j}a_{0,j}, a_{3,j}a_{4,j} \in F_g$, $a_{4,j}a_{2,j} \in F_r$ and $a_{2,j}a_{1,j} \in M \cup F_g \cup F_r$. For $i \in \{1,2,3,4,5\}$, the edge $a_{i-1,j}a_{i,j}$ is called the *i-th edge of T_j*. In what follows, we divide the proof according to the type of S.

S is of type 1. Since S is of type 1, for each $j = 0,\ldots,s-1$, we have $a_{3,j} = \operatorname{cv}_1(T_j) = \operatorname{tr}(T_{j+1}) = a_{4,j+1}$, and hence, by Lemma 4, we have $a_{2,j} = \operatorname{cv}_2(T_j) =$

$\mathrm{aux}(T_{j+1}) = a_{1,j+1}$. Now, for each $j = 0, \ldots, s-1$, let $T_j' = a_{2,j+1}a_{3,j}a_{4,j}a_{1,j}a_{2,j}a_{0,j+1}$ (see Figure 3.12). Note that $T_j' = T_j - a_{1,j}a_{0,j} + a_{1,j+}a_{0,j+1} - a_{2,j}a_{3,j} + a_{2,j-1}a_{3,j-1} - a_{4,j}a_{2,j} + a_{4,j+1}a_{2,j+1}$. More specifically, we have $a_{2,j+1}a_{3,j} = a_{4,j+1}a_{5,j+1}$ is the 5-th edge of T_{j+1}; $a_{3,j}a_{4,j}$ is the 4-th edge of T_j; $a_{4,j}a_{1,j} = a_{2,j-1}a_{3,j-1}$ is the 3rd edge of T_{j-1}; $a_{1,j}a_{2,j}$ is the 2nd edge of T_j; $a_{2,j}a_{0,j+1} = a_{1,j+1}a_{0,j+1}$ is the 1st edge of T_{j+1}. Clearly, T_j' is a trail of length 5. Moreover, since, for each $i \in \{1,2,3,4,5\}$, the element T_j' contains the i-th edge of an element of S, and, if $j \neq j'$, the elements T_j' and $T_{j'}'$ contain the i-th edge of different elements of S, the set $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_j \colon j = 0, \ldots, s-1\}\big) \cup \{T_j' \colon j = 0, \ldots, s-1\}$ is a decomposition of G into trails of length 5. We may regard $\mathcal{D}'$ as the decomposition obtained by reversing the direction of two components of F_g, namely, the green edges in S, and applying the same strategy used in Proposition 3.

In order to prove that T_j' is a path, we show that $a_{2,j+1}, a_{0,j+1} \notin \{a_{3,j}, a_{4,j}, a_{1,j}, a_{2,j}\}$. Note that, since, for each $j \in \{0, \ldots, s-1\}$, T_j is a path, we have $a_{i,j} \neq a_{i',j}$ for every $i \neq i'$. Since G is a simple graph, we have $a_{2,j+1} \notin \{a_{3,j}, a_{4,j}, a_{2,j}\}$ and $a_{0,j+1} \notin \{a_{3,j}, a_{4,j}, a_{1,j}, a_{2,j}\}$; and if $a_{2,j+1} = a_{1,j}$, then $a_{4,j+1}a_{2,j+1}$ and $a_{4,j-1}, a_{2,j-1}$ are two distinct red in-edges of $a_{1,j}$, a contradiction.

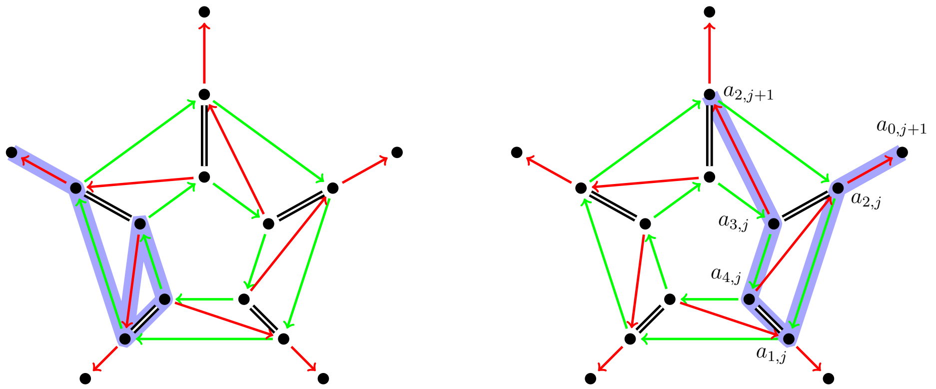


Figure 3.12: Exchange of edges between the elements an A-chain of type 1 with five elements in the proof of Claim 4.

We claim that $\mathcal{D}'$ is an admissible decomposition. Indeed, the only vertices of the elements of S that can be connection vertices of elements in $\mathcal{D}'$ are the vertices $a_{0,j}$, for $j = 0, \ldots, s-1$. But a hanging edge at a vertex $a_{0,j}$ is in $T_{j'} \in \mathcal{D}$ if and only if $a_{0,j} = a_{3,j'}$ for some $j' \neq j$, and, in this case $a_{3,j'}$ is not a connection vertex in $\mathcal{D}'$ because all edges incident to it are in elements of $\{T_j' \colon j = 0, \ldots, s-1\}$. Therefore, Definition 3(a) holds for $\mathcal{D}'$. Moreover, since T_j' is a path, for $j = 0, \ldots, s-1$, Definition 3(b) holds for $\mathcal{D}'$. Also, every edge of M in an element of S is the middle edge of an element T_j', for some $j \in \{0, \ldots, s-1\}$, and hence Definition 3(c) holds for $\mathcal{D}'$. Finally, $\mathcal{D}$ and $\mathcal{D}'$ have the same number of open chain, and hence Definition 3(d) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is an admissible decomposition of G such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - s$, a contradiction to the minimality of $\mathcal{D}$.

