ON THE COMPLEXITY OF THE SET GRAPH RECOGNITION PROBLEM RESTRICTED TO COGRAPHS AND SPLIT GRAPHS

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SOBRE A COMPLEXIDADE DO PROBLEMA DE RECONHECIMENTO DE SET GRAPHS RESTRITO A COGRAFOS E GRAFOS SPLIT

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Setembro/2022

Orientador: Márcia Rosana Cerioli

Programa: Engenharia de Sistemas e Computação

Nesta dissertação, tratamos do *problema de reconhecimento de set graphs*, isto é, o problema de decidir se um dado grafo é, ou não, um *set graph*. Este problema, também conhecido como *problema da orientação extensional acíclica* (EAO), foi provado ser NP-completo por A. Tomescu em 2012. Como é de costume, parte da pesquisa sobre o problema é determinar se esse problema pode ser resolvido em tempo polinomial quando restrito a certas classes de grafos, e assim, fazer um mapeamento da complexidade do problema quando restrito a cada classe de grafos. Primeiro, apresentamos o tema, junto a uma coletânea de resultados relevantes ao seu desenvolvimento. Segundo, desenvolvemos algumas ferramentas auxiliares para o reconhecimento de *set graphs*. Definimos as orientações extensionais-por-camadas acíclicas e um parâmetro chamado *set-deficiency*, que mede o quão distante um grafo está de ser um set graph. Terceiro, usamos as ferramentas descritas na elaboração de um algoritmo de tempo polinomial para reconhecer *set graphs* na classe de *cografos*. Quarto, provamos que o reconhecimento de *set graphs* restrito à classe dos *grafos split* é NP-completo.

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ON THE COMPLEXITY OF THE SET GRAPH RECOGNITION PROBLEM RESTRICTED TO COGRAPHS AND SPLIT GRAPHS

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In this dissertation, we study the *set graph recognition problem*, i.e., the problem of deciding whether a given graph is a *set graph* or not. This problem, also known as the *extensional acyclic orientation problem* (EAO), was proved to be NP-complete by A. Tomescu in 2012. As usual, part of the research on this topic is to determine whether EAO becomes solvable by polynomial-time algorithms when restricted to certain graph classes. First, we present the context of the problem, with a collection of results that are relevant to this line of research. Second, we develop some auxiliary tools for the recognition of set graphs. We define the *layered extensional acyclic orientations* and a graph parameter called *set-deficiency*, that measures how far a graph is from being a set graph. Third, we apply the developed tools to reach a polynomial-time algorithm for recognizing set graphs in the class of *cographs*. Fourth, we prove that the recognition of set graphs restricted to *split* graphs is NP-complete.

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Chapter 1

Introduction

In this dissertation, we concern ourselves with the class of *set graphs*. In particular, we study the SET GRAPH RECOGNITION problem, i.e., the problem of deciding whether a given graph is a set graph or not.

A graph is a set graph if it admits an *extensional acyclic orientation* (eao). An orientation of a graph is *extensional* if distinct vertices have distinct out-neighborhoods, and is *acyclic* if it has no cycles.

The SET GRAPH RECOGNITION problem, also known as the EXTENSIONAL ACYCLIC ORIENTATION problem (EAO), was first considered by A. Tomescu in [14]. Together with E. Omodeo and A. Policriti, A. Tomescu proved that EAO is NP-complete, even if restricted to bipartite graphs with exactly two leaves; the EAO can be solved in polynomial time in the classes of trees, unicyclic graphs, graphs admitting hamiltonian paths, claw-free graphs, complete multipartite graphs, block graphs and (apple,$K_{2,3}$,dart,$K_{1,1,3}$)-free graphs; and can be solved in linear time in classes of graphs with bounded *treewidth* (see [14] and [11]).

In [10], M. Milanič and A. Tomescu noted that it would be interesting to determine the time complexity of the EAO in certain classes of perfect graphs such as the threshold graphs, split graphs, cographs and chordal graphs. In this dissertation, we follow this line of investigation and determine the complexity of EAO in all of these suggested graph classes. More specifically, we prove that EAO is polynomial in the class of cographs, which contains the threshold graphs; and we prove that the restriction of EAO to split graphs is NP-complete. Since the split graphs are a subclass of the chordal graphs, the EAO is also NP-complete in the class of chordal graphs.

