A New Computational Approach for the Fermat-Weber Problem and Extensions

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Abstract

In this technical note we propose a new computational approach for the Fermat-Weber problem and two extensions of this problem.

1 Introduction

A more detailed work on the Fermat-Weber problem [7] can be found in [2], [6]. In a didactic way, we show in the next section an optimization model presented in [3] and we comment on numerical results for 1,000 points in R^3 with this formulation. In section 3 we present the multi-source Weber problem as a DC programming formulation. Finally, in section 4 we introduce a discrete multi-source Weber problem and its formulation

2 A Mathematical Formulation

Given $a^i \in R^n, i \in P = \{1, 2, ..., p\}$, the Fermat-Weber [7] optimization problem (FWP) requires finding a point $x \in R^n$, that minimizes the sum of weighted Euclidean distances to all points in P. Let $w_i \geq 0,\ i \in P$ a weight associated with the Euclidean distance from x to a^i.

The Euclidean distance between x and a^i is

$$||a^i - x||_2 = \sqrt{\sum_{k=1}^{n} (a_k^i - x_k)^2},\ i \in P.$$

Thus (FWP) can be written as follows:

$$(FWP): \quad \min \sum_{i=1}^{p} w_i \sqrt{\sum_{k=1}^{n} (a_k^i - x_k)^2}.$$

Where $x = (x_1\ x_2\ ...\ x_n)^\top$ is the vector with the variables. (FWP) is a convex optimization problem, but it is not smooth.

Another way of representing (FWP) :

$$(FWP1): \quad \min \sum_{i=1}^{p} w_i z_i,$$

subject to:

$$z_i \geq \sqrt{\sum_{k=1}^{n} (a_k^i - x_k)^2}, \quad i = 1, 2, ..., p. \tag{1}$$

Each constraint i in (1) is known as a second order cone, which can be also written:

$$z_i^2 \geq \sum_{k=1}^{n} (a_k^i - x_k)^2, \quad z_i \geq 0, \quad i = 1, 2, ..., p. \tag{2}$$

You can also write:

$$(FWP2): \quad \min \sum_{i=1}^{p} w_i z_i, \tag{3}$$

subject to:

$$z_i^2 \geq \sum_{k=1}^{n} (a_k^i - x_k)^2, \quad i = 1, 2, ..., p, \tag{4}$$

$$z_i \geq 0, \quad i = 1, 2, ..., p, \tag{5}$$

This last formulation $(FWP2)$ is a smooth and convex optimization problem, it was presented in [3], however, no computational results were achieved.

The constraints (4) and (5) define a second order cone, where interior point methods developed in [9] can be used. The commercial software XPRESS [8] can recognize automatically constraints (4) and (5) as a second order cone, and uses an interior point algorithm [9] for solving $(FWP2)$. We did some computational toy tests for $p = 1,000$ and $n = 3$ with XPRESS, the CPU time is negligible.

3 Multi-source Weber Problem (MWP)

Given $a^i \in R^n, i \in P = \{1, 2, ..., p\},$ geometrical position of customers and q a number of facilities whose we have to find their physical locations $x^j \in R^n,\ j = 1, 2, ..., q$. We know $d_i \in R_+,\ i = 1, 2, ..., p$ the given demand required by the i^{th} customer, and $o_j \in R_+,\ j = 1, 2, ..., q,$ the maximum offering of the j^{th} facility.

$$(MWP): \quad \min \sum_{i=1}^{p} \sum_{j=1}^{q} w_{ij} \sqrt{\sum_{k=1}^{n} (a_k^i - x_k^j)^2}, \tag{6}$$

subject to:

$$\sum_{j=1}^{q} w_{ij} \geq d_i,\ i = 1, 2, ..., p, \quad \sum_{i=1}^{p} w_{ij} \leq o_j,\ j = 1, 2, ..., q. \tag{7}$$

$$w_{ij} \geq 0,\ i = 1, 2, ..., p, \quad j = 1, 2, ..., q, \tag{8}$$

$$x_k^j \in R,\ k = 1, 2, ..., n, \quad j = 1, 2, ..., q. \tag{9}$$

Where w_{ij} denotes the unknown allocation from the j^{th} facility to the i^{th} customer, we assume that the involved transportation costs are proportional to the corresponding distances, see ([5]).

We suppose $\sum_{i=1}^{p} d_i \leq \sum_{j=1}^{q} o_j$, thus the set of constraints (7) and (8) will not be empty.

(MWP) is neither convex, nor smooth.

3.1 A DC programming formulation

Since $w_{ij} \geq 0,\ i = 1, 2, ..., p, \quad j = 1, 2, ..., q$ we can also write (6) as following:

$$(MWP): \quad \min \sum_{i=1}^{p} \sum_{j=1}^{q} w_{ij} z_{ij}, \tag{10}$$

subject to:

$$z_{ij} \geq \sqrt{\sum_{k=1}^{n} (a_k^i - x_k^j)^2}, i = 1, 2, ..., p,\ j = 1, 2, ..., q. \tag{11}$$

(11) will be replaces by

$$z_{ij}^2 \geq \sum_{k=1}^{n} (a_k^i - x_k^j)^2,\ z_{ij} \geq 0\ i = 1, 2, ..., p,\ j = 1, 2, ..., q. \tag{12}$$

We define
$w_{ij} = t_{ij} - v_{ij},\ z_{ij} = t_{ij} + v_{ij},\ t_{ij} \in R,\ v_{ij} \in R,\ i = 1, 2, ..., p,\ j = 1, 2, ..., q.$
Thus $w_{ij} z_{ij} = t_{ij}^2 - v_{ij}^2, i = 1, 2, ..., p,\ j = 1, 2, ..., q.$

$$(DC: MWP): \quad \min \sum_{i=1}^{p} \sum_{j=1}^{q} (t_{ij}^2 - v_{ij}^2), \tag{13}$$

subject to:

$$w_{ij} = t_{ij} - v_{ij},\ z_{ij} = t_{ij} + v_{ij},\ i = 1, 2, ..., p,\ j = 1, 2, ..., q, \tag{14}$$

$$z_{ij}^2 \geq \sum_{k=1}^{n}(a_k^i - x_k^j)^2,\ z_{ij} \geq 0\ i = 1, 2, ..., p,\ j = 1, 2, ..., q, \tag{15}$$

$$t_{ij} \in R,\ v_{ij} \in R,\ i = 1, 2, ..., p,\ j = 1, 2, ..., q, \tag{16}$$

$$\sum_{j=1}^{q} w_{ij} \geq d_i,\ i = 1, 2, ..., p,\quad \sum_{i=1}^{p} w_{ij} \leq o_j,\ j = 1, 2, ..., q, \tag{17}$$

$$w_{ij} \geq 0,\ i = 1, 2, ..., p,\quad j = 1, 2, ..., q, \tag{18}$$

$$x_k^j \in R,\ k = 1, 2, ..., n,\quad j = 1, 2, ..., q. \tag{19}$$

We minimize the difference of two convex functions known as DC programming in a convex set, see [4], $(DC : MWP)$ can be solved using branch-and-bound techniques, [1].

4 A Discrete Multi-source Weber Problem

We propose another approach, Discrete Multi-source Weber Problem $(DMWP)$.
We will define for each $i = 1, 2, ..., p$, and $\forall j,\ w_{ij} \in S_i = \{s_1^i, s_2^i, ..., d_i\}$,
where $0 = s_1^i < s_2^i < ... < s_{|S_i|}^i = d_i$.
We consider

$$M = \max_{1 \leq i < j \leq p} ||a^i - a^j||_2.$$

From (6):

$$w_{ij}\sqrt{\sum_{k=1}^{n}(a_k^i - x_k^j)^2} = \sqrt{\sum_{k=1}^{n} w_{ij}^2 (a_k^i - x_k^j)^2} \tag{20}$$

We define $t_{ijk} = w_{ij}(a_k^i - x_k^j)$. We will be able to write:

$$t_{ijk} \geq -M(1 - y_{ijl}) + s_l^i(a_k^i - x_k^j), l = 1, .., |S_i|, k = 1, .., n, i = 1, .., p, j = 1, .., q, \tag{21}$$

$$t_{ijk} \leq s_l^i(a_k^i - x_k^j) + M(1 - y_{ijl}), l = 1, .., |S_i|, k = 1, .., n, i = 1, .., p, j = 1, .., q, \tag{22}$$

$$\sum_{l=1}^{|S_i|} y_{ijl} = 1, i = 1, 2, ..., p,\ j = 1, 2, ..., q, \tag{23}$$

$$y_{ijl} \in \{0, 1\}, i = 1, 2, ..., p, j = 1, 2, ..., q, l = 1, 2, ..., |S_i|, \tag{24}$$

$$w_{ij} = \sum_{j=1}^{p} s_l^i y_{ijl},\ i = 1, 2, ..., p,\ l = 1, 2, ..., |S_i|, \tag{25}$$

$$\sum_{j=1}^{q} w_{ij} \geq d_i,\ i = 1, 2, ..., p,\quad \sum_{i=1}^{p} w_{ij} \leq o_j,\ j = 1, 2, ..., q. \tag{26}$$

From (20), (21), (22), (23), (24):

$w_{ij}\sqrt{\sum_{k=1}^{n}(a_k^i - x_k^j)^2} = \sqrt{\sum_{k=1}^{n} w_{ij}^2(a_k^i - x_k^j)^2} = \sqrt{\sum_{k=1}^{n} t_{ijk}^2}.$

Thus we can write:

$$(DMWP): \quad \min \sum_{i=1}^{p}\sum_{j=1}^{q} z_{ij}, \tag{27}$$

subject to: (21), (22), (23), (24), (25), (26), and

$$z_{ij} \geq \sqrt{\sum_{k=1}^{n} t_{ijk}^2}, \ i = 1, 2, ..., p, \ j = 1, 2, ..., q. \tag{28}$$

Constraints (28) can be replaced by

$$z_{ij}^2 \geq \sum_{k=1}^{n} t_{ijk}^2, \ i = 1, 2, ..., p, \ j = 1, 2, ..., q, \tag{29}$$

$$z_{ij} \geq 0, \ i = 1, 2, ..., p, \ j = 1, 2, ..., q. \tag{30}$$

$$x_k^j \in R, \ k = 1, 2, ..., n, \quad j = 1, 2, ..., q. \tag{31}$$

As we have seen above, the constraints (29), (30) form a second order cone [9].

Finally, we will present a new optimization model whose continuous relaxation is a convex and smooth optimization problem, which we would like to solve using XPRESS:

$$(DMWP): \quad \min \sum_{i=1}^{p}\sum_{j=1}^{q} z_{ij},$$

Subject to:

$$t_{ijk} \geq -M(1 - y_{ijl}) + s_l^i(a_k^i - x_k^j), l = 1, .., |S_i|, k = 1, .., n, i = 1, .., p, j = 1, .., q,$$

$$t_{ijk} \leq s_l^i(a_k^i - x_k^j) + M(1 - y_{ijl}), l = 1, .., |S_i|, k = 1, .., n, i = 1, .., p, j = 1, .., q,$$

$$\sum_{l=1}^{|S_i|} y_{ijl} = 1, i = 1, 2, ..., p, \ j = 1, 2, ..., q,$$

$$y_{ijl} \in \{0, 1\}, i = 1, 2, ..., p, j = 1, 2, ..., q, l = 1, 2, ..., |S_i|,$$

$$w_{ij} = \sum_{j=1}^{p} s_l^i y_{ijl}, \ i = 1, 2, ..., p, \ l = 1, 2, ..., |S_i|,$$

$$\sum_{j=1}^{q} w_{ij} \geq d_i,\ i = 1, 2, ..., p,\quad \sum_{i=1}^{p} w_{ij} \leq o_j,\ j = 1, 2, ..., q.$$

$$z_{ij}^2 \geq \sum_{k=1}^{n} t_{ijk}^2,\ i = 1, 2, ..., p,\ j = 1, 2, ..., q,$$

$$z_{ij} \geq 0,\quad i = 1, 2, ..., p,\quad j = 1, 2, ..., q,$$

$$x_k^j \in R,\ k = 1, 2, ..., n,\quad j = 1, 2, ..., q.$$

Note: In order for the $(DMWP)$ optimization problem to have a solution, we guarantee that:$\sum_{j=1}^{q} o_j \geq \sum_{i=1}^{p} d_i$.

References

[1] Le Thi Hoai An and Pham Dinh Tao. A branch and bound method via d.c. optimization algorithms and ellipsoidal technique for box constrained nonconvex quadratic problems. *Journal of Global Optimization*, 13:171–206, 1998.

[2] J. Brimberg. The Fermat—Weber location problem revisited. *Mathematical Programming*, 71:71–16, 1995.

[3] R. Chandrasekaran and A. Tamir. Algebraic optimization Fermat—Weber location problem. *Mathematical Programming*, 46:219–224, 1990.

[4] T. Q. Phong. An algorithm for solving general D.C. programming problems. *Operations Research Letters*, 15(2):73–79, 1994.

[5] A. Raeisi Dehkordi. The optimal solution set of the multi-source Weber problem. *Bulletin of the Iranian Mathematical Society*, 45:495–514, 2019.

[6] H. Venceslau, M. Karam-Venceslau, A. E. Xavier, and N. Maculan. A geometric perspective of the Weiszfeld algorithm for solving the Fermat—Weber problem. *RAIRO Operations Research*, 50:157–173, 2016.

[7] A. Weber. *Über den Standort der Industrien (1909) - English translation: The Theory of the Location of Industries.* Chicago, Chicago University Press, 1929.

[8] XPRESS. http://www.maths.ed.ac.uk/hall/xpress/.

[9] G. Xue and Y. Ye. An efficient algorithm for minimizing a sum of euclidean norms with applications. *SIAM Journal on Optimization*, 4(7):1017–1036, 1997.