S is of type 2. Since S is of type 2, for each $j = 0, \dots, s-1$, we have $a_{2,j} = \mathrm{cv}_2(T_j) = \mathrm{tr}(T_{j+1}) = a_{4,j+1}$. Now, for each $j = 0, \dots, s-1$, let $T'_j = a_{0,j}a_{1,j}a_{2,j}a_{3,j}a_{4,j}a_{4,j-1}$ (see Figure 3.13). Clearly, T'_j is a trail of length 5. Note that $T'_j = T_j - a_{4,j}a_{2,j} + a_{4,j-1}a_{2,j-1}$, i.e., T'_j is the element obtained from T_j by exchanging its 5-th edge by the 5-th edge of T_{j-1}. Thus, the set $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_j \colon j = 0, \dots, s-1\}\big) \cup \{T'_j \colon j = 0, \dots, s-1\}$ is a decomposition of G into trails of length 5. We may regard $\mathcal{D}'$ as the decomposition obtained by reversing the direction of one component of F_r and applying the same strategy used in Proposition 3. In order to prove that T'_j is a path, we show that $a_{4,j-1} \notin \{a_{0,j}, a_{1,j}, a_{2,j}, a_{3,j}, a_{4,j}\}$. Note that, since, for each $j \in \{0, \dots, s-1\}$, T_j is a path, we have $a_{i,j} \neq a_{i',j}$ for every $i \neq i'$. Since G is a simple graph, we have $a_{4,j-1} \notin \{a_{2,j}, a_{3,j}, a_{4,j}\}$; also, by Lemma 2, we have $a_{4,j-1} \neq a_{0,j}$; and if $a_{4,j-1} = a_{1,j}$, then $a_{2,j}a_{1,j}$ and $a_{3,j-1}a_{4,j-1}$ are two distinct green in-edges of $a_{4,j-1}$, a contradiction.

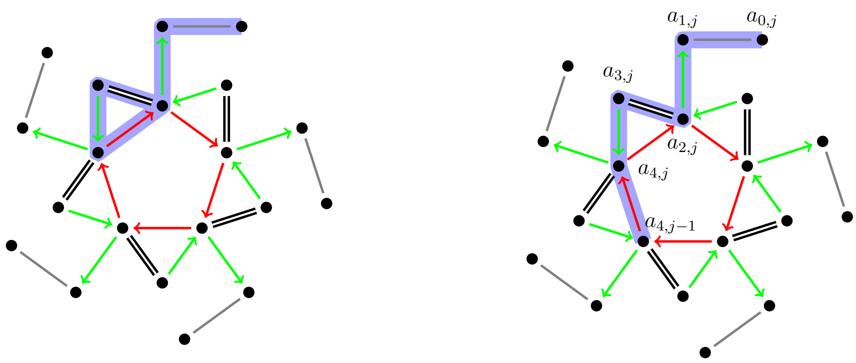


Figure 3.13: Exchange of edges between the elements an A-chain of type 2 with five elements in the proof of Claim 4.

We claim that $\mathcal{D}'$ is an admissible decomposition. Indeed, the only vertices that have hanging edges in $\mathcal{D}$ and may not have hanging edges in $\mathcal{D}'$ are the vertices $a_{3,j}$ and $a_{4,j} = a_{2,j-1}$, for $j = 0, \dots, s-1$, but these vertices are connection vertices of the elements in S, and hence can't be connection vertices of elements in $\mathcal{D}'$. Therefore, Definition 3(a) holds for $\mathcal{D}'$. Moreover, since T'_j is a path, for $j = 0, \dots, s-1$, Definition 3(b) holds for $\mathcal{D}'$. Also, every edge of M in an element of S is either the middle edge or an end edge of an element T'_j, for some $j \in \{0, \dots, s-1\}$, and hence Definition 3(c) holds for $\mathcal{D}'$. Finally, $\mathcal{D}$ and $\mathcal{D}'$ have the same number of open chain, and hence Definition 3(d) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is an admissible decomposition of G such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - s$, a contradiction to the minimality of $\mathcal{D}$. □

Claim 5. *Every A-chain contains at least four elements*

Proof. Let S be an A-chain in $\mathcal{D}$ with at most three elements. First, note that if S contains two elements, then G contains a parallel edge, a contradiction. Therefore, we may assume that S contains three elements T_1, T_2 and T_3. By Claim 4, we may assume $\mathrm{tr}(T_1) = \mathrm{cv}_2(T_3)$, $\mathrm{tr}(T_2) = \mathrm{cv}_1(T_1)$ and $\mathrm{tr}(T_3) = \mathrm{cv}_i(T_2)$, for $i \in \{1, 2\}$. In what follows, we divide the proof depending on whether $i = 1$ or $i = 2$.

Let $T_1 = a_0a_1a_2a_3a_4a_5$, $T_2 = b_0b_1b_2b_3b_4b_5$ and $T_3 = c_0c_1c_2c_3c_4c_5$ be the elements of S where $a_4 = \mathrm{tr}(T_1) = \mathrm{cv}_2(T_3) = c_2$, $b_4 = \mathrm{tr}(T_2) = \mathrm{cv}_1(T_1) = a_3$ and $c_4 = \mathrm{tr}(T_3) = \mathrm{cv}_i(T_2)$.

Case $\mathbf{i = 1}$. In this case, $c_4 = \mathrm{tr}(T_3) = \mathrm{cv}_1(T_2) = b_3$. Since $b_3 = c_4$, we have $b_2 = c_1$ and $c_0 = a_1$. Put $T_1' = a_0a_1a_2a_3a_4c_1$, $T_2' = b_0b_1b_2b_4b_3c_2$, $T_3' = c_0c_1c_4c_3c_2a_2$ and $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2, T_3\}\big) \cup \{T_1', T_2', T_3'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. We claim that T_1', T_2' and T_3' are all paths. By Lemma 3, T_2' is a path. Since G is simple, $c_1 \notin \{a_2, a_3, a_4\}$ and $a_2 \notin \{c_1, c_2, c_3, c_4\}$. By Lemma 2, we have $c_1 \neq a_0$ and $a_2 \neq c_0$. Therefore, T_3' is of type a path. Finally, if $c_1 = a_1$, then c_2c_1 and a_2a_1 are two green out-edges at a_1, a contradiction. Therefore, T_1' is a path.

To check that $\mathcal{D}'$ is an admissible decomposition first note that $\mathrm{hang}_{\mathcal{D}'}(\mathrm{cv}_i(T_j)) \geq \mathrm{hang}_{\mathcal{D}}(\mathrm{cv}_i(T_j))$ for every trail $T_j \in \mathcal{D}'$. Thus, definition 3(a) holds for $\mathcal{D}$. As seen above, the new elements are all paths. Thus, definition 3(b) holds. Since $\mathcal{D}$ is admissible and the new elements are paths, the elements of type A are still partitioned into A-chains. Thus, 3(d) holds for $\mathcal{D}'$. Also, since T_1 is an element of type A, we have $a_2 = a_3 + g + r$, and hence Definition 3(c) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is an admissible decomposition of G such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 3$, a contradiction to the minimality of $\mathcal{D}$.

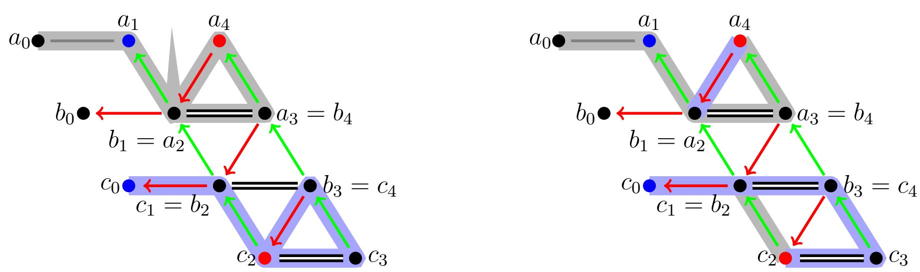


Figure 3.14: Exchange performed in the proof of Claim 5 in the case $\mathrm{tr}(T_3) = \mathrm{cv}_1(T_2)$.

Case $\mathbf{i = 2}$. In this case, $c_4 = \mathrm{tr}(T_3) = \mathrm{cv}_2(T_2) = b_2$. Put $T_1' = a_0a_1a_2a_4a_3b_5$, $T_2' = b_0b_1b_4b_3b_2c_2$, $T_3' = c_0c_1c_2c_3c_4b_1$ and let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2, T_3\}\big) \cup \{T_1', T_2', T_3'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. In what follows, we prove that T_1', T_2' and T_3' are all paths. By Lemma 3, T_1' is a path. Since G is simple, we have $c_2 \notin \{b_1, b_2, b_3, b_4\}$ and $b_1 \notin \{c_2, c_3, c_4\}$. By Lemma 2, $b_2 \neq a_0$, $c_2 \neq b_0$, $b_1 \neq c_0$. Therefore, T_2' is a path. If $b_1 = c_1$, then b_2b_1 and c_2c_1 are two green in-edges at c_1, a contradiction. Therefore, T_3' is also a path.

To check that $\mathcal{D}'$ is an admissible decomposition first note that $\mathrm{hang}_{\mathcal{D}'}(\mathrm{cv}_i(T_j)) \geq$

$\mathrm{hang}_{\mathcal{D}}(\mathrm{cv}_i(T_j))$ for every trail $T_j \in \mathcal{D}'$. Thus, definition 3(a) holds for $\mathcal{D}$. As seen above, the new elements are all paths. Thus, definition 3(b) holds. Since $\mathcal{D}$ is admissible and the new elements are paths, the elements of type A are still partitioned into A-chains. Thus, 3(d) holds for $\mathcal{D}'$. Also, since T_1 and T_3 are elements of type A, we have $a_2 = a_3 + g + r$, and hence Definition 3(c) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is an admissible decomposition of G such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 3$, a contradiction to the minimality of $\mathcal{D}$. □

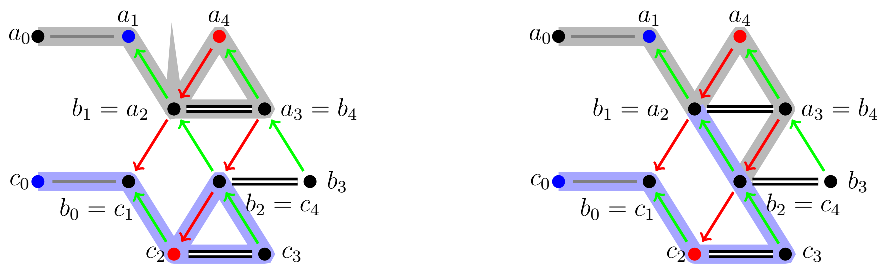


Figure 3.15: Exchange performed in the proof of Claim 5 in the case $\mathrm{tr}(T_3) = \mathrm{cv}_2(T_2)$

Claim 6. *$\mathcal{D}$ contains an open chain.*

Proof. Suppose, for a contradiction, that there is no open chain in $\mathcal{D}$. Since $\tau(\mathcal{D}) > 0$, $\mathcal{D}$ contains an A-chain $S = T_0, T_1, \ldots, T_{s-1}$. By Claim 4, S is a mixed A-chain. Then we can find three consecutive elements in S, say T_j, T_{j+1}, T_{j+2}, such that $\mathrm{cv}_2(T_j) = \mathrm{tr}(T_{j+1})$ and $\mathrm{cv}(T_{j+1}) = \mathrm{tr}(T_{j+2})$. Since S is cyclic, i.e., $\mathrm{tr}(T_0) = \mathrm{cv}_i(T_{s-1})$, for some $i \in \{1, 2\}$, we may suppose, without loss of generality, that $j = 0$. By Claim 5, there is an element $T_3 \in \mathcal{D}$ such that $\mathrm{tr}(T_3) = \mathrm{cv}_i(T_2)$, for some $i \in \{1, 2\}$. In what follows, the proof is divided according to i. Let $T_0 = a_0a_1a_2a_3a_4a_5$, $T_1 = b_0b_1b_2b_3b_4b_5$, $T_2 = c_0c_1c_2c_3c_4c_5$, and $T_3 = d_0d_1d_2d_3d_4d_5$, where $a_5 = a_2$, $b_5 = b_2$, $c_5 = c_2$, $d_5 = d_2$, and $a_2a_3, b_2b_3, c_2c_3, d_2d_3 \in M$. By the choice of T_0, T_1, and T_2, we have $b_4 = \mathrm{tr}(T_1) = \mathrm{cv}_2(T_0) = a_2$, $c_4 = \mathrm{tr}(T_2) = \mathrm{cv}_1(T_1) = b_3$. The exchanges of edges performed here are analogous to the exchanges performed on the proof of Claim 2 of Lemma 5 for elements of type A.

Case $\mathrm{tr}(\mathbf{T_3}) = \mathrm{cv}_\mathbf{1}(\mathbf{T_2})$. In this case, we have $d_4 = c_3$ and, by Lemma 4, $d_2 = c_1$. Put $T_1' = b_0b_1b_2b_4b_3c_2$, $T_2' = c_0c_1c_4c_3c_2d_0$, $T_3' = c_1d_1d_2d_3d_4d_2$ (see Figure 3.16), and let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2, T_3\}\big) \cup \{T_1', T_2', T_3'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. By Lemma 3, T_1' is an element of type C. In what follows, we prove that T_2' is a path and T_3' is an element of type A. Since G is simple, we have $d_0 \notin \{c_1, c_2, c_3, c_4\}$, and $c_1 \notin \{d_1, d_2, d_4\}$. By Lemma 2, $d_0 \neq c_0$, and hence, T_2' is a path. If $c_1 = d_3$, then b_2b_3 and d_2d_3 are two edges of M incident to c_1, a contradiction. Therefore, T_3' is an element of type A.

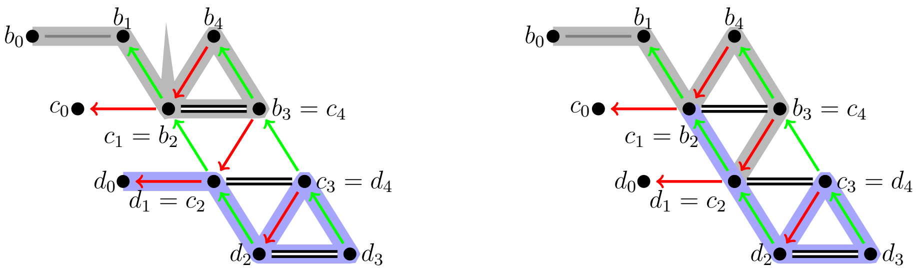


Figure 3.16: Exchange performed in the proof of Claim 6 in the case $\mathrm{tr}(T_3) = \mathrm{cv}_1(T_2)$
.

To check that $\mathcal{D}'$ is an admissible decomposition first note that $\mathrm{hang}_{\mathcal{D}'}(v) \geq \mathrm{hang}_{\mathcal{D}}(v)$ for every $v \in V(G) \setminus \{a_2\}$. Note also that $b_4 b_2$ is a hanging edge at a_2 in $\mathcal{D}$ but not in $\mathcal{D}'$. However, (T_1, T_2') is an exceptional pair. Thus, since c_3 is not a connection vertex in $\mathcal{D}'$, the element T_3' is free. Therefore, $S' = T_3', \ldots T_{s-1} T_1 T_2'$ is an open chain, and hence Definition 3(a) holds for $\mathcal{D}'$. Since, T_1' and T_2' are paths and T_3' is an element of type A, Definition 3(b) holds for $\mathcal{D}'$. Since no edge of M that is not a hanging edge in $\mathcal{D}$ is a hanging edge in $\mathcal{D}'$, Definition 3(c) holds for $\mathcal{D}'$. Finally, note that an element T of type A in $\mathcal{D} \setminus \{T_1, T_2, T_3\}$ is either in an A-chain of $\mathcal{D}$ different from S, which implies that T is in an A-chain of $\mathcal{D}'$, or is in S, which implies that T is in S'. Thus, Definition 3(d) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is an admissible decomposition of G such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 2$, a contradiction to the minimality of $\mathcal{D}$.

Case $\mathrm{tr}(\mathbf{T_3}) = \mathrm{cv}_\mathbf{2}(\mathbf{T_2})$. In this case, we have $d_4 = c_2$. Put $T_1' = b_0 b_1 b_2 b_4 b_3 c_2$, $T_2' = c_0 c_1 c_4 c_3 c_2 d_2$, $T_3' = c_1 d_4 d_3 d_2 d_1 d_0$ (see Figure 3.17), and let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_1, T_2, T_3\}\big) \cup \{T_1', T_2', T_3'\}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. By Lemma 3, T_1' is an element of type C. In what follows, we prove that T_2' and T_3' are paths. Since G is simple, we have $d_2 \notin \{c_1, c_2, c_3, c_4\}$, and $c_1 \notin \{d_2, d_3, d_4\}$. By Lemma 2, $d_2 \neq c_0$, and $c_1 \neq d_0$. Therefore, T_2' is a path. If $c_1 = d_1$, then $d_2 d_1$ and $c_2 c_1$ are two green in edges of c_1. Therefore, T_3' is a path.

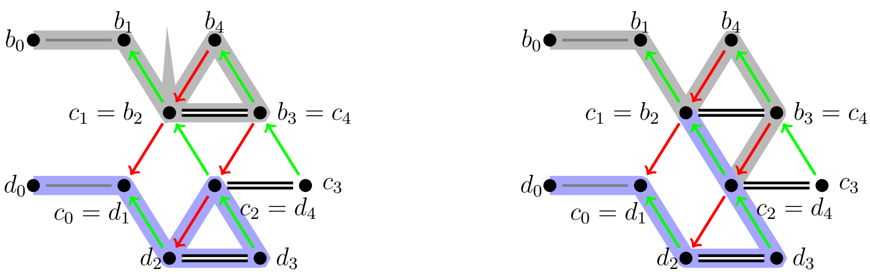


Figure 3.17: Exchange performed in the proof of Claim 6 in the case $\mathrm{tr}(T_3) = \mathrm{cv}_2(T_2)$
.

To check that $\mathcal{D}'$ is an admissible decomposition first note that $\mathrm{hang}_{\mathcal{D}'}(v) \geq \mathrm{hang}_{\mathcal{D}}(v)$ for every $v \in V(G) \setminus \{a_2\}$. Note also that $b_4 b_2$ is a hanging edge at a_2 in

$\mathcal{D}$ but not in $\mathcal{D}'$. However, (T_1, T_2') is an exceptional pair. Thus, since d_2 and d_3 are not connection vertices in $\mathcal{D}'$, the element T_4 (or T_1, if $s = 4$) is free. Therefore, $S' = T_4, \ldots T_{s-1} T_1 T_2'$ is an open chain, and hence Definition 3(a) holds for $\mathcal{D}'$. Since, T_1', T_2' and T_3' are paths, Definition 3(b) holds for $\mathcal{D}'$. Since no edge of M that is not a hanging edge in $\mathcal{D}$ is a hanging edge in $\mathcal{D}'$, Definition 3(c) holds for $\mathcal{D}'$. Finally, note that an element T of type A in $\mathcal{D} \setminus \{T_1, T_2, T_3\}$ is either in an A-chain of $\mathcal{D}$ different from S, which implies that T is in an A-chain of $\mathcal{D}'$, or is in S, which implies that T is in S'. Thus, Definition 3(d) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is an admissible decomposition of G such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 3$, a contradiction to the minimality of $\mathcal{D}$. $\square$

Let $S = T_0, T_1, \ldots, T_{s-1}$ be an open chain in $\mathcal{D}$. Let $T_j = a_{0,j} a_{1,j} a_{2,j} a_{3,j} a_{4,j} a_{5,j}$, for $j \in \{0, \ldots, s-1\}$, where $a_{2,j} a_{3,j} \in M$ and $a_{5,j} = a_{2,j}$ for $j \in \{0, \ldots, s-2\}$.

Claim 7. *T_1 is an element of type A and* $\operatorname{tr}(T_1) = \operatorname{cv}_1(T_0)$.

Proof. Suppose, for a contradiction, that T_1 is not an element of type A or $\operatorname{tr}(T_1) = \operatorname{cv}_2(T_0)$. We claim that T_1 does not contain a hanging edge at $\operatorname{cv}_1(T_0)$. Indeed, if T_1 is not an element of type A, then, by the definition of open chain, T_1 is an element of type C, and hence, by Remark 1, T_1 does not have a hanging edge at $\operatorname{cv}_1(T_0)$; and if T_1 is an element of type A for which $\operatorname{tr}(T_1) = \operatorname{cv}_2(T_0)$, then we have $a_{4,1} = \operatorname{tr}(T_1) = \operatorname{cv}_2(T_0) = a_{2,0}$, and hence, if $a_{1,1} = a_{3,0}$, then we have $a_{4,1} + r + g = a_{1,1} = a_{3,0} = a_{2,0} - r - g$, which implies that $2g + 2r = 0$, a contradiction. Therefore, T_1 does not contain a hanging edge at $\operatorname{cv}_1(T_0)$.

However, by Definition 3(a), there is a hanging edge at $\operatorname{cv}_1(T_0)$. Thus, let $T = u_0 u_1 u_2 u_3 u_4 u_5$ be an element of $\mathcal{D} \setminus \{T_0, T_1\}$ that contains a hanging edge, say $u_1 u_0$, at $\operatorname{cv}_1(T_0)$. Note that all the edges incident to $a_{2,0}$ are in $E(T_0) \cup E(T_1)$. Let $T_0' = a_{0,0} a_{1,0} a_{2,0} a_{4,0} a_{3,0} u_0$ and $T' = a_{2,0} u_1 u_2 u_3 u_4 u_5$ and put $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_0, T'\}\big) \cup \{T_0', T'\}$. By Lemma 3, T_0' is a path; and since all the edges incident to $a_{2,0}$ are in $E(T_0) \cup E(T_1)$, we have $a_{2,0} \notin \{u_1, u_2, u_3, u_4, u_5\}$, and hence if T is a path (resp. an element of type A), then T' is a path (resp. an element of type A). Thus Definition 3(b) holds for $\mathcal{D}'$.

To check that $\mathcal{D}'$ is an admissible decomposition first note that $\operatorname{hang}_{\mathcal{D}'}(v) \geq \operatorname{hang}_{\mathcal{D}}(v)$ for every $v \in V(G) \setminus \{a_{4,0}\}$. Thus, since T_0 is a free element, $a_{4,0}$ is not a connection vertex in $\mathcal{D}$, and hence $a_{4,0}$ is not a connection vertex in $\mathcal{D}'$. Note also that T_1 is either an element of type C or a free element of type A, and hence $S' = T_1, \ldots, T_{s-1}$ is an open chain. Thus, Definition 3(a) holds for $\mathcal{D}'$. Also, since T_0 is an element of type A, we have $a_{2,0} = a_{3,0} + g + r$, and hence Definition 3(c) holds for $\mathcal{D}'$. Finally, note that an element T of type A in $\mathcal{D} \setminus \{T_0\}$ is either in an A-chain of $\mathcal{D}$ different from S, which implies that T is in an A-chain of $\mathcal{D}'$, or is in S, which implies that T is in S'. Thus, Definition 3(d) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$

is an admissible decomposition of G such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 1$, a contradiction to the minimality of $\mathcal{D}$. $\square$

By Claim 7, we have $s \geq 3$, and hence, there is an element T_2 in S. Note that, by Lemma 4, since $\operatorname{tr}(T_1) = \operatorname{cv}_1(T_0)$, we have $\operatorname{aux}(T_1) = \operatorname{cv}_2(T_0)$. This implies that $a_{1,1}a_{0,1} \in F_r$ because all the edges incident to $a_{1,1} = \operatorname{cv}_2(T_0)$ are in $E(T_0) \cup E(T_1)$.

Claim 8. *T_2 is of type A.*

Proof. Suppose, for a contradiction, that T_2 is not of type A, then T_2 is an element of type C and (T_1, T_2) is an exceptional pair. Thus, we can write $T_2 = a_{0,2}a_{1,2}a_{2,2}a_{3,2}a_{4,2}a_{5,2}$ such that $a_{2,1} = a_{5,1} = a_{3,2}$, and $a_{2,2}a_{1,2}, a_{4,2}a_{3,2} \in F_g$, $a_{3,2}a_{2,2}, a_{4,2}a_{5,2} \in F_r$, $a_{1,2}a_{0,2} \in M_{g,r} \cup F_g \cup F_r$. We claim that $a_{0,1} = a_{1,2}$. Indeed, since $a_{1,1}a_{0,1} \in F_r$, we have $a_{0,1} = a_{1,1} + r = a_{2,1} + g + r$, but by the definition of type C, we have $a_{1,2} = a_{2,2} + g = a_{3,2} + r + g$. Thus, since $a_{3,2} = a_{2,1}$, we obtain $a_{0,1} = a_{1,2}$. Now, put $T_0' = a_{0,0}a_{1,0}a_{2,0}a_{4,0}a_{3,0}a_{2,1}$, $T_1' = a_{1,1}a_{4,1}a_{3,1}a_{2,1}a_{2,2}a_{0,1}$, and $T_2' = a_{0,2}a_{1,2}a_{1,1}a_{3,2}a_{4,2}a_{5,2}$, and put $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_0, T_1, T_2\}\big) \cup \{T_0', T_1', T_2'\}$. By Lemma 3, T_0' is a path; since G is a simple graph, $a_{2,2} \notin \{a_{1,1}, a_{4,1}, a_{3,1}, a_{2,1}, a_{2,2}, a_{0,1}\}$, and hence T_1' is a path; and since all edges incident to $a_{1,1}$ are in $E(T_0) \cup E(T_1)$, we have $a_{1,1} \notin V(T_2)$, which implies that T_2' is a path. Thus Definition 3(b) holds for $\mathcal{D}'$.

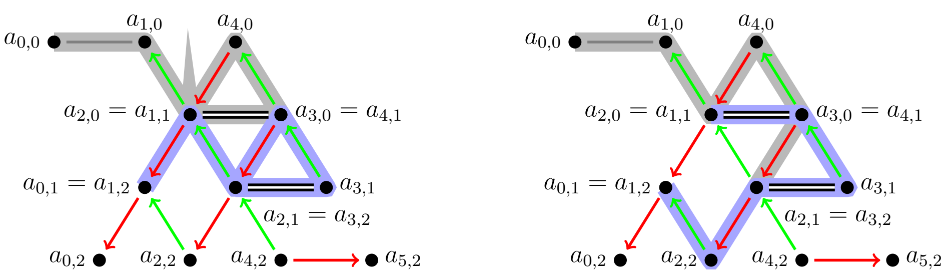


Figure 3.18: Exchange of edges between two elements of type A and another element of type E in the proof of Claim 8.

To check that $\mathcal{D}'$ is an admissible decomposition first note that $\operatorname{hang}_{\mathcal{D}'}(v) \geq \operatorname{hang}_{\mathcal{D}}(v)$ for every $v \in V(G) \setminus \{a_{4,0}, a_{1,1}, a_{3,1}\}$. Thus, since T_0 is a free element, $a_{4,0}$ is not a connection vertex in $\mathcal{D}$, and hence $a_{4,0}$ is not a connection vertex in $\mathcal{D}'$; and since the edges of M incident to $a_{1,1}$ and $a_{3,1}$ are in T_1', $a_{1,1}$ and $a_{3,1}$ are not connection vertices in $\mathcal{D}'$. Note also that no element of S is in $\mathcal{D}'$, and hence there are no open chains in $\mathcal{D}'$. Thus, Definitions 3(a) and 3(d) hold for $\mathcal{D}'$. Also, since T_0 is an element of type A, we have $a_{2,0} = a_{3,0} + g + r$, and hence Definition 3(c) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is an admissible decomposition of G such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 2$, a contradiction to the minimality of $\mathcal{D}$. $\square$

Now, by Claim 8, we have $s \geq 4$. In what follows, we divide the proof depending on whether $\mathrm{tr}(T_2) = \mathrm{cv}_1(T_1)$ or $\mathrm{tr}(T_2) = \mathrm{cv}_2(T_1)$. This proof is analogous to the proof of Claim 6.

Case $\mathbf{tr(T_2) = cv_1(T_1)}$. By Lemma 4, we have $a_{1,2} = \mathrm{aux}(T_2) = \mathrm{cv}_2(T_1) = a_{2,1}$. Put $T_0' = a_{0,0}a_{1,0}a_{2,0}a_{4,0}a_{3,0}a_{2,1}$, $T_1' = a_{0,1}a_{1,1}a_{4,1}a_{3,1}a_{2,1}a_{0,2}$, $T_2' = a_{1,1}a_{1,2}a_{2,2}a_{3,2}a_{4,2}a_{2,2}$ (see Figure 3.19), let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_0, T_1, T_2\}\big) \cup \{T_0', T_1', T_2'\}$, and let $S' = T_2', T_3, \ldots, T_{s-1}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. By Lemma 3, T_0' is an element of type C. In what follows, we prove that T_1' is a path and T_2' is an element of type A. Since G is simple, we have $a_{0,2} \notin \{a_{1,1}, a_{2,1}, a_{3,1}, a_{4,1}\}$, and $a_{1,1} \notin \{a_{1,2}, a_{2,2}, a_{4,2}\}$. By Lemma 2, $a_{0,2} \neq a_{0,1}$, and hence, T_1' is a path. If $a_{1,1} = a_{3,2}$, then $a_{2,0}a_{3,0}$ and $a_{2,2}a_{3,2}$ are two edges of M incident to $a_{1,1}$, a contradiction. Therefore, T_2' is an element of type A.

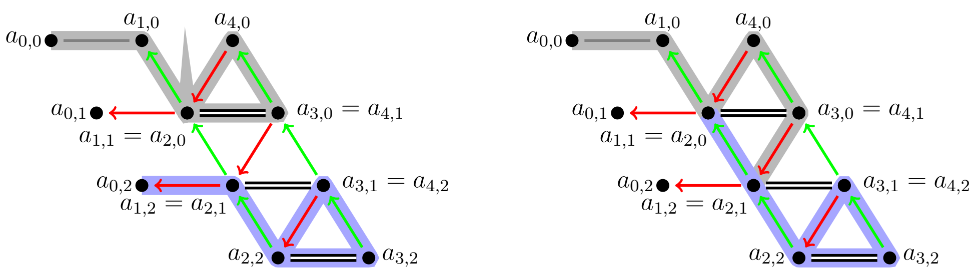


Figure 3.19: Exchange performed in the proof of Claim 6 in the case $\mathrm{tr}(T_3) = \mathrm{cv}_1(T_2)$
.

To check that $\mathcal{D}'$ is an admissible decomposition first note that $\mathrm{hang}_{\mathcal{D}'}(v) \geq \mathrm{hang}_{\mathcal{D}}(v)$ for every $v \in V(G) \setminus \{a_{4,0}\}$, but since T_0 is free, $a_{4,0}$ is not a connection vertex in $\mathcal{D}$, and hence $a_{4,0}$ is not a connection vertex in $\mathcal{D}'$. Thus, since $a_{3,1}$ is not a connection vertex in $\mathcal{D}'$, the element T_2' is free. Therefore, S' is an open chain, and hence Definition 3(a) holds for $\mathcal{D}'$. Since, T_0' and T_1' are paths and T_2' is an element of type A, Definition 3(b) holds for $\mathcal{D}'$. Since no edge of M that is not a hanging edge in $\mathcal{D}$ is a hanging edge in $\mathcal{D}'$, Definition 3(c) holds for $\mathcal{D}'$. Finally, note that an element T of type A in $\mathcal{D} \setminus \{T_0, T_1, T_2\}$ is either in an A-chain of $\mathcal{D}$ different from S, which implies that T is in an A-chain of $\mathcal{D}'$, or is in S, which implies that T is in S'. Thus, Definition 3(d) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is an admissible decomposition of G such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 2$, a contradiction to the minimality of $\mathcal{D}$.

Case $\mathbf{tr(T_3) = cv_2(T_2)}$. Put $T_0' = a_{0,0}a_{1,0}a_{2,0}a_{4,0}a_{3,0}a_{2,1}$, $T_1' = a_{0,1}a_{1,1}a_{4,1}a_{3,1}a_{2,1}a_{2,2}$, $T_2' = a_{0,2}a_{1,2}a_{2,2}a_{3,2}a_{4,2}a_{1,1}$ (see Figure 3.20), let $\mathcal{D}' = \big(\mathcal{D} \setminus \{T_0, T_1, T_2\}\big) \cup \{T_0', T_1', T_2'\}$, and let $S' = T_3, \ldots, T_{s-1}$. Note that $\mathcal{D}'$ is a decomposition of G into trails of length 5. By Lemma 3, T_1' is an element of type C. In what follows, we prove that T_1' and T_2' are paths. Since G is simple, we have $a_{2,2} \notin \{a_{1,1}, a_{2,1}, a_{3,1}, a_{4,1}\}$, and $a_{1,1} \notin \{a_{2,2}, a_{3,2}, a_{4,2}\}$. By Lemma 2, $a_{2,2} \neq a_{0,1}$, and $a_{1,1} \neq a_{0,2}$. Therefore, T_2'

is a path. If $a_{1,1} = a_{1,2}$, then $a_{2,2}a_{1,2}$ and $a_{2,1}a_{1,1}$ are two green in edges of $a_{1,1}$. Therefore, T_2' is a path.

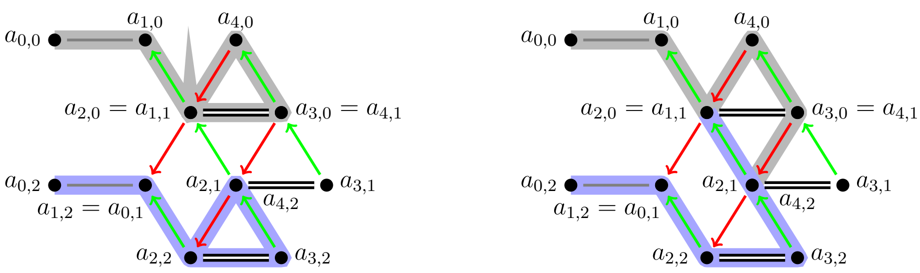


Figure 3.20: Exchange performed in the proof of Claim 6 in the case $\mathrm{tr}(T_3) = \mathrm{cv}_2(T_2)$
.

To check that $\mathcal{D}'$ is an admissible decomposition first note that $\mathrm{hang}_{\mathcal{D}'}(v) \geq \mathrm{hang}_{\mathcal{D}}(v)$ for every $v \in V(G) \setminus \{a_{4,0}\}$, but since T_0 is free, $a_{4,0}$ is not a connection vertex in $\mathcal{D}$, and hence $a_{4,0}$ is not a connection vertex in $\mathcal{D}'$. Thus, since $a_{2,1}$ is not a connection vertex in $\mathcal{D}'$, the element T_2' is free. Therefore, S' is either an open chain or contains only one element of type C, and hence Definition 3(a) holds for $\mathcal{D}'$. Since, T_0', T_1' and T_2' are paths, Definition 3(b) holds for $\mathcal{D}'$. Since no edge of M that is not a hanging edge in $\mathcal{D}$ is a hanging edge in $\mathcal{D}'$, Definition 3(c) holds for $\mathcal{D}'$. Finally, note that an element T of type A in $\mathcal{D} \setminus \{T_0, T_1, T_2\}$ is either in an A-chain of $\mathcal{D}$ different from S, which implies that T is in an A-chain of $\mathcal{D}'$, or is in S, which implies that T is in S'. Thus, Definition 3(d) holds for $\mathcal{D}'$. Therefore, $\mathcal{D}'$ is an admissible decomposition of G such that $\tau(\mathcal{D}') = \tau(\mathcal{D}) - 3$, a contradiction to the minimality of $\mathcal{D}$. This concludes the proof. $\square$

Recall that a $\{g, r\}$-graph G is a 5-regular graph that contains the Cayley graph $X(\Gamma, S)$, where $S = \{g, -g, r, -r\}$. Thus, since S is closed under taking inverses, G is also a $\{g, -r\}$-, $\{-g, r\}$-, $\{-g, -r\}$-graph, which yields the following corollary of Theorem 3.

Corollary 4. *Every $\{g, r\}$-graph for which $2g + 2r \neq 0$ or $2g - 2r \neq 0$ admits a P_5-decomposition.*

Finally, to show that every $\{g, r\}$-graph admits a P_5-decomposition, we consider the case when $2g + 2r = 0$ and $2g - 2r = 0$. As a corollary of Proposition 1 and Lemma 1 we obtain the following result.

Theorem 4. *Every $\{g, r\}$-graph such that $2g + 2r = 0$ and $2g - 2r = 0$ admits an $M_{g,r}$-centered decomposition.*

Proof. Let G be a $\{g, r\}$-graph for which $2g + 2r = 0$ and $2g - 2r = 0$ and put $M = M_{g,r}$. Note that we also have $4g = 4r = 0$. Let u be a vertex of G, and let H be the component of $G \setminus E(M)$ that contains u. In what follows, we prove that H is a copy of $K_{4,4}$. By the commutative property of SCG, if $v \in V(H)$, we have $v = u+ig+jr$, where $i, j \in \mathbb{N}$. Since $4g = 4r = 0$, we may assume $i, j \in \{0, 1, 2, 3\}$. Moreover, since $2g - 2r = 0$ (and hence $2g = 2r$), we may assume $j \in \{0, 1\}$. Therefore, there are eight vertices in H, namely, $V(H) = \{u, u+g, u+2g, u+3g, u+r, u+2g+r, u+3g+r\}$. We claim that H is bipartite. Indeed, suppose that there is an odd cycle C in H. Then, there is there is an element $x \in V(C)$ such that $x+ig+jr = x$, where $i, j \in \mathbb{N}$. Note that $i+j$ can be obtained from the length of C by ignoring pairs of edges with the same color and different directions. Since C is odd, one between i and j is odd. Suppose, without loss of generality, that i is odd, and hence j is even. Note that, since $2g = 2r$, we have $jr = jg$. Therefore, $(i + j)g = ig + jr = 0$. Let $s \in \{1, 3\}$ be such that $i + j = 4q + s$ for some $q \in \mathbb{N}$. Then we have $0 = (i + j)g = 4qg + sg$, which implies $sg = 0$. Thus, if $s = 1$, then $g = 0$; and if $s = 3$, then $g = 4g - sg = 0$, a contradiction to the definition of SCG. Thus, since H is 4-regular, H is a copy of $K_{4,4}$. Now, since every component of $G \setminus E(M)$ is isomorphic to $K_{4,4}$, by using induction on the number of components of $G \setminus E(M)$, one can prove that G is an M-constructable graph, and hence by Corollary 3, G admits an M-centered decomposition as desired. □

The main result of this dissertation is a straightforward consequence of Corollary 4 and Theorem 4.

Theorem 5. *Every $\{g, r\}$-graph admits a P_5-decomposition.*

Chapter 4

Conclusion and future works

In this dissertation, we verified Conjecture 3 for (i) $(2k+1)$-regular graphs containing a spanning $2k$-regular power of cycle, and (ii) 5-regular graphs containing a spanning 4-regular Cayley graph. We believe that the techniques developed here can be extended for a more general class of graphs, such as *Schreier Coset Graph*. We mention that Schreier coset graphs is a useful tool for deciding whether certain groups are infinite, and also proving theorems about groups generated by permutations. (See [5]).

Let G be a group and let H be a subgroup of G. For some $s \in G$, we define the *right coset* of H corresponding to s as the set $Hs = \{hs \colon h \in H\}$. *Left coset* can be defined analogously. Let $g_1, \ldots, g_r$ be a sequence in G whose members generate G, the *Schreier Right Coset Graph* (SRCG) is defined as follows. Its vertex set is the set of right cosets of H in G, for each coset H_i and each generator g_i there is an edge from H_i to the right coset $H_i g_i$. Note that a Cayley Graph is an SRCG where $H = \{id\}$ i.e., SRCG is a generalization of a Cayley "color" graph using cosets of some specified subgroup as vertices instead of group elements. In 1977, Gross [12] showed that every connected regular graph of even degree is an SRCG. This implies that, if we extend our result for 5-regular graphs that contain any SRCG, we verify the conjecture for all 5-regular graphs containing a perfect matching.

Finally, we can also explore others graphs containing special spanning Cayley graphs. For instance, a natural step is to examine 7-regular graphs containing a spanning 6-regular Cayley graph or a spanning 4-regular Cayley graph. Also, note that the definitions of simple commutative generator and $\{g, r\}$-graph are equivalent to Cayley graphs under the restriction of the equation $g + r = r + g$ for every pair of generators. Therefore, we plan to explore Cayley graphs generated by others equations, such as $g + r \neq r + g$, which would extend our result for 5-regular graphs containing every spanning 4-regular Cayley graph.

References

[1] Fábio Botler, Guilherme Oliveira Mota, Marcio Takashi Iura Oshiro, and Yoshiko Wakabayashi. Decomposing regular graphs with prescribed girth into paths of given length. *European Journal of Combinatorics*, 66:28–36, 2017.

[2] Fábio Botler, Guilherme Oliveira Mota, and Yoshiko Wakabayashi. Decompositions of triangle-free 5-regular graphs into paths of length five. *Discrete Mathematics*, 338(11):1845–1855, 2015.

[3] André Bouchet and Jean-Luc Fouquet. Trois types de décompositions d'un graphe en chaînes. In *North-Holland Mathematics Studies*, volume 75, pages 131–141. Elsevier, 1983.

[4] Arthur Cayley. The theory of groups: graphical representation. *American Journal of Mathematics*, 2:174–176, 1878.

[5] Marston Conder. Schreier coset graphs and their applications. *Research Institute for Mathematical Sciences*, pages 169–175, 1992.

[6] Michelle Edwards and Lea Howard. A survey of graceful trees. *Atl. Electron. J. Math.*, 1(1):5–30, 2006.

[7] Joshua Erde. Decomposing the cube into paths. *Discrete Math.*, 336:41–45, 2014.

[8] Odile Favaron, François Genest, and Mekkia Kouider. Regular path decompositions of odd regular graphs. *Journal of Graph Theory*, 63(2):114–128, 2010.

[9] John Frederick Fink. On the decomposition of n-cubes into isomorphic trees. *J. Graph Theory*, 14(4):405–411, 1990.

[10] Chris Godsil and Gordon F Royle. *Algebraic graph theory*, volume 207. Springer Science & Business Media, 2013.

[11] Solomon W. Golomb. How to number a graph. In *Graph theory and computing*, pages 23–37. Academic Press, New York, 1972.

[12] Jonathan L Gross. Every connected regular graph of even degree is a schreier coset graph. *Journal of Combinatorial Theory, Series B*, 22(3):227–232, 1977.

[13] Roland Häggkvist. Decompositions of complete bipartite graphs. *Surveys in combinatorics*, 141:115–147, 1989.

[14] Michael S. Jacobson, Mirosław Truszczyński, and Zsolt Tuza. Decompositions of regular bipartite graphs. *Discrete Math.*, 89(1):17–27, 1991.

[15] Anton Kotzig. Aus der theorie der endlichen regulären graphen dritten und vierten grades. *Casopis Pest. Mat*, 82:76–92, 1957.

[16] Mekkia Kouider and Zbigniew Lonc. Path decompositions and perfect path double covers. *Australasian Journal of Combinatorics*, 19:261–274, 1999.

[17] Julius Petersen. Die theorie der regulären graphs. *Acta Mathematica*, 15(1):193–220, 1891.

[18] Alexander Rosa. On certain valuations of the vertices of a graph, theory of graphs (internat. symposium, rome, july 1966), 1967.