The polynomial-time algorithm for recognizing set graphs in the class of cographs is based on the following general strategy. We generalize the extensional acyclic orientation to the broader concept of a *layered extensional acyclic orientation* (leao). In terms of leaos, we define the graph parameter *set-deficiency*, that measures how far a graph is from being a set graph. Then, we reduce the problem of obtaining an extensional acyclic orientation, for a given graph, to the problem of obtaining an optimal *layered extensional acyclic orientation*. A well-known characterization of the cographs states that a graph is a cograph if and only if every non-trivial induced subgraph is either disconnected or has a disconnected complement. So, we present how optimal leaos can be constructed, for a given graph, by combining optimal leaos of smaller graphs, whenever the given graph is disconnected or has a disconnected complement. The results lead us to a polynomial-time algorithm for constructing optimal leaos for cographs. As a corollary, we show that the SET GRAPH RECOGNITION problem can be solved in polynomial-time when restricted to the class of cographs.

To prove that EAO remains NP-complete when restricted to the class of split graphs, we present a polynomial reduction from the TOTAL ORDERING PROBLEM (TOP) to EAO. The TOTAL ORDERING PROBLEM was proven to be NP-complete by Opatrny [12]. However, the main proof given by Opatrny depends on unproved claims for which complete references are lacking. Thus, we also present a complete and detailed proof that TOP is NP-complete, filling in an outline of an alternative proof also by Opatrny [12].

1.1 Overview

In the last section of Chapter 1, we present the notations and conventions that we use throughout this text.

In Chapter 2, we introduce the class of set graphs. We present examples and explore some well-known sufficient or necessary conditions for being a set graph.

In Chapter 3, we define what a *layered extensional acyclic orientation* (leao) is, and we define the *set-deficiency* of a graph, which measures how far a graph is from being a set graph. Then, we recall the definition of *module* (of a graph), and prove some basic results about the leaos and the set-deficiencies of subgraphs induced by modules.

In Chapters 4 and 5, we show how to construct minimum leaos for graphs by combining minimum leaos of certain induced subgraphs. Respectively, in Chapter 4 we construct a minimum leao for a graph G that has a disconnected complement by combining minimum leaos of two subgraphs G_1 and G_2 such that G is the join $G_1 \wedge G_2$; and in Chapter 5, we construct minimum leaos for disconnected graphs by combining minimum leaos of each connected component of these graphs, provided these minimum leaos satisfy the special condition of having exactly one sink. Then, in Section 5.3, we briefly discuss the generality of this special condition. Additionally, in Section 5.4, we conclude by showing how the presented results can be applied for solving the SET GRAPH RECOGNITION problem in polynomial time in the class of cographs.

In Chapter 6, we prove that recognizing set graphs in the class of split graphs is NP-complete. This proof consists of a polynomial-time reduction from the TOTAL ORDERING PROBLEM (TOP), which was proven to be NP-complete in 1979 by Opatrny [12]. Unfortunately, the main proof given by Opatrny depends on unproved claims for which references are lacking (see Section 6.1). Thus, we also give a detailed proof of the NP-completeness of the TOTAL ORDERING PROBLEM, filling an outline, also given by Opatrny in [12], of an alternative polynomial-time reduction from 3-SAT to TOP.

In Chapter 7, we conclude with some remarks and some perspectives into possible future extensions of this work.

1.2 Definitions and Notational Conventions

We assume the reader has familiarity with the basic concepts of Graph Theory, Computational Complexity Theory, and Set Theory as presented, for example, in the books [3], [7], and [6], respectively. Next, we review some concepts and fix some notations that will be used throughout the text. Other local usage concepts will be reviewed as needed in later sections.

For every set X, we denote by $\mathcal{P}(X)$ the power set of X, i.e., the set of all subsets of X. The natural numbers include 0, i.e., $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. For all $m, n \in \mathbb{N}$, we denote by $[m, n]$ the set of integers between m and n, including m and n, i.e., $[m, n] = \{i \in \mathbb{N} : m \leq i \leq n\}$; and we denote by $[n]$ the set of the first n positive integers, i.e., $[n] = [1, n] = \{1, \dots, n\}$. For every set X and every $n \in \mathbb{N}$, we denote by $\binom{X}{n}$ the set of subsets of X with cardinality n, i.e., $\binom{X}{n} = \{Y \in \mathcal{P}(X) : |Y| = n\}$.

Formally, a graph, G, is a pair (V, E), where V is a non-empty finite set of vertices, and $E \subseteq \binom{V}{2}$ is the set of edges of G. Given, any graph G, we also denote the set of vertices of G by $V(G)$ and the set of edges of G by $E(G)$.

Definition 1. *Given a vertex $x \in V(G)$, the neighborhood of x, denoted by $N_G(x)$, is the set of vertices adjacent to x in G, i.e., $N_G(x) := \{y \in V(G) : \{x, y\} \in E(G)\}$. Given a set $X \subseteq V(G)$, we denote by $N_G(X)$ the union of the neighborhoods of the vertices in X, i.e., $N_G(X) := \bigcup\{N_G(x) : x \in X\}$. When there is no ambiguity, we write simply $N(x)$ and $N(X)$ instead of $N_G(x)$ and $N_G(X)$.*

Definition 2. *Given a vertex $x \in V(G)$, the degree of x, denoted by $d(x)$, is the number of vertices adjacent to x in G, i.e., $d(x) := |N(x)|$. The maximum degree of G and the minimum degree of G are denoted, respectively, by $\Delta(G)$ and $\delta(G)$, i.e., $\Delta(G) := \max\{d(v) : v \in V\}$ and $\delta(G) := \min\{d(v) : v \in V\}$.*

Definition 3. *Given a graph $G = (V, E)$ and a subset of vertices $X \subseteq V$, the subgraph of G induced by X, denoted by $G[X]$, is the graph $(X, E \cap \mathcal{P}(X))$. We denote the induced subgraph $G[V \setminus X]$ simply by $G \setminus X$.*

Definition 4. *Given a graph $G = (V, E)$, the complement of G, denoted by $\overline{G}$, is the graph $(V, \overline{E})$, where $\overline{E} = \binom{V}{2} \setminus E$.*

We will also need the corresponding concepts and notations for graph orientations.

Definition 5. *Given a graph $G = (V, E)$, a set $D \subseteq V^2$ is an orientation of G if there is a bijection $dir : E \to D$ such that, for every edge $e = \{x, y\} \in E$, either $dir(e) = (x, y)$ or $dir(e) = (y, x)$.*

Usually, given a graph G, an orientation D of G, and two vertices $x, y \in V(G)$, we write $xy \in D$, or $x \to y$ when D is implicit, instead of $(x, y) \in D$.

Definition 6. *Given an orientation D and a vertex x, the out-neighborhood of x in D is $N_D^+(x) = \{y \in V(G) : xy \in D\}$; and the in-neighborhood of x in D is $N_D^-(x) = \{y \in V(G) : yx \in D\}$. When there is no ambiguity, we write simply $N^+(x)$ and $N^-(x)$ instead of $N_D^+(x)$ and $N_D^-(x)$.*

Definition 7. *Given an orientation D, a vertex x is a sink if $N_D^+(x) = \emptyset$, and is a source if $N_D^-(x) = \emptyset$.*

Definition 8. *Given a graph $G = (V, E)$, an orientation D of G, and a subset of vertices $X \subseteq V$, the restriction of D to the subgraph $G[X]$, denoted by $D[X]$ or $D|_X$, is the orientation $D \cap X^2$ of $G[X]$.*

Chapter 2

Set Graphs or Extensional Acyclic Orientations

In this chapter, we review the definition of *set graphs*, give some illustrative examples, and present some known results. Our main objective here is to give the reader some familiarity with the basic concepts and results that will be used throughout the text.

Set graphs were first defined by A. Tomescu in his Ph.D. Thesis [14].

Definition 9. *Given a graph G and an orientation D of G, a directed cycle in D is a finite sequence of vertices $x_1x_2\dots x_k$ such that $k > 1$, $x_1 = x_k$ and $x_ix_{i+1} \in D$ for every $i \in [k-1]$. We say that D is acyclic if D has no directed cycles.*

Definition 10. *An orientation D of a graph G is extensional if, for all $x, y \in V(G)$ with $x \neq y$, $N^+(x) \neq N^+(y)$. If, otherwise, two distinct vertices $x, y \in V(G)$ are such that $N^+_D(x) = N^+_D(y)$, we say that x and y collide.*

Definition 11. *A graph G is a set graph if G admits an extensional acyclic orientation (an eao).*

For example, the graphs in Figure 2.1 are set graphs because they admit extensional acyclic orientations – see Figure 2.2. Some graphs are not set graphs. For

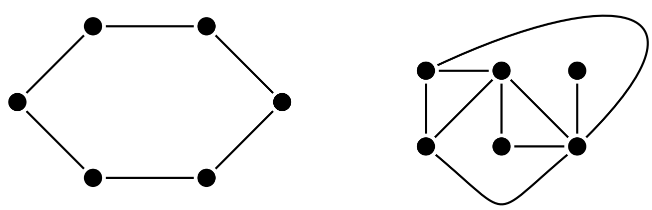

Figure 2.1: Set Graphs

instance, no disconnected graph is a set graph.

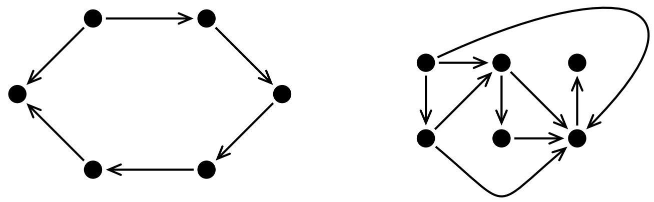

Figure 2.2: Extensional acyclic orientations of the set graphs in Figure 2.1

Lemma 12. *Every set graph is connected.*

Proof. Let G be a set graph. Then, G has an extensional acyclic orientation D. Since D is acyclic, each connected component of G has a sink. Since every sink has the same empty out-neighborhood and D is extensional, D has at most one sink. Therefore, G has exactly one connected component. $\square$

Some connected graphs are not set graphs. For instance, the complete bipartite graph $K_{1,3}$ (see Figure 2.3) is a connected graph that does not admit an eao (see Theorem 13, cf. [10]).

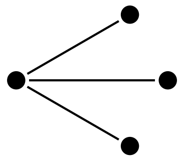

Figure 2.3: $K_{1,3}$

For all $m, n \in \mathbb{N}$, we denote by $K_{m,n}$ the complete bipartite graph $(X \cup Y, E)$ where $X \cap Y = \emptyset$, $|X| = m$, $|Y| = n$ and $E = \{\{x, y\} : x \in X, y \in Y\}$.

Theorem 13. *$K_{1,1}$ and $K_{1,2}$ are set graphs, but, for every $n \geq 3$, $K_{1,n}$ is not a set graph.*

Proof. Let $V(K_{1,n}) = \{a, b_1, b_2, \ldots, b_n\}$ with $d(a) = n$ and $d(b_i) = 1$ for every $i \in [n]$, i.e., a is the center of the star $K_{1,n}$. If $n = 1$, then $D = \{(b_1, a)\}$ is an eao of $K_{1,1}$. If $n = 2$, then, $D = \{(b_1, a), (a, b_2)\}$ is an eao of $K_{1,2}$. If $n \geq 3$, suppose, for a contradiction, that $K_{1,n}$ has an extensional orientation D. Then, by extensionality, $N^+(b_1), N^+(b_2), \ldots, N^+(b_n)$ are n distinct subsets of $\{a\}$. But $\{a\}$ has only two subsets – a contradiction. $\square$

Set graphs appear naturally from the very conception of sets. But the class of set graphs, and its recognition problem have only recently been taken as primary objects of study. From the point of view of the ZFC set theory, all members of a set are also sets (cf. [6]). Thus, the membership relation, $\in$, is a binary relation on any set X, yielding a directed graph with vertex set X. A set X is <u>transitive</u> if every element of X is also a subset of X. Set graphs are the underlying graphs of the digraphs that represent the reverse of the membership relation on *finite transitive sets*. The use of the reverse of the membership relation, instead of the membership relation itself, is historical and has no significant mathematical consequences. There is a natural correspondence between extensional acyclic orientations and finite transitive sets. The condition of extensionality corresponds to the property of sets, also called extensionality: two sets are equal if and only if they have the same elements. Similarly, given an extensional orientation D, of a graph G, two vertices are the same if and only if they have the same out-neighbors. The condition of acyclicity corresponds to the property of sets, of being well founded: there is no infinite descending sequence $\cdots \in x_n \in \cdots \in x_2 \in x_1$ in the membership relation. Any infinite walk on an orientation of a finite graph would require the orientation to have a cycle. In Lemmas 14 and 15, we show in detail that the two properties of extensionality and acyclicity are sufficient for characterizing the membership relation of finite transitive sets.

Lemma 14. *Given a finite transitive set X, there is a set graph G with an eao D and a bijection $f : V(G) \to X$ such that, for all $x, y \in V(G)$, $xy \in D$ if and only if $f(y) \in f(x)$, i.e., D is isomorphic to the reversed membership relation on X.*

Proof. Let $G = (X, \{\{x,y\} \in \binom{X}{2} : x \in y \text{ or } y \in x\})$, and $D = \{(x,y) \in X^2 : y \in x\}$. D is acyclic because a cycle in the membership relation would violate the Axiom of Regularity, postulated in the ZFC set theory (cf. [6]). Next, we prove that D is extensional. Let $x, y \in X$ be such that $x \neq y$. Since x and y are distinct sets, there exists $z \in x \setminus y$ or there exists $z \in y \setminus x$. Assume w.l.o.g. that $z \in x \setminus y$. Since $z \in x \in X$ and X is transitive, $z \in X$. Since $(x, z) \in X^2$ and $z \in x$, by definition, $xz \in D$. Moreover, since $z \notin y$, by the definition of D, $yz \notin D$. Thus, $z \in N^+_D(x) \setminus N^+_D(y)$. Therefore, D is extensional. Finally, we prove that D is isomorphic to the reversed membership relation on X. Let $f : X \to X$, be the identity function, i.e., $f(x) = x$ for every $x \in X$. Then, f is a bijection. Let $x, y \in X$. Suppose $xy \in D$. Then, by the definition of D, $f(y) = y \in x = f(x)$. So $f(y) \in f(x)$. Conversely, suppose $f(y) \in f(x)$. Then, $y \in x$, and consequently, $xy \in D$. Therefore, $xy \in D$ if and only if $f(y) \in f(x)$ for every $x, y \in X$. This completes the proof. □

Lemma 15. *Given a set graph G with an eao D, there is a finite transitive set X and a bijection $f : V(G) \to X$ such that, for all $x, y \in V(G)$, $xy \in D$ if and only if $f(y) \in f(x)$, i.e., the reversed membership relation on X is isomorphic to D.*

Proof. We prove, by induction on n, the stronger statement that if $n \leq |V(G)|$, then there is a subset $P \subseteq V(G)$ with $|P| = n$ such that

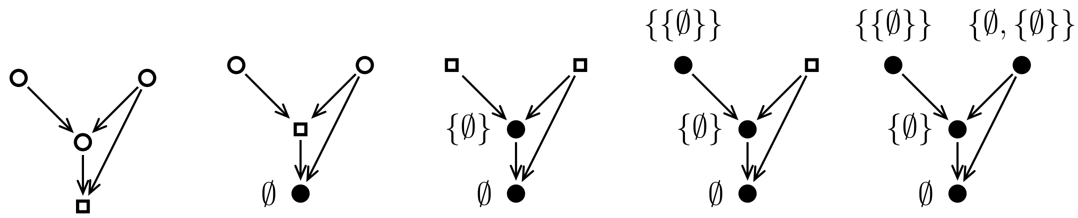

Figure 2.4: Illustration of the construction of a transitive set isomorphic to a given extensional acyclic orientation

(i) For every $x \in P$, $N_D^+(x) \subseteq P$;

(ii) There is a transitive set X and a bijection $f : P \to X$ such that, for all $x, y \in P$, $xy \in D$ if and only if $f(y) \in f(x)$. In this case, as usual, we say that f is an isomorphism between $D[P]$ and the reversed membership relation on X.

Base: Suppose $n = 1$. Since D is acyclic, it has a sink s. Let $P = \{s\}$. Since s is a sink, $N_D^+(s) = \emptyset \subseteq P$, so condition (i) is satisfied. Let $X = \{\emptyset\}$ and $f : P \to X$ defined by $f(s) = \emptyset$. Since $\emptyset \subseteq X$, by definition, X is transitive. Since, by the definition of orientation, $ss \notin D$, and $\emptyset \notin \emptyset$, f is an isomorphism.

Induction Hypothesis: Suppose that for every $n < |V(G)|$ there is a set $Q \subset V(G)$, with $|Q| = n$, satisfying conditions (i) and (ii) with a transitive set X_Q and isomorphism $f_Q : Q \to X_Q$.

Step: Let s be a sink of the acyclic orientation $D[V(G) \setminus Q]$. Let $P = Q \cup \{s\}$. By the IH, for every $x \in Q$, $N_D^+(x) \subseteq Q \subseteq P$. Besides, since s is a sink of $D[V(G) \setminus Q]$, $N_D^+(s) \subseteq Q \subseteq P$. Thus, condition (i) is satisfied.

Since $N_D^+(s) \subseteq Q$, every vertex of $N_D^+(s)$ is in the domain of f_Q. Then, we can define $x_s := \{f_Q(v) : v \in N_D^+(s)\}$. Furthermore, let $X := X_Q \cup \{x_s\}$.

To prove X is transitive, let $x \in X$. Either $x \in X_Q$ or $x = x_s$. If $x \in X_Q$, $x \subseteq X_Q \subseteq X$ because, by IH, X_Q is transitive. If $x = x_s$, then $x = \{f_Q(w) : w \in N_D^+(s)\} \subseteq X_Q \subseteq X$. In any case, $x \subseteq X$. Thus, X is transitive.

Let $f : P \to X$ be such that, for all $x \in P$,

$$f(v) = \begin{cases} f_Q(v) & \text{if } v \in Q, \\ x_s & \text{if } v = s. \end{cases}$$

Note that, since $s \notin Q$, f is well-defined.

To prove f is injective, let $x, y \in P$ be such that $f(x) = f(y)$. We consider four cases:

Case 1: Suppose $x, y \in Q$. Then, $f_Q(x) = f(x) = f(y) = f_Q(y)$, and thus $x = y$ because, by IH, f_Q is injective.

Case 2: Suppose $x = s$ and $y \in Q$. We prove that this case is contradictory. Let s_Q be a sink of the acyclic orientation $D[Q]$. Then, by (i), $N_D^+(s_Q) = \emptyset$. Since D is extensional, s_Q is the only sink of D. Thus, the unique sink of D is in Q, and therefore, s is not a sink of D. We will prove that $N_D^+(s) = N_D^+(y)$. Let $w \in N_D^+(s)$. Then, by definition of x_s, $f_Q(w) \in x_s = f(s) = f(y)$. Since $y \in Q$, $f(y) = f_Q(y)$. Then, $f_Q(w) \in f_Q(y)$, and since f_Q is an isomorphism, $yw \in D$. So, $w \in N_D^+(y)$. Thus $N_D^+(s) \subseteq N_D^+(y)$. Conversely, let $w \in N_D^+(y)$. Since f_Q is an isomorphism, $f_Q(w) \in f_Q(y)$. Then, $f_Q(w) \in f_Q(y) = f(y) = f(s) = x_s$. By the definition of x_s, there is $w' \in N_D^+(s)$ such that $f_Q(w') = f_Q(w)$. Since f_Q is injective, $w' = w$. Thus, $w \in N_D^+(s)$. Therefore, $N_D^+(s) = N_D^+(y)$, contradicting the extensionality of D.

Case 3: Suppose $x \in Q$ and $y = s$. Analogously to Case 2, this case is contradictory.

Case 4: Suppose $x = s$ and $y = s$. Then, trivially, $x = y$.

Thus, f is injective.

By definition, f is surjective. To prove that f is surjective, let $x \in X$. Since $X = X_Q \cup \{x_s\}$, we consider two cases.

Case 1: Suppose $x \in X_Q$. Since, by IH, f_Q is surjective, there is $v \in Q$ such that $f_Q(v) = x$. Then, by definition, $f(v) = f_Q(v) = x$.

Case 2: Suppose $x \in x_s$. Then, by definition, $f(s) = x_s = x$.

In any case, there exists $v \in P$ such that $f(v) = x$. Hence, f is surjective. Therefore, f is a bijection.

To prove f is an isomorphism, let $x, y \in P$. We consider four cases:

Case 1: Suppose $x, y \in Q$. Then, since, by IH, f_Q is an isomorphism, $xy \in D$ if and only if $f_Q(y) \in f_Q(x)$. Since $x, y \in Q$, by definition, $f(x) = f_Q(x)$ and $f(y) = f_Q(y)$. Thus, $xy \in D$ if and only if $f(x) = f(y)$.

Case 2: Suppose $x = s$ and $y \in Q$. We will prove $sy \in D$ if and only if $f(y) \in f(s)$. Suppose $sy \in D$. Since $y \in N_D^+(s)$, by definition, $f_Q(y) \in x_s = f(s)$. Since $y \in Q$, by definition, $f(y) = f_Q(y) \in f(s)$. Conversely, suppose $f(y) \in f(s)$. By definition, $f(s) = x_s$. Then, by definition of x_s, since $f(y) \in x_s$, there exists $w \in N_D^+(s)$ such that $f_Q(w) = f(y)$. Since $w \in N_D^+(s) \subseteq Q$, by definition, $f(w) = f_Q(w)$. Since $f(w) = f_Q(w) = f(y)$, and f is injective, $w = y$. Then, $y = w \in N_D^+(s)$. Therefore, $sy \in D$.

Case 3: Suppose $x \in Q$ and $y = s$. We will prove that $xs \in D$ if and only if $f(s) \in f(x)$, by proving that $xs \notin D$ and $f(s) \notin f(x)$. Since $x \notin Q$ and, by condition (i), $N_D^+(x) \subseteq Q$, we have $s \notin N_D^+(x)$. Thus, $xs \notin D$. Suppose, for a contradiction, $f(s) \in f(x)$. Since $x \in Q$, by definition, $f(s) = f_Q(s)$. Then, $f(s) \in f_Q(x)$. Since $f(s) \in f_Q(x) \in X_Q$ and, by IH, X_Q is transitive, $f(s) \in X_Q$. Since, by IH, f_Q is a bijection, there exists $w \in Q$ such that $f_Q(w) = f(s)$. Then, we have $f(w) = f_Q(w) = f(s)$. But, we have already proved that f is injective. So, $s = w \in Q$, a contradiction. Thus $f(s) \notin f(x)$. Since $xs \notin D$ and $f(s) \notin f(x)$, we have $xs \notin D$ if and only if $f(s) \notin f(x)$.

Case 4: Suppose $x = y = s$. Then, by the definition of orientation, $ss \notin D$. And, by the Axiom of Regularity, of ZFC set theory (cf. [6]), $f(s) \notin f(s)$.

Thus, f is an isomorphism. □

Figure 2.4 illustrates the proof of Lemma 15. We start with an acyclic extensional orientation, and at each step, we paint in black the vertices for which f is already defined (the set P), and we represent by squares the sinks of the orientations restricted to the remaining vertices (the sinks of $D[G(V) \setminus P]$, including s). Specifically, at each step, we pick a vertex s for which every out-neighbor is already assigned to a set, by f, and we assign it to the set $f[N_D^+(s)]$.

Set graphs play an important role in the interface between *sets* and *graphs*. It is often convenient to view graphs from a set theory perspective and, vice versa, to view sets from a graph theory perspective. This connection between sets and graphs has been helpful in approaching many kinds of problems of both fields. For instance, much like *Prüfer sequences* can be used for counting labeled trees [13], A. Tomescu used extensional acyclic orientations for counting transitive sets [14]. And conversely, in [10], Milanič and Tomescu gave a simpler proof of a known theorem from graph theory (stating that every connected claw-free graph with an even number of vertices admits a perfect matching) by leveraging their result that connected claw-free graphs admit extensional acyclic orientations, and thus have the structure of transitive sets.

One of the main problems concerning set graphs is to decide whether a given graph is a set graph. Now, we present some conditions related to this problem.

First, keep in mind the following property of acyclic orientations:

Lemma 16 (M. Milanič, A. Tomescu [10]). *Given an acyclic orientation D of a graph G and two distinct vertices $x, y \in V(G)$, if there is a path in D from x to y, then $N_D^+(x) \neq N_D^+(y)$.*

Proof. Suppose there is a path $x_1 x_2 \ldots x_p$ in D such that $x = x_1$ and $y = x_p$. If it were the case that $N_D^+(x) = N_D^+(y)$, then D would have the cycle $x_2 x_3 \ldots x_p x_2$, contradicting that D is acyclic. Therefore, $N_D^+(x) \neq N_D^+(y)$. □

Recall that a Hamiltonian path, on a graph, is a path that contains every vertex of the graph, without repetition. A simple sufficient condition for being a set graph is the following:

Lemma 17 (M. Milanič, A. Tomescu [10])**.** *If a graph G has a Hamiltonian path, then it is a set graph.*

Proof. Let $G = (V, E)$ be a graph with a Hamiltonian path $v_1 \dots v_n$. Define the orientation $D := \{v_i v_j \in V^2 : i < j \text{ and } \{v_i, v_j\} \in E\}$. D is acyclic because directed edges of D can only move forward in the Hamiltonian path. D is extensional because given two vertices $v_i, v_j \in V$, with $i < j$, $v_i v_{i+1} \dots v_j$ is a path in D, and thus, by Lemma 16, $N_D^+(v_i) \neq N_D^+(v_j)$. □

However, the converse of Lemma 17 does not hold. In Figure 2.5, below, we have a set graph (with an eao) that does not admit a Hamiltonian path:

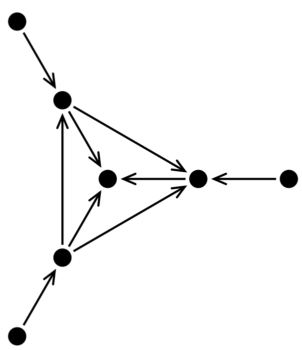

Figure 2.5: A set graph that does not admit a Hamiltonian path

If a graph has a Hamiltonian path, the vertices of degree one (the leaves) must be endpoints of the Hamiltonian path. Thus, graphs admitting Hamiltonian paths have at most two vertices of degree one. Therefore, the set graph in Figure 2.5, which has three vertices of degree one, does not admit a Hamiltonian path.

The set graphs that admit Hamiltonian paths likely do not have a simple characterization, as the problem of recognizing graphs admitting Hamiltonian paths (HP) restricted to the class of set graphs is NP-complete. In fact, the restriction of HP to *line graphs* is NP-complete [2], *line graphs* are a subclass of $K_{1,3}$*-free* graphs (cf. [1]), and connected $K_{1,3}$-free graphs are a subclass of set graphs [14] (we prove the latter inclusion in Theorem 27).

By analyzing some features of the out-neighborhoods of eaos, we reach, in Lemma 22 a necessary condition for being a set graph. First, let us define what a nested family of sets is, and prove some intermediary lemmas.

Definition 18. *A family of sets $\mathcal{F}$ is* nested *if, for all $X, Y \in \mathcal{F}$, $X \subseteq Y$ or $Y \subseteq X$, i.e., inclusion is a linear order on $\mathcal{F}$.*

Lemma 19 (M. Milanič, A. Tomescu [10]). *Let D be an acyclic orientation of a graph $G = (V, E)$ and let $u, v \in V$ be such that $N(u) = N(v)$. Then, $\{N_D^+(u), N_D^+(v)\}$ is nested, i.e., $N_D^+(u) \subseteq N_D^+(v)$ or $N_D^+(v) \subseteq N_D^+(u)$.*

Proof. Suppose $N(u) = N(v)$. Assume, for a contradiction, that there exist $w \in N_D^+(u) \setminus N_D^+(v)$ and $z \in N_D^+(v) \setminus N_D^+(u)$. Since $N(u) = N(v)$, we have $w \in N(v)$ and $z \in N(u)$. Thus, $w \in N_D^-(v)$ and $z \in N_D^-(u)$. But then, we have the cycle $wvzuw$ in D, contradicting that D is acyclic. □

Corollary 20 (M. Milanič, A. Tomescu [10]). *Let D be an eao of a graph $G = (V, E)$ and let $u, v \in V$ be distinct vertices. If $N(x) = N(y)$, then $|N_D^+(u)| \neq |N_D^+(v)|$.*

Proof. Suppose $N(x) = N(y)$. By Lemma 19, $\{N_D^+(u), N_D^+(v)\}$ is nested. Since $N_D^+(u)$ and $N_D^+(v)$ are finite sets such that $N_D^+(u) \subseteq N_D^+(v)$ or $N_D^+(v) \subseteq N_D^+(u)$, if they had the same cardinality, they would be equal, contradicting the extensionality of D. Thus, $|N_D^+(u)| \neq |N_D^+(v)|$. □

A necessary condition for being a set graph is known as the *same neighbors condition*:

Definition 21 (A. