An affine scaling algorithm for biobjective linear programming

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Abstract

Given a biobjective linear programming problem, we develop an affine scaling algorithm with min-max direction and demonstrate its convergence for an efficient solution. We implement the algorithm for some minor issues in literature.

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Statements and Declarations

The authors declare no competing interests.

1 Introduction

The biobjective linear programming problem is presented in many applications such as the linear programming problem (monocriterion), in the sense that a new objective is necessary.

According to Clímaco et al. [7], there are some classifications of methods of resolution of a multicriterion problem. Basically, there are methods that try to find the group of efficient solutions (see, for example, Bornstein et al. [6]) and also, those that try to find an efficient solution adequate for the decision maker amongst all others. We will follow this idea.

We work with this perspective by using the relative interior of the feasible set (primal, dual or primal-dual) of the multiobjective linear programming problem that can be found in Arbel & Oren [3, 4], in which is developed primal and primal-dual interior point algorithms, respectively, both using a specific utility function to be optimized; in Aghezzaf & Ouaderhman [1], in which is developed an interior point method using, also, a specific utility function to be optimized; in Fonseca et al. [11], in which is developed a dual feasible infeasible-interior-point algorithm for the network flow problem.

Given a biobjective linear programming problem, we develop an affine scaling algorithm with min-max direction and demonstrate its convergence to an efficient solution. We implement the algorithm for some minor issues in literature.

This work is divided into the following sections: section 2 presents the biobjective linear programming problem and some hypothesis; the algorithm and its convergency is introduced in section 3; in section 4, we present some implementations; and in section 5, our final considerations are given.

2 The problem

Consider the following multiobjective linear programming problem

$$(PLMO) \quad \begin{array}{ll} \text{minimize} & z = Cx \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array}$$

where $A \in R^{m\times n}$, $b \in R^m$ and $C \in R^{p\times n}$, with $0 < m < n$.

Denote

$$X = \{x \in R^n; Ax = b, x \geq 0\}$$

the feasible set and

$$X^0 = \{x \in X; x > 0\}$$

the set of feasible interior points.

Consider $i = 1, \dots, p$. A solution $x^1 \in X$ is considered an efficient solution when there is no other solution $x^2 \in X$ such that $C_i x^2 \leq C_i x^1$, for every i and the inequality is strict for at least one *i*. A point in objective space $z = C\bar{x} \in Z \subset R^p$ is said nondominated solution when $\bar{x}$ is an efficient solution.

In a multiobjective problem, a compromise solution satisfying the decision maker, within the efficient solutions, is to be selected.

Next, we enunciate a proposition that states that a single optimal solution matches with an efficient solution in a multiobjective problem.

Proposition 2.1 *Consider* $i, k = 1, \dots, p$ *and given scalar values* $w_i, i \neq k$ *for some k. If* x^**is the single optimal solution of the linear optimization problem*

$$
\begin{array}{lll}
(PLMO)_k & \textit{minimize} & z_k = C_k x \\
 & \textit{subject to} & x \in X \\
 & & C_i x \leq w_i, i = 1, \dots, p,\ i \neq k,
\end{array}
$$

so x^**is an efficient solution to the problem (PLMO).*

Proof Clímaco et al. [7].

The problem we propose to solve is the biobjective linear programming problem

$$
\begin{array}{lll}
(P) & \textit{minimize} & z_c = c^T x \\
 & \textit{minimize} & z_d = d^T x \\
 & \textit{subject to} & Ax = b \\
 & & x \geq 0,
\end{array}
$$

where $A \in R^{m\times n}, b \in R^n$ and $d \in R^n$, with a rank of A equal to *m*, for $0 < m < n$.

Observe that, in case of $m = n$, the problem (*P*) is reduced to a problem of solving linear systems. The matter of *rank(A) = m* enables the use of the projection matrix by guaranteeing the inverse of matrix (AA^T).

Still, the biobjective linear programming problem (*P*) can be written as it follows:

$$
\begin{array}{lll}
(P') & \textit{minimize} & f(x) = (f_1(x), f_2(x)) \\
 & \textit{subject to} & x \in X,
\end{array}
$$

where

$$f_1(x) = c^T x \quad \textit{and} \quad f_2(x) = d^T x,$$

with

$$f: R^n \to R^2 \quad \textit{and} \quad f_i: R^n \to R,\ i = 1, 2.$$

The gradients $\nabla f_1(x) = c$ and $\nabla f_2(x) = d$.

We can also rewrite the definition of efficient solution (Pareto optimal) for problem (P'), in this way: a decision vector $x^* \in X$ is Pareto optimal if there is no other decision vector $x \in X$ such that $f_i(x) \leq f_i(x^*)$ for some *i*, *i* = 1, 2.

In this work, we will consider the following hypothesis:

(*H*1) the feasible set X is bounded;

(*H*2) the set of feasible interior points X^0 is nonempty;

($H3$) it is given an feasible interior point x^0;

($H4$) the vectors $c, d \notin R(A^T)$, where $R(A^T)$ denotes the row space of A;

($H5$) the monocriteria problems in z_c or in z_d both have a single optimal solution; and

($H6$) the monocriteria problems in z_c or in z_d are nondegenerate.

The hypothesis ($H1$) and ($H2$) guarantee the existence of optimal solutions to the monocriteria problems both in z_c and in z_d. The hypothesis ($H3$) enables the use of an algorithm that develops in the relative interior of the feasible set. The hypothesis ($H4$) dismisses the optimal solutions to each monocriterion problem as any feasible point. The hypothesis ($H5$) guarantees the existence of efficient solutions, and the hypothesis ($H6$) is used to demonstrate the convergence of the algorithm.

3 The algorithm and its convergence

Consider problem (P). The idea of the affine scaling method is based on Dikin [7]. In other words, from a biobjective linear programming point of view, we can say that the Dikin method consists in walking by the interior of a polyhedral set X through feasible interior points generated by the solution of a sequence of subproblems, $(P_k), k = 0, 1, \ldots$, knowing:

$$\begin{array}{lll} (P_k) & \textit{minimize} & z_c = c^T x \\ & \textit{minimize} & z_d = d^T x \\ & \textit{subject to} & Ax = b \\ & & (x - x^k)^T X_k^{-2} (x - x^k) \leq 1, \end{array}$$

where $x^k \in X^0$ is the center of the largest simple ellipsoid and $X_k = diag(x^k)$.

Next, we will enunciate a proposition that states that we can disregard the restriction $x \geq 0$ of the problem (P_k).

Proposition 3.1 *Consider $x^k \in X^0$ and $X_k = diag(x^k)$. So, the ellipsoid*

$$\{x \in R^n; (x - x^k)^T X_k^{-2} (x - x^k) \leq 1\}$$

is contained within the non-negative orthant of R^n.

Proof Barnes [5].

Now, we enunciate a master algorithm for the problem (P), based on Dikin's idea for a linear programming problem.

Algorithm 3.2 *Master*

Data: $x^0 \in X^0$.
k := 0.
REPEAT
Obtain $x^{k+1} \in X$ *resolving subproblem* (P_k).
k := k + 1.
UNTIL 'convergence'.

Consider *k = 0, 1, ...* Observing the master algorithm, the question now is: how to solve subproblem (P_k)? For each *k*, solving subproblem (P_k) is just as hard as the original problem (*P*). However, we will develop a strategy to obtain a single solution to each (P_k), so that, in the end, we will have an efficient solution for problem (*P*). That is what will be done for the remaining of this section.

The following result solves to the problem of minimizing a linear function in a ball with equality constraints.

Proposition 3.3 *Suppose that* $u \in R^n$ *is not in row space of A. So, the problem*

$$\begin{array}{ll} \text{minimize} & u^T h \\ \text{subject to} & Ah = 0 \\ & \|h\| \leq 1, \end{array}$$

has the solution,

$$\hat{h} = -\frac{u_p}{\|u_p\|},$$

where u_p *is a vector u projected in the null space of matrix A*, denoted by $N(A)$.

Proof Gonzaga [12].

We denote c_p and d_p as vectors *c* and *d* projected in the null space of matrix *A*, respectively.

We want to develop a strategy to obtain a direction along which the values of the objective functions in z_c and z_d decrease. An alternative is to find a direction with the shortest distance from to origin to the convex hull of vectors c_p and d_p (see, for example, Menezes [14]), knowing that:

$$\begin{array}{lll} (ED) & \text{minimize} & \frac{1}{2}\|v\|^2 \\ & \text{subject to} & v = tc_p + (1-t)d_p \\ & & t \in [0,1]. \end{array}$$

Using the Karush-Kuhn-Tucker conditions for the optimization problem (*ED*), we obtain: for $c_p - d_p \neq 0$,

$$\begin{aligned} &for \quad t = 0, \; {d_p}^T(c_p - d_p) \geq 0; \\ &for \quad t = 1, \; c_p^T(c_p - d_p) \leq 0; \\ &for \quad t \in (0,1), \; t = \frac{-d_p^T(c_p - d_p)}{(c_p - d_p)^T(c_p - d_p)}. \end{aligned}$$

If $c_p - d_p = 0$, then $v = c_p$ (or $v = d_p$).

Next, we will elaborate a procedure over problem (*ED*). Consider β the optimal value of (*ED*) and $\hat{h}$ a solution. If $\beta = 0$, then a decrease direction does not exist for both $c^T\hat{h}$ and $d^T\hat{h}$, and there may be an indeterminacy of the choice of $\hat{h}$. To avoid this indeterminacy, the search procedure will be as it follows.

Procedure 3.4 *About* (*ED*).
Solve (*ED*).
If $\hat{v} \neq 0, then\ \hat{h} = \frac{-\hat{v}}{||\hat{v}||}$
Else, $\hat{h} = 0$.

The problem of finding directions in (*ED*) means finding the shortest distance to the convex hull of vectors c_p and d_p, of which the result is a vector $\hat{v}$ for $\hat{v} \neq 0$. Then, take $\hat{h} = -\frac{\hat{v}}{||\hat{v}||}$ if $\hat{v} \neq 0$. Note that the directions obtained are in $N(A)$.

Notice that, in virtue of procedure 3.4 and proposition 3.3, the vector u_p is vector $\hat{v}$, for $\hat{v} \neq 0$.

Our strategy to solve subproblem (P_k), $k = 0,1, ...$, is in the following result.

Theorem 3.5 *Suppose that* $c \in R^n$ *and* $d \in R^n$ *are not in the row space of A. So, problem* (P_k), $k = 0,1, ...$, *has the solution*

$$\hat{x} = X_k\left(e - \frac{\bar{v}}{||\bar{v}||}\right),$$

where $x^k \in X^0, \; X_k = diag(x^k)$, *e denotes the vector of ones in* $R^n, \; \bar{c} = X_k c, \; \bar{d} = X_k d,$ $\bar{A} = AX_k, \; P_{\bar{A}} = I - \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A}$ *is the projection matrix in null space of* $\bar{A}$ *and*

$$\bar{v} = P_{\bar{A}}\left(\hat{t}\bar{c} + (1 - \hat{t})\bar{d}\right) = \hat{t}c_p + (1 - \hat{t})d_p,$$

$\hat{t} \in [0,1]$ *is obtained by procedure 3.5 for* $\bar{v} \neq 0.$ *In case of* $\bar{v} = 0$ *on procedure 3.4, then do monocriterion optimization over* $\bar{c}$ *or* $\bar{d}$.

Proof Consider the problem (P_k), *k = 0, 1, …. By making a scale change*

$$x = X_k \bar{x},$$

where $X_k = diag(x^k)$ and $\bar{x} \in R^n$, we transform the ellipsoid with center in x^k in a ball with center in point $e = [1,1,\ldots,1]^T \in R^n$, because point $x^k = X_k e$. In other words, substituting $x = X_k \bar{x}$ in problem (P_k), we obtain the following biobjective nonlinear programming problem:

$$
\begin{array}{lll}
(\bar{P}_k) & minimize & \bar{c}^T \bar{x} \\
 & minimize & \bar{d}^T \bar{x} \\
 & subject\ to & \bar{A}\bar{x} = b \\
 & & \|\bar{x} - e\| \leq 1,
\end{array}
$$

where $\bar{c} = X_k c,\ \bar{d} = X_k d,\ \bar{A} = AX_k$ and $\bar{x} \in R^n$. We know that problem $(\bar{P}_k)$ is problem (P_k) with a scale change defined by X_k^{-1}. In that way, vector e is in interior feasible point to subproblem $(\bar{P}_k)$, k = 0, 1, …, because

$$\bar{A}e = (AX_k)e = A(X_k e) = Ax^k = b \ \text{ and } \ \|e - e\| = 0 \leq 1.$$

Taking a feasible direction $h = \bar{x} - e$, and since $\bar{c}^T e$ and $\bar{d}^T e$ are constant, we have the problem

$$
\begin{array}{ll}
minimize & \bar{c}^T h \\
minimize & \bar{d}^T h \\
subject\ to & \bar{A}h = 0 \\
 & \|h\| \leq 1,
\end{array}
$$

of which the solution is

$$\hat{h} = -\frac{\bar{v}}{\|\bar{v}\|},$$

according to procedure 3.4, for $\bar{v} \neq 0$. If $\bar{v} = 0$, then do monocriteria optimization over $\bar{c}$ or $\bar{d}$.

So, for $\bar{v} \neq 0$,

$$\bar{x} = e + \hat{h} = e - \frac{\bar{v}}{\|\bar{v}\|},$$

is solution for $(\bar{P}_k)$. Finally, the solution to problem $(\bar{P}_k)$ is obtained by rescaling the solution in $(\bar{P}_k)$,

$$\hat{x} = X_k \bar{x} = X_k \left(e - \frac{\bar{v}}{\|\bar{v}\|}\right).$$

This ends the demonstration.

Now, consider the following proposition.

Proposition 3.6 *Let S be a nonempty closed convex set of R^n. A point $y_x \in S$ is the projection of $x \in R^n$ onto S if and only if*

$$(x - y_x)^T(y - y_x) \leq 0$$

for all $y \in S$.

Proof Hiriart-Urruty and Lemaréchal [13].

Consider procedure 3.4 with $\bar{v}$ not null and the last proposition. Taking x as the origin, $y_x = \bar{v}$ and $S = \{tc_p + (1-t)d_p \in R^n; t \in [0,1]\}$, for any $y \in S$, we obtain

$$0 < \bar{v}^T\bar{v} \leq \bar{v}^T y.$$

In this manner, like in the previous theorem, sequences $(c^T x^k)$ and $(d^T x^k)$ are strictly decreasing monotonous sequences, because $\hat{h} = -\frac{\bar{v}}{\|\bar{v}\|}$.

Consider the subproblem (P_k), k = 0, 1, …. The ellipsoid with axes parallel to the coordinate axes is the largest possible simple ellipsoid in the positive orthant. We must discredit the possibility of obtaining a non-efficient solution in this subproblem with a null coordinate.

Theorem 3.7 *Consider $\hat{x}$ a solution to subproblem* (P_k), whatever k, k = 0, 1, …. *If* $\hat{x}_j = 0$ *for some j, j* = 1, 2, …, *n, then $\hat{x}$ is an efficient solution to problem* (P).

Proof Fix arbitrarily k, k = 0, 1, …, *and consider the subproblem* (P_k). Suppose $\hat{x}$ as a solution to (P_k) such as $\hat{x}_j = 0$ for some j, j = 1, 2, …, n. According to the previous theorem, if $\bar{v} = 0$ we must execute monocriterion optimization. Thus, the following result; according to Saigal [16]. By the definition of projection matrix and scale change $\bar{c} = X_k c$, $\bar{d} = X_k d$ and $\bar{A} = AX_k$, for $X_k = diag(x^k)$ with $x^k \in X^0$, we have that, for $\hat{t} \in [0,1]$ and $\bar{v} \neq 0$,

$$\bar{v} = (I - \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A})\big(\hat{t}\bar{c} + (1-\hat{t})\bar{d}\big).$$

Developing this equality,

$$\bar{v} = X_k[\hat{t}c + (1-\hat{t})d - A^T(AX_k^2A^T)^{-1}AX_k^2(\hat{t}c + (1-\hat{t})d)].$$

Taking

$$y = (AX_k^2A^T)^{-1}AX_k^2(\hat{t}c + (1-\hat{t})d) \quad \text{and} \quad s = (\hat{t}c + (1-\hat{t})d) - A^Ty, \tag{1}$$

we obtain

$$\bar{v} = X_k s. \tag{2}$$

So, by using the previous theorem,

$$\hat{x} = X_k\left(e - \frac{\bar{v}}{\|\bar{v}\|}\right) = X_k\left(e - \frac{X_k s}{\|X_k s\|}\right) = X_k e - \frac{X_k^2 s}{\|X_k s\|}.$$

Thus,

$$\hat{x} = x^k - \frac{X_k^2 s}{\|X_k s\|}.$$

Then, making use of the hypothesis,

$$0 = \hat{x}_j = x_j^k - \frac{\left(x_j^k\right)^2 s_j}{\|X_k s\|}.$$

So, $x_j^k s_j = \|X_k s\|$. Then, $x_i^k s_i = 0$ for every $i \neq j$. Since $x_i^k > 0$ by definition of X_k, we obtain $s_i = 0$ for every $i \neq j$. Thus, $s \geq 0$. Taking y and s as according to (1) and considering the condition $(\hat{x})^T s = 0$, for $t \in (0,1)$, it follows by the sufficient conditions of Karush-Kuhn-Tucker for efficient solutions (Pareto optimal), see Miettinen [15], that $\hat{x}$ is an efficient solution to problem (P). Still, for $\hat{t} = 0$ or $(1-\hat{t}) = 0$ in (1), by the Karush-Kuhn-Tucker optimality conditions, $\hat{x}$ is an optimal solution for monocriteria problem (P). And, according to proposition 2.1, $\hat{x}$ is an efficient solution to problem (P). This puts end to the demonstration.

Now we are ready to enunciate the algorithm.

Algorithm 3.8 *Biobjective Dikin.*

Data: x^0 *an initial feasible interior point and* $\epsilon > 0$.
k:=0.
REPEAT
Scaling:

$X_k := diag(x^k)$.
$\bar{A} := AX_k$.
$\bar{c} := X_k c$.
$\bar{d} := X_k d$.
Projection:
$P_{\bar{A}} := I - \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A}$.
$c_p := P_{\bar{A}}\bar{c}$.
$d_p := P_{\bar{A}}\bar{d}$.
Procedure 3.4: $\bar{v}$.
Test:
If $\bar{v} = 0$, *then do monocriteria optimization.*
Direction:
$\bar{h} := -\frac{\bar{v}}{\|\bar{v}\|}$.
Rescaling:
$h^k := X_k\bar{h}$.
New point:
$x^{k+1} := x^k + h^k$.
$k := k + 1$.
UNTIL $x_j^k < \epsilon$ *for some j = 1, ..., n.*

Notice in this algorithm that the monocriterion optimization must be decided *a priori* about $\bar{c}$ or $\bar{d}$.

Definition 3.9 *A sequence* (z^k) *in* R^n *converges quasi-Fejér to* $S \subset R^n$ *if, for every* $z \in S$ *exists a sequence of non-negative numbers* (ε_k) *such that*

$$\|z^{k+1} - z\|^2 \leq \|z^k - z\|^2 + \varepsilon^k \text{ and } \sum_{k=1}^{\infty} \varepsilon^k < \infty.$$

The following lemma helps to demonstrate the convergence of the algorithm.

Lemma 3.10 *Consider the hypothesis* (*H1*) − (*H5*). *Let* (x^k) *be the sequence generated by algorithm 3.8. Then*

$$\|x^{k+1} - \hat{x}\|^2 \leq \|x^k - \hat{x}\|^2 + \|x^{k+1} - x^k\|^2,$$

where $\hat{x} \in \{x \in X; f(x) \leq f(x^k), for\ all\ k \in N\}$.

Proof Suppose the hypothesis (*H1*) − (*H5*). Consider procedure 3.4 in algorithm 3.8 with $\bar{v} \neq 0$. Consider problem (*P'*) equivalent to problem (*P*). By definition, f_i, *i=1,2,* is a convex and differentiable function. According to proposition 3.6, the sequence $(f(x^k))$ is decreasing. Because function *f* is defined in a compact feasible set, the set

$$E = \{x \in X; f(x) \leq f(x^k), for\ all\ k \in N\}$$

is nonempty. Take $\hat{x} \in E$. In virtue of procedure 3.4 and scale change in algorithm, for each $k \in N$ exists $\lambda^k = (t_k, 1 - t_k)^T, t_k \in [0,1]$, such that

$$h^k = -\left({}^1/_{\tau_k}\right) X_k P_{AX_k} X_k \sum_{i=1}^{2} \lambda_i^k \nabla f_i(x^k),$$

where

$$\tau_k = \left\| P_{AX_k} X_k \sum_{i=1}^{2} \lambda_i^k \nabla f_i(x^k) \right\|.$$

This way, from the non-negativity of λ_i^k and convexity of each f_i, $i = 1, 2$, we have that

$$\begin{aligned} (\lambda^k)^T f(\hat{x}) &\geq \textstyle\sum_{i=1}^{2} \lambda_i^k \left[f_i(x^k) + \left(\nabla f_i(x^k)\right)^T (\hat{x} - x^k)\right] \\ &\geq \textstyle\sum_{i=1}^{2} \lambda_i^k \left[f_i(x^k) - \left(\nabla f_i(x^k)\right)^T (x^k - \hat{x})\right] \end{aligned}$$

$$\textstyle\sum_{i=1}^{2} \lambda_i^k \left[\left(\nabla f_i(x^k)\right)^T (x^k - \hat{x})\right] \geq (\lambda^k)^T (f(x^k) - f(\hat{x})).$$

Adding $-(A^T y)^T (x^k - \hat{x})$, for some $y \in R^m$ such as (1), in both sides of this last inequality, we have

$$(\sum_{i-1}^{2} \lambda_i^k \left[\left(\nabla f_i(x^k)\right) - A^T y)^T (x^k - \hat{x})\right] \geq (\lambda^k)^T (f(x^k) - f(\hat{x})) - (A^T y)^T (x^k - \hat{x}).$$

Since $(x^k - \hat{x}) \in N(A)$ and $A^T y \in R(A^T)$, by definition of set E and again using (1) and non-negativity of λ^k, we get

$$s^T (x^k - \hat{x}) \geq (\lambda^k)^T (f(x^k) - f(\hat{x})) \geq 0. \quad (3)$$

In this point, consider the angles in the interval $[0, \pi/2]$, θ for vectors s and $(x^k - \hat{x})$, θ' for vectors $(x^k - \hat{x})$ and $X_k^2 s$ and γ for vectors s and $X_k^2 s$. If $\theta' = \theta - \gamma$, then the angle is an acute angle by definition of cosine. Suppose that $\theta' = \theta + \gamma$. Elaborating

$$\cos(\theta') = \cos(\theta + \gamma) = \cos(\theta)\cos(\gamma) - \sin(\theta)\sin(\gamma)$$

and, keeping in mind (3), we get $(X_k^2 s)^T(x^k - \hat{x}) \geq 0$. Then, using (2) and the definition of h^k in algorithm,

$$(h^k)^T(x^k - \hat{x}) \leq 0. \qquad (4)$$

This way,

$$\begin{aligned}\|x^{k+1} - \hat{x}\|^2 &= (x^{k+1} - \hat{x} + x^k - x^k)^T(x^{k+1} - \hat{x} + x^k - x^k) \\ &= \|x^k - \hat{x}\|^2 + 2(x^{k+1} - x^k)^T(x^k - \hat{x}) + \|x^{k+1} - x^k\|^2 \\ &\leq \|x^k - \hat{x}\|^2 + \|x^{k+1} - x^k\|^2,\end{aligned}$$

in which the inequality stems from (4). This finalizes the demonstration.

The next result refers to the convergence of the biobjective Dikin algorithm to an efficient solution to problem (*P*).

Theorem 3.11 *Considerer the hypothesis* (*H*1) $-$ (*H*6). *Let* (x^k) *be the sequence generated by algorithm 3.8. Then, sequence* (x^k) *converges to* $\hat{x}$*, where* $\hat{x}$ *is an efficient solution to problem* (*P*).

Proof Suppose the hypothesis (*H*1) $-$ (*H*6). Consider procedure 3.4 in algorithm 3.8. For $\bar{v} = 0$, the result is as it follows; according to Dikin [9] and proposition 2.1. For $\bar{v} \neq 0$, the result is as it follows; according to the previous lemma and theorem 2.10 in Drummond & Svaiter [10].

4 Implementations

The programming language used was Octave and the computer was a CORE I5 processor, 2X2GB memory and HD 500GB.

Given the following biobjective optimization problems; according to Clímaco et al. [7] and Arbel [2]. Also, given an initial feasible interior point. Next, we present the number of iterations and the result of the execution of algorithm 3.8. Here, the tolerance designated is $\epsilon = 10^{-8}$ and the result has four significant digits.

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$$\begin{array}{lll}(P_1) & \textit{minimize} & z_c = -25x_1 - 20x_2 \\ & \textit{minimize} & z_d = -x_1 - 8x_2 \\ & \textit{subject to} & x_1 + x_2 + x_3 \quad = 50\end{array}$$

$$
\begin{aligned}
2x_1 + x_2 + x_4 &= 80 \\
2x_1 + 5x_2 + x_5 &= 220 \\
x_1, \dots, x_5 &\geq 0.
\end{aligned}
$$

Take the initial interior feasible point $x^0 = [10, 10, 30, 50, 150]^T$. On $k = 6$ iterations, the result is

$$
\hat{x} = \begin{bmatrix} 2.4333e+00 \\ 4.3027e+01 \\ 4.5400e+00 \\ 3.2107e+01 \\ 1.3644e-10 \end{bmatrix}.
$$

Now, take initial feasible interior point $x^0 = [30, 10, 10, 10, 110]^T$. On $k = 5$ iterations, the result is

$$
\hat{x} = \begin{bmatrix} 2.0000e+00 \\ 3.0000e+01 \\ 6.5496e-17 \\ 1.0000e+01 \\ 3.0000e+01 \end{bmatrix}.
$$

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$$
\begin{aligned}
(P_2) \quad & \textit{minimize} && z_c = -3x_1 - x_2 \\
& \textit{minimize} && z_d = -x_1 - 4x_2 \\
& \textit{subject to} && -x_1 + x_2 + x_3 = 2 \\
& && x_1 + x_2 + x_4 = 7 \\
& && x_1 + 2x_2 + x_5 = 10 \\
& && x_1, \dots, x_5 \geq 0.
\end{aligned}
$$

Take initial feasible interior point $x^0 = [1, 1, 2, 5, 7]^T$. On $k = 18$ iterations, the result is

$$
\hat{x} = \begin{bmatrix} 4.0000e+00 \\ 3.0000e+00 \\ 3.0000e+00 \\ 7.0440e-09 \\ 9.8616e-09 \end{bmatrix}.
$$

Problem 1 in [2]

$$
\begin{aligned}
(P_3) \quad & \textit{minimize} && z_c = -x_1 \\
& \textit{minimize} && z_d = -x_2 \\
& \textit{subject to} && x_1 + x_2 + x_3 = 10 \\
& && x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

Take initial feasible interior point $x^0 = [1, 1, 8]^T$. On $k = 6$ iterations, the result is

$$\hat{x} = \begin{bmatrix} 5.0000e+00 \\ 5.0000e+00 \\ 1.0588e-21 \end{bmatrix}.$$

Problem 2 in [2]

$$\begin{array}{llll} (P_4) & \textit{minimize} & z_c = -x_1 & \\ & \textit{minimize} & z_d = -x_2 & \\ & \textit{subject to} & 8x_1 + 6x_2 - x_3 & = 112 \\ & & 5x_1 + 7x_2 - x_4 & = 96 \\ & & x_1 + x_2 + x_5 & = 18 \\ & & x_1, \dots, x_5 \geq 0. & \end{array}$$

Take initial feasible interior point $x^0 = [13, \frac{9}{2}, 19, \frac{1}{2}, \frac{1}{2}]^T$. On $k = 5$ iterations, the result is

$$\hat{x} = \begin{bmatrix} 1.3250e+01 \\ 4.7500e+00 \\ 2.2500e+01 \\ 3.5000e+00 \\ 1.2143e-10 \end{bmatrix}.$$

5 Concluding remarks

Given a biobjective linear programming problem, we present an algorithm and demonstrate its convergence to an efficient solution. We suggest implementations for practical problems for a better assessment of this algorithm's practical contribution.

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