TESSELLATIONS ON GRAPHS: THEORY, ALGORITHMS, AND COMPLEXITY

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TESSELAÇÕES DE GRAFOS: TEORIA, ALGORITMOS, E COMPLEXIDADE

Alexandre Santiago de Abreu

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Orientadores: Franklin de Lima Marquezino
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Programa: Engenharia de Sistemas e Computação

Devido ao avanço tecnológico recente, a computação quântica tem ganhado notoriedade. Neste paradigma, o conceito de passeio quântico é fundamental para o desenvolvimento de algoritmos para computadores quânticos. Dentre os modelos de passeios quânticos existentes, destaca-se o modelo escalonado, proposto por Portugal et al., que inclui o modelo de Szegedy como um caso particular, além de parte do modelo com moeda. O modelo escalonado utiliza o conceito de tesselações em grafos para gerar os operadores quânticos de evolução que regem os movimentos do caminhante sobre o grafo. Tesselações em grafos possuem ainda um interessante valor para teoria de grafos visto que o parâmetro *tessellation cover number* $T(G)$ se relaciona com diversos outros parâmetros presentes na literatura tais como *chromatic number*, *chromatic index*, *total chromatic number*, *independent set number*, e *clique number*. Além disso, os problemas que se relacionam com $T(G)$, tais como t-TESSELLABILITY, GOOD TESELLABLE RECOGNITION, e TOTAL GOOD TESSELLABLE RECOGNITION têm relações profundas com problemas clássicos em teoria dos grafos que envolvem coloração de grafos. Neste trabalho apresentamos resultados em teoria de grafos para os problemas relacionados com tesselações em grafos citados acima, tais como complexidade computacional destes problemas para várias classes de grafos, limites inferior e superior para $T(G)$, e o valor de $T(G)$ para diversas classes de grafos. Além disso, apresentamos um modelo de passeio quântico baseado em *total tessellation cover*, sendo este pioneiro no uso de vértices e arestas como localidades possíveis para o caminhante, simultaneamente.

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Due to recent technological advances, quantum computing has gained notoriety. In this paradigm, the concept of quantum walk is fundamental for the development of algorithms for quantum computers. Among the existing quantum walk models, the staggered model, proposed by Portugal et al. stands out, since it includes the Szegedy model as a particular case, and part of the coined model. The staggered model uses the concept of tessellations on graphs to generate the quantum evolution operators that perform the walker's movements on the graph. Tessellations on graphs also have an interesting value for graph theory, since the parameter *tessellation cover number* $T(G)$ is related to several other parameters present in the literature such as *chromatic number*, *chromatic index*, *total chromatic number*, *independent set number*, and *clique number*. In addition, problems related to $T(G)$, such as t-TESSELLABILITY, GOOD TESELLABLE RECOGNITION, and TOTAL GOOD TESSELLABLE RECOGNITION have deep relations with classic problems in graph theory involving graph coloring. In this work we present results in graph theory for the problems related to tessellations on graphs mentioned above, such as computational complexity, lower and upper bounds for $T(G)$, and the value of $T(G)$ for several graph classes. Furthermore, we present a quantum walk model based on *total tessellation cover*, being pioneer in the use of vertices and edges as possible locations for the walker, simultaneously.

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Chapter 1

Introduction

The concept of *random walks* is the mathematical modeling of a particle, i.e., a walker, walking over a graph through successive random steps. This concept has been important in the development of algorithms to solve problems in computation, physics, psychology [42], and biology [20]. In random walks we have a stochastic matrix operator that is applied in each step of the walker, and acts over the state that represents the walker position over the graph. We can see this operator as a coin, i.e., its action is like a coin toss whose result indicates the direction of the next step of the walker.

Nowadays, *quantum computation* has gained notoriety. Feymanm [30] was pioneer in showing that quantum systems could be used to perform computational tasks more efficiently than the usual computational paradigm. Indeed, before the existence of the first quantum hardware, quantum algorithms more efficient than their classical counterparts were developed. Grover [33] algorithm can find a desired element in a list with N elements in $O(\sqrt{N})$ steps, while its classical counterpart would need $O(N)$ steps to find the same element. Shor [56] algorithm can be used to solve the prime number factoring problem in polynomial time, while in classical computation, this problem is still considered hard to be solved.

In quantum computation there is the concept of *quantum walks* [13], which is similar to the concept of random walks, however, the first one must obey the quantum mechanics postulates, which define how the physical system must be, the evolution of the quantum system, the composition of quantum states, and the measurement of a quantum state to obtain the output of the quantum algorithms. This concept is also important in the development of quantum algorithms [12, 15–19, 23, 29, 35, 39, 43, 53], and in the development of tools to simulate quantum systems and algorithms [4, 26, 36, 40].

There are some important quantum walk models. The first one is the *coined model* that is similar to the previous description of random walks. In this model, the quantum operator is composed by the coin operator, which acts like a coin, and the space operator, that will "shift" the walker according to the result of the coined operator. We also have the *Szegedy model* [57], which performs a quantum walk without a coin operator, and differently of other quantum walk models, this one uses the edges of the graph as locations to the walker. In the middle of last decade, a more general quantum walk model, known as *staggered model* [47–49], was proposed by Portugal et al., generalizing the Szegedy one as a particular case.

The staggered model is defined by an evolution operator that is described by a product of local unitary matrices obtained from a *graph tessellation cover*. A *tessellation* is defined as a partition of the vertices of a graph into vertex disjoint cliques. A tessellation cover is defined as a set of tessellations whose union of them covers the edge set of the graph. The staggered model requires that each edge of the graph is covered at least once, since an uncovered edge would play no role in the quantum walk. To understand the possibilities of the staggered model, it is important to introduce the t-TESSELLABILITY problem, which aims to decide whether a given graph can be covered by t tessellations. This problem is interesting since it is related to important problems in the literature, such as k-COLORABILITY. Moreover, several graph classes have the *tessellation cover number* $T(G)$, which is the size of a smallest tessellation cover of a graph G, well related with important parameters in graph theory, such as maximum degree $\Delta(G)$, chromatic number $\chi(G)$, chromatic index $\chi'(G)$, total chromatic number $\chi_T(G)$, and independent set number $\alpha(G)$.

1.1 Related works

The problem of tessellations on graphs was firstly studied independently by Duchet [27] in 1979 as EQUIVALENCE COVERING. Later, in 1986 Alon [14] proved that the equivalence number, denoted by $eq(G)$ is lower bounded by $\log_2(\frac{|V(G)|}{d})$, where $|V(G)|$ is the number of vertices of graph G and d is the maximum degree of the complement graph of G, denoted by G^c. Following, in 1995 Blokhuis et al. [21] showed that deciding that $eq(G) \leq k$, for an integer k is $\mathcal{NP}$-complete for split graphs. At the beginning of the last decade, in 2010, Esperet et al. [28] improved the current best bounds to the equivalence number of a line graph.

With the appearance of staggered model, this theme came across again as tessellation on graphs, where $eq(G)$ and $T(G)$ are actually the same parameter. In this context, I brought in my Master thesis [3] the first results of last decade with respect of graph theory, in 2017. In that work it was established that $T(G) \leq \chi(K(G))$, where $K(G)$ is the clique graph of graph G, I proved that for *w-wheel* graphs $T(G) = \lceil \frac{w}{2} \rceil$, and for *windmill graphs* $T(G) = \chi(K(G))$. I also revisited an important quantum algorithm for element distinctness proposed by Ambainis [15] and

I presented it in staggered model. The complexity of $T(G)$ was considered by Posner et al. [50], who showed that k-TESSELLABILITY is $\mathcal{NP}$-complete for line graphs of triangle-free graphs.

1.1.1 Contributions

Recently, Abreu et al. [9] improved the upper bound established in [3] by showing that $T(G) \leq \min\{\chi'(G), \chi(K(G))\}$. Furthermore, we showed the value for $T(G)$ for several graph classes, and showed hardness results for the t-TESSELLABILITY problem for several graph classes. These results are presented in Chapter 2, Section 2.1, and they can be seen in more details in Appendix A. Abreu et al. [6] introduced the problems of k-STAR SIZE and GOOD TESSELLABLE GRAPH RECOGNITION, and related them with the t-TESSELLABILITY problem by presenting graph classes with certain behaviors. The problem definitions and the results are presented in Chapter 2, Section 2.2, and they can be seen in more details in Appendix B. Abreu et al. [8] related $T(G)$ with total chromatic number, denoted by $\chi_T(G)$. We presented a contrast between hardness results for the problems of k-EDGE-COLORABILITY, k-TOTAL-COLORABILITY, t-TESSELLABILITY, and a new problem introduced in that work, called by t-TOTAL-COLORABILITY. Moreover, we presented the *total staggered quantum walk model*. The problem definitions and the results are presented in Chapter 2, Section 2.3, and they can be seen in more details in Appendix C. Chapter 3 discusses works in progress corresponding to Appendices D and E presented at conferences [2, 7].

Chapter 2

Tessellations on graphs: Related problems and applications

Along this chapter we introduce the main results corresponding to three full papers, each one in one section. Section 2.1 introduces the results of the paper attached in Appendix A, which contains the first results about tessellations on graphs after the staggered quantum walk model was proposed by Portugal et al. [49] in 2015. Section 2.2 introduces the results of the paper attached in Appendix B, which contains new problems related to tessellations on graphs. Section 2.3 introduces the results of the paper attached in Appendix C, which contains a new problem related to tessellations on graphs, and a new quantum walk model based on this new problem and the related parameter.

2.1 The graph tessellation cover number: chromatic bounds, efficient algorithms and hardness

An extended abstract containing the following results was presented in LATIN 2018, The 13th Latin American Theoretical Informatics Symposium [5], and then, the full paper was published in Theoretical Computer Science C, "Natural Computing", TCS-C [9], in 2020, which is attached in Appendix A.

We start with two main definitions.

Definition 2.1 *[9]* A *tessellation* $\mathcal{T}$ is a partition of the vertices of a graph into cliques, called *tiles*. An edge *belongs* to the tessellation $\mathcal{T}$ if and only if its endpoints belong to the same clique in $\mathcal{T}$. The set of edges belonging to $\mathcal{T}$ is denoted by $\mathcal{E}(\mathcal{T})$.

Definition 2.2 *[9]* Given a graph G with edge set $E(G)$, a *tessellation cover* of size t of G is a set of t tessellations $\mathcal{T}_1, ..., \mathcal{T}_t$, whose union $\cup_{i=1}^{t} \mathcal{E}(\mathcal{T}_i) = E(G)$. A graph G is called *t-tessellable* if there is a tessellation cover of size at most t. The

t-TESSELLABILITY PROBLEM aims to decide whether a graph G is t-tessellable. The *tessellation cover number* $T(G)$ is the size of a smallest tessellation cover of G.

We improve the previous upper bound for $T(G)$ presented in [1, 3] by showing that $T(G)$ is at most the minimum between the chromatic index $\chi'(G)$ of the graph and the chromatic number of its clique graph $\chi(K(G))$.

Theorem 2.3 *[9] If G is a graph, then $T(G) \leq \min\{\chi'(G), \chi(K(G))\}$.*

We conclude that if G is a triangle-free graph, then $T(G) = \chi'(G) = \chi(K(G)) = \chi(L(G))$ since an edge-coloring induces a tessellation cover, where $L(G)$ is the line graph of G. Moreover, t-TESSELLABILITY is polynomial-time solvable for bipartite graphs and for $\{\text{triangle}, \text{proper major}\}$-free graphs, and there are also polynomial-time algorithms to obtain a minimum tessellation cover for these graph classes [31, 58]. In contrast, from [34], we conclude that t-TESSELLABILITY of triangle-free graphs for $t \geq 3$ is $\mathcal{NP}$-complete. Particularly, t-TESSELLABILITY for $t \geq 3$ is $\mathcal{NP}$-complete even for unichord-free graphs with girth at least 15, for $\Delta \geq 3$ [38], which are triangle-free.

We are able to present graph classes for which (i) $T(G) = \chi'(G)$ with $\chi(K(G))$ arbitrarily large [3], (ii) $T(G) = \chi(K(G))$ with $\chi'(G)$ arbitrarily large, in Theorem 2.4 (example depicted in Figure 2.1), and (iii) $T(G) = 3$ with both upper bounds arbitrarily large, in Theorem 2.5 (example depicted in Figure 2.2).

Theorem 2.4 *[9] Let G_p be a star-octahedral graph. Then:*

1. *$T(G_p) = \Delta(G_p) = \chi'(G_p) = \chi(K(G_p)) = 2p$, for $p \in \{2, 3\}$, and;*
2. *$T(G_p) = \Delta(G_p) = \chi'(G_p) = 2p$ and $\chi(K(G_p)) = 2^{p-1} + 1$, for $p \geq 4$.*

Theorem 2.5 *[9] Let $E_{3,p}$ be a $(3, p)$-extended wheel graph. Then, $T(E_{3,p}) = 3$ for $p \geq 2$.*

The t-TESSELLABILITY problem aims to decide whether there is a tessellation cover of the graph with t tessellations. We found that t-TESSELLABILITY is polynomial-time solvable for bipartite, $\{\text{triangle, proper major}\}$-free, threshold, and diamond-free K-perfect graphs, and it is $\mathcal{NP}$-complete for triangle-free for $t \geq 3$, unichord-free for $t \geq 3$, planar for $t = 3$, biplanar for $t \geq 3$, chordal $(2, 1)$-graphs for $t \geq 4$, $(1, 2)$-graphs for $t \geq 4$, and diamond-free with diameter at most five for $t = 3$. We improved the complexity of 2-TESSELLABILITY problem to linear time.

A graph G is called *extremal* if $T(G) = \min\{\chi'(G), \chi(K(G))\}$. Next, we define a property of the cliques on a tessellation called exposed maximal clique.

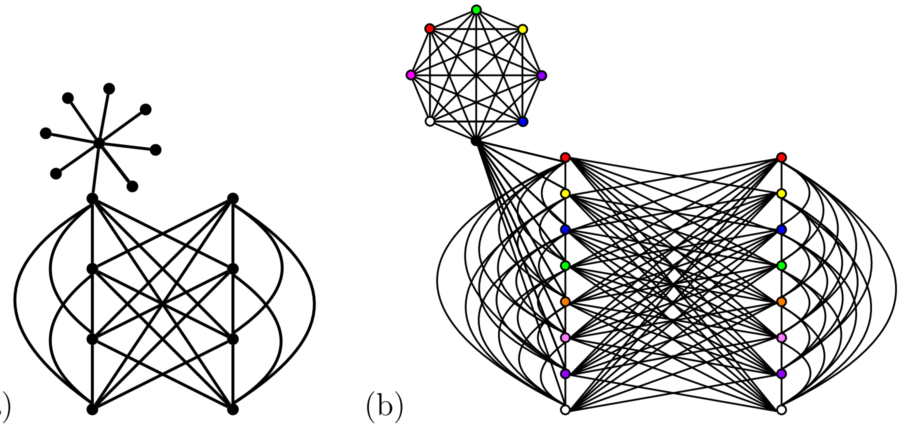


Figure 2.1: (a) The star-octahedral graph G_4, i.e., the coalescence between the octahedral graph O_4 and the star graph S_8. (b) The clique graph $K(G_4)$. Notice that $T(G_4) = \chi'(G_4) = 8$, while $\chi(K(G_4)) = 9$.

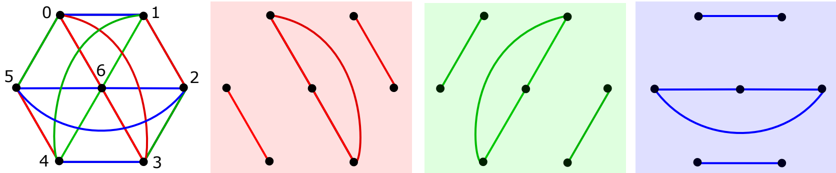


Figure 2.2: An example of the $(3, 2)$-extended wheel graph. Notice that the tessellations applied in this graph are $\mathcal{T}_1 = \{\{0, 3, 6\}, \{1, 2\}, \{4, 5\}\}, \mathcal{T}_2 = \{\{1, 4, 6\}, \{2, 3\}, \{5, 0\}\}$, and $\mathcal{T}_3 = \{\{2, 5, 6\}, \{3, 4\}, \{0, 1\}\}$.

Definition 2.6 *[9]* A maximal clique K of a graph G is said to be *exposed* by a tessellation cover $\mathcal{C}$ if $E(K) \not\subseteq \mathcal{E}(\mathcal{T})$ for all $\mathcal{T} \in \mathcal{C}$, that is, the edges of K are not covered by any single tessellation of $\mathcal{C}$.

Lemma 2.7 *[9] A graph G admits a minimum tessellation cover with no exposed maximal cliques if and only if $T(G) = \chi(K(G))$.*

Now, we consider diamond-free graphs whose clique-graphs are diamond-free, and any two maximal cliques intersect in at most one vertex.

Theorem 2.8 *[9] If G is a diamond-free graph with $\chi(K(G)) = \omega(K(G))$, then $T(G) = \chi(K(G))$.*

A graph is *K-perfect* if its clique graph is perfect [22]. Since a diamond-free K-perfect graph G satisfies the premises of Theorem 2.8, we conclude that $T(G) = \chi(K(G))$.

Threshold graphs can be constructed from an empty graph by adding either an isolated vertex or a universal vertex. If a threshold graph G is connected, then G has a universal vertex, and by construction, its clique graph is a complete graph.

Theorem 2.9 *[9] If G is a connected threshold graph, then $T(G) = \chi(K(G))$.*

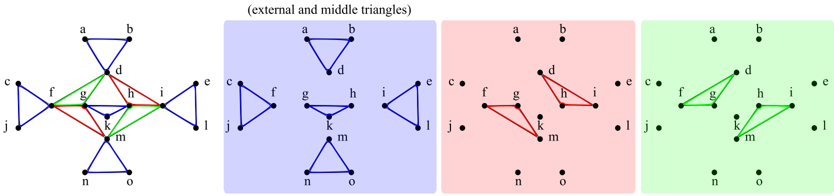


Figure 2.3: The 3-tessellable *graph-gadget* of Lemma 2.10. Each tessellation is depicted separately. The external vertices are a, b, c, e, j, l, n, o, and the internal vertices are the remaining ones.

Now, we focus on presenting $\mathcal{NP}$-completeness of the t-TESSELLABILITY problem of planar graphs with maximum degree $\Delta(G) \leq 6$ for $t = 3$ in Theorem 2.11, biplanar graphs for $t \geq 3$ in Theorem 2.12, chordal $(2,1)$-graphs for $t \geq 4$ in Theorem 2.13, $(1,2)$-graphs for $t \geq 4$ in Theorem 2.14, and diamond-free graphs with diameter at most five for $t \geq 3$ in Theorem 2.15.

Lemma 2.10 *[9] Any tessellation cover of size 3 of the* graph-gadget *depicted in Figure 2.3 contains a tessellation that covers the middle and the external triangles.*

Using the gadget depicted in Figure 2.3 we can provide a polynomial transformation from the $\mathcal{NP}$-complete 3-COLORABILITY of planar graphs with maximum degree four [31] to 3-TESSELLABILITY of planar graphs with maximum degree six. Then the next theorem follows.

Theorem 2.11 *[9]* 3-TESSELLABILITY *of planar graphs with $\Delta(G) \leq 6$ is $\mathcal{NP}$-complete.*

The next theorem shows that t-TESSELLABILITY, with any fixed $t \geq 3$, of biplanar graphs is $\mathcal{NP}$-complete. Curiously, the polynomiality of Δ-EDGE COLORABILITY for planar graphs with $\Delta(G) \geq 8$ suggests that t-TESSELLABILITY for planar graphs might be polynomial-time solvable for large enough t, in contrast with t-TESSELLABILITY of biplanar graphs.

Theorem 2.12 *[9] t-*TESSELLABILITY *of biplanar graphs for $t \geq 3$ is $\mathcal{NP}$-complete.*

We are able to show a polynomial transformation from the $\mathcal{NP}$-complete 3-COLORABILITY [31] to 4-TESSELLABILITY of chordal $(2,1)$-graphs. This hardness proof can be generalized for any fixed $t \geq 4$. Then, we have the following theorems.

Theorem 2.13 *[9] The t-*TESSELLABILITY *of chordal $(2,1)$-graphs is $\mathcal{NP}$-complete, for any fixed $t \geq 4$.*

Theorem 2.14 *[9] The t-*TESSELLABILITY *of $(1,2)$-graphs is $\mathcal{NP}$-complete, for any fixed $t \geq 4$.*

Considering the NAE 3-SAT problem, we can provide a polynomial transformation from this $\mathcal{NP}$-complete problem [31] to 3-TESSELLABILITY of diamond-free graphs with diameter at most five.

Theorem 2.15 *[9]* 3-TESSELLABILITY *of diamond-free graphs with diameter at most five is $\mathcal{NP}$-complete.*

To close this section, we show that we can solve the 2-TESSELLABILITY problem in linear time by improving the previous Peterson's algorithm [44], whose idea is to group true twin vertices of a same clique of a line graph G. These true twin vertices represent multiedges in the bipartite multigraph H, where $G = L(H)$. Then, the algorithm removes all those true twin vertices in each group but one. The last step of the algorithm is to verify if a graph is a line graph of a bipartite graph by using the Roussopoulos' linear-time algorithm [52]. Our improvement consists in showing a faster way to remove true twin vertices belonging to a clique of a graph using its modular decomposition.

Theorem 2.16 *[9]* 2-TESSELLABILITY *can be solved in linear time.*

Tables 2.1 and 2.2 summarize the results of the paper [9] in Appendix A.

Table 2.1: Extremal graph classes and tight upper bounds.

Graph class	$T(G) \leq \min\{\chi'(G), \chi(K(G))\}$
Bipartite	$T(G) = \chi'(G) = \Delta(G)$
Triangle-free	$T(G) = \chi'(G)$
Unichord-free with girth ≥ 15	$T(G) = \chi'(G) = \Delta(G)$
$Wd_{p,q}$	$T(Wd_{p,q}) = \chi(K(Wd_{p,q})) = q$
$G_p, p \in \{2,3\}$	$T(G_p) = \chi'(G_p) = \chi(K(G_p)) = 2p$
G_p, any p	$T(G_p) = \chi'(G_p) = 2p$
$E_{3,p}$	$T(G) = 3$
Diamond-free K-perfect	$T(G) = \chi(K(G)) = \omega(K(G))$
Threshold	$T(G) = \chi(K(G)) = \lvert S\rvert + 1$

Table 2.2: The complexity of the t-TESSELLABILITY problem for graph classes

t	Graph class	Complexity
$t = 2$	Generic	Linear
$t = 3$	Planar, $\Delta(G) \leq 6$	$\mathcal{NP}$-complete
	Diamond-free, diameter $= 5$	$\mathcal{NP}$-complete
$t \geq 3$	Threshold	Polynomial
	Bipartite	Polynomial
	{triangle, proper major}-free	Polynomial
	Diamond-free K-perfect	Polynomial
	Unichord-free with girth ≥ 15	$\mathcal{NP}$-complete
	Triangle-free	$\mathcal{NP}$-complete
	Biplanar	$\mathcal{NP}$-complete
$t \geq 4$	Chordal $(2, 1)$-graphs	$\mathcal{NP}$-complete
	$(1, 2)$-graphs	$\mathcal{NP}$-complete

2.2 The Tessellation Cover Number of Good Tessellable Graphs

The full paper [6] was submitted to Theoretical Computer Science IITG-Silver Jubilee, and it is under review. The content of this work is available in *ArXiv*, and it is attached in Appendix B.

In this work we define the *star number* of a graph G, denoted by $is(G)$, which is the number of edges of a maximum induced star of G. Notice that $T(G) \geq is(G)$, since the number of edges of a maximum induced star of G is a lower bound on $T(G)$. We say that a graph G is *good tessellable* if $T(G) = is(G)$. In this context, we introduce the GOOD TESSELLABLE RECOGNITION problem (GTR), which aims to decide whether a graph G is good tessellable. We also introduce the k-STAR SIZE problem, which aims to decide whether $is(G) \geq k$, for an integer k. We analyze the combined behavior of the computational complexity of the problems below.

k-STAR SIZE

Instance: Graph G and integer k.

Question: $is(G) \geq k$?

t-TESSELLABILITY

Instance: Graph G and integer t.

Question: $T(G) \leq t$?

GTR

Instance: Graph G.

Question: $T(G) = is(G)$?

As byproduct, we obtain graph classes that obey the corresponding computational behaviors described in Table 2.3.

Notice that all graph classes studied in [9] and presented in Sec. 2.1 obey behavior (a), since for those classes $is(G)$ is fixed and equal to t. Posner et al. [50] studied graphs that obey behavior (b), since for those graphs $is(G) = 2$ and 3-TESSELLABILITY is $\mathcal{NP}$-complete.

Table 2.3: Computational complexities of k-STAR SIZE, t-TESSELLABILITY, and GTR problems and examples of corresponding graph classes.

Behavior \ Problem	k-STAR SIZE	t-TESSELLABILITY	GTR
(a)	$\mathcal{P}$	$\mathcal{NP}$-complete	$\mathcal{NP}$-complete
(b)	$\mathcal{P}$	$\mathcal{NP}$-complete	$\mathcal{P}$
(c)	$\mathcal{NP}$-complete	$\mathcal{P}$	$\mathcal{NP}$-complete
(d)	$\mathcal{NP}$-complete	$\mathcal{P}$	$\mathcal{P}$
(e)	$\mathcal{NP}$-complete	$\mathcal{NP}$-complete	$\mathcal{P}$

Graph classes that obey behaviors (c) and (d) are provided by Construction 2 below. This result comes from Theorem 2.17 that shows that GTR is $\mathcal{NP}$-complete for graphs of Construction 2 (I), which have a known tessellation cover number.

Construction 1 *[6]* Let i be a non-negative integer and G a graph. The $[i, G]$-graph is obtained as follows. Add i vertices to graph G, and then add a universal vertex.

Construction 2 *[6]* Let the Mycielski graph be denoted by M_j for $j \geq 2$. Let i be a non-negative integer and G a graph with $V(G) = \{v_1, \ldots, v_n\}$. We construct a graph $H = H_1 \cup H_2$ as follows. Add i disjoint copies $G_1, \ldots, G_i$ of G to H_1, such that $V(G_j) = \{v_1^j, \ldots, v_n^j\}$ for $1 \leq j \leq i$, where v_k^j represents the same vertex v_k of G for $1 \leq k \leq n$. Add to H_1 all possible edges between pairs of vertices that represent the same vertex of G. Add a vertex u to H_1 adjacent to all v_k^j for $1 \leq j \leq i$ and $1 \leq k \leq n$. Now, we consider two possibilities: either (I) H_2 is $[|V(G)| - 3, M_3^c]$-graph of Construction 1 (example depicted in Figure 2.4) or (II) H_2 is $[|V(G)| - 3, M_4^c]$-graph of Construction 1. Denote the universal vertex of H_2 by u'.

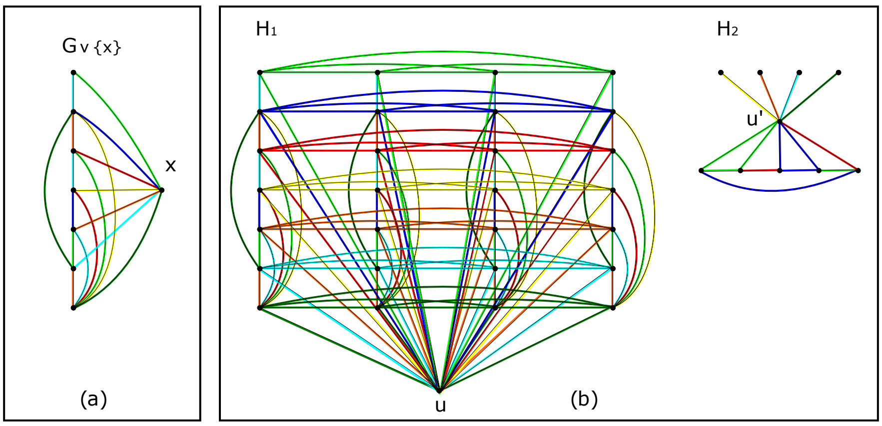


Figure 2.4: (a) An edge-coloring of $G \vee \{x\}$. (b) Example of a graph $H_1 \cup H_2$ of Construction 2 (I) obtained from graph G.

Theorem 2.17 *[6] k-STAR SIZE and GTR are $\mathcal{NP}$-complete for graphs of Construction 2 (I).*

Now we focus in showing that t-TESSELLABILITY remains $\mathcal{NP}$-complete even if the gap between $T(G)$ and $is(G)$ is large. This result leads us to provide a graph class that obeys behavior (e), depicted in Construction 4. By using universal graphs $G = [u \cup G']$, we show that $is(G)$ is a tight lower bound for the tessellation cover number.

Lemma 2.18 *[6] If G is a t-tessellable graph, then*

$$\max_{v \in V(G)} \{is(G[v \cup N_G(v)])\} \leq \max_{v \in V(G)} \{\chi(G^c[N_G(v)])\} \leq t.$$

Let $u \notin V(G)$ be a vertex. If $G \vee \{u\}$ is a t-tessellable graph, then

$$is(G \vee \{u\}) = \alpha(G) \leq \chi(G^c) \leq t.$$

From Lemma 2.18 and from the fact that we can use $\chi(G^c)$ tessellations to cover a partition P of the vertices of G, we have that

$$\chi(G^c) \leq T(G \vee \{u\}) \leq \chi(G^c) + \Delta(G) + 1, \tag{2.1}$$

which allows us to conclude that there is no universal graph such that the gap between $T(G \vee \{u\})$ and $\chi(G^c)$ is larger than $\Delta(G) + 1$. In particular, if $\chi(G^c) \geq 2\Delta(G) + 1$, then by Theorem 2.19 below $T(G \vee \{u\}) = \chi(G^c)$.

Theorem 2.19 *[6] A graph $G \vee \{u\}$ with $\theta(G) \geq 2\Delta(G)+1$ has $T(G \vee \{u\}) = \theta(G)$.*

The gap between $T(G)$ and $is(G)$ can be arbitrarily large for certain graphs, for instance, a subclass of universal graphs described next. We also show that k-STAR SIZE and t-TESSELLABILITY are $\mathcal{NP}$-complete for graphs of Construction 4, for which GTR is in $\mathcal{P}$, obeying behavior (e).

Construction 3 *[6]* Let $G = (V, E)$ be a graph. Obtain $S_2(G)$ by subdividing each edge of G two times, so that each edge $vw \in E(G)$ becomes a path v, x_1, x_2, w, where x_1 and x_2 are new vertices. Let $L(S_2(G))$ be the line graph of $S_2(G)$. Add a universal vertex u to $L(S_2(G))$, that is, consider the graph $L(S_2(G)) \vee \{u\}$.

First, we show that there is a connection between $T(H)$ of a graph H of Construction 3 on G with the size of a maximum stable set of G.

Theorem 2.20 *[6] If $G = (V, E)$ is a graph with $|E(G)| \geq 4$ and $H = (L(S_2(G)) \vee \{u\})$ is obtained from Construction 3 on G, then $T(H) = |V(G)| + |E(G)| - \alpha(G)$.*

By Theorem 2.20 and the fact that deciding whether $\alpha(G) \geq k$ is $\mathcal{NP}$-complete [31], we have the following result for the graphs of Construction 3.

Construction 4 *[6] Let H_1 be the graph obtained from Construction 2 (I) on a given graph G_1 and a non-negative integer i. Let H_2 be the graph obtained from Construction 3 on the graph $G_2 \vee K_{3|V(G_1)|}$ of a given graph G_2. Let u and u' be the two universal vertices of the two connected components of H_1. Add $is(H_2)$ degree-1 vertices to H_1 adjacent to u and $is(H_2)$ degree-1 vertices adjacent to u'. Consider $H_1 \cup H_2$.*

Theorem 2.21 *[6] k-STAR SIZE and t-TESSELLABILITY are $\mathcal{NP}$-complete for graphs of Construction 4, for which* GTR *is in $\mathcal{P}$.*

In Table 2.3 we have omitted triples. The behavior $(\mathcal{P}, \mathcal{P}, \mathcal{P})$ is obeyed by Threshold graphs and bipartite graphs [5, 9] since GTR problem always has answer YES and the other two problems are solved in polynomial time. The behavior ($\mathcal{P}, \mathcal{P}, \mathcal{NP}$-complete) is unreachable since if both k-STAR SIZE and t-TESSELLABILITY are in $\mathcal{P}$, so is GTR. The behavior ($\mathcal{NP}$-complete, $\mathcal{NP}$-complete, $\mathcal{NP}$-complete) is obeyed by the union of graphs G_1 and G_2 so that G_1 is in a graph class that obey behavior (a) and G_2 is in a graph class that obey behavior (c).

2.3 Total tessellation cover and quantum walk

The full paper [8] was submitted to the 46th International Workshop on Graph-Theoretic Concepts in Computer Science, Lecture Notes in Computer Science. It is available in *ArXiv*, and it is attached in Appendix C.

We start this section by presenting *total tessellation cover* and *total tessellation cover number* definitions.

Definition 2.22 *[8]* Let $G = (V, E)$ be a graph and Σ a non-empty label set. A *total tessellation cover* comprises a proper vertex coloring and a tessellation cover of G both with labels in Σ such that, for any vertex $v \in V$, there is no edge $e \in E$ incident to v so that e belongs to a tessellation with label equal to the color of v.

Definition 2.23 *[8]* The *total tessellation cover number* $T_t(G)$ of a graph G is the minimum size of the set of labels Σ for which G has a total tessellation cover. The k-TOTAL TESSELLABILITY problem aims to decide whether a given graph G has $T_t(G) \leq k$.

We establish bounds on $T_t(G)$, which is the smallest number of tessellations required in a total tessellation cover of G, in special, we highlight $T_t(G) \geq \omega(G)$, where $\omega(G)$ is the size of a maximum clique of G, and $T_t(G) \geq is(G) + 1$.

Since a total coloring of a graph G induces a total tessellation cover,

$$T_t(G) \leq \chi_t(G), \tag{2.2}$$

and in particular, we have that $T_t(G) = \chi_t(G)$ for triangle-free graphs. Hence, $(\Delta + 1)$-TOTAL TESSELLABILITY is hard even when restricted to regular bipartite graphs [41]. By definition,

$$\max\{\chi(G), T(G)\} \leq T_t(G) \leq \chi(G) + T(G), \tag{2.3}$$

and it implies that $T_t(G) \geq \omega(G)$.

Using these bounds, we define the good total tessellable graphs with either $T_t(G) = \omega(G)$ or $T_t(G) = is(G) + 1$. The k-TOTAL TESSELLABILITY problem aims to decide whether a given graph G has $T_t(G) \leq k$. We show that k-TOTAL TESSELLABILITY is in $\mathcal{P}$ for good total tessellable graphs. We establish the $\mathcal{NP}$-completeness of the following problems when restricted to the following classes: $(is(G) + 1)$-TOTAL TESSELLABILITY for graphs with $\omega(G) = 2$; $\omega(G)$-TOTAL TESSELLABILITY for graphs G with $is(G)+1 = 3$; k-TOTAL TESSELLABILITY for graphs G with $\max\{\omega(G), is(G)+1\}$ far from k; and 4-TOTAL TESSELLABILITY for graphs G with $\omega(G) = is(G) + 1 = 4$. As a consequence, we establish hardness results for bipartite graphs, line graphs of triangle-free graphs, universal graphs, planar graphs, and $(2, 1)$-chordal graphs.

Lemma 2.24 *[8] If $\chi(G) \geq 3T(G)$, then $T_t(G) = \chi(G)$.*

From the lemma above, we can improve the upper bound of Eq. (2.3) as follows

$$T_t(G) \leq \max\{\chi(G), T(G) + \lceil 2\chi(G)/3 \rceil\}. \tag{2.4}$$

This equation shows that $\chi(G) \geq 3T(G)$ implies $T_t(G) = \chi(G)$, or $\chi(G) \leq 3T(G)$ implies $T(G) \leq T_t(G) \leq 3T(G)$. If we consider $\chi(G) = 3$, Eq. (2.4) implies that $T(G) \leq T_t(G) \leq T(G) + 2$.

Lemma 2.25 *[8] $T_t(G) \geq \max_{v \in V(G)}\{\chi(G^c[N(v)])\} + 1 \geq \max_{v \in V(G)}\{\omega(G^c[N(v)])\} + 1 = is(G) + 1$.*

Since $T_t(G) \geq is(G) + 1 \geq k + 1$, every graph G such that $T_t(G) = T(G) = k$ is $K_{1,k}$-free. Furthermore, in the total tessellation cover there is no tile of size k. If we consider $k = 3$ the total tessellation cover of G induces a total coloring of G.

We established bounds for the total tessellation cover number for some graph classes. For bipartite graphs, $T(G) = \Delta(G)$ and $T_t(G) > T(G)$. For triangle-free graphs, $T_t(G) = T(G)$ if $\chi'(G) = \chi_t(G) = \Delta + 1$. Thus deciding whether $T_t(G) = T(G) = \Delta(G) + 1$ is $\mathcal{NP}$-complete from the proof that $(\Delta + 1)$-TOTAL COLORABILITY is $\mathcal{NP}$-complete for triangle-free snarks [55], which are graphs with $\chi'(G) = \Delta + 1$. If a graph G has $T_t(G) = T(G) = k$, we conclude that G has no induced subgraph $K_{1,k}$ because $T_t(G) \geq is(G) + 1 \geq k + 1$, and there is no tile of size k in any tessellation of a total tessellation cover. If $T_t(G) = T(G) = 3$, then G is $K_{1,3}$-free and there is no clique of size three in any tessellation. Therefore, the total tessellation cover of G induces a total coloring of G, and the only graphs for which $T_t(G) = T(G) = 3$ are the odd cycles with n vertices such that $n \equiv 0 \pmod 3$.

We now define the concept of *good total tessellable graphs*.

Definition 2.26 *[8]* A graph G is *good total tessellable* if either $T_t(G) = \omega(G)$ or $T_t(G) = is(G) + 1$. We say that G is *Type I* (resp. *Type II*) if $T_t(G) = \omega(G)$ (resp. $T_t(G) = is(G) + 1$).

Using the *Lovász number* [32], which is a real number such that $\omega(G^c) \leq \vartheta(G) \leq \chi(G^c)$, we are able to show that k-TOTAL TESSELLABILITY is in $\mathcal{P}$ if we know beforehand that the graph is either good total tessellable Type I or Type II, since the integer nearest to $\vartheta(G)$ can be determined in polynomial time.

The following theorems show that k-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete for the following cases: line graph of triangle-free graphs with $k = \omega(G) \geq 9$ and $is(G) + 1 = 3$; universal graphs with k very far apart from both $is(G) + 1$ and $\omega(G)$; planar graphs with $k = 4 = \omega(G) = is(G) + 1$; and $(2, 1)$-chordal graphs with $k = is(G) + 1 = \omega(G) + 3$.

Theorem 2.27 *[8]* k-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for line graphs $L(G)$ of 3-colorable k-regular triangle-free graphs G for any $k \geq 9$.*

Theorem 2.28 *[8]* k-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for universal graphs.*

Theorem 2.29 *[8]* 4-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for planar graphs.*

Theorem 2.30 *[8]* k-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for chordal graphs.*

Table 2.4 summarizes the results of paper [8] of Appendix C by making a contrast between the computational complexities of decision problems related to the parameters $\chi'(G), \chi_t(G), T(G)$, and $T_t(G)$.

	$\chi'(G)$	$T(G)$
$[2\|V(G)\|, G^c]$	$\mathcal{P}$	$\mathcal{NP}$-c
Line of Bipartite	$\mathcal{NP}$-c	$\mathcal{P}$

	$T(G)$	$X_t(G)$
Bipartite	$\mathcal{P}$	$\mathcal{NP}$-c
$[2\|V(G)\|, G^c]$	$\mathcal{NP}$-c	$\mathcal{P}$

	$\chi'(G)$	$\chi_t(G)$
$G \cup K_{\Delta(G)+1}$, Δ even	$\mathcal{P}$	$\mathcal{NP}$-c
$G \cup K_{\Delta(G)+1}$, Δ odd	$\mathcal{NP}$-c	$\mathcal{P}$

	$T(G)$	$T_t(G)$
Bipartite	$\mathcal{P}$	$\mathcal{NP}$-c
$G \cup K_{3\Delta(G)}$	$\mathcal{NP}$-c	$\mathcal{P}$

	$\chi'(G)$	$T_t(G)$
$[2\|V(G)\|, G^c]$	$\mathcal{P}$	$\mathcal{NP}$-c
Line of Bipartite, $\omega(G) \geq 6$	$\mathcal{NP}$-c	$\mathcal{P}$

	$\chi_t(G)$	$T_t(G)$
$G \cup K_{\Delta(G)+1}$, Δ odd	$\mathcal{P}$	$\mathcal{NP}$-c
Open	$\mathcal{NP}$-c	$\mathcal{P}$

Table 2.4: Computational complexities of parameters $\chi'(G), \chi_t(G), T(G)$, and $T_t(G)$.

To close this chapter, we present the total staggered quantum walk model, which is the first quantum walk model to use both vertices and edges as locations to the walker. A quantum walk models the walk of a particle, called by *walker*, over a graph, and the walker is represented by a unitary vector. Let $G = (V, E)$ be a simple graph so that $|V(G)| = n$ and $|E(G)| = m$. Let $\mathcal{H}^{n+m}$ be a $(n+m)$-dimensional Hilbert space, whose computational basis is the set $\{|v\rangle, v \in V(G)\} \cup \{|vw\rangle, vw \in E(G)\}$. We represent the state vector of the walker by $|v\rangle$ if the walker is located on a vertex, or $|vw\rangle$, if the walker is located on an edge vw. Then, we define a generic state by

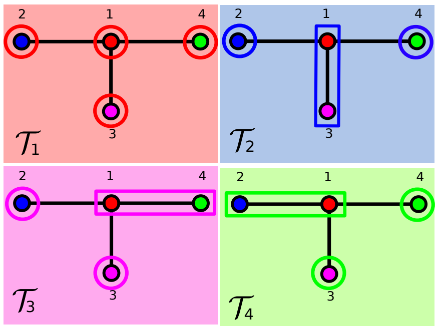
$$|\psi\rangle = \sum_{v \in V(G)} a_v \, |v\rangle + \sum_{vw \in E(G)} b_{vw} \, |vw\rangle \,, \tag{2.5}$$

where the coefficients a_v and b_{vw} are complex numbers that obey the normalization constraint

$$\sum_{v \in V(G)} |a_v|^2 + \sum_{vw \in E(G)} |b_{vw}|^2 = 1. \tag{2.6}$$

A quantum walk model on graphs must provide a recipe to build local unitary operators based on the graph structure. We can represent such operators as unitary matrices, i.e., $UU^\dagger = U^\dagger U = I$, where U represents a matrix, $U^\dagger$ is the transpose conjugated matrix of U, and I is the identity matrix. Moreover, such operators must be reversible, i.e., $U|\psi_0\rangle = |\psi_1\rangle$, and $U^\dagger|\psi_1\rangle = |\psi_0\rangle$. In our proposal, the evolution operator that drives the quantum walk is obtained from a total tessellation cover, which provides a tessellation cover $\{\mathcal{T}_1, ..., \mathcal{T}_k\}$ and a compatible proper vertex coloring.

Figure 2.5 depicts an example of a total tessellation cover. Note that the colors of the vertices in 2-tiles are different from the tile color. On the other hand, there are 1-tiles that contain a vertex with the tile color.



$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 2.5: (Left) A total tessellation cover of a claw graph G. (Right) The evolution operator of a total staggered quantum walk on G with $\lambda = \pi/2$.

We can write the operator generically as

$$U = \begin{bmatrix} \frac{\sin^2\lambda}{\sqrt{3}} & \cos\lambda & 0 & 0 & b\sin\lambda & c\sin\lambda & c\sin\lambda \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & \frac{a}{\sqrt{3}} + c\sin\lambda & \frac{a}{\sqrt{3}} - b\sin\lambda & \frac{a}{\sqrt{3}} + c\sin\lambda \\ \frac{\sin^2\lambda}{\sqrt{3}} & 0 & \cos\lambda & 0 & c\sin\lambda & c\sin\lambda & -b\sin\lambda \\ -\frac{a}{\sqrt{3}} & \sin\lambda & 0 & 0 & -b\sin\lambda & -c\sin\lambda & -c\sin\lambda \\ a - \frac{a}{\sqrt{3}} & 0 & 0 & 0 & \frac{\sin^2\lambda}{\sqrt{3}} - c\cos\lambda & \frac{\sin^2\lambda}{\sqrt{3}} - b\cos\lambda & \frac{\sin^2\lambda}{\sqrt{3}} - c\cos\lambda \\ -\frac{a}{\sqrt{3}} & 0 & \sin\lambda & 0 & -c\cos\lambda & -c\cos\lambda & -b\cos\lambda \end{bmatrix} \tag{2.7}$$

where $a = \cos\lambda\sin\lambda$, $b = \frac{-2-\cos\lambda}{3}$, $c = \frac{1-\cos\lambda}{3}$, $d = \cos^2\lambda + \frac{\sin^2\lambda}{\sqrt{3}}$, and $0 \leq \lambda \leq 2\pi$. Each tessellation $\mathcal{T}_j$ is associated with a Hermitian matrix H_j. Since H_j is local, the action of H_j on the state of a walker that is located on a vertex v drives the walker to the neighborhood of v and to the edges incident to v.

The dynamic of this quantum walk driven by H_j must obey the following locality rules, as described in [8]:

1. If the walker is located on a vertex v that belongs to a 1-tile of $\mathcal{T}_j$, there are two cases: (i) If the color of the vertex is equal to the color of tessellation $\mathcal{T}_j$, the walker hops to v and to the edges incident to v; and (ii) if the color of the vertex is different from the color of tessellation $\mathcal{T}_j$, the walker stays put.

2. If the walker is located on a vertex v that belongs to a tile of $\mathcal{T}_j$ of size at least 2, the walker hops to the vertices in such a tile.

3. If the walker is located on an edge that belongs to tessellation $\mathcal{T}_j$, the walker stays put.

4. If the walker is located on an edge that does not belong to tessellation $\mathcal{T}_j$, there are two cases: (i) If there is an incident vertex v whose color is equal to

the color of tessellation $\mathcal{T}_j$, the walker hops to v and to the edges incident to v; and (ii) otherwise, the walker stays put.

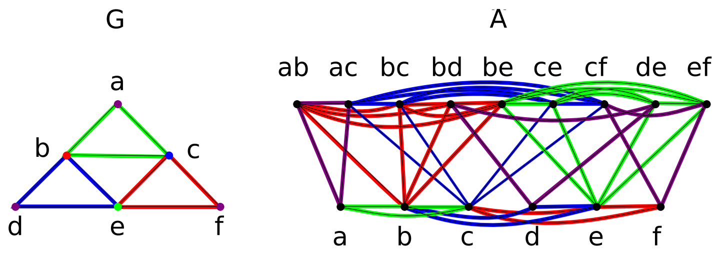


Figure 2.6: Total tessellation cover of a graph G and the associated tessellation cover of $A = \text{Tot}(G)$.

It is possible to simulate a total staggered quantum walk on a graph G with a staggered quantum walk on its total graph $\text{Tot}(G)$, defined below.

Definition 2.31 *The total graph $Tot(G)$ of G has $V(Tot(G)) = V(G) \cup E(G)$ and $E(Tot(G)) = E(G) \cup \{u\ uw \mid u \in V(G), uw \in E(G)\} \cup \{uv\ vw \mid uv \in E(G)$ and $vw \in E(G)\}$.*

Let $A = \text{Tot}(G)$, $A[E(G)] = Y$ and $A[V(G)] = X$. Subgraph Y is isomorphic to the line graph $L(G)$ of G, and X is isomorphic to the original G. We define the clique $K_v = \{v\} \cup \{vw \mid vw \in E(G)\}$ of A.

We first consider a total tessellation cover of a graph G, and we define an associated tessellation cover of A as follows. Assign the labels of the edges of G to the respective edges of X and assign the color of each vertex v of G to the edges of $A[K_v]$. By this way, we relate the total tessellation cover of G to the tessellation cover of A, as well as their respective evolution operators generated from these tessellation covers. To simulate the total staggered quantum walk on G with the staggered quantum walk on A, we consider the vertices of G as the corresponding vertices of X in A, and the edges of G as the corresponding vertices of Y in A. Fig. 2.6 depicts a total tessellation cover of a graph G and the associated tessellation cover of $A = \text{Tot}(G)$.

Chapter 3

Conclusion

The staggered model proposal [49] in 2015 gave to the studies about the tessellation cover number a great motivation due to the applicability of this concept in the generation of quantum operators to perform quantum walks using that model. Since the less quantum operators needed to be implemented, the less complex is the quantum implementation, the knowledge about how many tessellations are needed to cover a graph is important in quantum walk context, and the results obtained in the present thesis can help the studies in quantum computation [10, 11, 24, 25, 37, 45, 46, 54]. We summarize the results, discuss current work, and propose open questions.

In Section 2.1 we investigate the tessellation cover number for graph classes whose $T(G)$ reaches one of the upper bounds. Those graphs are called by *extremal graphs* and they are fundamental for the development of quantum walks in the staggered model, since those results lead us to a better understanding about the complexity of the unitary operators necessary to express the evolution of staggered quantum walks. Besides the extremal graph classes presented, we also improve the known algorithm to recognize line graphs of bipartite multigraphs [51], for 2-tessellable graphs [48], and graphs G such that $K(G)$ is bipartite [44], to linear-time.

Naturally, an interesting question arises: Does every graph have a minimum tessellation cover such that every tessellation contains a maximal clique? Although the intuition says that in most cases the answer is true, using a minimization model proposed by Abreu et al. [2] for the optimization version of t-TESSELLABILITY problem, we found a surprising example of a graph, which is depicted in Figure 3.1, with all minimum tessellation covers requiring a tessellation without maximal cliques. This minimization model was presented in *Congresso Nacional de Matemática Aplicada e Computacional*, in 2018 (attached in Appendix D), and we are currently working on improvements for this model [2], while we study the different quantum walk dynamics over a same graph G resulting from the use of two different minimum tessellation covers for G.

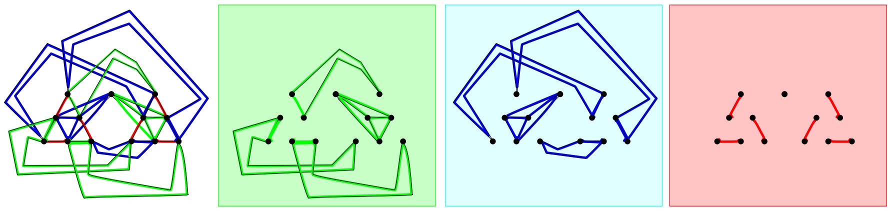


Figure 3.1: 3-tessellable graph. Rightmost tessellation does not contain a maximal clique.

In Section 2.2 we proposed a tight lower bound for $T(G)$, which implicitly appeared in the previous hardness proofs of [9]. It is important to analyze the gap between $T(G)$ and $is(G)$, since in the quantum walk context, it is advantageous to implement physically as few operators as possible in order to reduce the complexity of the quantum system. We presented graph classes for which $T(G) = is(G)$. These graphs are called by *good tessellable graphs*, and it is related with a new problem proposed, the GOOD TESSELLABLE GRAPHS RECOGNITION (GTR), and we analyzed the computational complexity behaviors for several graph classes with respect of the problems: t-TESSELLABILITY, GTR, and k-STAR SIZE. From this work, an interesting research topic is the extension of the concept of good tessellable graphs to *perfect tessellable graphs*, the graphs G for which $T(H) = is(H)$ for any induced subgraph H of G.

In Section 2.3 we have defined the total tessellation cover on a graph G. From this concept we propose the total staggered quantum walk model, which is the first quantum walk model to use both vertices and edges as possible locations for the walker. An open problem is to search for graphs with at least 3 vertices satisfying $T_t(G) = 3T(G)$ and $T_t(G) > \chi(G)$. Furthermore, it is interesting to define graph classes with $T_t(G) = T(G) = k$ for $k \geq 4$, since for $k = 3$ the only such graphs are the odd cycles C_n with $n \equiv 0 \bmod 3$. Another open problem is to find a threshold for $T_t(G)$ for which all planar graphs are Type II.

In Appendix E, we show results presented in the 8th Latin-American Workshop on Cliques in Graphs, whose proceedings were published at *Matemática Contemporânea* [7]. In that work we analyzed the t-TESSELLABILITY problem in graphs with few induced P_4. We proved that adding true-twin vertices in a graph G results in a graph G' such as $T(G) = T(G')$ for any graph G. Moreover, we presented a polynomial time algorithm for t-TESSELLABILITY of quasi-threshold graphs. Furthermore, the concept of t-TESSELLABILITY COMPLETION on G was introduced, and it aims to decide whether there is a tessellation cover $\mathcal{T}$ of G with t tessellations given by a partial tessellation cover $\mathcal{T}'$ of G, such that $\mathcal{T}'$ is part of $\mathcal{T}$.

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Appendix A

Manuscript “The graph tessellation cover number: chromatic bounds, efficient algorithms and hardness”

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The graph tessellation cover number: Chromatic bounds, efficient algorithms and hardness ☆

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ABSTRACT

A tessellation of a graph is a partition of its vertices into vertex disjoint cliques. A tessellation cover of a graph is a set of tessellations that covers all of its edges, and the tessellation cover number is the size of the smallest tessellation cover. These concepts are motivated by their application to quantum walk models, in special, the evolution operator of the staggered model is obtained from a graph tessellation cover. We show that the minimum between the chromatic index of the graph and the chromatic number of its clique graph, which we call chromatic upper bound, is tight with respect to the tessellation cover number for star-octahedral and windmill graphs; whereas for $(3, p)$-extended wheel graphs, the tessellation cover number is 3 and the chromatic upper bound is $3p$. The t-TESSELLABILITY problem aims to decide whether there is a tessellation cover of the graph with t tessellations. Using graph classes whose tessellation cover numbers achieve the chromatic upper bound, we obtain that t-TESSELLABILITY is polynomial-time solvable for bipartite, {triangle, proper major}-free, threshold, and diamond-free K-perfect graphs; whereas is $\mathcal{NP}$-complete for triangle-free for $t \geq 3$, unichord-free for $t \geq 3$, planar for $t = 3$, biplanar for $t \geq 3$, chordal $(2, 1)$-graphs for $t \geq 4$, $(1, 2)$-graphs for $t \geq 4$, and diamond-free with diameter at most five for $t = 3$. We improve the complexity of 2-TESSELLABILITY problem to linear time.

1. Introduction

Random walks play an important role in Computer Science mainly in the area of algorithms and it is expected that quantum walks, which are the quantum counterpart of random walks, will play at least a similar role in Quantum Computation. In fact, the interest in quantum walks has grown considerably in the last decades, especially because they can be used to build quantum algorithms that outperform their classical counterparts [2].

Recently, the staggered quantum walk model [3] was proposed. This model is defined by an evolution operator, which is described by a product of local unitary matrices obtained from a *graph tessellation cover*. A *tessellation* is a partition of the

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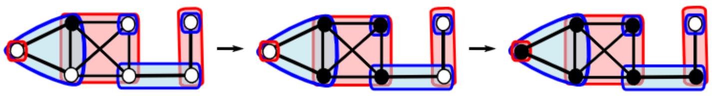

Fig. 1. The spreading of a walker subject to locality across a 2-tessellable graph. At each step, the walker may be observed at filled vertices that represent non-zero amplitudes, meaning that a measurement of the position can reveal the walker at one of those vertices. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

vertices of a graph into vertex disjoint cliques, and a tessellation cover is a set of tessellations so that the union covers the edge set of the graph. To cover the entire edge set is important because an edge that would not be in the tessellation cover would play no role in the quantum walk dynamics. In order to fully understand the possibilities of the staggered model, it is fundamental to introduce the t-TESSELLABILITY problem. This problem aims to decide whether a given graph can be covered by t tessellations.

The simplest evolution operators are the product of few local unitary matrices and, to obtain a non-trivial quantum walk, at least two matrices (corresponding to 2-tessellable graphs) are required [3]. There is a recipe to build a local unitary matrix based on a tessellation. Each clique in a tessellation is associated with a unit vector, and the set of those unit vectors spans a subspace of the model's Hilbert space. A subspace has an associated orthogonal projection Π, which is used to define the local unitary operator $(2\Pi - I)$ associated with the tessellation. Each clique of the partition establishes a neighborhood around which the walker can move under the action of the local unitary matrix. The evolution operator of the quantum walk is the product of the unitary operators associated with the tessellations of a tessellation cover. Fig. 1 depicts an example of how a quantum walker could spread across the vertices of a graph, given a particular tessellation cover, where the filled vertices represent that the probability of finding the walker is non-zero. Note that after each step the walker spreads across the cliques in the corresponding tessellation.

The study of tessellations in the context of Quantum Computing was proposed by Portugal et al. [3] with the goal of obtaining the dynamics of quantum walks. Portugal analyzed the 2-tessellable case in [4], showing that a graph is 2-tessellable if and only if its clique graph is bipartite, and examples for the t-tessellable case are available in [5]. The present paper is the first systematic attempt to study the graph tessellation cover as a branch of Graph Theory. Our aim is the study of graph classes whose tessellation cover number is close or equal to chromatic upper bounds, efficient algorithms, and hardness.

In Section 2, we establish a chromatic upper bound as the minimum between the chromatic index of the graph and the chromatic number of its clique graph, and we present infinite families of star-octahedral graphs and windmill graphs, showing that this bound is tight. We also present the infinite family of extended wheel graphs whose tessellation cover number is far from the chromatic upper bound. We describe the tessellation cover number for the classes of bipartite graphs and {triangle, proper major}-free graphs, and we prove that t-TESSELLABILITY for these classes is polynomial-time solvable, while is $\mathcal{NP}$-complete for triangle-free graphs, when $t \geq 3$. In Section 3, we present extremal graph classes, i.e., classes whose tessellation cover numbers reach the chromatic upper bound. Such classes are useful to establish hardness results in Section 4. We obtain proofs of $\mathcal{NP}$-completeness for t-TESSELLABILITY problem of planar graphs for $t = 3$, biplanar graphs for $t \geq 3$, chordal $(2,1)$-graphs for $t \geq 4$, $(1,2)$-graphs for $t \geq 4$, and diamond-free graphs with diameter at most five for $t = 3$. Moreover, we describe a linear-time algorithm for 2-TESSELLABILITY by improving the algorithm proposed by Peterson [6] for line graph of bipartite multigraph recognition. In Section 5, we summarize in Table 1 the extremal graph classes analyzed in Section 2.2 and in Section 3, whereas in Table 2 the complexity of the t-TESSELLABILITY problem for the graph classes analyzed in Section 4. Moreover, we leave open questions and discuss future work, such as whether every minimum tessellation cover contains tessellations with at least one maximum clique, and whether two minimum tessellation covers in a same graph have different quantum walk dynamics.

2. Preliminaries on the tessellation cover number

In this section, we present the main definitions of this paper, we introduce the chromatic upper bound, and we show that this bound is tight by presenting infinite families of graphs whose tessellation cover numbers achieve this chromatic upper bound. On the other hand, we present an infinite family of graphs whose tessellation cover number is far from the chromatic upper bound.

2.1. Definitions and upper bounds

A *clique* is a subset of vertices of a graph such that its induced subgraph is complete, and a *d-clique* is a clique of size d. The size of a maximum clique of a graph G is denoted by $\omega(G)$. The *clique graph* $K(G)$ is the intersection graph of the maximal cliques of G. A *partition of the vertices of a graph into cliques* is a collection of vertex disjoint cliques, where the union of these cliques is the vertex set. Clique graphs play a central role in tessellation covers. See [7] for an extensive survey on clique graphs and [8] for omitted graph theory terminologies.

Definition 1. A *tessellation* $\mathcal{T}$ is a partition of the vertices of a graph into cliques. An edge *belongs* to the tessellation $\mathcal{T}$ if and only if its endpoints belong to the same clique in $\mathcal{T}$. The set of edges belonging to $\mathcal{T}$ is denoted by $\mathcal{E}(\mathcal{T})$.

Definition 2. Given a graph G with edge set $E(G)$, a *tessellation cover* of size t of G is a set of t tessellations $\mathcal{T}_1, \ldots, \mathcal{T}_t$, whose union $\cup_{i=1}^{t} \mathcal{E}(\mathcal{T}_i) = E(G)$. A graph G is called *t-tessellable* if there is a tessellation cover of size at most t. The t-TESSELLABILITY PROBLEM aims to decide whether a graph G is t-tessellable. The *tessellation cover number* $T(G)$ is the size of a smallest tessellation cover of G.

A *coloring* (resp. an *edge-coloring*) of a graph is a labeling of the vertices (resp. edges) with colors such that no two adjacent vertices (resp. adjacent edges) have the same color. A *k-colorable* (resp. *k-edge-colorable*) graph is the one which admits a *coloring* (resp. an *edge-coloring*) with at most k colors. The *chromatic number* $\chi(G)$ (resp. *chromatic index* $\chi'(G)$) of a graph G is the smallest number of colors needed to color the vertices (resp. edges) of G.

Note that an edge-coloring of a graph G induces a tessellation cover of G. Each color class induces a partition of the vertex set into disjoint cliques of size two (vertices incident to edges of that color) and cliques of size one (vertices not incident to edges of that color), which forms a tessellation. Moreover, a coloring of $K(G)$ induces a tessellation cover of G. As presented in [5], two vertices of the same color in $K(G)$ correspond to disjoint maximal cliques of G and every edge of G is in at least one maximal clique. So, each color in $K(G)$ defines a tessellation in G by possibly adding cliques of size one (vertices that do not belong to the maximal cliques of G related to vertices of $K(G)$ with that color), such that the union of these tessellations is the edge set of G. Hence, we have the *chromatic upper bound*, denoted by $\text{cub}(G)$, as the minimum between $\chi'(G)$ and $\chi(K(G))$.

Theorem 1. *If G is a graph, then $T(G) \le \text{cub}(G) = \min\{\chi'(G), \chi(K(G))\}$.*

Portugal [4] characterized the 2-tessellable graphs as those whose clique graphs are bipartite graphs. Note that if $K(G)$ is bipartite, then $\chi(K(G)) = 2$, while $\chi'(G)$ may be arbitrarily large due to the fact that this parameter is related to the maximum degree $\Delta(G)$. In order to characterize t-tessellable graphs, for $t \ge 3$, we find graph classes such that $T(G) = 3$, with $\chi'(G)$ and $\chi(K(G))$ arbitrarily large, and graph classes whose tessellation cover number reaches the chromatic upper bound of Theorem 1, i.e., $T(G) = \chi'(G)$ but $\chi(K(G))$ arbitrarily large; and $T(G) = \chi(K(G))$ but $\chi'(G)$ arbitrarily large, some of those examples were described in [5], and further developed in Section 2.2.

An interesting case occurs for a triangle-free graph. Note that any of its tessellations can only be formed by cliques of size two or one. Hence, we have that if G is a triangle-free graph, then $T(G) = \chi'(G) = \chi(K(G)) = \chi(L(G))$, where $L(G)$ is the line graph of G. Therefore, t-TESSELLABILITY is polynomial-time solvable for bipartite graphs and for {triangle, proper major}-free graphs, and there are also polynomial-time algorithms to obtain a minimum tessellation cover for these graph classes [9,10]. On the other hand, it is known that Δ-EDGE COLORABILITY of triangle-free graphs for $\Delta \ge 3$ is $\mathcal{NP}$-complete [11]. Therefore, t-TESSELLABILITY of triangle-free graphs for $t \ge 3$ is also $\mathcal{NP}$-complete. Similarly, we know that Δ-EDGE COLORABILITY of regular unichord-free graphs with girth at least 15 for $\Delta \ge 3$ is $\mathcal{NP}$-complete [12]. As this graph class is triangle-free, we conclude that the same hardness proof holds for t-TESSELLABILITY for $t \ge 3$.

2.2. Three infinite families

We present three infinite families of graphs G that illustrate some interesting situations: (i) $T(G) = \chi'(G)$ with $\chi(K(G))$ arbitrarily large; (ii) $T(G) = \chi(K(G))$ with $\chi'(G)$ arbitrarily large; and (iii) $T(G) = 3$ with both upper bounds arbitrarily large. Note that the first two situations are illustrated by families of graphs whose tessellation cover numbers achieve the chromatic upper bound.

A *coalescence* [13] of disjoint graphs G_1 and G_2 is obtained by identifying a vertex of G_1 with another vertex of G_2. The first family of *star-octahedral graphs* G_p is the coalescence of the graphs S_{2p} and O_p — where S_{2p} is the star graph with $2p$ leaves and O_p is the p dimensional octahedral graph defined by the $(2p-2)$-regular graph with $2p$ vertices — by identifying a leaf of S_{2p} into any vertex of O_p. Fig. 2 depicts the star-octahedral graph G_4.

Next, we establish that the tessellation cover number of star-octahedral graph G_p is equal to its chromatic index.

Theorem 2. *Let G_p be a star-octahedral graph. Then:*

1. $T(G_p) = \Delta(G_p) = \chi'(G_p) = \chi(K(G_p)) = 2p$, *for* $p \in \{2, 3\}$, *and;*
2. $T(G_p) = \Delta(G_p) = \chi'(G_p) = 2p$ *and* $\chi(K(G_p)) = 2^{p-1} + 1$, *for* $p \ge 4$.

Proof. We know that the clique graph of the octahedral graph O_p is the octahedral graph $O_{2^{p-1}}$ [14]. Moreover, as the star-octahedral graph has only one vertex with maximum degree, we know that $\chi'(G_p) = 2p$. The tessellation cover number of S_{2p} is equal to $2p$, hence $T(G_p) \ge 2p$. The proof is divided into two cases:

1. Consider $p = 2$. We know that $K(O_2) = O_2$ and $K(S_4) = K_4$, which are induced subgraphs of $K(G_2)$. Since G_2 has a vertex $v \in V(S_4)$ identified with a vertex of O_2, it follows that $K(G_2)$ has a vertex $u \in V(K(S_4))$ that is a neighbor of two vertices of $K(O_2)$. Hence, the largest maximal clique of $K(G_2)$ has size 4, and $\chi(K(G_2)) = 2p = 4$. Moreover, $\Delta(G_2) = \Delta(S_4) = \chi'(G_2) = 2p = 4$. Then, from Theorem 1, it follows that $T(G_2) = \chi'(G_2) = \chi(K(G_2)) = 2p$. The proof is analogous for $p = 3$.

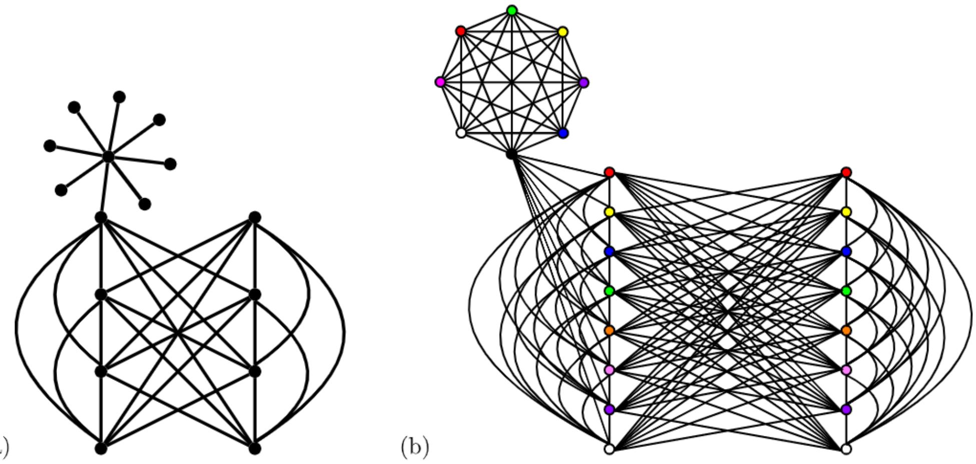

Fig. 2. (a) The star-octahedral graph G_4, i.e., the coalescence between the octahedral graph O_4 and the star graph S_8. (b) The clique graph $K(G_4)$. Notice that $T(G_4) = \chi'(G_4) = 8$, while $\chi(K(G_4)) = 9$.

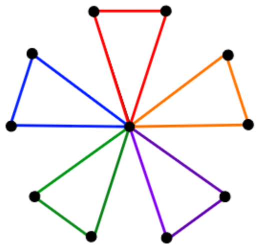

Fig. 3. The windmill graph $\mathrm{Wd}_{3,5}$, composed by 5 copies of the complete graph K_3. Notice that this graph has $T(W_{3,5}) = \chi(K(\mathrm{Wd}_{3,5})) = 5$, since its clique graph $K(\mathrm{Wd}_{3,5})$ is the complete graph K_5.

2. Notice that each vertex of O_p belongs to 2^{p-1} maximal cliques. Since G_p has a vertex $v \in V(S_{2p})$ identified with a vertex of O_p, it follows that v belongs to $2^{p-1}+1$ maximal cliques. Hence, $\chi(K(G_p)) \geq \omega(K(G_p)) \geq 2^{p-1}+1$. One can obtain a $(2^{p-1}+1)$-coloring of $K(G_p)$ as displayed in Fig. 2(b). Hence, $\chi(K(G_p)) = 2^{p-1}+1$.

As $\Delta(G_p) = \Delta(S_{2p}) = 2p$, then $\chi'(G_p) = 2p$. From Theorem 1, it follows that $T(G_p) \leq \min\{\chi'(G_p), \chi(K(G_p))\} = \chi'(G_p) = 2p$, since $2p \leq 2^{p-1}+1$ for $p \geq 4$. As $T(S_{2p}) = \chi'(S_{2p}) = 2p \leq T(G_p)$, we conclude that $T(G_p) = \chi'(G_p) = 2p$. □

The second family of *windmill graphs* $\mathrm{Wd}_{p,q}$ is obtained by identifying q copies of the complete graph K_p at a universal vertex. Note that $T(\mathrm{Wd}_{p,q}) = \chi(K(\mathrm{Wd}_{p,q})) = q$ for $p \geq 2$ and $\chi'(\mathrm{Wd}_{p,q}) = (p-1)q$ [5]. Fig. 3 depicts an example of the windmill graph $\mathrm{Wd}_{3,5}$, that is composed by 5 copies of the complete graph K_3. Clearly, its clique graph $K(\mathrm{Wd}_{3,5})$ is a complete graph K_5, then $\chi(K(\mathrm{Wd}_{3,5})) = 5$, which is equal to the tessellation cover number of $\mathrm{Wd}_{3,5}$. On the other hand $\chi'(\mathrm{Wd}_{3,5}) = 10$.

The third family of (k, p)*-extended wheel graphs* $E_{k,p}$, for $k \geq 3$ and $p \geq 2$, is defined by adding to the wheel graph W_{kp} (defined by a cycle C_{kp}, $V(C_{kp}) = \{0, 1, 2, \ldots, kp-1\}$, after adding a universal vertex with label kp.) the following edges: $\{ki, kj\}$, $\{ki+1, kj+1\}$, ..., $\{ki+k-1, kj+k-1\}$, for $0 \leq i < j < p$. When focusing on the case $k = 3$, we show that $T(E_{3,p}) = 3$, $\chi'(G) = 3p$, and $\chi(K(E_{3,p})) = 3p+3$. The class $E_{3,p}$ comprises 3-tessellable graphs with arbitrarily large chromatic index, whose clique graphs have arbitrarily large chromatic numbers. It shows that the tessellation cover number does not necessarily depend neither on $\chi'(G)$ nor on $\chi\big(K(G)\big)$.

Lemma 1. *The maximal cliques of $E_{3,p}$ are 3-cliques or $(p+1)$-cliques. The number of maximal cliques is $3p+3$. The maximal cliques are the 3-cliques of the spanning wheel W_{3p}, plus three new $(p+1)$-cliques. All maximal cliques share the vertex with label $3p$, which is the universal vertex.*

Proof. Each of the $3p$ vertex sets $\{0, 1, 3p\}, \{1, 2, 3p\}, \ldots, \{3p-2, 3p-1, 3p\}, \{3p-1, 0, 3p\}$ is a maximal clique K_3 because it induces a triangle of the spanning wheel graph and the vertex set $\{i : 0 \leq i < 3p\}$ contains no maximal clique of size 3 in $E_{3,p}$.

Now consider the three sets of vertices $\{0, 3, 6, ..., 3p-3, 3p\}$, $\{1, 4, 7, ..., 3p-2, 3p\}$, and $\{2, 5, 8, ..., 3p-1, 3p\}$, each of them with cardinality $p+1$. We claim that each one is a maximal clique K_{p+1}. Consider the set $\{0, 3, 6, ..., 3p-3, 3p\}$ (analogous for the other ones). All vertices in this set are adjacent because every pair of vertices is either $\{3i, 3j\}$ for some $0 \leq i, j < p$ or $\{3i, 3p\}$ for some $0 \leq i < p$. In the first case, these edges were added to W_{3p} to define $E_{3,p}$, and in the second case the edges belong to the spanning wheel graph. If a new vertex is added, it must have the form $3i+1$ or $3i+2$

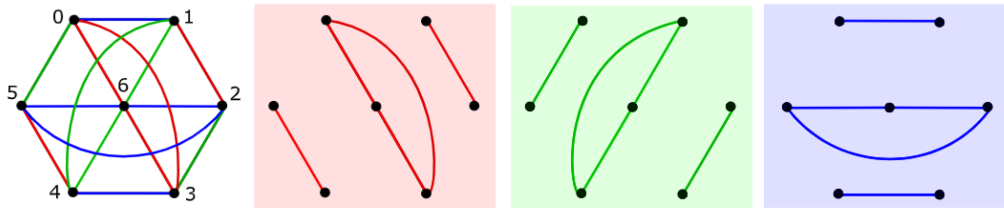

Fig. 4. An example of the $(3, 2)$-extended wheel graph. Notice that the tessellations applied in this graph are $\mathcal{T}_1 = \{\{0, 3, 6\}, \{1, 2\}, \{4, 5\}\}$, $\mathcal{T}_2 = \{\{1, 4, 6\}, \{2, 3\}, \{5, 0\}\}$, and $\mathcal{T}_3 = \{\{2, 5, 6\}, \{3, 4\}, \{0, 1\}\}$, as described in the proof of Theorem 3.

for some $0 \le i < p$ and it will not be adjacent to all vertices of set $\{0, 3, 6, ..., 3p - 3, 3p\}$. Hence, there are three maximal cliques of size $p + 1$ in $E_{3,p}$. Then, the total number of maximal cliques is $3p + 3$ and all of them share the vertex with label $3p$. □

Theorem 3. *Let $E_{3,p}$ be a $(3, p)$-extended wheel graph. Then, $T(E_{3,p}) = 3$ for $p \ge 2$.*

Proof. Let us show that $E_{3,p}$ is 3-tessellable by describing explicitly three tessellations that cover the edges of $E_{3,p}$. The tessellations are the following ones:

$$\mathcal{T}_1 = \{\{0, 3, 6..., 3p - 3, 3p\}, \{1, 2\}, \{4, 5\}, ..., \{3p - 5, 3p - 4\}, \{3p - 2, 3p - 1\}\};$$
$$\mathcal{T}_2 = \{\{1, 4, 7, ..., 3p - 2, 3p\}, \{2, 3\}, \{5, 6\}, ..., \{3p - 4, 3p - 3\}, \{3p - 1, 0\}\};$$
$$\mathcal{T}_3 = \{\{2, 5, 8, ..., 3p - 1, 3p\}, \{3, 4\}, \{6, 7\}, ..., \{3p - 3, 3p - 2\}, \{0, 1\}\}.$$

Let us show that $\mathcal{T}_1$ is a well defined tessellation (analogous for the other ones) by checking each item of the following list: (1) Each vertex set in $\mathcal{T}_1$ must induce a clique, (2) the vertex sets in $\mathcal{T}_1$ must be pairwise disjoint, and (3) the union of the vertex sets in $\mathcal{T}_1$ must be the vertex set of $E_{3,p}$. Item (1) holds by Lemma 1, since the set {0,3,6,...,3p-3,3p} is a clique, and the remaining sets define edges of the spanning wheel. Item (2) holds since the set $\{0, 3, 6, ..., 3p - 3, 3p\}$ is comprised of vertices that are multiple of 3 while the remaining sets are disjoint and contain no multiple of 3. Item (3) holds since the union of the sets in $\mathcal{T}_1$ is the vertex set. Since no edge belongs to more than one tessellation and each tessellation covers $p(p + 3)/2$ edges, the union $\mathcal{E}(\mathcal{T}_1) \cup \mathcal{E}(\mathcal{T}_2) \cup \mathcal{E}(\mathcal{T}_3)$ covers $3p(p + 3)/2$ edges, which is the number of edges of $E_{3,p}$. It is not possible to cover the edges of $E_{3,p}$ with less than three tessellations because if $T(E_{3,p}) = 2$ then $\chi(K(E_{3,p})) = 2$ [3]. However, the chromatic number of the clique graph of $E_{3,p}$ is $3p + 3$. Then, $T\big(E_{3,p}\big) = 3$ for $p \ge 2$. □

Fig. 4 depicts the $(3, 2)$-extended wheel graph $E_{3,2}$.

It is straightforward to extend those results and to prove that $T(E_{k,p}) \le k$, $\chi'(G) = kp$, and $\chi\big(K(E_{k,p})\big) = k(p + 1)$. Therefore, we are able to provide examples of classes of k-tessellable graphs with arbitrarily large chromatic index, whose clique graphs have arbitrarily large chromatic number for any $k \ge 3$.

3. Extremal graph classes

In this section, we show extremal graph classes, by presenting constructions that force the tessellation cover number of some graphs to be equal to the chromatic upper bound. An *extremal graph* is a graph whose tessellation cover number is equal to the chromatic upper bound of Theorem 1. We are particularly interested in constructing graphs with tessellation cover number corresponding or close to the chromatic upper bound. Note that the family of star-octahedral, windmill, triangle-free, bipartite, {triangle, proper major}-free, and unichord-free graphs with girth at least 15, analyzed in Section 2, are examples of extremal graph classes. For the sake of convenience, we may omit one-vertex cliques inside tessellations in our proofs.

Construction 1. Let H be obtained from a graph G by adding a star with $\chi'(G)$ leaves and identifying one of these leaves with a minimum degree vertex of G. See Fig. 5.

The tessellation cover number of H, obtained from Construction 1 on a non-regular graph G, is equal to $\chi'(G)$, i.e., $T(H) = \chi'(H) = \chi'(G)$. For regular graphs, if $\chi'(G) = \Delta(G) + 1$, then $T(H) = \chi'(H) = \chi'(G)$. Otherwise, $T(H) = \chi'(H) = \chi'(G) + 1$. Construction 1 also implies that every non-regular graph G is a subgraph of a graph H with $T(H) = \chi'(H) = \chi'(G)$.

Additionally, Construction 2 in diamond-free graphs G forces the tessellation cover number of the obtained graph H to be equal to the chromatic number of the clique graph $\chi(K(G))$. First, we define a property of the cliques on a tessellation called exposed maximal clique. Such a property helps us with particular cases of diamond-free graphs.

Fig. 5. Example of Construction 1. $T(G) = 3$, and $T(H) = \chi'(H) = \chi'(G) = 4$.

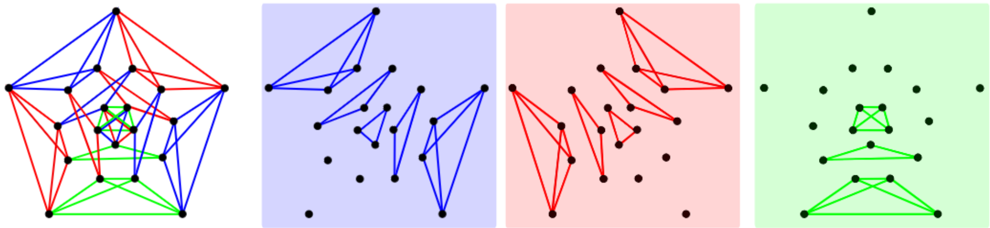

Fig. 6. Example of a 3-tessellable graph G whose clique graph is the Mycielskian of a C_5, with $\chi(K(G)) = 4$ but $T(G) = 3$. Each tessellation is depicted separately.

Definition 3. A maximal clique K of a graph G is said to be *exposed* by a tessellation cover $\mathcal{C}$ if $E(K) \nsubseteq \mathcal{E}(\mathcal{T})$ for all $\mathcal{T} \in \mathcal{C}$, that is, the edges of K are not covered by any single tessellation of $\mathcal{C}$.

Lemma 2. *A graph G admits a minimum tessellation cover with no exposed maximal cliques if and only if $T(G) = \chi(K(G))$.*

Proof. Given a minimum tessellation cover $\mathcal{C} = \{\mathcal{T}_1, \ldots, \mathcal{T}_t\}$ of G, if there are no exposed maximal cliques in G, then $\mathcal{C}$ induces a coloring of $K(G)$. In fact, suppose that C_v is a maximal clique of G associated with vertex $v \in V(K(G))$. If C_v is covered by tessellation $\mathcal{T}_i$ then v receives color c_i (if C_v is covered by more than one tessellation, we have more than one choice for coloring v). Using the definitions of tessellation and tessellation cover, we conclude that this method produces a coloring of K(G) with colors $c_1, \ldots, c_t$, which implies that $\chi(K(G))$ is at most t. And Theorem 1 implies the equality $\chi(K(G)) = t = T(G)$.

Conversely, the proof of Theorem 1 describes a minimum tessellation cover of size t with no exposed maximal clique when $\chi(K(G)) = T(G) = t$. $\square$

In the remaining part of this section we consider diamond-free graphs, which have the following properties [6]: (1) their clique-graphs are diamond-free, and (2) any two maximal cliques intersect in at most one vertex.

Theorem 4. *If G is a diamond-free graph with $\chi(K(G)) = \omega(K(G))$, then $T(G) = \chi(K(G))$.*

Proof. Let $d = \chi\big(K(G)\big) = \omega(K(G))$. Hence, there is a complete graph K_d, where $V(K_d) = \{v_1, \ldots, v_d\}$ in $K(G)$. Let $C_{v_1}, \ldots, C_{v_d}$ be the maximal cliques in G, such that each C_{v_i} is associated with vertex v_i in $K(G)$.

Since G is diamond-free, the cliques $C_{v_1}, \ldots, C_{v_d}$ compose an induced subgraph H and these cliques share exactly one vertex in G, that is universal in H, because any two maximal cliques of a diamond-free graph intersect in at most one vertex and each edge belongs to exactly one maximal clique. Since $\chi(K(G)) = d$, this coloring induces a tessellation cover with d tessellations in H, that is optimal for H, then $T(G) \geq \chi(K(G))$. By Theorem 1 $T(G) \leq \chi(K(G))$, then $T(G) = \chi(K(G))$. $\square$

A graph is *K-perfect* if its clique graph is perfect [15]. Since a diamond-free K-perfect graph G satisfies the premises of Theorem 4, we have $T(G) = \chi(K(G))$. Note that the size of the clique graph of a diamond-free graph is polynomially bounded by the size of the original graph [6]. Moreover, there is a polynomial-time algorithm to obtain an optimal coloring of $K(G)$ with $\omega(K(G))$ colors [16] and, by Theorem 1, a coloring of $K(G)$ with t colors yields that G is t-tessellable. Thus, both the tessellation cover number and a minimum tessellation cover of diamond-free K-perfect graphs are obtained in polynomial time.

Interestingly, there are diamond-free graphs whose clique graphs have chromatic number greater than the tessellation cover number. Fig. 6 illustrates an example of a 3-tessellable diamond-free graph whose clique graph has chromatic number 4 (the clique graph $K(G)$ is the Grötzsch graph, i.e. Mycielskian of a 5-cycle graph). Note that any minimum tessellation cover of this graph necessarily has an exposed maximal clique. Moreover, this graph shows that the upper bound of Theorem 5 is tight.

Lemma 3. *Let G be a 3-tessellable diamond-free graph. If C_1 and C_2 are two maximal cliques of G with a common vertex, then C_1 and C_2 cannot be both exposed by a minimum tessellation cover.*

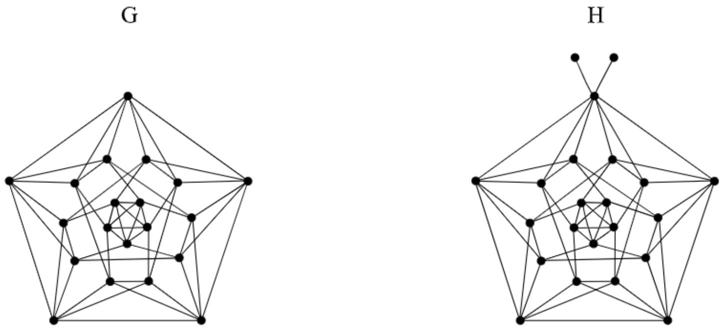

Fig. 7. Example of Construction 2. $T(G)=3$ and $\chi(K(G))=4$, but $T(H)=4$ and $\chi(K(H))=4$.

Proof. For the sake of the contradiction, assume that $v \in V(C_1 \cap C_2)$, and that C_1 and C_2 are both exposed maximal cliques. Since G is diamond-free, the vertex v is the only vertex in the intersection between cliques C_1 and C_2. Let $\mathcal{C}$ be a minimum tessellation cover with size at most 3. Let us focus on cliques of size at least two in tessellations that cover edges of C_1. In a tessellation, no clique of size at least two covers all edges from v to its neighbors because C_1 is an exposed maximal clique. Then, v belongs to at least two cliques of size at least two in different tessellations, where each one covers a proper subset of the edges of C_1. The same is true for C_2, that is, v belongs to at least two cliques of size at least two in different tessellations, where each one covers a proper subset of the edges of C_2. This means that at least four cliques of size at least two intersect on v and all of them must belong to different tessellations. This contradiction shows that C_1 and C_2 cannot be both exposed by a minimum tessellation cover, if G is a diamond-free graph. □

Theorem 5. *If G is a diamond-free graph with $T(G)=3$, then $3 \le \chi(K(G)) \le 4$.*

Proof. By Theorem 1, we have that $3 \le \chi(K(G))$. Given a minimum tessellation cover $\mathcal{C}=\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$ of G, Lemma 3 implies that the set of vertices in $K(G)$ that are associated with the exposed maximal cliques in G is a stable set in $K(G)$. $K(G)$ can be colored with four colors in the following way: the vertices in $K(G)$ that correspond to exposed cliques have color c_4; the vertices in $K(G)$ that correspond to a maximal clique fully contained in $\mathcal{T}_i$ have color c_i. This coloring shows that $\chi\big(K(G)\big) \le 4$. □

Now we present a construction which forces the tessellation cover number of a graph H, obtained from Construction 2 on a diamond-free graph G, to be $T(H)=\chi(K(H))=\chi(K(G))$. If G has $T(G) < \chi(K(G))$, then there is no vertex of G that belongs to $\chi(K(G))$ maximal cliques. The graph H obtained from G by Construction 2 satisfies $\chi(K(H))=\chi(K(G))$ and contains a vertex that belongs to $\chi(K(G))$ maximal cliques, which implies $T(H)=\chi(K(G))$.

Construction 2. Let H be obtained from a graph G by iteratively adding pendant vertices to a vertex of G until it belongs to $\chi(K(G))$ maximal cliques. See Fig. 7.

Construction 2 implies that every diamond-free graph G is a subgraph of a graph H with $T(H)=\chi(K(H))=\chi(K(G))$. Note that this construction is not restricted to diamond-free graphs and it can also be applied several times to vertices that only belong to one maximal clique. The hardness proofs of Theorems 8 and 9 rely on this result.

We finish this section showing that threshold graphs are extremal graphs by proving that the tessellation cover numbers of these graphs achieve the chromatic upper bound. The class of threshold graphs is hereditary and self-complementary [17]. We can describe a threshold graph as $G=(C \cup S, E)$, where C represents a maximum clique of G, S represents an independent set of G with nested neighborhood, and E represents the edge set of G. Threshold graphs can be constructed from an empty graph by repeatedly adding either an isolated vertex or a universal vertex. Considering a connected threshold graph $G=(C \cup S, E)$, in the clique graph $K(G)$, C is represented by vertex v_C, and each maximal clique containing vertices $v_i \in S, i \in \{1, \ldots, |S|\}$ is represented by a vertex v_{S_i}. Since there exists a universal vertex $u \in V(G)$, there are no disjoint maximal cliques in G, and the clique graph $K(G)$ is a complete graph with $|S|+1$ vertices. Moreover, note that its chromatic index $\chi'(G)$ can be arbitrarily large.

Theorem 6. *If $G=(C \cup S, E)$ is a connected threshold graph, then $T(G)=\chi(K(G))$.*

Proof. Since G does not have disjoint maximal cliques and its clique graph is the complete graph with size $|S|+1$, then $\chi(K(G))=|S|+1$. Hence, by Theorem 1, $T(G) \le |S|+1$. By the fact of C is a maximal clique in G, there is no vertex $v \in S$ such that v is neighbor of all vertices in C, otherwise there would exist a maximal clique C', greater than C, containing the clique C and the vertex v. The vertices of S in a threshold graph G must have a nested neighborhood [18]. Hence, there is at least one vertex $w \in V(C)$ such that w is not neighbor of any vertex in S. Since G has a universal vertex u, then G has

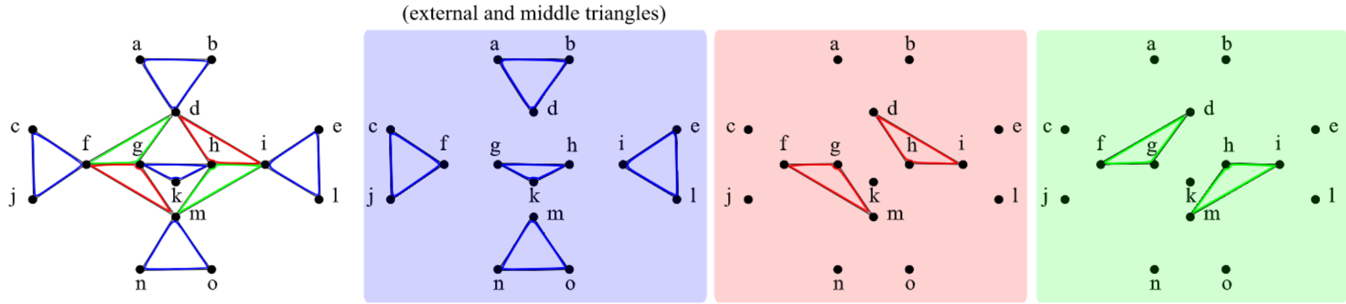

Fig. 8. The 3-tessellable *graph-gadget* of Lemma 5. Each tessellation is depicted separately. The external vertices are a, b, c, e, j, l, n, o, and the internal vertices are the remaining ones.

an induced star subgraph with $|S|+1$ leaves centered in u, all vertices of S and $w \in V(C)$ as its leaves. Therefore, such an induced star requires at least $|S|+1$ tessellations, i.e., $T(G) \geq |S|+1$.

We conclude that $T(G) = |S|+1 = X(K(G))$. □

Since to construct and to color the complete clique graph $K(G)$ can be done in polynomial-time for a threshold graph G, we conclude that t-TESSELLABILITY is polynomial-time solvable.

4. Computational complexity

Now, we focus on the computational complexity of t-TESSELLABILITY by firstly proving that the problem is in $\mathcal{NP}$. In Section 4.1, we use extremal graph classes obtained in the previous section to show $\mathcal{NP}$-completeness of planar graphs with maximum degree $\Delta(G) \leq 6$ for $t = 3$, biplanar graphs for $t \geq 3$, chordal $(2,1)$-graphs for $t \geq 4$, $(1,2)$-graphs for $t \geq 4$, and diamond-free graphs with diameter at most five for $t \geq 3$. In Section 4.2, we efficiently solve 2-TESSELLABILITY in linear time.

Lemma 4. *t-TESSELLABILITY is in $\mathcal{NP}$.*

Proof. Let G be an instance for t-TESSELLABILITY. If $t \geq \Delta(G)+1$, then by Theorem 1 and the well-known Vizing's theorem on Δ-EDGE COLORABILITY, the answer is always *YES*. When $t \leq \Delta(G)$, consider a certificate for t-TESSELLABILITY, which consists of at most t tessellations that cover the edge set $E(G)$. Note that each of these tessellations has at most $|E(G)|$ edges. One can easily verify in polynomial time if the at most $|E(G)|$ edges in each of the at most $t \leq \Delta(G)$ tessellations form disjoint cliques in G and if the at most $|E(G)|\Delta(G)$ edges in these tessellations cover $E(G)$. □

4.1. $\mathcal{NP}$-completeness

We remarked in Section 2.1 that the 3-TESSELLABILITY problem is $\mathcal{NP}$-complete for the triangle-free graphs. This result comes from the result presented by Koreas [11], who proved that Δ-EDGE COLORABILITY problem of triangle-free graphs with maximum degree three is $\mathcal{NP}$-complete. Since Δ-EDGE COLORABILITY of unichord-free graphs with girth at least 15 (which are triangle-free) for $\Delta \geq 3$ is $\mathcal{NP}$-complete [12], t-TESSELLABILITY for $t \geq 3$ is also $\mathcal{NP}$-complete for this graph class.

In this section, we present the $\mathcal{NP}$-completeness of the t-TESSELLABILITY problem of planar graphs with maximum degree $\Delta(G) \leq 6$ for $t = 3$ in Theorem 7, biplanar graphs for $t \geq 3$ in Theorem 8, chordal $(2,1)$-graphs for $t \geq 4$ in Theorem 9, $(1,2)$-graphs for $t \geq 4$ in Theorem 10, and diamond-free graphs with diameter at most five for $t \geq 3$ in Theorem 11.

A graph is *planar* if it can be embedded in the plane such that no two edges cross each other. We show a polynomial transformation from the $\mathcal{NP}$-complete 3-COLORABILITY of planar graphs with maximum degree four [9] to 3-TESSELLABILITY of planar graphs with maximum degree six.

Lemma 5. *Any tessellation cover of size 3 of the* graph-gadget *depicted in Fig. 8 contains a tessellation that covers the middle and the external triangles.*

Proof. Consider any tessellation cover of size 3 for the *graph-gadget* of Fig. 8. For the sake of the contradiction, assume that the triangle $\{a, b, d\}$ is exposed, needing to be covered by 3 tessellations, one tessellation for each one of its edges. However, the remaining neighborhood of vertex d does not induce a clique, needing at least other 2 tessellations to be covered, a contradiction with the fact that the graph-gadget is 3-tessellable.

Now, without loss of generality, assume that the triangle $\{a, b, d\}$ is covered by tessellation 1. If we cover the triangle $\{d, g, h\}$ with tessellation 2, we will need more two tessellations to cover the edges $\{d, f\}$ and $\{d, i\}$, a contradiction. Therefore, we need to cover the triangle $\{d, f, g\}$ with tessellation 3 and the triangle $\{d, i, h\}$ with tessellation 2.

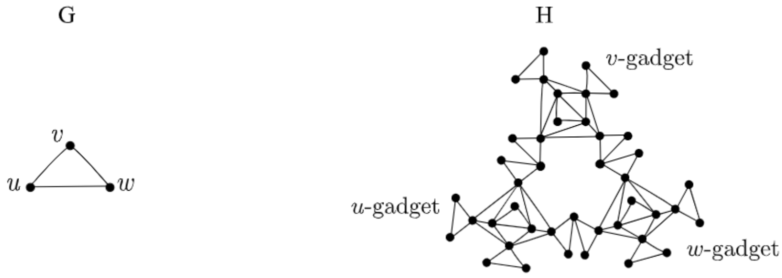

Fig. 9. Example of Construction 3.

Now, the middle triangle $\{g, h, k\}$ needs to be covered by tessellation 1, otherwise, we need the edge $\{g, h\}$ in tessellation 1 and as the set of vertices $\{g, k, m\}$ does not induce a clique, there will be two edges to be covered in the neighborhood of g with only one remaining tessellation, a contradiction. Next, we need to cover the triangles $\{f, g, m\}$ and $\{h, i, m\}$ with tessellations 2 and 3, respectively.

Finally, to obtain a 3-tessellation of this graph, the other external triangles $\{c, f, j\}$, $\{e, i, l\}$, and $\{m, n, o\}$ must be covered by tessellation 1. □

Construction 3. Let graph H be obtained from a graph G by local replacements of each vertex u of G for a graph-gadget of Fig. 8 denoted by u-gadget. Each edge uv of G represents the intersection of the u-gadget with the v-gadget by identifying two external vertices of external triangles of those graph-gadgets. See Fig. 9.

Theorem 7. 3-TESSELLABILITY *of planar graphs with* $\Delta(G) \leq 6$ *is* $\mathcal{NP}$*-complete.*

Proof. Let G be an instance graph of 3-COLORABILITY of planar graphs with $\Delta(G) \leq 4$ and H be obtained by Construction 3 on G. Notice that applying Construction 3 on a planar graph with $\Delta(G) \leq 4$ results on a planar graph with $\Delta(H) \leq 6$.

Suppose that G is 3-colorable. Then, H is 3-tessellable because the middle and the external triangles of a v-gadget can be covered by the tessellation related to the color of v and the remaining triangles of the v-gadget can be covered by the other two tessellations.

Suppose that H is 3-tessellable. Then, G is 3-colorable because the color of v in G can be related to the tessellation that covers the middle triangle of the v-gadget. This assignment is a 3-coloring because by Lemma 5 all external triangles of the v-gadget belong to the same tessellation of the middle triangle. The external triangles of the v-gadget are connected to the external triangles of the graph-gadgets of the neighborhood of v. Then, the tessellations of the latter external triangles must differ from the external triangles of the v-gadget. This implies that the neighborhood of vertex v receives different colors from the color of v. □

The next construction allows us to show a hardness proof of t-TESSELLABILITY, with any fixed $t \geq 3$ of biplanar graphs. A graph $G = (V, E)$ is biplanar if we can partition the edge set E into at most two sets E_1 and E_2 such that $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are planar graphs. Biplanar graphs are known as graphs of thickness ≤ 2. The polynomiality of Δ-EDGE COLORABILITY for planar graphs with $\Delta(G) \geq 8$ suggests that t-TESSELLABILITY for planar graphs might be polynomial-time solvable for large enough t. On the other hand, by Theorem 8, we know that t-TESSELLABILITY of biplanar graphs remains $\mathcal{NP}$-complete, even for a large value of t.

Construction 4. Let t be an integer and H be a graph obtained from a graph G as follows. Initially H is equal to G. Add a star S_t with t leaves. Add three paths P^1, P^2, and P^3 with $2|V(G)| + 1$ vertices each one. Identify the first vertex in each one of these paths with three different leaves of S_t. Let $V(P^1) = \{p_{1,1}, p_{1,2}, \dots, p_{1,2|V(G)|+1}\}$, $V(P^2) = \{p_{2,1}, p_{2,2}, \dots, p_{2,2|V(G)|+1}\}$, and $V(P^3) = \{p_{3,1}, p_{3,2}, \dots, p_{3,2|V(G)|+1}\}$. For each edge of type $(p_{i,2j+1}, p_{i,2j+2})$ (for $1 \leq i \leq 3$ and $0 \leq j \leq |V(G)| - 1$), add $t - 1$ vertices adjacent to both endpoints of the edge and, for each of these $t - 1$ vertices, add $t - 1$ pendant vertices adjacent to it. Add a stable set $U = \{u_1, u_2, \dots, u_{|V(G)|}\}$ and relate each one of these vertices with $V(G) = \{v_1, v_2, \dots, v_{|V(G)|}\}$. For each vertex $u_k \in U$ add the edges: $(u_k, p_{1,2k})$, $(u_k, p_{1,2k+1})$, $(u_k, p_{2,2k})$, $(u_k, p_{2,2k+1})$, $(u_k, p_{3,2k})$, and $(u_k, p_{3,2k+1})$. For each vertex $u_k \in U$ add the edge (u_k, v_k). For each vertex u_k, add the vertices $w_{k,m}$ (for $1 \leq m \leq t - 3$) adjacent to both u_k and v_k, and add $t - 1$ pendant vertices for each of these $w_{k,m}$ (for $1 \leq m \leq t - 3$) vertices. See Fig. 10.

Theorem 8. t-TESSELLABILITY *of biplanar graphs for* $t \geq 3$ *is* $\mathcal{NP}$*-complete.*

Proof. Let G be an instance graph for 3-COLORABILITY of planar graphs with $\Delta(G) \leq 4$, B be the graph obtained from Construction 3 on G, and H be the graph obtained from Construction 4 on B. We claim that H is t-tessellable (for $t \geq 3$) if and only if B is 3-tessellable. Therefore, the $\mathcal{NP}$-completeness follows immediately from Theorem 7.

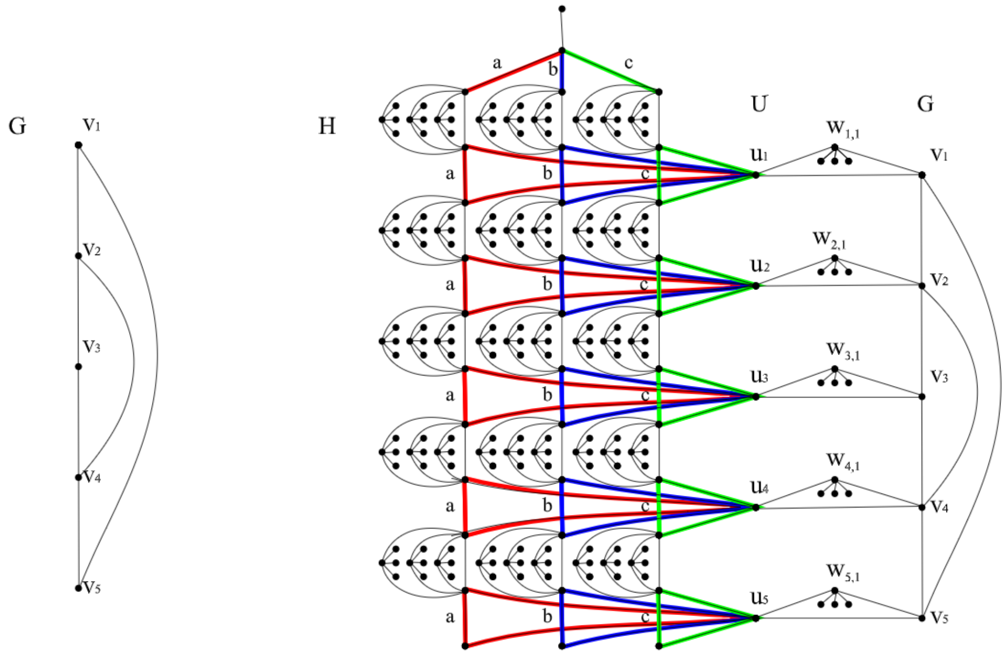

Fig. 10. Example of Construction 4, for $t = 4$. The graph H is biplanar since we can partition its edges into two planar graphs as follows. The edges of G, S_t, P^1, P^2, P^3, the triangles connected to these paths, and the pendant vertices incident to these vertices define a planar graph and the remaining edges (incident to vertices u_i and $w_{k,m}$) define other planar graph. Colors a, b, c highlight three tessellations.

Consider the case when H is t-tessellable. Let a, b, and c be three tessellations used to cover the edges of S_t which have one of their endpoints identified with P^1, P^2, and P^3, respectively. The $t-1$ triangles incident to the edges $(p_{1,2j-1}, p_{1,2j})$, for $1 \le j \le |V(G)|$, are not exposed since there are $t-1$ pendant vertices incident to a vertex in each of the triangles, which force them to be covered by $t-1$ tessellations. Therefore, for $j = 1$, the tessellation a cannot cover the corresponding $t-1$ triangles which implies that edge $(p_{1,2}, p_{1,3})$ must be covered by tessellation a, which in turn implies that all edges $(p_{1,2j}, p_{1,2j+1})$, for $1 \le j \le |V(G)|$, must be covered by tessellation a. The same holds with tessellation b and edges $(p_{2,2j}, p_{2,2j+1})$, and tessellation c and edges $(p_{3,2j}, p_{3,2j+1})$ (for $1 \le j \le |V(G)|$). Moreover, as the vertices $p_{1,2j-1}$ $(1 \le j \le |V(G)|)$ are in $t-1$ tessellations because the triangles incident to the edges $(p_{1,2j-1}, p_{1,2j})$, the triangles with vertices $u_k, p_{1,2k}, p_{1,2k+1}$ need to be not exposed and use the tessellation a. The same holds for tessellation b and the triangles with the vertices $u_k, p_{2,2k}, p_{2,2k+1}$, and for tessellation c and the triangles with the vertices $u_k, p_{3,2k}, p_{3,2k+1}$.

Now, as the $t-3$ vertices $w_{k,m}$ are not exposed (because they have $t-1$ pendant vertices incident to them), the triangles they are part with vertices of U and vertices of B need to be covered by a single tessellation. There are $t-3$ such $w_{k,m}$ vertices incident to each vertex of B and they are part of $t-3$ tessellations different from a, b, and c. Therefore, all vertices of B are part of these $t-3$ tessellations and it remains only three tessellations (a, b, and c) to cover the edges of the original graph B, i.e., if H is t-tessellable, then B is 3-tessellable.

Conversely, if B is 3-tessellable, we can cover the edges of the triangles in vertices $w_{k,m}$ with $t-3$ tessellations not used in B. Now, we can cover the edges of triangles $u_k, p_{1,2k}, p_{1,2k+1}$ with one of the three remaining tessellations a, b or c. Without loss of generality, let it be the tessellation a, the triangles $u_k, p_{2,2k}, p_{2,2k+1}$ be covered by tessellation b and the triangles $u_k, p_{3,2k}, p_{3,2k+1}$ be covered by tessellation c. Now we can cover the $t-1$ triangles which uses the edges of type $(p_{2,2j-1}, p_{2,2j})$ $(1 \le j \le |V(G)|)$ with the $t-1$ tessellations different from the one used to cover the edges $(p_{2,2j}, p_{2,2j+1})$. The remaining edges of pendant vertices are trivially covered by the non-used tessellations. Therefore, if B is 3-tessellable, then H is t-tessellable. □

A graph is (k, ℓ) if its vertex set can be partitioned into at most k stable sets and at most ℓ cliques. Next, we show a polynomial transformation from the $\mathcal{NP}$-complete 3-COLORABILITY [9] to 4-TESSELLABILITY of chordal $(2, 1)$-graphs, and then we generalize this proof for any fixed $t \ge 4$. This proof is based on a result of Bodlaender et al. [19] for 3-$L(0, 1)$-COLORABILITY of split graphs.

Construction 5. Let H be a graph obtained from a non-bipartite graph G as follows. Initially $V(H) = V(G) \cup E(G)$ and $E(H) = \emptyset$. Add edges to H so that the $E(G)$ vertices induce a clique. For each $e = vw \in E(G)$, add to H edges ve and we. For each vertex $v \in V(H) \cap V(G)$, add three pendant vertices adjacent to v. Add a vertex u adjacent to all $E(G)$ vertices. Add three pendant vertices adjacent to u. Denote all pendant vertices by V_2. See Fig. 11.

Theorem 9. *The t-TESSELLABILITY of chordal $(2, 1)$-graphs is $\mathcal{NP}$-complete, for any fixed $t \ge 4$.*

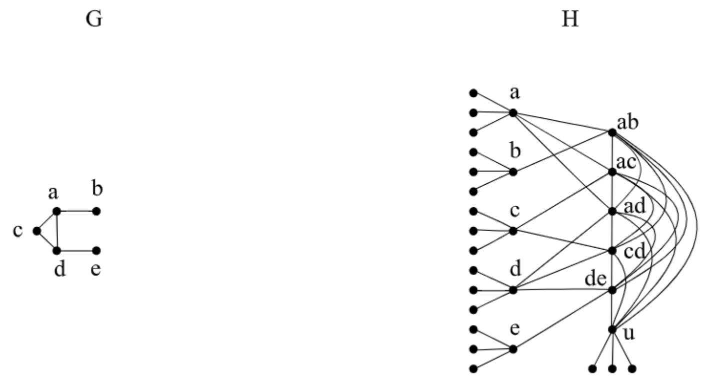

Fig. 11. Example of Construction 5.

Proof. Firstly, we show that the 4-TESSELLABILITY is $\mathcal{NP}$-complete for chordal (2, 1)-graphs, and then, we extend to the cases when $t \geq 4$, for any fixed t.

Let G be a non-bipartite instance graph of 3-COLORABILITY. We show that G is 3-colorable if and only if H, obtained by Construction 5 on G, is 4-tessellable.

Let f be a 3-coloring of G. Consider the following tessellation cover: for any vertex $v \in V(G)$, cover the maximal clique it belongs with vertices of $E(G)$ in H with tessellation $f(v)$, and use tessellation 4 to cover the maximal clique of vertices of $E(G)$ with vertex u. Now, cover the three pendant vertices of each vertex of $V(G)$ and u with their 3 non incident tessellations. Note that all edges of H were covered and two maximal cliques between vertices of $V(G)$ and $E(G)$ in H can only share a vertex in $E(G)$. However, if the maximal cliques of these vertices share a vertex in $E(G)$, it means these two vertices are adjacent in G and, therefore, their maximal cliques are covered by different tessellations.

Conversely, consider a tessellation cover of H with 4 tessellations. We need the maximal clique given by u and the vertices of $E(G)$ not be exposed. Additionally, the tessellation used to cover it cannot cover any other maximal clique between vertices of $V(G)$ and $E(G)$. Therefore, there are only three remaining tessellations to cover them.

Each vertex of $V(G)$ in H has 4 maximum cliques incident to them sharing only one vertex. Thus, all maximal cliques incident to them must not be exposed. Note that if two vertices v and w of G are adjacent, then their related maximal cliques between vertices of $V(G)$ and $E(G)$ share a vertex vw. Therefore, the colors of $f(v)$ and $f(w)$, which are related to the tessellations that cover these maximal cliques, are different.

This proof holds for t-TESSELLABILITY, with $t \geq 5$, of chordal (2, 1)-graphs. The idea is to use the same proof considering $(t-1)$-COLORABILITY of G instead of 3-COLORABILITY, and adding the necessary number of pendant vertices to H to force all its maximal cliques not to be exposed.

Note that the vertices of H can be partitioned into one clique and two stable sets: The vertices in H related with $E(G)$ and vertex u define a clique, the vertices in H related with $V(G)$ define a stable set, and the pendant vertices define another stable set. Moreover, clearly H is chordal as the induced graph by the vertices related with $E(G)$ and $V(G)$ is a split graph (a subclass of chordal graph), and the addition of pedant vertices does not create any cycles in the graph, i.e., H is chordal. □

Construction 6. Let H' be a graph obtained from the graphs G and H of Construction 5 by transforming the stable set S of H corresponding to $V(G)$ into a clique, removing one pendant vertex of each vertex of S, and adding a vertex u' adjacent to all vertices of S with three new pendant vertices adjacent to it. See Fig. 12.

Theorem 10. *The t-TESSELLABILITY of (1, 2)-graphs is $\mathcal{NP}$-complete, for any fixed $t \geq 4$.*

Proof. Firstly, we show that the 4-TESSELLABILITY is $\mathcal{NP}$-complete for (1, 2)-graphs, and then, we extend to the cases when $t \geq 4$, for any fixed t.

Consider the graph H', obtained from Construction 6 on graph H of Theorem 9 for 4-TESSELLABILITY. Clearly, H' is a (1, 2)-graph. We will show that H is 4-tessellable if and only if H' is 4-tessellable.

In the 4-tessellation cover given by the proof of Theorem 9, an edge of a pendant vertex of each of $V(G)$'s vertices is covered by tessellation 4 (the same tessellation of the maximal clique of u and the vertices of $E(G)$). Define 3 tessellations of H' using the first three tessellations of H. Now, we cover the edges in the maximal clique of $V(G)$'s vertices and u' with tessellation 4 and the three remaining edges of the pendant vertices incident to u' with tessellations 1, 2, and 3.

Consider a tessellation cover of H' with 4 tessellations. First, the maximal clique of vertices of $V(G)$ and u' must be covered by the same tessellation of the maximal clique of vertices of $E(G)$ and u. For the sake of the contradiction, assume these two maximal cliques are covered by different tessellations. Therefore, now there are only two available tessellations to cover maximal cliques between vertices of $V(G)$ and $E(G)$ in H'. However, these maximal cliques are related to a coloring of vertices of G and if we could obtain a tessellation cover of them using only two colors, then G would be a bipartite graph (which we exclude from the 3-COLORABILITY instance graphs), a contradiction.

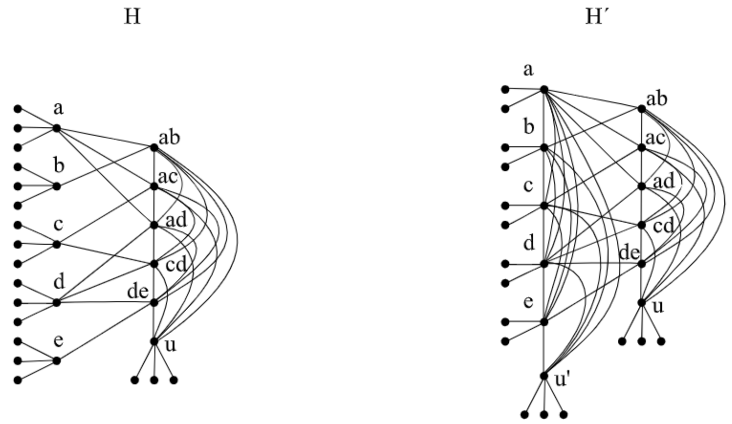

Fig. 12. Example of Construction 6.

Now we obtain a tessellation cover of H with the same number of tessellations as follows. We remove the edges of the maximal clique of $V(G)$ and u' (which are all covered by the tessellation 4). Then, we remove the vertex u' and its pendant vertices. Moreover, we add a pendant vertex to each vertex of $V(G)$ with the tessellation 4 covering their edges.

This proof holds for t-TESSELLABILITY with $t \geq 5$ of $(1, 2)$-graphs considering the t-TESSELLABILITY of chordal $(2, 1)$-graphs with $t \geq 5$ presented in Theorem 9. Moreover, the $\mathcal{NP}$-completeness of $(t-1)$-COLORABILITY for non-$(t-2)$-colorable graphs, for $t \geq 5$, holds by the following facts: (1) an edge coloring of a graph Γ is equivalent to a vertex coloring of its line graph $L(\Gamma)$; (2) the k-EDGE-COLORABILITY problem is $\mathcal{NP}$-complete for any fixed $k = \Delta(\Gamma) \geq 3$ [20], and; (3) the line graph of a graph Γ is non-$(\Delta(\Gamma) - 1)$-colorable because a vertex of degree $\Delta(\Gamma)$ of Γ implies a clique of size $\Delta(\Gamma)$ in $L(\Gamma)$. □

Next, we show a polynomial transformation from the $\mathcal{NP}$-complete problem NAE 3-SAT [9] to 3-TESSELLABILITY of diamond-free graphs with diameter at most five. The NAE 3-SAT problem consists in finding, in a given set U of literals and a set C of clauses all of size three, if we can assign true/false values to each literal in U satisfying all clauses in C, with the restriction that in each clause at least one literal must have true value and at least one literal must have false value. Hereinafter, we do not consider clauses with two repeated variable, i.e., if there is a clause (v_1, v_1, v_2), we create a new instance I' with a new variable x and exchange the previous clause for two clauses (x, v_1, v_2) and $(\overline{x}, v_1, v_2)$, such that I is satisfiable if and only if I' is satisfiable.

This proof is given in two phases: given an instance I of NAE 3-SAT we construct a graph B for which we show that there is a 3-coloring of B if and only if I is satisfiable; subsequently, we show that there is a construction of a diamond-free graph G with diameter at most five for which G is 3-tessellable if and only if $K(G)$ is 3-colorable and $K(G)$ is isomorphic to B.

Construction 7. Let B be a graph obtained from an instance of NAE 3-SAT as follows. For each variable v of I, include a P_2 with vertices v and $\overline{v}$ in B. Moreover, add a vertex u adjacent to all P_2's vertices. And, for each clause $\{a \vee b \vee c\}$ of I, add a triangle with vertices T_a, T_b, T_c in B and three edges aT_a, bT_b, and cT_c. See Fig. 13.

Lemma 6. *Let B be obtained from Construction 7 on a* NAE 3-SAT *instance I. Then B is 3-colorable if and only if I is satisfiable.*

Proof. If B is 3-colorable, then there are no three vertices connected to a clause's triangle with the same color. Moreover, without loss of generality, the color 1 given to the vertex u in a 3-coloring cannot be used in any vertex of a P_2. Besides, each one of the literal vertices v and $\overline{v}$ of a P_2 receives either the color 2 or 3. Assume without loss of generality that a literal is true if its color is 2, and false otherwise. Therefore, the above assignment of values to literals gives a satisfiable solution to the instance.

Conversely, if I is satisfiable, then one may assign color 2 to each literal vertex which is true and color 3 to its negation. Moreover, vertex u receives color 1. Since there are no three literal vertices with the same color adjacent to the clause triangles, one may assign colors to the vertices of the triangles in a 3-coloring where one vertex of the triangle adjacent to a vertex with color 2 receives color 3, and one vertex adjacent to a vertex with color 3 receives color 2. The remaining vertex receives color 1. □

Next, we construct a graph G whose clique graph $K(G)$ is isomorphic to graph B obtained from Construction 7.

Construction 8. Let G be obtained from the graph B (of Construction 7), which is isomorphic to the clique graph $K(G)$ of G, as follows. For each clause's triangle in B, add a star with three leaves in G, where each of those leaves represents a literal of this clause. Next, all P_2's triangles in B are represented in G by a clique C of size the number of P_2's. Each vertex of this

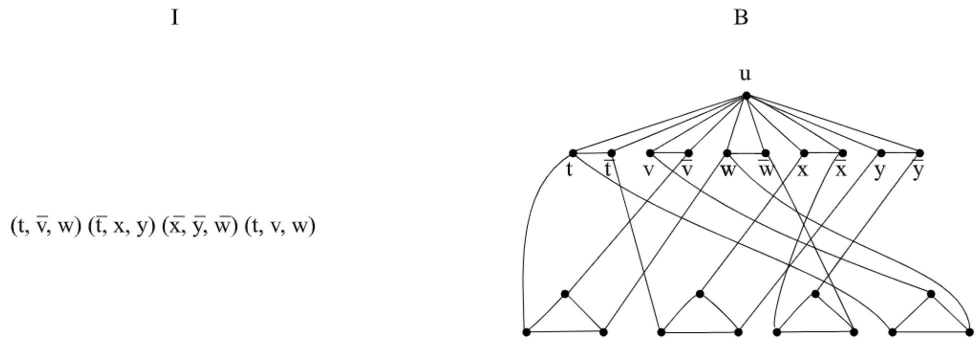

Fig. 13. Example of Construction 7.

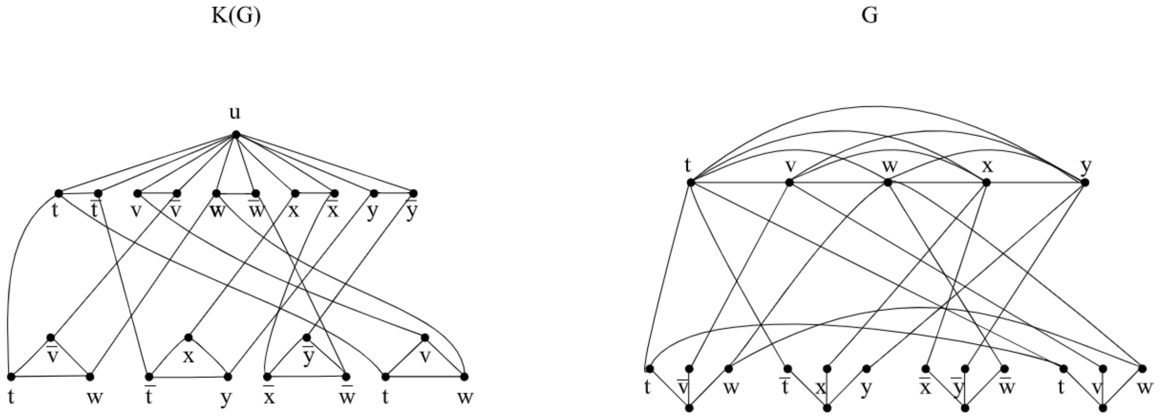

Fig. 14. Example of Construction 8, where $K(G)$ is isomorphic to graph B (Construction 7).

clique C represents a variable of I (of Construction 7). For each vertex v of C include the edges of two other cliques (one for each literal of the variable v) composed by the leaves of the stars which represent the literals v and $\overline{v}$ and the vertex v of C, as depicted in Fig. 14.

Lemma 7. *Let B be obtained by Construction 7 on a* NAE 3-SAT *instance I and G be obtained by Construction 8 on B, such that B is isomorphic to $K(G)$. Then G is 3-tessellable if and only if $K(G)$ is 3-colorable.*

Proof. If G is 3-tessellable, then we need one tessellation to cover the clique composed by the vertices related with the variables, whose size is the number of variables. Therefore, the other two tessellations are used by at most two maximal cliques of each variable, which represent their literals. Moreover, the star of three leaves of each clause also needs to be covered by 3 tessellations. Note that these maximal cliques represent vertices in $K(G)$ and the tessellations represent their colors. Therefore, $K(G)$ is 3-colorable.

If $K(G)$ is 3-colorable, by Theorem 1 G is 3-tessellable. □

Clearly, the graph G obtained from Construction 8 is diamond-free with diameter at most five. Therefore, by Lemmas 6 and 7, the next theorem follows.

Theorem 11. 3-TESSELLABILITY *of diamond-free graphs with diameter at most five is $\mathcal{NP}$-complete.*

4.2. 2-TESSELLABILITY

Portugal [4] showed that a graph G is 2-tessellable if and only if $K(G)$ is a bipartite graph. Moreover, Peterson [6] showed that $K(G)$ is bipartite if and only if G is the line graph of a bipartite multigraph. Hence, determine if G is 2-tessellable is equivalent to verifying if G is the line graph of a bipartite multigraph.

Protti and Szwarcfiter [21] showed an $O(n^2m)$ time algorithm to decide if the clique graph of a given graph is bipartite. Moreover, Peterson [6] showed an $O(n^3)$ time algorithm to decide if G is the line graph of a bipartite multigraph.

A vertex u is *true twin* of a vertex v of a graph G if u and v have the same closed neighborhood in G. The key idea of Peterson's algorithm is to group true twin vertices of a same clique of a line graph G. These true twin vertices represent multiedges in the bipartite multigraph H, where $G = L(H)$. Then, it removes all those true twin vertices in each group but one, and the resulting graph is a line graph of a bipartite simple graph if and only if $K(G)$ is a bipartite graph. To verify if a graph is a line graph of a bipartite graph, the Roussopoulos' linear-time algorithm is used [22].

We improve Peterson's algorithm [6], by showing a faster way to remove true twin vertices belonging to a clique of a graph using its modular decomposition. Throughout this section, we use notations of modules of a graph given in [23]. In a graph G, a subset S of $V(G)$ is a emphmodule if all elements of S have the same set of neighbors among vertices that are in $V(G)\backslash S$. We say that S is a *strong module* if for every module S', $S \cap S' = \emptyset$, or $S \subseteq S'$, or $S' \subseteq S$ holds. A strong module $S \subsetneq V(G)$ is a maximal strong module if the only strong module properly containing S is $V(G)$.

Let $\mathcal{F}$ be the family of bipartite multigraphs obtained by adding multiple edges to C_4, S_n or P_4. In order to make a modular decomposition of a graph G, we only consider graphs G which are not line graphs of a graph in $\mathcal{F}$. If G is a line graph of a graph in $\mathcal{F}$, we can consider this case separately, and easily achieve linear time. Note that there are bipartite multigraphs with a same line graph. Therefore, we only consider the ones which maximize the number of multiple edges. Moreover, we only consider connected graphs, since the tessellation cover number of a disconnected graph is the maximum among the parameter on its connected components.

Lemma 8. *Let H be a bipartite multigraph not in $\mathcal{F}$ and $L(H)$ be its line graph. Two edges e_1 and e_2 with same endpoints in H represent vertices in a same maximal strong module of $L(H)$.*

Proof. By hypothesis, we consider H a bipartite multigraph. Therefore, we do not consider the cases H has an induced cycle of odd size, including triangles and other complete graphs. For the sake of the contradiction, assume there are such two edges e_1 and e_2 of H and that its related vertices in $L(H)$ are in different maximal strong modules M_1 and M_2. Since the vertices associated to e_1 and e_2 are adjacent in $L(H)$, there are all edges between vertices of M_1 and M_2 in $L(H)$.

(Case 1) Assume there is a vertex outside M_1 and M_2. Therefore, without loss of generality, there is a vertex $e_3 \notin (M_1 \cup M_2)$ such that $e_3 \in N(w_1)$ for all $w_1 \in M_1$ and $e_3 \notin N(w_2)$ for all $w_2 \in M_2$, otherwise, $M_1 \cup M_2$ would be a maximal strong module, a contradiction with the fact that M_1 and M_2 were maximal. However, e_1 and e_2 are multiedges with same endpoints of H, i.e., there cannot be another edge e_3 in H which shares an endpoint with e_1 but does not share one endpoint with e_2, as e_1 and e_2 have the same endpoint vertices, a contradiction.

(Case 2) Assume there is no vertex outside M_1 and M_2, and neither M_1 nor M_2 induces cliques in $L(H)$. Note that all vertices in M_1 and M_2 are adjacent to e_2 and e_1, respectively. Let w_1 and w'_1 be two non-adjacent vertices of M_1 and w_2 and w'_2 be two non-adjacent vertices of M_2. Therefore, w_1 and w'_1 are edges in H that share vertices of e_2 in H, as both of them are non-adjacent vertices which are adjacent to e_2 in $L(H)$. Note that w_2 and w'_2 must be adjacent to w_1, w'_1, and e_1, i.e., they must have the same endpoints as e_1 and e_2 in H. However, w_2 and w'_2 cannot be incident to the same endpoints in H since they are not adjacent in $L(H)$, a contradiction.

(Case 3) Assume there is no vertex outside M_1 and M_2, and M_1 or M_2 induces cliques in $L(H)$. If M_1 and M_2 induces cliques in $L(H)$, then $L(H)$ is a complete graph and H is a multigraph of a P_2 (which is a star multigraph), a contradiction with the fact that H is not a graph in $\mathcal{F}$. Otherwise, without loss of generality, let M_1 induces a clique in $L(H)$. Note that the vertices of M_1 represent multiedges with same endpoints in H, since H is a bipartite multigraph that maximizes the number of multiple edges. Moreover, all vertices of M_2 are adjacent to e_1 (and to e_2). Therefore, the vertices of M_2 in $L(H)$ represent edges in H incident to one of the endpoints of e_1 and e_2 (and two of those edges in different endpoints do not share other same endpoint, or H would not be a bipartite graph). However, this is a contradiction as H is a multigraph of a P_4 which is a graph in $\mathcal{F}$. $\square$

Lemma 9. *Let H be a bipartite multigraph not in $\mathcal{F}$ and $L(H)$ be its line graph. Any maximal strong module in a modular decomposition of $L(H)$ with size less than $|V(L(H))|$ induces a clique in $L(H)$.*

Proof. By hypothesis, we consider H a bipartite multigraph. Therefore, we do not consider the cases H has an induced cycle of odd size, including triangles and other complete graphs. For the sake of the contradiction, assume there is such strong module M_k of $L(H)$, which is not a clique. Note that $|M_k| \geq 2$, and since M_k does not induce a clique, then there are two vertices w_k and w'_k in M_k that are not adjacent. Therefore, as w_k and w'_k are not adjacent in $L(H)$, then w_k and w'_k have different endpoint in H.

(Case 1) All vertices of M_k in $L(H)$ share both endpoints in H with edges w_k or w'_k. As $L(H)$ is connected, there is a vertex i outside M_k that shares one endpoint with w_k and other endpoint with w'_k. Moreover, there could be another vertex j outside M_k that shares one endpoint with w_k and other endpoint with w'_k different from the endpoints of i. However, all other vertices outside M_k must share the same endpoints of i or j, otherwise there would be an edge $\cancel{k}$ in H which shares an endpoint with i (or j) and did not share an endpoint with w_k and w'_k, a contradiction. Therefore, H is a multigraph of a C_4 or a P_4, which are in $\mathcal{F}$, a contradiction.

(Case 2) There is a vertex of M_k in $L(H)$ whose endpoints do not coincide with edge w_k nor with edge w'_k.

(Case 2.a) There is a vertex x of M_k that shares one endpoint with w_k and the other with w'_k in H. As $L(H)$ is a connected graph, there is a vertex i outside of M_k adjacent to all vertices in M_k. Note that i must share both endpoints with x in H. Assume there is a vertex j such that no vertex in M_k is adjacent to j in $L(H)$, therefore as $L(H)$ is connected, there is a vertex l adjacent to all vertices of M_k, that is adjacent to a vertex l', where l' is adjacent to no vertex in M_k.

Table 1
Extremal graph classes and tight upper bounds.

Graph class	$T(G) \leq \min\{\chi'(G), \chi(K(G))\}$	Reference
Bipartite	$T(G) = \chi'(G) = \Delta(G)$	Sec. 2.1
Triangle-free	$T(G) = \chi'(G)$	Sec. 2.1
Unichord-free with girth ≥ 15	$T(G) = \chi'(G) = \Delta(G)$	Sec. 2.1
$Wd_{p,q}$	$T(Wd_{p,q}) = \chi(K(Wd_{p,q})) = q$	Sec. 2.2, [5]
G_p, $p \in \{2,3\}$	$T(G_p) = \chi'(G_p) = \chi(K(G_p)) = 2p$	Theorem 2
G_p, any p	$T(G_p) = \chi'(G_p) = 2p$	Theorem 2
$E_{3,p}$	$T(G) = 3$	Theorem 3
Diamond-free K-perfect	$T(G) = \chi(K(G)) = \omega(K(G))$	Theorem 4
Threshold	$T(G) = \chi(K(G)) = \|S\| + 1$	Theorem 6

However, this is a contradiction, because l shares both endpoints with i, while l' shares one endpoint with l, but l' does not share any endpoint with neither w_k nor w'_k. Therefore, all vertices outside M_k are adjacent to all vertices in M_k, and H is a P_4 multigraph in $\mathcal{F}$.

(Case 2.b) Each vertex y of M_k shares precisely one endpoint either with w_k or with w'_k in H. Note that with a similar reasoning of (Case 2.a) all vertices outside M_k must be adjacent to all vertices of M_k, and H is a P_4 multigraph in $\mathcal{F}$.

(Case 2.c) There is a vertex z of M_k that shares no endpoint with neither w_k nor with w'_k in H. As $L(H)$ is connected, there is a vertex i adjacent to all vertices in M_k, however, i has only two endpoints and it must share at least one endpoint with all edges of z, w_k and w'_k in H, which have distinct endpoints, a contradiction. $\square$

Theorem 12. 2-TESSELLABILITY *can be solved in linear time.*

Proof. First, we use McConnell and Spinrad's linear-time algorithm to obtain a modular decomposition of G. By Lemmas 8 and 9, we know that the strong modules in any modular decomposition of a line graph of a bipartite multigraph $H \notin \mathcal{F}$ induce cliques. Moreover, the vertices of these cliques in $L(H)$ are related to edges of H with same endpoints.

Then, we check if each of at most $O(|V(G)|)$ strong modules induces cliques in G, which can be done in $O(|V(G)| + |E(G)|)$. Otherwise, we know that G is not a line graph of a bipartite multigraph. Next, we remove all true twins vertices in each strong modules but one, obtaining the graph G'. This step is related to remove all multiedges of H which share same endpoints. Therefore, the graph G is a line graph of a bipartite multigraph H if the resulting graph G' is a line graph of a simple bipartite graph H'.

Finally, we use Roussopoulos' linear-time algorithm to determine if G' is a line graph, and if so, obtain its root graph H' whose line graph is isomorphic to G'. Note that verifying if H' is a bipartite graph can be done in linear time by using a breadth-first search (because the size of the root graph of G' is asymptotically bounded by the size of G'). $\square$

5. Concluding remarks and discussion

We investigate the tessellation cover number for extremal graph classes, which are fundamental for the development of quantum walks in the staggered model. These results help to understand the complexity of the unitary operators necessary to express the evolution of staggered quantum walks. We establish tight upper bounds for the tessellation cover number of a graph G related to the chromatic parameters $\chi'(G)$ and $\chi(K(G))$, and we determine graph classes which reach these upper bounds. This study provides tools to distinguish several classes for which the t-TESSELLABILITY problem is efficiently tractable (bipartite graphs, {triangle, proper major}-free graphs, diamond-free K-perfect graphs, and threshold graphs) from others where the problem is $\mathcal{NP}$-complete for $t \geq 3$ (planar graphs, triangle-free graphs, chordal $(2,1)$-graphs, $(1,2)$-graphs, and diamond-free graphs with diameter at most five). We also establish the t-TESSELLABILITY $\mathcal{NP}$-completeness for biplanar graphs. Moreover, we improve to linear-time the known algorithm to recognize line graphs of bipartite multigraphs [21], and consequently, for 2-tessellable graphs [4], and graphs G such that $K(G)$ is bipartite [6]. Table 1 and Table 2 summarize the extremal graph classes and the complexity of the t-TESSELLABILITY problem, respectively, for the graph classes studied in this paper.

We establish an interesting complexity dichotomy between Δ-EDGE COLORABILITY and t-TESSELLABILITY: Δ-EDGE COLORABILITY of planar graphs with $\Delta(G) \geq 8$ is in $\mathcal{P}$ [24], while t-TESSELLABILITY for $t \geq 3$ is $\mathcal{NP}$-complete, (Theorem 7 replacing each of the four non external triangles that share two vertices of external triangles by K_4's) and; Δ-EDGE COLORABILITY of line graph of bipartite graphs for $\Delta \geq 3$ is $\mathcal{NP}$-complete [25], while t-TESSELLABILITY is in $\mathcal{P}$ (Theorem 12). We have not managed yet to establish the same dichotomy between k-COLORABILITY OF CLIQUE GRAPH and t-TESSELLABILITY.

Regarding (k,ℓ)-graph classes, since any (k,ℓ)-graph is a $(k+1,\ell)$-graph and a $(k,\ell+1)$-graph, the $\mathcal{NP}$-completeness of t-TESSELLABILITY for $(1,2)$-graphs and $(2,1)$-graphs imply that the problem is $\mathcal{NP}$-complete for (k,ℓ)-graphs with $k+\ell \geq 3$ and $\min\{k,\ell\} \geq 1$ for $t \geq 4$. We are currently working on the complexity of t-TESSELLABILITY for split graphs that are a super class of threshold graphs of Theorem 6, $(k,0)$-graphs with $k \geq 3$, and $(0,\ell)$-graphs with $\ell \geq 2$.

A question that naturally arises is whether every graph has a minimum tessellation cover such that every tessellation contains a maximal clique. Although we believe in most cases the answer is true, we have computationally found a surprising example of a graph, which is depicted in Fig. 15, with all minimum tessellation covers requiring a tessellation without

Table 2
The complexity of the t-TESSELLABILITY problem for graph classes.

t	Graph class	Complexity	Reference
$t = 2$	Generic	Linear	Theorem 12
$t = 3$	Planar, $\Delta(G) \leq 6$	$\mathcal{NP}$-complete	Theorem 7
	Diamond-free, diameter = 5	$\mathcal{NP}$-complete	Theorem 11
$t \geq 3$	Threshold	Polynomial	Sec. 3
	Bipartite	Polynomial	Sec. 2.1
	{triangle, proper major}-free	Polynomial	Sec. 2.1
	Diamond-free K-perfect	Polynomial	Sec. 3
	Unichord-free with girth ≥ 15	$\mathcal{NP}$-complete	Sec. 2.1
	Triangle-free	$\mathcal{NP}$-complete	Sec. 2.1
	Biplanar	$\mathcal{NP}$-complete	Theorem 8
$t \geq 4$	Chordal $(2, 1)$-graphs	$\mathcal{NP}$-complete	Theorem 9
	$(1, 2)$-graphs	$\mathcal{NP}$-complete	Theorem 10

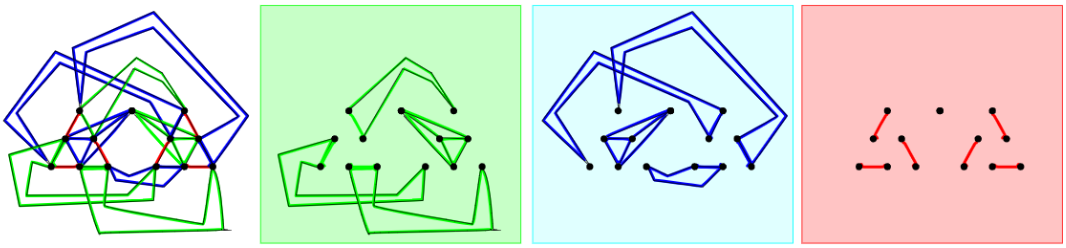

Fig. 15. 3-tessellable graph. Rightmost tessellation does not contain a maximal clique.

maximal cliques. We are currently trying to establish an infinite family of graphs for which this property does not hold and to establish other graph classes where it holds. The computational verification was performed through a reduction from t-TESSELLABILITY problem to SET-COVERING problem (the description of SET-COVERING is available at [9]), where the finite set is the edge set of the input graph, and the family of subsets consists of the edge subsets corresponding to all possible tessellations of the input graph. Another interesting issue is that two minimum tessellation covers may present different quantum walk dynamics. Therefore, we intend to study the different tessellation covers using the same number of tessellations, which may result in simpler quantum walks and more efficient quantum algorithms. More recently, a general partition-based framework for quantum walks has been proposed [26].

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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Appendix B

Manuscript "The tessellation cover number of good tessellable graphs"

The Tessellation Cover Number of Good Tessellable Graphs

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Abstract

A tessellation of a graph is a partition of its vertices into vertex disjoint cliques. A tessellation cover of a graph is a set of tessellations that covers all of its edges, and the tessellation cover number, denoted by $T(G)$, is the size of a smallest tessellation cover. The t-TESSELLABILITY problem aims to decide whether a graph G has $T(G) \leq t$ and is $\mathcal{NP}$-complete for $t \geq 3$. Since the number of edges of a maximum induced star of G, denoted by $is(G)$, is a lower bound on $T(G)$, we define good tessellable graphs as the graphs G such that $T(G) = is(G)$. The GOOD TESSELLABLE RECOGNITION (GTR) problem aims to decide whether G is a good tessellable graph. We show that GTR is $\mathcal{NP}$-complete not only if $T(G)$ is known or $is(G)$ is fixed, but also when the gap between $T(G)$ and $is(G)$ is large. As a byproduct, we obtain graph classes that obey the corresponding computational complexity behaviors.

1 Introduction

It is known that there is a strong connection between the areas of graph theory and quantum computing. For instance, algebraic graph theory provides many tools to analyze the time-evolution of the continuous-time quantum walk, because its evolution operator is directly defined in terms of the graph's adjacency matrix. Recently, a new discrete-time quantum walk model has been defined by using the concept of graph tessellation cover [10]. Each tessellation in the cover is associated with a unitary operator and the full evolution operator is the matrix product of those operators. For practical applications, it is interesting to characterize graph classes that admit small-sized covers. Accordingly, we establish a new lower bound on tessellation cover that is described next.

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Throughout this paper we only consider undirected and simple graphs. A *tessellation* of a graph G is a partition of its vertices into vertex disjoint cliques. A *tessellation cover* of G is a set of tessellations that covers all of its edges. The *tessellation cover number* of G, denoted by $T(G)$, is the size of a smallest tessellation cover of G. If G admits a tessellation cover of size t, then G is *t-tessellable*. The t-TESSELLABILITY problem aims to decide whether G is t-tessellable. We disregard cliques of size one in a tessellation since they play no role in our proofs. The *star number*, denoted by $is(G)$, is the number of edges of a maximum induced star of G. Notice that $T(G) \geq is(G)$, since any two edges of an induced star cannot be covered by a same tessellation. We say that G is *good tessellable* if $T(G) = is(G)$, and the GOOD TESSELLABLE RECOGNITION (GTR) problem aims to decide whether a graph is good tessellable.

The known results about the tessellation cover number up to now were related to upper bounds on T(G) and the complexities of the t-TESSELLABILITY problem [2, 1, 11]. Abreu et al. [2] verified that $T(G) \leq \min\{\chi'(G), \chi(K(G))\}$, and they proved that t-TESSELLABILITY is in $\mathcal{P}$ for quasi-threshold, diamond-free K-perfect graphs, and bipartite graphs. On the other hand, they showed that the problem is $\mathcal{NP}$-complete for triangle-free graphs, unichord-free graphs, planar graphs with $\Delta \leq 6$, $(2,1)$-chordal graphs, $(1,2)$-graphs, and diamond-free graphs with diameter at most five. Surprisingly, all the hardness results presented by Abreu et al. [2] for t-TESSELLABILITY aim to decide whether $t = is(G)$, i.e., if the instance graph is good tessellable. Therefore, all their $\mathcal{NP}$-complete proofs for t-TESSELLABILITY also hold for GTR. The only previous $\mathcal{NP}$-completeness result for t-TESSELLABILITY for non good tessellable graphs was presented by Posner et al. [11] for line graphs of triangle-free graphs (where $t = 3$ and $is(G) = 2$).

We recently discovered that the concept of tessellation cover of graphs has been independently studied in the literature for a same problem, named as EQUIVALENCE COVERING by Duchet [4] in 1979. Since the tessellation cover number $T(G)$ and the equivalence covering number $eq(G)$ are the same parameter, we highlight the common results, as follows: $\chi'(G)$ is an upper bound for $T(G)$ [2] and for $eq(G)$ [3]; if G is triangle-free, then $T(G) = \chi'(G)$ [2] and $eq(G) = \chi'(G)$ [5]; if G is triangle-free, then 3-TESSELLABILITY of line graphs $L(G)$ is $\mathcal{NP}$-complete [11] and to decide whether $eq(G) \leq 3$ for the same class is $\mathcal{NP}$-complete as well [5]; if G is $(2,1)$-chordal, then t-TESSELLABILITY is $\mathcal{NP}$-complete for $t \geq 4$ [2], whereas EQUIVALENCE COVERING is $\mathcal{NP}$-complete for $(1,1)$-graphs [3].

Contributions

We propose the GTR problem, which aims to decide whether a graph is good tessellable. We analyze the combined behavior of the computational complexity of the problems t-TESSELLABILITY, GTR, and k-STAR SIZE. Clearly, these three problems belong to $\mathcal{NP}$.

k-STAR SIZE	t-TESSELLABILITY	GTR
Instance: Graph G and integer k.	**Instance:** Graph G and integer t.	**Instance:** Graph G.
Question: $is(G) \geq k$?	**Question:** $T(G) \leq t$?	**Question:** $T(G) = is(G)$?

In order to highlight our results, we define graph classes using triples that specify the computational complexities of k-STAR SIZE, t-TESSELLABILITY, and GTR, summarized in Table 1.

Table 1: Computational complexities of k-STAR SIZE, t-TESSELLABILITY, and GTR problems and examples of corresponding graph classes.

Behavior \ Problem	k-STAR SIZE	t-TESSELLABILITY	GTR	Examples
(a)	$\mathcal{P}$	$\mathcal{NP}$-complete	$\mathcal{NP}$-complete	[2, 1]
(b)	$\mathcal{P}$	$\mathcal{NP}$-complete	$\mathcal{P}$	[11] Sec. 3
(c)	$\mathcal{NP}$-complete	$\mathcal{P}$	$\mathcal{NP}$-complete	Sec. 2
(d)	$\mathcal{NP}$-complete	$\mathcal{P}$	$\mathcal{P}$	Sec. 2
(e)	$\mathcal{NP}$-complete	$\mathcal{NP}$-complete	$\mathcal{P}$	Sec. 3

All graph classes for which Abreu et al. [2] presented hardness proofs for t-TESSELLABILITY obey behavior (a), since for those classes $is(G)$ is fixed and equal to t. The graphs studied by Posner et al. [11] obey behavior (b), since for those graphs $is(G) = 2$ and 3-TESSELLABILITY is $\mathcal{NP}$-complete. In Section 3, we present additional examples that obey behavior (b) with $T(G)$ arbitrarily larger than a non fixed $is(G)$. Graphs of Construction 2.2 (I) in Section 2 are examples that obey behavior (c), since $T(G)$ is known but k-STAR SIZE is $\mathcal{NP}$-complete for $k = T(G)$, which implies that GTR is $\mathcal{NP}$-complete. Graphs of Construction 2.2 (II) in Section 2 are examples that obey behavior (d), because k-STAR SIZE is $\mathcal{NP}$-complete for $k = T(G) - 1$, $T(G)$ is known, and $T(G) > is(G)$, which implies GTR is in $\mathcal{P}$. Graphs of Construction 3.2 in Section 3 are examples that obey behavior (e), since it is known that $T(G) > is(G)$, which implies GTR is in $\mathcal{P}$, and we construct graphs so that k-STAR SIZE and t-TESSELLABILITY are $\mathcal{NP}$-complete.

Notice that there are omitted triples in Table 1. Threshold graphs and bipartite graphs are examples of graph classes that obey the behavior $(\mathcal{P}, \mathcal{P}, \mathcal{P})$ [2]. There are no graphs that obey the behavior $(\mathcal{P}, \mathcal{P}, \mathcal{NP}$-complete$)$, since if both k-STAR SIZE and t-TESSELLABILITY are in $\mathcal{P}$, so is GTR. Graph classes obtained by the union of graphs G_1 and G_2 so that G_1 is in a graph class that obey behavior (a) and G_2 is in a graph class that obey behavior (c) are examples satisfying the behavior $(\mathcal{NP}$-complete$, \mathcal{NP}$-complete$, \mathcal{NP}$-complete$)$.

Notation and graph theory terminologies

Given a graph $G = (V, E)$, the neighborhood $N(v)$ (or $N_G(v)$) of a vertex $v \in V$ of G is given by $N(v) = \{u \mid uv \in E(G)\}$. $\Delta(G)$ is the size of a maximum neighborhood of a vertex of G. We say that a vertex u of G is universal if $|N(u)| = |V(G)| - 1$. A graph is *universal* if it has a universal vertex. A *clique* of G is a subset of V with all possible edges between its vertices. An *independent set* of G is a subset of V with no edge between any of its vertices. A *matching* of G is a subset of edges of E without a common endpoint. A *k-coloring* of G is a partition of V into k independent sets. A *k-clique cover* of G is a partition of V into k cliques. A *k-edge coloring* of G is a partition of E into k matchings.

The parameters $\alpha(G)$, $\omega(G)$, and $\mu(G)$ are the size of a maximum independent set, the size of the maximum clique, and the size of the maximum matching of a graph G, respectively. The *chromatic number* $\chi(G)$ (*chromatic index* $\chi'(G)$) is the minimum k for which G admits a k-coloring (k-edge coloring), and the *clique cover number* $\theta(G)$ is the minimum k for which G admits a k-clique cover. Note that $\theta(G) = \chi(G^c)$ and $\alpha(G) = \omega(G^c)$, where G^c denotes the *complement* of G for which $V(G^c) = V(G)$ and $E(G^c) = \{xy | x \in V(G), y \in V(G), x \neq y\} \setminus E(G)$. The k-COLORABILITY (k-EDGE COLORABILITY) aims to decide whether a graph G has $\chi(G) \leq k$ ($\chi'(G) \leq k$). The k-INDEPENDENT SET problem aims to decide whether a graph G has $\alpha(G) \geq k$.

The *line graph* $L(G)$ of a graph G is the graph such that each edge of $E(G)$ is a vertex of $V(L(G))$, and two vertices of $V(L(G))$ are adjacent if their corresponding edges in G have a common endpoint. The *clique graph* $K(G)$ of a graph G is the graph such that each maximal clique of G is a vertex of $V(K(G))$, and two vertices of $V(K(G))$ are adjacent if their corresponding maximal cliques in G have a common vertex. $S_k(G)$ is the graph obtained from G by subdividing k times each edge $e = xy \in E(G)$, i.e., each edge $e = xy$ is replaced by a path $(x, v_1, v_2, \ldots, v_k, y)$.

The *union* $G \cup H$ of two graphs G and H has $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The *join* $G \vee H$ of two graphs G and H has $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{vw \mid v \in V(G)$ and $w \in V(H)\}$. An *induced subgraph* $H = (V_H, E_H)$ of a graph $G = (V_G, E_G)$ has $V_H \subseteq V_G$ and $E_H = \{vw \mid v \in V(H), w \in V(H)$, and $vw \in E(G)\}$. $G[S]$ is the induced subgraph of G by the set of vertices $S \subseteq V(G)$.

2 Graphs with known $T(G)$

We prove in this section that GTR is $\mathcal{NP}$-complete for graphs of Construction 2.2 (I), which have a known tessellation cover number. Using this result, we provide a graph class that obeys the behavior (c) and another graph class that obeys behavior (d). Note that if the tessellation cover number of G is upper bounded by a constant, then we obtain $is(G)$ in polynomial time using a brute force algorithm.

The Mycielski graph M_j for $j \geq 2$ has chromatic number j, maximum clique size 2, and is defined as follows. $M_2 = K_2$ and for $j > 2$, M_j is obtained from M_{j-1} with vertices $v_1, \ldots, v_{|V(M_{j-1})|}$ by adding vertices $u_1, \ldots, u_{|V(M_{j-1})|}$ and one more vertex w. Each vertex u_i is adjacent to all vertices of $N_{M_{j-1}}(v_i) \cup \{w\}$.

Construction 2.1. Let i be a non-negative integer and G a graph. The (i, G)-graph is obtained as follows. Add i vertices to graph G, and then add a universal vertex.

Construction 2.2. Let i be a non-negative integer and G a graph with $V(G) = \{v_1, \ldots, v_n\}$. We construct a graph $H = H_1 \cup H_2$ as follows. Add i disjoint copies $G_1, \ldots, G_i$ of G to H_1, such that $V(G_j) = \{v_1^j, \ldots, v_n^j\}$ for $1 \leq j \leq i$, where v_k^j represents the same vertex v_k of G for $1 \leq k \leq n$. Add to H_1 all possible edges between pairs of vertices that represent the same vertex of G. Add a vertex u to H_1 adjacent to all v_k^j for $1 \leq j \leq i$ and $1 \leq k \leq n$. Now, we consider two possibilities: either (I) H_2 is $(|V(G)| - 3, M_3^c)$-graph of Construction 2.1 or (II) H_2 is $(|V(G)| - 3, M_4^c)$-graph of Construction 2.1. Denote the universal vertex of H_2 by u'.

Figure 1 provides an example of a graph of Construction 2.2 (I). In (a) we have an edge coloring of the graph $G \vee \{x\}$ with $|V(G)|$ colors. In (b) we have the graph $H = H_1 \cup H_2$ and a tessellation cover of H with $|V(G)|$ tessellations.

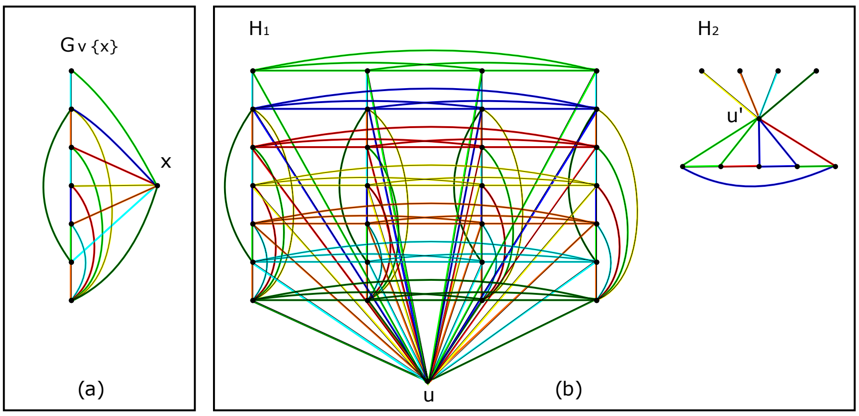

Figure 1: (a) An edge-coloring of $G \vee \{x\}$. (b) Example of a graph $H_1 \cup H_2$ of Construction 2.2 (I) obtained from graph G.

We now verify that the graphs of Construction 2.2 (I) obey the behavior (c) by showing that $T(H)$ is a known fraction of the number of vertices of H and by deciding whether $is(H) \geq k$ is $\mathcal{NP}$-complete for $k = T(H)$. This also implies that the graphs of Construction 2.2 (II) obey the behavior (d), since we have increased $T(H)$ by one unit when we have replaced M_3^c by M_4^c in H_2. In this case $T(H) > is(H)$ and GTR is in $\mathcal{P}$ with answer always no, whereas to decide whether $is(H) \geq k$ remains $\mathcal{NP}$-complete for $k = T(H) - 1$.

Theorem 2.1. *k-STAR SIZE and GTR are $\mathcal{NP}$-complete for graphs of Construction 2.2 (I).*

Proof. Let G be a graph without a universal vertex and an instance of the q-COLORABILITY problem, a well-known $\mathcal{NP}$-complete problem [6]. Consider the graph $H = H_1 \cup H_2$ of Construction 2.2 (I) on G with $i = q$.

We need 3 tessellations to cover the edges of $M_3^c \vee \{u'\}$, and another $|V(G)|-3$ tessellations to cover the remaining edges of the pendant vertices, thus, by construction, $T(H_2) = |V(G)|$. Moreover, since $\alpha(M_3^c) = 2$, then $is(H_2) = |V(G)| - 1$.

We define a tessellation cover of H_1 with $|V(G)|$ tessellations as follows. Consider an optimum edge-coloring of the graph $G \vee \{x\}$. Since G has no universal vertex, x is the unique universal vertex and we know that $\chi'(G \vee \{x\}) = \Delta(G \vee \{x\}) = |V(G)|$ [8]. Now, when we remove x and the edges incident to it, this edge-coloring is a tessellation cover of G with $|V(G)|$ tessellations, where for each vertex there is a distinct unused tessellation. We now use this tessellation cover to each copy of G in H_1. Next, we entirely cover each clique between vertices that represent the same vertex of G and the edges incident to u with the available tessellation for this clique. Therefore, $T(H_1) \leq |V(G)|$.

We have $T(H) = \max\{T(H_1), T(H_2)\} = |V(G)| = \frac{|V(H)|-1}{q}$ and $is(H) = \max\{is(H_1), is(H_2)\}$. Since $is(H_2) = |V(G)| - 1$, H is good tessellable if and only if $is(H_1) = |V(G)|$. Poljak [9] proved that a graph G admits a q-coloring if and only if $\alpha(H_1 \setminus \{u\}) = |V(G)|$. Since $is(H) = \alpha(H_1 \setminus \{u\})$, deciding whether H is good tessellable is equivalent to deciding whether G is q-colorable. $\square$

3 Universal graphs

The local behavior of tessellation covers given by Lemma 3.1 below motivates us to study universal graphs in this section, since the induced subgraph $G[\{v\} \cup N(v)]$ is a universal graph. We prove that t-TESSELLABILITY remains $\mathcal{NP}$-complete even if the gap between $T(G)$ and $is(G)$ is large. Using this proof, we provide a graph class that obeys behavior (e).

Given a t-tessellable graph G and a vertex $v \in V(G)$, we consider the relation between $\chi(G^c[N_G(v)])$ and the cliques of those t tessellations that share a vertex v. Note that these cliques cover all edges incident to v in any tessellation cover of G. Moreover, the vertices of the neighborhood of v in a same tessellation are a clique in G and, therefore, they are an independent set in G^c. The independent sets in G^c given by these cliques of $N_G(v)$ may share some vertices, and we can choose whichever color class they belong in such coloring of $G^c[N_G(v)]$. Therefore, for any vertex v of G, $\chi(G^c[N_G(v)]) \leq t$. Since $is(G[v \cup N_G(v)]) = \omega(G^c[N_G(v)])$, $is(G[v \cup N_G(v)]) = \omega(G^c[N_G(v)]) \leq \chi(G^c[N_G(v)]) \leq t$, and we have the following result.

Lemma 3.1. *If G is a t-tessellable graph, then*

$$\max_{v \in V(G)} \{is(G[v \cup N_G(v)])\} \leq \max_{v \in V(G)} \{\chi(G^c[N_G(v)])\} \leq t.$$

Let $u \notin V(G)$ be a vertex. If $G \vee \{u\}$ is a t-tessellable graph, then

$$is(G \vee \{u\}) = \alpha(G) \leq \chi(G^c) \leq t.$$

We start this subsection by showing that

$$\chi(G^c) \leq T(G \vee \{u\}) \leq \chi(G^c) + \Delta(G) + 1. \tag{1}$$

The lower bound is given by Lemma 3.1. The upper bound holds because we can use $\chi(G^c)$ tessellations to cover a partition P of the vertices of G in cliques $p_1, p_2, \ldots, p_i$ by assigning a different tessellation for each p_j for $1 \leq j \leq i$. Moreover, the edges between u and the vertices of p_j maintain the same tessellation of p_j. The remaining edges between vertices of $p_1, p_2, \ldots, p_i$ are covered by cliques of size two with the non used $\Delta(G)+1$ tessellations following an edge coloring of such edges. Thus, there is no universal graph such that the gap between $T(G \vee \{u\})$ and $\chi(G^c)$ is larger than $\Delta(G) + 1$. In particular, if $\chi(G^c) \geq 2\Delta(G) + 1$, then by Theorem 3.1 below $T(G \vee \{u\}) = \chi(G^c)$.

Theorem 3.1. *A graph $G \vee \{u\}$ with $\theta(G) \geq 2\Delta(G)+1$ has $T(G \vee \{u\}) = \theta(G)$.*

Proof. Note that $\theta(G) = \chi(G^c)$. Consider a graph $G \vee \{u\}$. By Lemma 3.1, $T(G \vee \{u\}) \geq \chi(G^c)$. Now we prove that $T(G \vee \{u\}) \leq \chi(G^c)$. Since $\chi(G^c) \geq 2\Delta(G) + 1$, there is a tessellation cover of $G \vee \{u\}$ with $\chi(G^c)$ tessellations as follows. Use the $\chi(G^c)$-coloring of G^c as a guide to define a partial tessellation of the edges of G with $\chi(G^c)$ tessellations in so that each color class of G^c induces a clique of G entirely covered by the tessellation related to this color class. Moreover, we extend the tessellations so that the edges uw are covered by the corresponding tessellation of the color class of w. Now, the remaining edges in $G \vee \{u\}$ are the ones between vertices of G that are not in a same color class in the coloring of G^c. The maximum number of tessellations incident to the endpoints of an edge xy is $2\Delta(G)$ because 2 tessellations come from the edges ux and uy, and $2\Delta(G) - 2$ come from the edges of G incident to x and y. Therefore, $T(G \vee \{u\}) \leq \chi(G^c)$ because it is possible to greedily assign tessellations of cliques of size two to these edges.

□

Corollary 3.1. *A graph $G \vee \{u\}$ with $is(G \vee \{u\}) = \alpha(G \vee \{u\}) \geq 2\Delta(G) + 1$ has $T(G \vee \{u\}) = \chi(G^c)$. Moreover, if H is a $(2\Delta(G) + 1, G)$-graph of Construction 2.1 on G with $2\Delta(G) + 1$ pendant vertices to u, then $T(H) = \theta(G) = \chi(G^c) + 2\Delta(G) + 1$.*

Proof. Note that if $\alpha(G \vee \{u\}) \geq 2\Delta(G)+1$, then $\chi(G^c) \geq \omega(G^c) = \alpha(G \vee \{u\}) \geq 2\Delta(G) + 1$ and, by Theorem 3.1, $T(G \vee \{u\}) = \chi(G^c)$. Consider now the graph H. Since each pendant vertex added to u in H does not modify $\Delta(G)$, and it increases $is(H)$ by one unit, $is(H) \geq 2\Delta(G) + 1$. Moreover, each of those pendant vertices is a universal vertex in G^c, increasing $\chi(G^c)$ by one unit. Thus, $T(H) = \chi(H^c) = \chi(G^c) + 2\Delta(G) + 1$. □

Good tessellable universal graphs

A universal graph $G \vee \{u\}$ is good tessellable if $T(G \vee \{u\}) = is(G \vee \{u\})$. In this case, by Lemma 3.1, $T(G \vee \{u\}) = \chi(G^c) = is(G \vee \{u\})$. Therefore, if

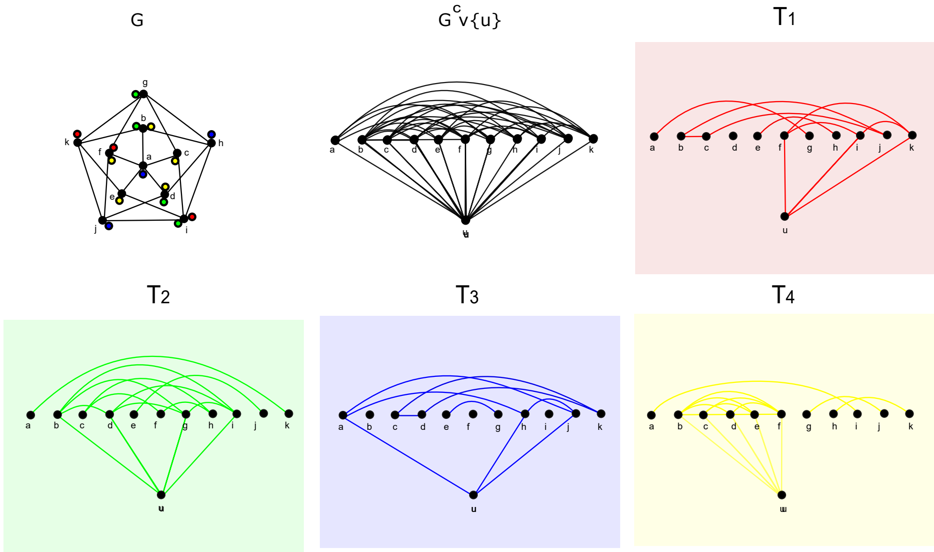

Figure 2: A Tessellation cover of $M_j^c \vee \{u\}$ with 4 tessellations and possible 4-colorings of M_4 guided by this tessellation cover.

$G \vee \{u\}$ has $T(G \vee \{u\}) > \chi(G^c)$, then it is not a good tessellable graph. By Corollary 3.1, if $\alpha(G \vee \{u\}) \geq 2\Delta(G)+1$, then $T(G \vee \{u\}) = \chi(G^c)$, and $G \vee \{u\}$ is good tessellable when $\chi(G^c) = \omega(G^c) = is(G \vee \{u\})$.

The computational complexity of GTR of a subclass of universal graphs depends on the restrictions used to define the subclass. On the one hand, perfect graphs G with $\alpha(G) \geq 2\Delta(G)+1$ can be recognized in polynomial time [7], and the addition of a universal vertex results in a good tessellable universal graph. On the other hand, planar graphs G with $\Delta(G) \leq 4$ and $\alpha(G) \geq 2\Delta(G)+1 = 9$ for which to decide whether $\chi(G) = \omega(G) = 3$ is $\mathcal{NP}$-complete [6].

Graphs with arbitrary gap between $T(G)$ and $is(G)$

We start by showing that the gap between $T(G)$ and $is(G)$ can be arbitrarily large for graphs G composed by the join of the complement of Mycielski graphs with a vertex u.

Since the Mycielski graph M_j is triangle-free [12], the graph M_j^c has no independent set of size three and $is(M_j^c \vee \{u\}) = 2$. Moreover, $\chi(M_j) = j$ [12], and by Lemma 3.1, $T(M_j^c \vee \{u\}) \geq \chi((M_j^c)^c) \geq j$. Fig. 2 depicts an example of the Mycielski graph M_4 and the relation between its 4-coloring and a minimal tessellation cover of $M_4^c \vee \{u\}$. Therefore, there is a graph $H = M_j^c \vee \{u\}$ with $is(H) = 2$ and $T(H) \geq j$ for $j \geq 3$.

Now, we describe a subclass of universal graphs for which the gap between $T(G)$ and $is(G)$ is very large. We also show that k-STAR SIZE and t-

TESSELLABILITY are $\mathcal{NP}$-complete for graphs of Construction 3.2, for which GTR is in $\mathcal{P}$.

Construction 3.1. Let $G = (V, E)$ be a graph. Obtain $S_2(G)$ by subdividing each edge of G two times, so that each edge $vw \in E(G)$ becomes a path v, x_1, x_2, w, where x_1 and x_2 are new vertices. Let $L(S_2(G))$ be the line graph of $S_2(G)$. Add a universal vertex u to $L(S_2(G))$, that is, consider the graph $L(S_2(G)) \vee \{u\}$.

First, we show that there is a connection between $T(H)$ of a graph H of Construction 3.1 on G with the size of a maximum stable set of G.

Theorem 3.2. *If $G = (V, E)$ is a graph with $|E(G)| \geq 4$ and $H = (L(S_2(G)) \vee \{u\})$ is obtained from Construction 3.1 on G, then $T(H) = |V(G)| + |E(G)| - \alpha(G)$.*

Proof. We claim that $T(H) = \chi((H \setminus \{u\})^c)$. By Lemma 3.1, $T(H) \geq \chi((H \setminus \{u\})^c)$. Now, we show that $T(H) \leq \chi((H \setminus \{u\})^c)$. Consider a partial tessellation cover of $H \setminus \{u\}$ with $\chi((H \setminus \{u\})^c)$ tessellations induced by a coloring of $(H \setminus \{u\})^c$, so that cliques of $H \setminus \{u\}$ are assigned to tessellations associated to different colors of $(H \setminus \{u\})^c$. Since $H \setminus \{u\} = L(S_2(G))$ is the line graph of a $S_2(G)$ graph, every vertex of $(H \setminus \{u\})^c$ has a maximum clique of size two and another maximum clique incident to it with an arbitrary size. Consider now a maximum clique K_a of size at least three which is not completely covered yet. The partial tessellation cover cannot have two cliques completely inside K_a (otherwise their merge would result in a coloring of the complement graph with less than its chromatic number). Therefore, the edges of K_a are partially covered at this moment with one clique, and the remaining cliques covering the vertices of K_a are the maximal cliques of size two that are incident to the vertices of K_a. Thus, if the partial tessellation cover of K_a has only maximal cliques of size two given by edges incident to K_a, then each edge e of K_a has at most two already used tessellations on cliques incident to their endpoints (the ones given to these maximal cliques of size two).

Poljak [9] proved that $\chi(L(S_2(G))^c) = |V(G)| + |E(G)| - \alpha(G)$. Since $|E(G)| \geq 4$ and $\alpha(G) \leq |V(G)|$, $|V(G)| + |E(G)| - \alpha(G) \geq 4$ and there is at least one available tessellation for each edge of K_a. We claim that these available tessellations for each edge are enough to extend this partial tessellation cover to all edges of K_a. First, pick an arbitrary tessellation for each edge. Since the endpoint vertices of any collection of edges of K_a on a same available tessellation do not have these tessellations incident to their endpoints, we cover the clique induced by these vertices with this tessellation.

Otherwise, K_a has a clique K_b in the partial tessellation cover and all the other vertices of K_a must be covered by maximal cliques of size two with edges outside K_a. So, modify this partial tessellation cover assigning the tessellation of K_b into all edges of K_a and remove the vertices of K_a from cliques of size two on this partial tessellation cover, i.e., now they are cliques of size one and K_a is entirely covered by the tessellation of K_b.

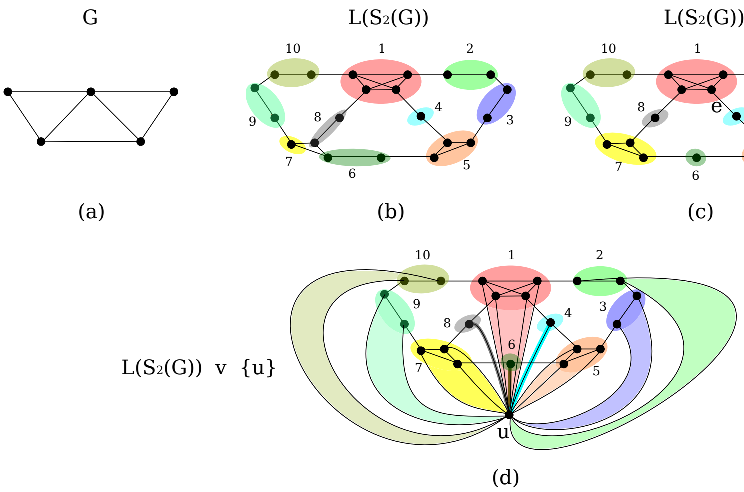

Figure 3: A tessellation cover of $H = L(S_2(G)) \vee \{u\}$ with $|V(G)|+|E(G)|-\alpha(G)$ tessellations.

The remaining uncovered edges of $H \setminus \{u\}$ in this partial tessellation cover are maximal cliques of size two. Now, if an edge is uncovered and it is incident to a maximal clique of size two or more, then we need this clique to be entirely covered by a single tessellation. Therefore, the maximum number of already used tessellations incident to the endpoints of a remaining edge is three. Since there are $|V(G)| + |E(G)| - \alpha(G) \geq 4$ tessellations, there is always an available tessellation for these edges. Finally, the edges incident to u are covered by the tessellations which covered the cliques of $H \setminus \{u\}$. Then, $T(H) \leq \chi((H \setminus \{u\})^c)$. □

Figure 3 depicts the proof of Theorem 3.2. In (a), we have graph G. In (b), we have a clique cover of $L(S_2(G))$. In (c), we modify the clique cover so that the clique with label 7 is covered by a new tessellation and at the same time we remove the vertices of the cliques of size two incident to the clique with label 7. Now the cliques with labels 6 and 8 have only one vertex each. Finally, in (d) we extend the partial tessellation cover of $L(S_2(G))$ to include the edges incident to u.

Since deciding whether $\alpha(G) \geq k$ is $\mathcal{NP}$-complete [6], by Theorem 3.2 we have the following result for the graphs of Construction 3.1.

Corollary 3.2. t-TESSELLABILITY *is $\mathcal{NP}$-complete for universal graphs.*

Proof. Let G be an instance graph of k-INDEPENDENT SET with $|E(G)| \geq 4$. We know that deciding whether $\alpha(G) \geq k$ is $\mathcal{NP}$-complete [6]. Consider the graph H of Construction 3.1 on G with $H = L(S_2(G)) \vee \{u\}$. By Theorem 3.2, $T(H) = |E(G)| + |V(G)| - \alpha(G)$. Therefore, deciding whether $\alpha(G) \geq k$ is equivalent to decide whether $T(H) \leq t = |E(G)| + |V(G)| - k$. □

Next, we show that there are graphs of Construction 3.1 for which the gap between $T(G)$ and $is(G)$ is very large, whereas t-TESSELLABILITY remains $\mathcal{NP}$-complete.

Theorem 3.3. *Let G be a graph and $H = L(S_2(G')) \vee \{u\}$ be obtained from Construction 3.1 on G', where G' is obtained from G by adding x universal vertices, with x polynomially bounded by the size of G. To decide whether $T(H) = k$ with $k \geq is(H) + c$, for $c = (O|V(G)|^d)$ and constant d, is $\mathcal{NP}$-complete.*

Proof. Consider a graph G and $L(S_2(G)) \vee \{u\}$ as described in Corollary 3.2. Note that $is(L(S_2(G)) \vee \{u\}) = \alpha(L(S_2(G))) = \mu(S_2(G))$. We claim that $\mu(S_2(G)) = |E(G)|+\mu(G)$. There are three edges in $S_2(G)$ between two adjacent vertices of G. In a maximum matching of $S_2(G)$, we need to select at least one of them, otherwise, we could include the middle edge to a maximum matching, which is a contradiction. Moreover, if there is only one edge and it is not a middle edge, then we obtain another maximum matching by replacing this edge by the middle edge. Clearly, we cannot choose three edges and in case we choose two edges, different from the middle edge. The case of two edges forces that both of them are incident to vertices of $S_2(G)$ associated to vertices of $V(G)$. Therefore, the maximum number of such selection of two edges in $S_2(G)$ is equal to the size of a maximum matching of G. For each edge in a maximum matching $\mu(G)$ of G we have two edges in the maximum matching in $S_2(G)$ and, for each other edge of G, we have one edge in the maximum matching of $S_2(G)$. Thus, $\mu(S_2(G)) = 2\mu(G) + |E(G)| - \mu(G) = |E(G)| + \mu(G)$.

By Theorem 3.2, $T(H) = |E(G)|+|V(G)|-\alpha(G)$. The addition of universal vertices to G does not modify $\alpha(G)$. The addition of each universal vertex may add one unit to $\mu(G)$ until it reaches $|V(G)|$. Then, we add one unit to $\mu(G)$ for each addition of two universal vertices. In that case, we start to increase the difference between $T(H) = |E(G)| + |V(G)| - \alpha(G)$ and $is(H) = |E(G)| + \mu(G)$, since for each two universal vertices we add to G, we increase $T(H)$ by two units and $is(H)$ by one unit. Therefore, we can arbitrarily enlarge the gap between $T(G)$ and $is(G)$. And, as long as the addition of these universal vertices are polynomially bounded by the size of G, it holds the same polynomial transformation of Corollary 3.2 from k-INDEPENDENT SET of G' (obtained from G by the addition of universal vertices) to t-TESSELLABILITY of $L(S_2(G')) \vee \{u\}$. □

Finally, we show that the graphs from Construction 3.2 below obey behavior (e).

Construction 3.2. *Let H_1 be the graph obtained from Construction 2.2 (I) on a given graph G_1 and a non-negative integer i. Let H_2 be the graph obtained from Construction 3.1 on the graph $G_2 \vee K_{3|V(G_1)|}$ of a given graph G_2. Let u and u' be the two universal vertices of the two connected components of H_1. Add $is(H_2)$ degree-1 vertices to H_1 adjacent to u and $is(H_2)$ degree-1 vertices adjacent to u'. Consider $H_1 \cup H_2$.*

Theorem 3.4. *k-STAR SIZE and t-TESSELLABILITY are $\mathcal{NP}$-complete for graphs of Construction 3.2, for which GTR is in $\mathcal{P}$.*

Proof. Let G_1 be an instance graph with no universal vertex of the well-known $\mathcal{NP}$-complete problem q-COLORABILITY [6]. Let G_2 be an instance graph of the well-known $\mathcal{NP}$-complete problem p-INDEPENDENT SET with $E(G_2) \geq 4$ [6]. Consider a graph $H = H_1 \cup H_2$ obtained from Construction 3.2 on G_1 and G_2 with $i = q$.

Since H_2 is obtained from Construction 3.1 on $G_2 \vee K_{3|V(G_1)|}$, by Theorem 3.3, $T(H_2) - is(H_2) > |V(G_1)|$. By Theorem 2.1, $1 \leq is(H_1) \leq T(H_1) = |V(G_1)|$. The parameter $is(H_2)$ can be obtained in polynomial time by applying a maximum matching algorithm [6] (see Theorem 3.3). And the addition of the degree-1 vertices to H_1 of Construction 3.2 implies that $1 + is(H_2) \leq is(H_1) \leq T(H_1) = |V(G_1)| + is(H_2)$.

Therefore, $H = H_1 \cup H_2$ is a graph that obeys $is(H_2) \leq is(H_1) \leq T(H_1) \leq T(H_2)$ with $T(H) = T(H_2)$ and $is(H) = is(H_1)$. The proof holds because GTR is in $\mathcal{P}$ with answer always no and both k-STAR SIZE on graphs H_1 of Construction 2.2 (I) (see Theorem 2.1) and t-TESSELLABILITY on graphs H_2 of Construction 3.1 (see Theorem 3.3) are $\mathcal{NP}$-complete. $\square$

4 Concluding remarks

The concept of tessellation cover of graphs appeared in a thesis by Duchet [4], and subsequently in [3, 5], as EQUIVALENCE COVERING. The known results about tessellation cover number of a graph up to now were related to upper bounds of the values of $T(G)$, and the complexities of the t-TESSELLABILITY problem [2]. In this work we focus on a different approach by analyzing the tessellation cover number $T(G)$ with respect to $is(G)$, one of its lower bounds, which implicitly appeared in the previous hardness proofs of [2].

The motivation to define the tessellation cover number comes from the analysis of the dynamics of quantum walks on a graph G in the context of quantum computation [10]. Since it is advantageous to implement physically as few operators as possible in order to reduce the complexity of the quantum system, it is important to analyze the gap between $T(G)$ and $is(G)$.

We have proposed the GOOD TESSELLABLE RECOGNITION problem (GTR), which aims to decide whether a graph G satisfies $T(G) = is(G)$, and we have analyzed the combined behavior of the computational complexities of the problems k-STAR SIZE, t-TESSELLABILITY, and GTR. We have defined graph classes corresponding to triples which specify the computational complexities of these problems, summarized in Table 1. We have defined graph classes in Construction 2.2 (I) and Construction 2.2 (II) that obey behaviors ($\mathcal{NP}$-complete, $\mathcal{P}$, $\mathcal{NP}$-complete) and ($\mathcal{NP}$-complete, $\mathcal{P}$, $\mathcal{P}$), respectively. Graphs that obey behavior ($\mathcal{NP}$-complete, $\mathcal{NP}$-complete, $\mathcal{P}$) are obtained using Construction 3.2. We also note that there are omitted triples in Table 1, which are either empty or easy to provide examples, as described in Section 1.

We are interested in the following two research topics: (i) The concept of good tessellable graphs can be extended to *perfect tessellable graphs*, the graphs G for which $T(H) = is(H)$ for any induced subgraph H of G. A natural open task is to establish the characterization by forbidden induced subgraphs and a polynomial-time recognition algorithm for perfect tessellable graphs. We conjecture that this class is exactly the {gem, W_4, odd cycles}-free graphs; (ii) We have already established relations between $T(G)$ with other well-known graph parameters such as the chromatic number and the maximum size of a stable set. We are currently investigating further relations such as those between $T(G)$ with the chromatic index and the total chromatic number.

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Appendix C

Manuscript “Total tessellation cover and quantum walk”

Total tessellation cover and quantum walk *

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Abstract. We propose the total staggered quantum walk model and the total tessellation cover of a graph. This model uses the concept of total tessellation cover to describe the motion of the walker who is allowed to hop both to vertices and edges of the graph, in contrast with previous models in which the walker hops either to vertices or edges. We establish bounds on $T_t(G)$, which is the smallest number of tessellations required in a total tessellation cover of G. We highlight two of these lower bounds $T_t(G) \geq \omega(G)$ and $T_t(G) \geq is(G) + 1$, where $\omega(G)$ is the size of a maximum clique and $is(G)$ is the number of edges of a maximum induced star subgraph. Using these bounds, we define the good total tessellable graphs with either $T_t(G) = \omega(G)$ or $T_t(G) = is(G) + 1$. The k-TOTAL TESSELLABILITY problem aims to decide whether a given graph G has $T_t(G) \leq k$. We show that k-TOTAL TESSELLABILITY is in $\mathcal{P}$ for good total tessellable graphs. We establish the $\mathcal{NP}$-completeness of the following problems when restricted to the following classes: $(is(G) + 1)$-TOTAL TESSELLABILITY for graphs with $\omega(G) = 2$; $\omega(G)$-TOTAL TESSELLABILITY for graphs G with $is(G) + 1 = 3$; k-TOTAL TESSELLABILITY for graphs G with $\max\{\omega(G), is(G) + 1\}$ far from k; and 4-TOTAL TESSELLABILITY for graphs G with $\omega(G) = is(G) + 1 = 4$. As a consequence, we establish hardness results for bipartite graphs, line graphs of triangle-free graphs, universal graphs, planar graphs, and $(2, 1)$-chordal graphs.

Keywords: Graph tessellation, Quantum walk, Graph coloring, Computational complexity.

1 Introduction

A *tessellation* of a graph $G = (V, E)$ is a partition of V into vertex disjoint cliques called *tiles*. A *k-tessellation cover* of G is a set of k tessellations that covers E. The *tessellation cover number* $T(G)$ of a graph G is the size of a minimum tessellation cover. The k-TESSELLABILITY problem aims to decide whether a given graph G has $T(G) \leq k$. The concept of tessellations on graphs was introduced in [1]. See [2] for basic definitions and notations in graph theory.

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Definition 1 Let $G = (V, E)$ be a graph and Σ a non-empty label set. A *total tessellation cover* comprises a proper vertex coloring and a tessellation cover of G both with labels in Σ such that, for any vertex $v \in V$, there is no edge $e \in E$ incident to v so that e belongs to a tessellation with label equal to the color of v.

An alternative way to characterize a tessellation is by describing the edges that belong to the tessellation. A k-tessellation cover of $G = (V, E)$ is a function h that assigns to each edge of E a nonempty subset in $\mathcal{P}(\Sigma)$, where $\Sigma = \{1, \ldots, k\}$, such that the set of edges having the same label corresponds to a tessellation, i.e., induces a partition of V into cliques. A *k-total tessellation cover* of a graph G simultaneously assigns labels in Σ to V as a proper vertex coloring f and labels in $\mathcal{P}(\Sigma) \setminus \emptyset$ to E as a tessellation cover with function h, such that each $uv \in E$ satisfies $f(u) \notin h(uv)$ and $f(v) \notin h(uv)$.

Definition 2 The *total tessellation cover number* $T_t(G)$ of a graph G is the minimum size of the set of labels Σ for which G has a total tessellation cover. The k-TOTAL TESSELLABILITY problem aims to decide whether a given graph G has $T_t(G) \leq k$.

Motivation. The quantum computation paradigm has gained popularity due to the recent advances in the physical implementation and in the development of quantum algorithms. There is an important concept, known as quantum walk, which is the mathematical modeling of a walk of a particle on a graph. This concept provides a powerful tool in the development of quantum algorithms [3]. Indeed, in the last decades the interest in quantum walks has grown considerably since quantum algorithms that outperform their classical counterparts employ quantum walks [4, 5]. In 2016, Portugal et al. proposed the staggered quantum walk model [1], which is more general than the previous quantum walk models [6] by containing the Szegedy model [7] and part of the flip-flop coined model [3]. The staggered quantum walk employs the concept of graph tessellation cover to obtain local unitary matrices such that their product results in the evolution operator for the quantum walk. There is a recipe to obtain a local unitary matrix from a tessellation. The staggered model requires at least two tessellations (corresponding to 2-tessellable graphs). In a tessellation, each clique establishes a neighborhood around which the walker can move under the action of the associated local unitary matrix. To define the evolution operator, one has to check whether the set of tessellations contains the whole edge set of the graph, since an uncovered edge would play no role in a quantum walk [1].

Related works. Abreu et al. [8, 9] proved that $\chi'(G)$ and $\chi(K(G))$ are upper bounds for $T(G)$, where $K(G)$ is the clique graph of G. They also proved the hardness of k-TESSELLABILITY for planar graphs, $(2, 1)$-chordal graphs, and $(1, 2)$-graphs and showed that 2-TESSELLABILITY is solved in linear time. Since $T(G) = \chi'(G)$ for triangle-free graphs, k-TESSELLABILITY is hard for this graph class [10]. Posner et al. [11] showed that k-TESSELLALBILITY is $\mathcal{NP}$-complete for line graphs of triangle-free graphs. Abreu et al. [12] proved that $is(G)$ is a lower bound for $T(G)$, where $is(G)$ is the number of edges in a maximum induced star of a graph G. They prove the hardness of k-TESSELLABILITY for universal graphs and the hardness of GOOD TESSELLABLE RECOGNITION, which aims to decide whether G is *good tessellable*, i.e., $T(G) = is(G)$. The concept of minimum tessellation cover was independently proposed as equivalence dimension in Duchet [13], and the relation between the two concepts was described in [12].

Contributions. This work is presented in the following sections. Section 2 contains a study on the bounds of the value of $T_t(G)$. Such bounds describe not only the number of operators required for a total staggered quantum walk model, but they also provide tools to analyse the computational complexity of k-TOTAL TESSELLABILITY, which is done in Section 3. Since $T_t(G) = \chi_t(G)$ for triangle-free graphs, the problem is hard even when restricted to bipartite graphs [14]. We show that k-TOTAL TESSELLABILITY is in $\mathcal{P}$ for good total tessellable graphs, and as a by product k-TESSELLABILITY is in $\mathcal{P}$ for good tessellable graphs. On the other hand, we show hardness results for the k-TOTAL TESSELLABILITY problem for line graphs of triangle-free graphs, universal graphs, planar graphs, and $(2,1)$-chordal graphs. As a consequence, the GOOD TOTAL TESSELLABILITY RECOGNITION problem is $\mathcal{NP}$-complete. Note that there are few results about the hardness of TOTAL COLORABILITY. Section 4 describes the total staggered quantum walk model, which drives a walker to hop both to vertices and edges. It also contains a description of the simulation of the total staggered quantum walk on a graph G in terms of a staggered quantum walk on the total graph of G. In Section 5, Table 1 presents the behavior analysis of the computational complexity related to the following parameters: $\chi'(G), \chi_t(G), T(G)$, and $T_t(G)$.

2 Bounds on $T_t(G)$

Since a total coloring of a graph G induces a total tessellation cover,

$$T_t(G) \leq \chi_t(G). \tag{1}$$

Particularly, for triangle-free graphs $T_t(G) = \chi_t(G)$ because the set of edges in each tessellation of any total tessellation cover is a matching. Hence, $(\Delta+1)$-TOTAL TESSELLABILITY is hard even when restricted to regular bipartite graphs [14]. Furthermore, by definition,

$$\max\{\chi(G), T(G)\} \leq T_t(G) \leq \chi(G) + T(G). \tag{2}$$

Note that the lower bound of Eq. (2) implies that $T_t(G) \geq \omega(G)$.

Lemma 1. *If* $\chi(G) \geq 3T(G)$*, then* $T_t(G) = \chi(G)$*.*

Proof. Let f be a proper vertex coloring and $\mathcal{C} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{T(G)}\}$ be a $T(G)$-tessellation cover for G. We define $\mathcal{C}'$ a tessellation cover for G with $3T(G)$ labels such that $\mathcal{C}'$ is compatible with f as follows. Each tessellation $\mathcal{T}'_i, 1 \leq i \leq 3T(G)$, of $\mathcal{C}'$ is associated with a color i. Since $\chi(G) \geq 3T(G)$ there are enough colors.

The edges of tessellations $\mathcal{T}'_{3j-2}, \mathcal{T}'_{3j-1}$, and $\mathcal{T}'_{3j}$ are given by the edges of the tessellation $\mathcal{T}_j, 1 \leq j \leq T(G)$, such that $\mathcal{T}'_{3j-2}$ (resp. $\mathcal{T}'_{3j-1}$, $\mathcal{T}'_{3j}$) consists of the edges of $\mathcal{T}_j$ that do not have an endpoint with color $3j-2$ (resp. $3j-1, 3j$). □

Using an argument similar to the one in the proof of Lemma 1, we can rewrite the upper bound of Eq. (2) as follows

$$T_t(G) \leq \max\{\chi(G), T(G) + \lceil 2\chi(G)/3 \rceil\}. \tag{3}$$

Eq.(3) says that $\chi(G) \geq 3T(G)$ implies $T_t(G) = \chi(G)$, or $\chi(G) \leq 3T(G)$ implies $T(G) \leq T_t(G) \leq 3T(G)$. In case $\chi(G) = 3$, Eq. (3) implies that $T(G) \leq T_t(G) \leq T(G) + 2$. An example of a graph G for which $T_t(G) = 3T(G) - 1$ and $T_t(G) > \chi(G)$ has $V(G) = \{v_1, v_2, v_3, v_4\} \cup \{u_1, u_2, u_3, u_4\} \cup \{w_1, w_2, w_3, w_4\}$, where $\{v_1, v_2, v_3, v_4\}$ and $\{u_1, u_2, u_3, u_4\}$ are maximal cliques and $\{v_i, u_i, w_i\}$ are triangles for $1 \leq i \leq 4$. In this case $T_t(G) = 5$, $\chi(G) = 4$ and $T(G) = 2$. Note that $T_t(G) = \chi(G) + T(G)$ requires that $\chi(G) \leq 2$, i.e., G is bipartite, which implies $T_t(G) = \chi_t(G)$ and $T_t(G)$ may assume only two values: $T_t(G) = \chi(G) + T(G) = \Delta(G) + 2$ or $T_t(G) = \chi(G) + T(G) - 1 = \Delta(G) + 1$.

Lemma 2. $T_t(G) \geq \max\limits_{v \in V(G)} \{\chi(G^c[N(v)])\} + 1 \geq \max\limits_{v \in V(G)} \{\omega(G^c[N(v)])\} + 1 = is(G) + 1$.

Proof. Consider a total tessellation cover of a graph G, a vertex v of G, and $G^c[N(v)]$, which is the complement graph of the graph induced by the neighborhood of v. In any tessellation, the endpoints of the edges that are incident to v and belong to the tessellation induce a clique, hence the vertices of this clique are a stable set in $G^c[N(v)]$. Therefore, the tessellations with edges incident to a vertex v induce a vertex coloring of $G^c[N(v)]$, and the number of these tessellations is at least $\chi(G^c[N(v)])$. Moreover, these tessellations have labels that are different from the color of vertex v. Therefore, $T_t(G) \geq \chi(G^c[N(v)]) + 1$. Note that $is(G[N[v]]) = \alpha(G[N(v)]) = \omega(G^c[N(v)])$ and $is(G) = \max\limits_{v \in V(G)} is(G[N[v]])$. □

Graphs with $T_t(G) = T(G) = k$ have no induced subgraph $K_{1,k}$ because $T_t(G) \geq is(G) + 1 \geq k + 1$. Moreover, there is no tile of size k in any tessellation of a total tessellation cover. If $T_t(G) = T(G) = 3$, then G is $K_{1,3}$-free and there is no clique of size three in any tessellation. Therefore, the total tessellation cover of G induces a total coloring of G, and the only graphs for which $T_t(G) = T(G) = 3$ are the odd cycles with n vertices such that $n \equiv 0 \mod 3$. For bipartite graphs, $T(G) = \Delta(G)$ and $T_t(G) > T(G)$. For triangle-free graphs, $T_t(G) = T(G)$ if $\chi'(G) = \chi_t(G) = \Delta + 1$. It follows that deciding whether $T_t(G) = T(G) = \Delta(G) + 1$ is $\mathcal{NP}$-complete from the proof that $(\Delta+1)$-TOTAL COLORABILITY is $\mathcal{NP}$-complete for triangle-free snarks [15], which are graphs with $\chi'(G) = \Delta + 1$.

3 Good Total Tessellable Graphs

Since the concept of good tessellable graphs introduced in [12] has provided keen insights into the hardness of finding minimum-sized tessellation covers, we define the concept of good total tessellable graphs in order to further explore hardness results related to total tessellation covers. In the quantum computation context, we are interested in graph classes which use as few color labels as possible because the number of operators is as low as possible. In this case, $T_t(G)$ must be close to the lower bounds.

Definition 3 A graph G is *good total tessellable* if either $T_t(G) = \omega(G)$ or $T_t(G) = is(G) + 1$. We say that G is *Type I* (resp. *Type II*) if $T_t(G) = \omega(G)$ (resp. $T_t(G) = is(G) + 1$).

Now we show that k-TOTAL TESSELLABILITY is in $\mathcal{P}$ if we know beforehand that the graph is either good total tessellable Type I or Type II.

The *Lovász number* $\vartheta(G)$ is a real number such that $\omega(G^c) \leq \vartheta(G) \leq \chi(G^c)$ [16]. We denote $\psi(G)$ the integer nearest to $\vartheta(G)$. The value of $\psi(G)$ can be be determined in polynomial time [16].

For Type I graphs, $T_t(G) = \omega(G)$. Since Eq. (2) implies that $\omega(G) \leq \chi(G) \leq T_t(G)$, we have $\omega(G) = \chi(G) = T_t(G) = \psi(G^c)$.

For Type II graphs, $T_t(G) = is(G)+1$. For any vertex $v \in V(G)$, $\omega(G^c[N(v)]) \leq \psi(G[N(v)]) \leq \chi(G^c[N(v)])$, and by Lemma 2, $T_t(G) \geq \psi(G[N(v)]) + 1$. Since $T_t(G) = is(G) + 1$, by Lemma 2 there is a vertex $u \in V(G)$ such that $T_t(G) = \omega(G^c[N(u)]) + 1$. In this case, $\omega(G^c[N(u)]) + 1 = \chi(G^c[N(u)]) + 1$, and we determine $\omega(G^c[N(u)])$ using $\psi(G^c[N(u)])$. Therefore, $T_t(G) = \max_{v \in V(G)} \{\psi(G[N(v)])\} + 1$.

The same method used to determine $T_t(G)$ for Type II graphs can be applied for good tessellable graphs in order to determine $T(G)$, where $T(G) = \max_{v \in V(G)} \{\psi(G[N(v)])\}$.

Hardness results. As presented in Section 2, $(\Delta + 1)$-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete for bipartite graphs, which have $is(G)+1 = \Delta+1$ and $\omega(G) = 2$. Now, we show that k-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete for the following cases: line graph of triangle-free graphs with $k = \omega(G) \geq 9$ and $is(G) + 1 = 3$; universal graphs with k very far apart from both $is(G) + 1$ and $\omega(G)$; planar graphs with $k = 4 = \omega(G) = is(G) + 1$; and $(2, 1)$-chordal graphs with $k = is(G) + 1 = \omega(G) + 3$.

Line graph of triangle-free graphs. Machado et al. [17] proved that k-EDGE COLORABILITY is $\mathcal{NP}$-complete for 3-colorable k-regular triangle-free graphs if $k \geq 3$. The key idea of the proof of Theorem 1 is to verify that $T_t(L(G)) = \chi'(G)$ when $k \geq 9$. The edges incident to any vertex v of graph G correspond to a clique of $L(G)$, whose size is the degree of v. If two vertices of G are non-adjacent, then the corresponding cliques in $L(G)$ share no vertices. Hence, we cover the edges of the cliques of $L(G)$ incident to the vertices of each of the three color class of the 3-coloring of G with a tessellation related to the color class because these cliques share no vertices. Therefore, since $T(L(G)) = 3$ and $\chi(L(G)) \geq 9 \geq 3T(G)$, by Lemma 1, $T_t(L(G)) = \chi(L(G)) = \chi'(G)$. Note that in this case $k = \omega(L(G))$ and $is(L(G)) + 1 = 3$.

Theorem 1. k-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for line graphs $L(G)$ of 3-colorable k-regular triangle-free graphs G for any $k \geq 9$.*

Universal graphs. Abreu et al. [12] reduced q-COLORABILITY to k-TESSELLABILITY for universal graphs. We present a similar argument to establish the $\mathcal{NP}$-completeness of k-TOTAL TESSELLABILITY for universal graphs. Let G be an instance of q-COLORABILITY. The key idea of the proof of Theorem 2 is to add to G^c a universal vertex u and $2|V(G)|$ pendant vertices adjacent to u, which defines the graph $[2|V(G)|, G^c]$ of G^c. Now, the total tessellation cover number of the constructed graph is given by $2|V(G)|+\chi(G)+1$, using labels $1, \ldots, \chi(G)$ to cover the edges incident to u that belong to the subgraph induced by $V(G^c \cup \{u\})$, labels $\chi(G)+1, \ldots, \chi(G)+2|V(G)|$ to cover the edges incident to the pendant vertices and labels $\chi(G)+1, \ldots, \chi(G)+|V(G)|$ are enough to cover the edges of G^c; assign to u the color $2|V(G)|+\chi(G)+1$, to the pendant vertices color 1, and to the remaining vertices colors $\chi(G)+|V(G)|+1, \ldots, \chi(G)+2|V(G)|$. The minimality follows from Lemma 2. Therefore, $T_t([2|V(G)|, G^c]) = 2|V(G)|+\chi(G)+1$.

Note that $is(C_5 \vee \{u\}) = 2$, $T_t(C_5 \vee \{u\}) = 4$, and any minimum total tessellation cover of $C_5 \vee \{u\}$ has at least three labels assigned to the edges incident to u and a fourth label assigned to u. Thus, $T_t([2|V(G)|, G^c \cup C_5]) = T_t([2|V(G)|, G^c]) + 3$; $is([2|V(G)|, G^c \cup C_5]) = is([2|V(G)|, G^c]) + 2$; and $\omega([2|V(G)|, G^c \cup C_5]) = \omega([2|V(G)|, G^c])$. Therefore, each addition of a C_5 increases the gap between the total tessellation cover number and both the sizes of a maximum induced star and a maximum clique. As long as the number of the C_5's is polynomially bounded by the size of G, k-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete even if k is far apart from $is(G)$ and $\omega(G)$.

Theorem 2. k-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for universal graphs.*

Planar graphs. We show that 4-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete when restricted to planar graphs G with $is(G)+1 = \omega(G) = 4$. We present a polynomial transformation from 3-COLORABILITY when restricted to planar graphs with maximum degree four [18] to 4-TOTAL TESSELLABILITY for planar graphs. Let G be an instance of such coloring problem. $G' = G \vee \{u\}$ has a 4-coloring if and only if the planar graph G has a 3-coloring. We define three gadgets as depicted in Fig. 1. The edges of the external triangles of the *Duplicator Gadget* are tiles of size three in a same tessellation. The edges of the external triangles of the *NotEqual Gadget* are tiles of size three in different tessellations. The *Shifter Gadget* forces triangles T_1 and T_4 to be tiles on a tessellation a, and triangles T_2 and T_3 to be tiles on a tessellation b different from a.

Each vertex v of G' is associated with a Duplicator Gadget such that the number of external triangles of the Duplicator Gadget of v is equal to $d_{G'}(v)$. If two vertices of G' are adjacent, we connect one external triangle of each Duplicator Gadget with a NotEqual Gadget. Thus, in a 4-total tessellation cover of the obtained graph H, the labels of the external triangles of the Duplicator

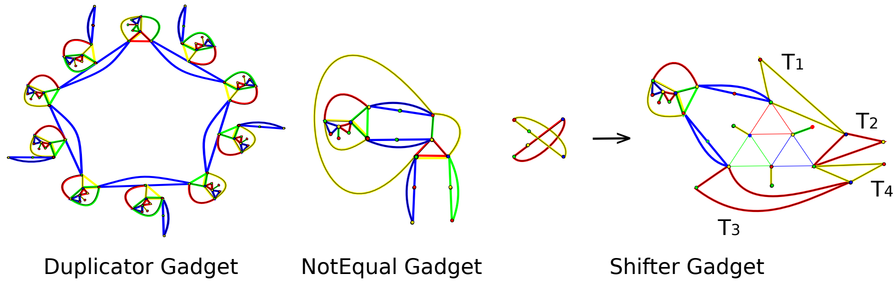

Fig. 1. Duplicator Gadget, NotEqual Gadget, and Shifter Gadget.

Gadget associated with a vertex v are equal to the color of v in a 4-coloring of G'. Now, we transform H into a planar graph H' by replacing each crossing triangles of H by a Shifter Gadget. Therefore, the planar graph H' has a 4-total tessellation cover if and only if G has a 3-coloring. Note that in this case $k = \omega(G) = 4 = is(G) + 1$.

Theorem 3. 4-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for planar graphs.*

(2, 1)-chordal graphs. A graph G is $(2, 1)$ if its vertex set can be partitioned into two stable sets and one clique. Since 3-EDGE COLORABILITY is $\mathcal{NP}$-complete for 3-regular graphs [17], 3-VERTEX COLORABILITY for 4-regular line graphs is also $\mathcal{NP}$-complete. Let G be a 4-regular line graph. We construct a graph H from G as follows. $V(H)$ contains a clique $\{e_0, \ldots, e_{|E(G)|-1}\}$ where each e_i, $0 \le i \le |E(G)| - 1$, is associated with a distinct edge of G. $V(H)$ contains an stable set $\{e'_0, \ldots, e'_{|E(G)|-1}\}$ such that each e'_i is adjacent to all e_j with $j \ne i$ and $j \ne i + 1 \mod |E(G)|$. $V(H)$ contains an stable set $\{v_0, \ldots, v_{|V(G)|-1}\}$, where each v_i, $0 \le i \le |V(G)| - 1$, is associated with a distinct vertex of G. Each $e_j \in \{e_0, \ldots, e_{|E(G)|-1}\}$ is adjacent to vertices $v_r, v_s \in \{v_0, \ldots, v_{|V(G)|-1}\}$ such that $e_j = v_r v_s$. $V(H)$ contains an stable set P comprising $(|V(G)| + |E(G)|)(|E(G)| + 1)$ pendant vertices such that each vertex of $\{v_0, \ldots, v_{|V(G)|-1}\} \cup \{e'_0, \ldots, e'_{|E(G)|-1}\}$ is adjacent to $|E(G)| + 1$ pendant vertices. By construction, H is $(2, 1)$ and chordal.

We claim that $T_t(H) = |E(G)| + 3$ if and only if $\chi(G) = 3$. Consider a 3-coloring c of G. Obtain a k-total tessellation cover of H with $k = |E(G)| + 3$ as follows. Assign colors in $\{1, \ldots, |E(G)|\}$ to the vertices of the clique $\{e_0, \ldots, e_{|E(G)|-1}\}$. Assign to vertex e'_i, for $1 \le i \le |E(G)|$, the same color of the vertex e_i. For $0 \le i \le |E(G)| - 1$, the tile with vertices $\{e'_i\} \cup \{e_j \mid j \ne i \text{ and } j \ne i + 1 \mod |E(G)|\}$ is in the tessellation with label $i + 2 \mod |E(G)|$. Note that if two vertices v_i and v_k of G are not adjacent, then the cliques $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$ and $\{v_k\} \cup \{e_j \mid v_k \text{ is endpoint of } e_j \text{in } G\}$ are disjoint. Thus, the tile with vertices $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$ is in the tessellation with label $c(v_i) + |E(G)|$. Finally, greedily assign colors and labels to the remaining vertices and edges of H. Consider a total tessellation cover of H with $k = |E(G)| + 3$ labels. Note that we require $|E(G)|$ tessellations to cover the edges between the vertices $\{e_0, \dots, e_{|E(G)|-1}\} \cup \{e'_0, \dots, e'_{|E(G)|-1}\}$ in any total tessellation cover of H. Moreover, a tile in each of those $|E(G)|$ tessellations contains $|E(G)| - 2$ vertices of the clique $\{e_0, \dots, e_{|E(G)|-1}\}$. Since each tile $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$, for $0 \le i \le |V(G)| - 1$, contains four vertices of the clique $\{e_0, \dots, e_{|E(G)|-1}\}$, there are only three tessellation labels used by the tiles $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$, for $0 \le i \le |V(G)| - 1$. Moreover, if two vertices v_i and v_k are adjacent in G, then the tiles $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$ and $\{v_k\} \cup \{e_j \mid v_k \text{ is endpoint of } e_j \text{ in } G\}$ share a vertex $e_j = v_i v_k$ in H and they are tiles belonging to different tessellations. Hence, we obtain a 3-coloring c of G as follows. Assign the label of the tile $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$ to the color of v_i in c.

Therefore, G has a 3-coloring if and only if H has a total tessellation cover with $|E(G)| + 3$ labels. Note that $k = is(H) + 1 = \omega(H) + 3 = |E(G)| + 3$.

Theorem 4. k-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for chordal graphs.*

4 The total staggered quantum walk model

We now show how to simulate a total staggered quantum walk on a graph G with a staggered quantum walk on its total graph $\mathrm{Tot}(G)$. The *total graph* $\mathrm{Tot}(G)$ of G has $V(\mathrm{Tot}(G)) = V(G) \cup E(G)$ and $E(\mathrm{Tot}(G)) = E(G) \cup \{u\ uw \mid u \in V(G),\ uw \in E(G)\} \cup \{uv\ vw \mid uv \in E(G) \text{ and } vw \in E(G)\}$. Let $A = \mathrm{Tot}(G)$, $A[E(G)] = Y$ and $A[V(G)] = X$. Subgraph Y is isomorphic to the line graph $L(G)$ of G, and X is isomorphic to the original G. We define the clique $K_v = \{v\} \cup \{vw \mid vw \in E(G)\}$ of A.

Consider a total tessellation cover of a graph G. Define an associated tessellation cover of A as follows. Assign the labels of the edges of G to the respective edges of X and assign the color of each vertex v of G to the edges of $A[K_v]$. We simulate the total staggered quantum walk on G with the staggered quantum walk on A by considering the vertices of G as the corresponding vertices of X in A, and the edges of G as the corresponding vertices of Y in A. Fig. 2 depicts a total tessellation cover of a graph G and the associated tessellation cover of $A = \mathrm{Tot}(G)$.

Consider the walker located on a vertex a of G. If we apply the operator H_j associated with the color of a, the walker hops to the edges incident to a (the edges ab and ac). If we apply an operator associated with the label of an edge incident to a, the walker hops to the vertices in the tile of the tessellation of the same label that contains a (the vertices b and c). The same happens by considering the walker located on a vertex a in X. If we apply the operator H_j associated with the labels of the edges of $A[K_a]$, the walker hops to the vertices ab and ac of Y, and if we apply the operator associated with the label of an edge of X incident to a, the walker hops to the vertices b and c of X. Consider the walker located on an edge ab of G. If we apply the operator associated with the color of a (or b), the walker hops to a (or b) and to the edges incident to it.

The same happens by considering the walker located on a vertex ab in Y. If we apply the operator associated with the labels of the edges of $A[K_a]$ (or $A[K_b]$), the walker hops to vertices of K_a (or K_b). Otherwise, the walker stays put in both G and A.

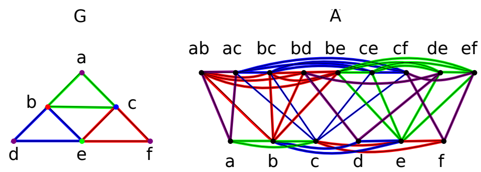

Fig. 2. Total tessellation cover of a graph G and the associated tessellation cover of A.

5 Concluding remarks

We have defined the total tessellation cover on a graph G and have used this concept to define the total staggered quantum walk model. This work strengthens the connection between quantum walk and graph coloring.

We have established examples of graphs for which $T_t(G)$ reaches the bounds of Section 2. We leave as an open problem to search for graphs with at least 3 vertices satisfying $T_t(G) = 3T(G)$ and $T_t(G) > \chi(G)$. Moreover, it would be interesting to define graph classes with $T_t(G) = T(G) = k$ for $k \geq 4$, since for $k = 3$ the only such graphs are the odd cycles C_n with $n \equiv 0 \mod 3$.

We have shown that 4-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete for planar graphs satisfying $is(G) + 1 = \omega(G) = 4$. This is important since the hardness of k-EDGE COLORABILITY and k-TOTAL COLORABILITY for planar graphs are still open. On the other hand, we know that planar graphs with large maximum degree have edge and total colorings as small as possible [19, 20]. We leave as an open problem to find a threshold for $T_t(G)$ for which all planar graphs are Type II.

Table 1 summarizes the computational complexities of EDGE-COLORABILITY cf. [17], TOTAL COLORABILITY cf. [21], TESSELLABILITY cf. [8], and TOTAL TESSELLABILITY. These four problems are in $\mathcal{P}$ when restricted to complete graphs, star graphs and trees, whereas for triangle-free graphs, the four problems are $\mathcal{NP}$-complete. We leave as an open problem to find a graph class for which TOTAL COLORABILITY is $\mathcal{NP}$-complete and TOTAL TESSELLABILITY is in $\mathcal{P}$. We have not identified this class because all known $\mathcal{NP}$-completeness proofs of TOTAL COLORABILITY are restricted to graph classes with $\chi_t(G) = T_t(G)$.

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	$\chi'(G)$	$T(G)$
$[2\|V(G)\|, G^c]$	$\mathcal{P}$	$\mathcal{NP}$-c
Line of Bipartite	$\mathcal{NP}$-c	$\mathcal{P}$

	$\chi'(G)$	$\chi_t(G)$
$G \cup K_{\Delta(G)+1}$, Δ even	$\mathcal{P}$	$\mathcal{NP}$-c
$G \cup K_{\Delta(G)+1}$, Δ odd	$\mathcal{NP}$-c	$\mathcal{P}$

	$\chi'(G)$	$T_t(G)$
$[2\|V(G)\|, G^c]$	$\mathcal{P}$	$\mathcal{NP}$-c
Line of Bipartite, $\omega(G) \geq 6$	$\mathcal{NP}$-c	$\mathcal{P}$

	$T(G)$	$X_t(G)$
Bipartite	$\mathcal{P}$	$\mathcal{NP}$-c
$[2\|V(G)\|, G^c]$	$\mathcal{NP}$-c	$\mathcal{P}$

	$T(G)$	$T_t(G)$
Bipartite	$\mathcal{P}$	$\mathcal{NP}$-c
$G \cup K_{3\Delta(G)}$	$\mathcal{NP}$-c	$\mathcal{P}$

	$\chi_t(G)$	$T_t(G)$
$G \cup K_{\Delta(G)+1}$, Δ odd	$\mathcal{P}$	$\mathcal{NP}$-c
Open	$\mathcal{NP}$-c	$\mathcal{P}$

Table 1. Computational complexities of parameters $\chi'(G), \chi_t(G), T(G)$, and $T_t(G)$.

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Appendix

Planar Graphs - Detailed proof of Theorem 3

The computational complexity of TOTAL COLORING for planar graphs is an open problem. On the other hand, we show in this section that 4-TOTAL TESSELLABILITY for planar graphs is $\mathcal{NP}$-complete. Since in this proof we use a generic graph G such that $is(G) + 1 = \omega(G) = 4$, we also prove that deciding whether a graph has both $T(G) = is(G) + 1$ and $T(G) = \omega(G)$ is $\mathcal{NP}$-complete even if restricted to planar graphs.

Lemma 3. *Let G be the graph with $V(G) = \{a_1, b_1, c_1, d_1\} \cup \{a_2, b_2, c_2, d_2\} \cup \{a_3, b_3, c_3, d_3\}$, where $\{a_1, b_1, c_1, d_1\}$ is a maximal clique and $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, $\{c_1, c_2, c_3\}$, and $\{d_1, d_2, d_3\}$ are triangles. Any total tessellation cover of G with four labels has the following property: The edges of three triangles are tiles on a same tessellation and the edges of the remaining triangle are a tile on a different tessellation.*

Proof. The proof follows after analyzing all possibilities of total tessellation covers with four labels. □

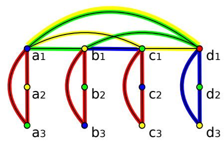

Fig. 3. A 4-total tessellation cover of the graph of Lemma 3.

Fig. 3 shows an example of a total tessellation cover of graph G described in Lemma 3. The edges of three triangles must have the same color (red) and the fourth triangle must have a different color (blue). In the first gadget of the Fig. 4, the two external triangles' edges must receive the same label, since the two internal triangles shares a vertex and they must receive different labels. In the second gadget, since the internal triangles' edges must receive the same label the external triangle' edges must receive different labels.

Lemma 4. *Any total tessellation cover of the graph G of Fig. 6 with four labels has the following property: The triangles T_1 and T_4 are tiles in a same tessellation, and the triangles T_2 and T_3 are tiles in a same tessellation, which is different from the tessellation that contains T_1 and T_4.*

Proof. Since there are three maximal cliques incident to the vertices b (resp. c and e), they are three tiles on different tessellations. Therefore, the three triangles of the Hajos subgraph of G are tiles on different tessellations. The Equal Gadget has its tiles incident to the vertices a and d on a same tessellation.

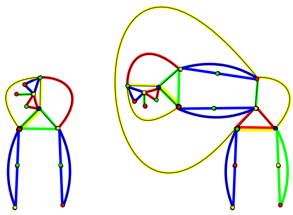

Fig. 4. Equal Gadget: edges of its two external triangles are covered by 3-tiles of a same tessellation in a 4-total tessellation cover. NotEqual Gadget: edges of its two external triangle are covered by 3-tiles of different tessellations in a 4-total tessellation cover.

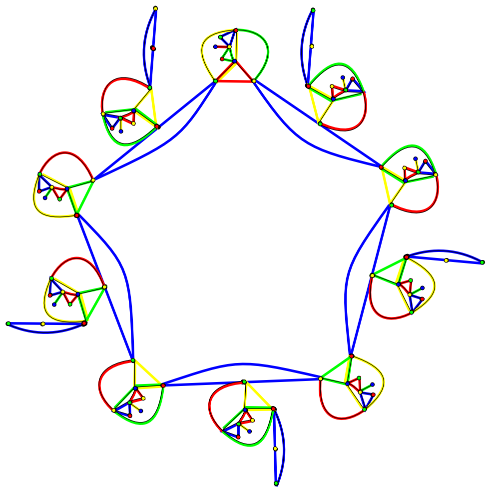

Fig. 5. Duplicator Gadget: it forces the five external triangles' edges to have the same label. Moreover, if the label of the triangle' edges is a, then its vertices have the next two consecutive labels $a+1$ and $a+2$ modulo 4 available to the vertices of four of these five triangles in a 4-total tessellation cover.

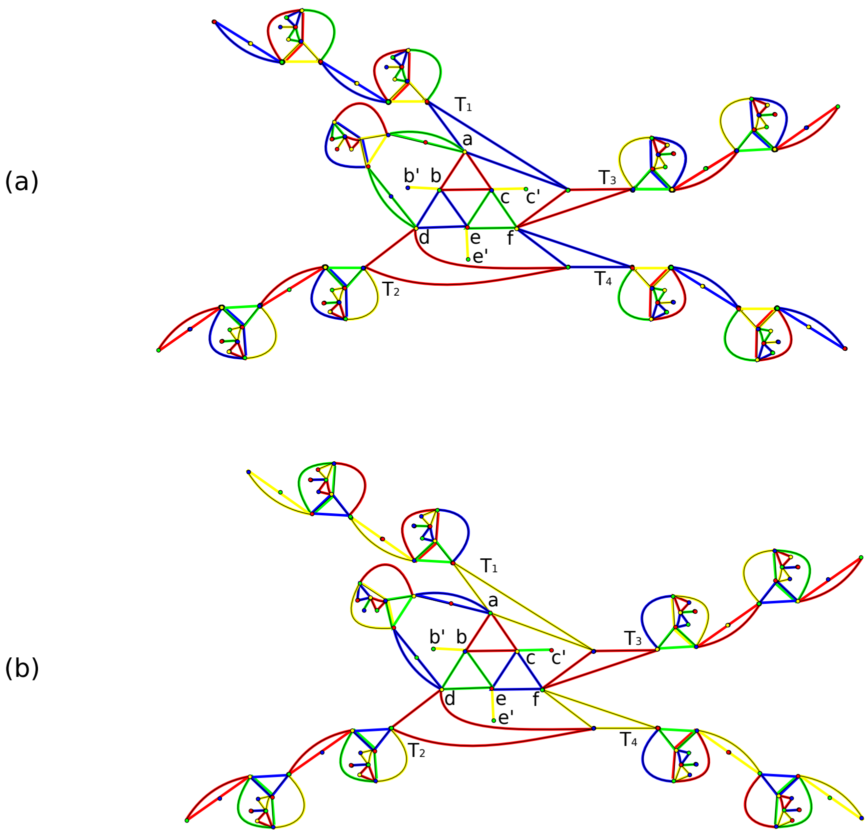

Fig. 6. Shifter Gadget: it shifts two crossing tiles in different tessellations such that the tiles get from one side to the other side without crossing edges and maintaining their 3-tiles tessellations in a 4-total tessellation cover.

Assume that the color of the vertex a is equal to the color of the vertex d. Since the color of vertex a is the same of the color of vertex d and the tiles of the Equal Gadget have a same tessellation different from the label of the color of a and d, it implies that T_1 is a tile on the same tessellation of the triangle with vertices $\{b, d, e\}$ and that T_2 is a tile on the same tessellation of the triangle with vertices $\{a, b, c\}$. Note that the colors of the vertices a (resp. d) and b are different and that they are also different from the labels of T_1 and T_2. Therefore the color of the vertices c and e must be the same labels of the tessellations of T_1 and T_2. Now, the triangle $\{c, e, f\}$ and the vertex f must receive two labels different from the labels used by the triangles $\{a, b, c\}$ and $\{b, d, e\}$. This implies that the triangles T_3 and T_4 are tiles with the same labels of the triangles T_1 and T_2. Since T_1 share a vertex with T_3 and T_2 share a vertex with T_4, the triangles T_1 and T_4 are tiles on a same tessellation and the triangles T_2 and T_4 are also tiles on a same tessellation different from the tessellation of T_1 and T_3. Therefore, if there is a total tessellation cover of G with 4 labels, the proof of the theorem holds. A total tessellation cover of G with 4 labels is depicted in Fig. 6 (a). Note that we obtain total tessellation covers of G with all possible combinations of two distinct labels of the four labels for T_1 and T_2 by replacing the color classes of G by the desired labels.

Assume that the color of the vertex a is different from the color of the vertex d. For the sake of contradiction assume we use only two labels to the colors of a, d and the tiles of the triangles $\{a, b, c\}$ and $\{b, d, e\}$. This implies that b receives a third label different from these two, and that there is only one available label to the color of the vertices c and e, a contradiction. We also cannot use four different labels to the vertices a, d and the triangles $\{a, b, c\}$ and $\{b, d, e\}$ or there would be no available label to the triangles of the Equal Gadget. Therefore, we have three different labels used in the colors of the vertices a, d and the labels of the tiles of the triangles $\{a, b, c\}$ and $\{b, d, e\}$. This implies that the color of the vertex b and the tiles of the Equal Gadget receive the same label. The color of the vertices a, b, and the label of the tile of the triangle $\{a, b, c\}$ are different from the labels of T_1. This implies that the color of c is equal to the label of the tile of T_1. The same holds for the vertex e and the label of the tile of T_2. Now the color of the vertex f and the label of the tile of the triangle $\{c, e, f\}$ must be different from the colors of c and e (i.e., the label of the tiles of T_1 and T_2). This implies that the label of the tiles T_3 and T_4 are the same labels of the tiles T_1 and T_2. Since T_1 share a vertex with T_3 and T_2 share a vertex with T_4, the triangles T_1 and T_4 are tiles on a same tessellation and the triangles T_2 and T_4 are also tiles on a same tessellation different from the tessellation of T_1 and T_3. Therefore, if there is a total tessellation cover of G with 4 labels, the proof of the theorem holds. A total tessellation cover of G with 4 labels is depicted in Fig. 6 (b). Note that we obtain total tessellation covers of G with all possible combinations of two distinct labels of the four labels for T_1 and T_2 by replacing the color classes of G by the desired labels. $\square$

Theorem 3 4-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for planar graphs.*

Proof. Let G be an instance of 3-COLORABILITY of planar graphs with degree at most four [18]. Add a universal vertex u to G so that $G \vee \{u\}$ has a 4-coloring if and only if G has a 3-coloring. We create a planar graph H from $G \vee \{u\}$ as follows. We replace each vertex of $G \vee \{u\}$ by a Duplicator Gadget with the degree of the vertex duplication in H. We replace each edge of $G \vee \{u\}$ by a NotEqual Gadget connecting the related triangles of the Duplicator Gadgets of the endpoints of the edge in H. The only crossing edges in H are from the triangles of the universal vertex and the triangles of the other Duplicator Gadget that have labels different from the one of the universal Gadget. We replace these crossing tiles with the Shifter Gadget.

We claim that the resulting planar graph H has a 4-total tessellation cover if and only if the graph G has a 3-coloring.

Consider a 4-total tessellation cover of H. If two vertices are adjacent in $G \vee \{u\}$, then the NotEqual Gadget forces the tiles of the external triangles of the respective Duplicator Gadgets of these two adjacent vertices to be on different tessellations. Therefore, we obtain a 4-coloring of $G \vee \{u\}$ by assigning the color of a vertex as the label of the tile of the external triangles of the Duplicator Gadget related to that vertex.

Consider a 4-coloring f of $G \vee \{u\}$. We obtain a 4-total tessellation cover of H as follows. Assign each tile of the external triangles of the Duplicator Gadget to the tessellation related to the color the vertex received in f. Label the remaining vertices and edges as described in Figure 5 by rotating the color classes labels to obtain the desired label.

Since we obtain the total tessellation cover of the Duplicator Gadget by rotating the color classes, we have that the label of a external triangle and a vertex of the degree two is related to consecutive colors. Therefore, if the label of the tile of the external triangle is 1 (resp. 2, 3, and 4), then there are two vertex of degree two in this external triangle with colors 2 and 3 (resp. 3 and 4, 4 and 1, 1 and 2). Now, for any two different tessellations of the tiles of the external triangles, we select one vertex of degree two of each so that we do not use all four labels in these two vertex and in the two tiles of the external triangles. By Lemma 3, there is a total tessellation cover with four labels of the NotEqual Gadget if we do not use all four labels on the two tiles of its external triangles and the two vertices of the K_4 of that external triangles.

We obtain a total tessellation cover with 4 labels of the Shifter Gadgets as described in Lemma 4. Note that, as depicted in Fig. 6, the two consecutive Equal Gadgets connected to the external triangles T_1 (resp. T_2, T_3, and T_4) allow us to assign colors to the vertices of the Shifter Gadgets so that the vertices of the last of their external triangles have the same colors of the vertices of the external triangles of the Duplicator Gadgets that they are related. □

Appendix D

Manuscript "Towards the Optimization of Graph Tessellation Covers"

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Towards the Optimization of Graph Tessellation Covers

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A tessellation is defined as a partition of the vertices of a graph into disjoint cliques. A tessellation cover of a graph is a set of tessellations that covers all of its edges. The tessellation cover number $T(G)$ is the size of the smallest tessellation cover of G. The t-TESSELLABILITY problem consists in deciding whether $T(G) \leq t$. This problem is motivated by quantum walks, which is the mathematical modeling of a particle moving through the vertices of a graph according to the postulates of quantum mechanics. In particular, in the staggered quantum walk model one needs to find a tessellation cover of a graph before defining the evolution operators for the quantum walker [2].

Abreu *et al.* proved that $T(G)$ is upper bounded by the minimum between the chromatic index of G and the chromatic number of its clique graph $K(G)$; presented extremal graphs whose tessellation number achieves one of these upper bounds; used these results to prove the $\mathcal{NP}$-completeness for some restricted graph classes; and presented a linear-time algorithm for the 2-TESSELLABILITY problem [1]. In this work, we study the optimization version of t-TESSELLABILITY with a formulation for the minimization problem, and discuss numerical results achieved with Gurobi API.

The formulation is as follows. Given a graph $G = \{V(G), E(G)\}$, this problem has the following premises: (P1) a tessellation must cover the vertex set $V(G)$; (P2) all cliques in a tessellation must be vertex disjoint cliques; and (P3) the edge set $E(G)$ must be covered by the union of the tessellations of G. All variables used in this formulation are binaries: T_i indicates whether tessellation i is used in the solution, $V_v^{i,j}$ indicates whether vertex v is used by clique j in tessellation i, and $E_m^{i,j}$ indicates whether edge m is used by clique j in tessellation i. As this problem is a minimization problem, we must minimize the objective function $\sum_{i=1}^{|E(G)|} T_i$. We need at most $|E(G)|$ tessellations in a solution, and at most $|V(G)|$ cliques in a tessellation.

First, we enunciate the following restrictions: (R1) $\sum_{j=1}^{|V(G)|} V_v^{i,j} = 1$ is needed to meet premises (P1) and (P2), because this restriction allows vertex V_v to compose a unique clique C_j in a tessellation T_i; (R2) $V_v^{i,j} + V_u^{i,j} \leq 1, \forall (v,u) \notin E(G)$ is needed to avoid that the model consider non-edges; (R3) $E_m^{i,j} \leq T_i$, because an edge m can only be used in the solution by a tessellation T_i in a clique C_j if T_i itself is in the solution; (R4)

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$\sum_i^{|E(G)|} E_m^{i,j} \geq 1$, is need to meet premise (P3). The next three restrictions guarantee that an edge can only compose a solution in a tessellation T_i if both incident vertices of this edge are in the same clique C_j in the tessellation T_i: (R5) $E_m^{i,j} \leq V_v^{i,j}$; (R6) $E_m^{i,j} \leq V_u^{i,j}$; (R7) $E_m^{i,j} \geq V_v^{i,j} + V_u^{i,j} - 1$, such that $E_m^{i,j} = (v, u) \in E$. Finally, to avoid symmetrical answers we still need a last restriction: (R8) $T_{i-1} \geq T_i$.

Our model was implemented in C with Gurobi API [4], and tested with input graphs already previously analyzed by Abreu *et al.* [1]. Every result obtained with our proposed formulation was consistent with those obtained in [1] via analytical calculations or exhaustive search.

A question that arises is whether every optimal tessellation cover must have tessellations with at least one maximal clique. Abreu *et al.* [1] showed a graph, depicted in Fig. 1 that uses in a optimal tessellation cover one tessellation that does not have a maximal clique. Using the formulation proposed in this work we found the same tessellation cover found by Abreu *et al.* and numerical simulations suggest that this tessellation cover is unique.

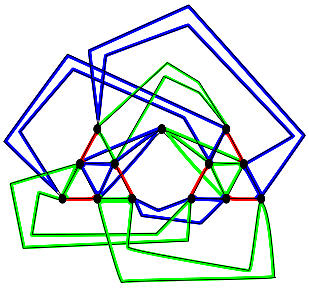

Figure 1: A non-trivial tessellation cover found by Gurobi using our proposed formulation.

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Appendix E

Manuscript "Tessellability and tessellability completion of graphs with few P_4"

Matemática Contemporânea, Vol. 46, 56–64

TESSELLABILITY and TESSELLABILITY COMPLETION of graphs with few P_4

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Daniel F. D. Posner

Abstract

A tessellation of a graph $G = (V, E)$ is a set of disjoint cliques that covers $V(G)$. A tessellation cover of G is a set of tessellations that covers $E(G)$. The tessellation cover number of G, denoted by $T(G)$, is the minimum size of a smallest tessellation cover of G. The t-TESSELLABILITY of G aims to decide whether $T(G) \leq t$. In this work, we present a polynomial time algorithm for t-TESSELLABILITY for quasi-threshold graphs. Next, we introduce the t-TESSELLABILITY COMPLETION of G, which aims to decide whether there is a tessellation cover $\mathcal{T}$ of G with t tessellations given a partial tessellation cover $\mathcal{T}'$ of G that must be part of $\mathcal{T}$. Finally, we compare the behavior of the computational complexity of t-TESSELLABILITY COMPLETION and k-EDGE PRECOLORING in some subclasses of graphs with few P_4, such as complete bipartite graphs, triangulated of complete graphs, and complete graphs.

2000 AMS Subject Classification: 05C15, 05C90, and 68Q17.

Key Words and Phrases: graph tessellations, graphs with few P_4's, quasi-threshold graphs.

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1 Introduction

Nowadays quantum computation receives a lot of attention from the scientific community. An important concept in this computational paradigm is the quantum walk. This concept is defined as a mathematical model of a particle's walk thought the edges of a graph. Recently, Portugal *et al.* [11] proposed the *Staggered Quantum Walk Model*, that includes Szegedy Model and an important part of Coined Model. The Staggered Model uses the concept of tessellations on graphs to generate the evolution operators that rules the corresponding quantum walk. Given a graph $G = (V, E)$, a *tessellation* is a set of disjoint cliques of G that covers $V(G)$. A set of tessellations $\mathcal{T} = \{C_1, \ldots, C_j\}$ is a *tessellation cover* when $\mathcal{T}$ covers $E(G)$. The size of a smallest tessellation cover in a graph G is denoted by $T(G)$. The t-TESSELLABILITY problem aims to decide whether a graph G has $T(G) \leq t$ [1].

Let $K(G)$ be the clique graph of G, i.e., the vertices of $K(G)$ are related to maximal cliques of G and two vertices are adjacent if the related maximal cliques are non-disjoint in G. Abreu *et al.* [1] proved that $T(G) \leq \min\{\chi(K(G)), \chi'(G)\}$, where $\chi(K(G))$ and $\chi'(G)$ denote the chromatic number and chromatic index of graphs $K(G)$ and G, respectively. They also showed NP-completeness proofs of the t-TESSELLABILITY problem for several graph classes. Moreover, they showed that this problem is polynomial-time solvable for *threshold graphs* $G = (C \cup S, E)$. A threshold graph G has $K(G)$ that is a complete graph, and $T(G) = \chi(K(G)) = |S| + 1$ (C is a largest maximum clique of G, $S = V \setminus C$ is a stable set of G).

Note that the computational complexity of t-TESSELLABILITY is still open for cographs, whereas it is polynomial time solvable for threshold graphs [1]. In this work, we present the tessellation cover number for *quasi-threshold* graphs and the polynomial-time algorithm for t-TESSELLABILITY for this graph class, in Sec. 2. We also present the definition of t-TESSELLABILITY COMPLETION relating it to k-EDGE PRECOL-

ORING, in Sec. 3. Finally, in Sec. 4, we present the concluding remarks.

2 Tessellability for quasi-threshold graphs

Note that the tessellation cover number of a disconnected graph is given by the maximum of the parameters of their connected components, i.e., if $G = G_1 \cup G_2$, then $T(G) = \max\{T(G_1), T(G_2)\}$.

A graph G is a *cograph*, *quasi-threshold*, *threshold* if G is $\{P_4\}$-free, $\{P_4, C_4\}$-free, and $\{P_4, C_4, 2K_2\}$-free, respectively [3]. Let G be a graph with a vertex u. The addition of a twin vertex v of u in G includes v in G with the same neighborhood of u, and there is an edge uv in $E(G)$ if v is a *true twin*. Otherwise, v is a *false twin*. Let G' be obtained from a graph G by adding a true twin v of $u \in V(G)$. The cliques containing u in G will become cliques in G' that also contain v. So, we can use the same cliques of tessellations that cover the edges incident to u in G to cover the edges incident to v in G'.

Lemma 1. *If G is a graph with a vertex u and G' is obtained from G by the addition of a true twin vertex v of u, then $T(G) = T(G')$.*

Quasi-threshold graphs can be recursively obtained by the following operations from a K_1: adding universal vertices, and; the union operation of two quasi-threshold graphs [12].

Theorem 1. *Let G be a quasi-threshold graph and G' be a quasi-threshold graph constructed by adding a universal vertex v to G. Hence, $T(G') = \sum_i T(C_i)$, where C_i is a connected component of G.*

Proof. The vertex v is universal, and we have two cases:

(I) Consider G connected. Therefore, G has a universal vertex u such that v and u are true twin vertices in G'. So, by Lemma 1, $T(G') = \sum_i T(C_i) = T(G)$.

(II) Consider G disconnected. Therefore each connected component C_i of G is a subgraph that is a quasi-threshold graph with a universal vertex u_i. So we can consider that vertices v and u_i are true twin vertices in each connected component C_i' (that is related to each C_i before adding

vertex v), thus the tessellation cover number in each connected component C'_i remains equal to $T(C_i)$. Since each connected component shares the vertex v in G', the cliques in each connected component share the vertex v. Then, to cover the incident edges of v we cannot use the same tessellations for each connected component, so $T(G') = \sum_i T(C_i)$. □

Every quasi-threshold graph is also a cograph, which have a *cotree*, that is a tree where the internal nodes represent operations of union or join, and the nodes that are leaves represent the vertices of the cograph [3]. We can construct the cotree of quasi-threshold graphs in such a way that every join operation occurs between a vertex and a quasi-threshold graph, and the cotree be binary where, w.l.o.g., the left side is a cotree and the right side is a leaf. Thereby, we are able to calculate the tessellation cover number of graphs of this class using its cotree by climbing this tree until the root. When the internal node of this cotree is a union operation, we know the value is the maximum among the parameters of the connected components. Otherwise, the internal node represents the join operation, so we use the result provided in Theorem 1. Note that the number of connected components in this situation is exactly the number of union operations in sequence until the next join operation in this cotree plus one. Therefore, we can calculate the tessellation cover number for quasi-threshold graphs in polynomial time.

3 Tessellability completion

We now introduce the t-TESSELLABILITY COMPLETION problem, which has a graph G and a partial tessellation cover $\mathcal{T}'$ of G as instance and aims to decide whether G has a tessellation cover $\mathcal{T}$ with t tessellations such that the tessellations of $\mathcal{T}'$ are part of $\mathcal{T}$. Note that in this work we consider that the cliques of tessellations of $\mathcal{T}'$ in $\mathcal{T}$ may expand, including new vertices. The k-EDGE PRECOLORING problem has a graph G and a partial edge coloring of G as instance and aims to decide whether G has an edge coloring with k colors such that the colors used in the partial edge coloring given by the instance are maintained.

The *Latin Square* problem has a $n \times n$ matrix M as instance and a set of elements of M with values in $\{1, ..., n\}$ and aims to decide whether it is possible to fill the remaining elements of M with values in $\{1, \ldots, n\}$ in such way that there is no repeated value in any line or column of M. Colbourn [4] proved that k-EDGE PRECOLORING is $\mathcal{NP}$-complete for complete bipartite graphs $K_{x,y}$ showing a polynomial transformation from the LATIN SQUARE. The idea of the proof is that the lines of M will be vertices of one stable set of the complete bipartite, the columns will be vertices of the other stable set. Moreover, the set of given values of M is related to the colors of the partial edge coloring of the complete bipartite graph such as if $M_{i,j} = \alpha$, then the edge ij of the complete bipartite graph receives the color α. It is not hard to verify that this complete bipartite graph has a n-edge coloring using that partial n-edge coloring if and only if the matrix can be filled with values in $\{1, \ldots, n\}$ given a set of elements of M already labeled. Figure 1(a) illustrates this construction.

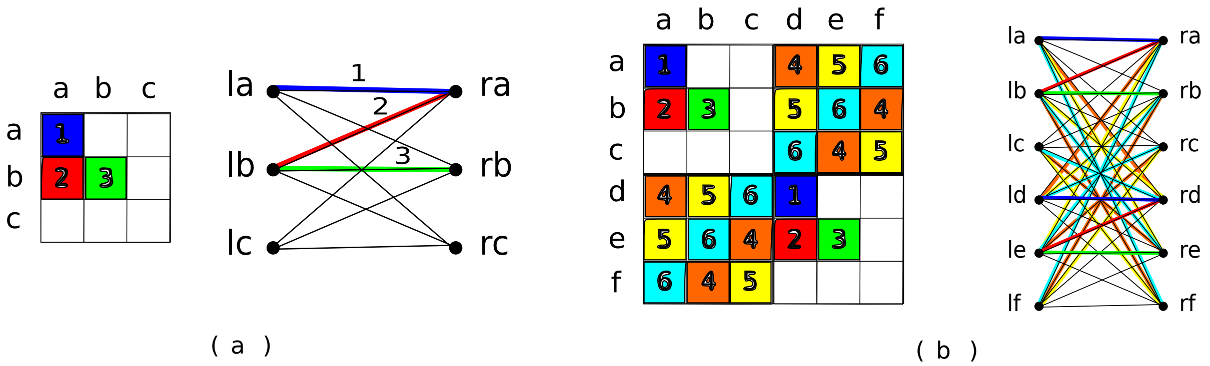

Figure 1: LATIN SQUARE and K-EDGE PRECOLORING on complete bipartite.

Bonomo et al. [2] proved that $(n-1)$-EDGE PRECOLORING of complete split (resp. complete) graphs is $\mathcal{NP}$-complete. The key idea of that proof is that given an instance I of LATIN SQUARE with a $n \times n$ matrix M, it is possible to create another instance I' with a $2n \times 2n$ matrix M' in such way I has a **YES** answer if and only if I' also has a **YES** answer. The matrix M' is obtained by adding two $n \times n$ elements in the top right and bottom left of M' with permutations of the values in $\{n+1, \ldots, 2n\}$ and by copying the values of M in the bottom right positions of M' (see Figure 1(b)). Moreover, given an instance I' of LATIN SQUARE with even n' we can construct a complete bipartite graph as described before. Next, we include all the missing edges of one clique (resp. two disjoint cliques) of size n' such that all these edges of the clique (resp. cliques) of even size n also appears in the partial edge coloring using colors in $\{n'+1, \ldots, 2n'-1\}$. Now, I' has a **YES** answer if and only if the $(2n'-1)$-EDGE PRECOLORING of this complete split graph of $2n'$ vertices (resp. complete graph of $2n'$ vertices) also is **YES**. Therefore, $(n-1)$-EDGE PRECOLORING is $\mathcal{NP}$-complete for complete bipartite graphs (a superclass of cographs), complete graphs, and complete split graphs.

In triangle-free graphs a tessellation cover behaves just like an edge coloring [1], the same holds for t-TESSELLABILITY COMPLETION and PARTIAL k-EDGE COLORING. Therefore, the computational complexity of k-EDGE PRECOLORING and k-TESSELLABILITY COMPLETION for triangle-free graphs are the same. Moreover, since k-EDGE PRECOLORING of Star graphs S_n is always **YES** for $k \geq \Delta(S_n) = n$ and **NO** otherwise, both k-EDGE PRECOLORING and k-TESSELLABILITY COMPLETION are in $\mathcal{P}$ for star graphs S_n. Marx [7] proved that k-EDGE PRECOLORING is $\mathcal{NP}$-complete for planar 3-regular bipartite graphs; bipartite outerplanar graphs; and bipartite series-parallel graphs. Thus, t-TESSELLABILITY COMPLETION is also hard for these graph classes.

Consider t-TESSELLABILITY COMPLETION for a complete graph G. If there is an edge without any available tessellation, then we know that the answer is **NO**. Otherwise, each edge has at least one available color and we obtain a tessellation cover of G by selecting one color for each unlabeled edge, and then covering all the endpoints of these unlabeled edges with a same color as a single clique in the tessellation related to this color, repeating this process for all colors.

The triangulated $TR(G)$ of a graph $G = (E, V)$ is obtained by adding to G, for each $e = uv \in E$, a vertex e_{uv} adjacent only to u and to v. Note that the $TR(K_n)$ of complete graphs K_n are split graphs. Let I

be an instance of $(n-1)$-EDGE PRECOLORING of a complete graph K_n with even n, which is $\mathcal{NP}$-complete [2]. Now, consider an instance I' of t-TESSELLABILITY COMPLETION of the graph $TR(K_n)$. Moreover, for each edge uv in the partial edge coloring of I we relate the triangle $e_{uv},\ u,\ v$ to a tessellation of the same label of the color of uv in the partial t-tessellation cover of $TR(K_n)$. Since $TR(K_n)$ has an induced star of size $n-1$, all the triangles with e-vertices incident to any vertex of the original clique K_n in $TR(K_n)$ need to be entirely covered by some tessellation. Note that each of these triangles of $TR(K_n)$ are related to an edge of K_n. Therefore, I has a **YES** answer if and only if I' also is **YES**.

Theorem 2. t-TESSELLABILITY COMPLETION *for Stars, Completes are in* $\mathcal{P}$ *whereas it is* $\mathcal{NP}$*-complete for Complete Bipartite and Triangulated Complete.*

Table 1: Computational Complexities Behaviors

	S_n	K_n	$K_{x,y}$	$TR(K_n)$	threshold	cograph
t-TESSELLATION COMPLETION	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{NP}-c$	$\mathcal{NP}-c$	Open	$\mathcal{NP}-c$
k-PARTIAL EDGE COLORABILITY	$\mathcal{P}$ [4]	$\mathcal{NP}$-c [4]	$\mathcal{NP}$-c [4]	$\mathcal{NP}$-c [4]	$\mathcal{NP}$-c [4]	$\mathcal{NP}$-c [4]
t-TESSELLABILITY	$\mathcal{P}$ [1]	$\mathcal{P}$ [1]	$\mathcal{P}$ [1]	$\mathcal{P}$ [1]	$\mathcal{P}$ [1]	Open
k- EDGE COLORABILITY	$\mathcal{P}$ [5]	$\mathcal{P}$ [5]	$\mathcal{P}$ [6]	$\mathcal{P}$ [5]	$\mathcal{P}$ [9]	Open

4 Final Remarks

In this work, we show that the tessellation cover number of quasi-threshold graphs is $T(G) = \sum_i T(C_i)$, where C_i is a connected component of G. Using these results we also prove that the t-TESSELLABILITY is polynomial-time solvable for quasi-threshold graphs.

There exist polynomial algorithms for k-EDGE COLORING restricted to complete graphs [5], complete bipartite graphs [6], complete split graphs [10], split indifference [8], and threshold graphs [9]. Similarly, in this work we have established polynomial time solutions for t-TESSELLABILITY COMPLETION restricted to star graphs and complete graphs. Moreover, we showed the hardness of t-TESSELLABILITY COMPLETION for complete bipartite graphs and triangulated complete graphs, a subclass of split graphs. Table 1 summarizes these results.

All the proofs for t-TESSELLABILITY $\mathcal{NP}$-complete also hold in the case of t-TESSELLABILITY COMPLETION. Therefore, it is only interesting to investigate graph classes for which t-TESSELLABILITY is in $\mathcal{P}$ or its computational complexity is open. We are close to establish a polynomial time algorithm for t-TESSELLABILITY COMPLETION restricted to line graphs of bipartite graphs, complete split graphs, and split indifference graphs, all graph classes for which we know t-TESSELLABILITY has linear time solution [1].

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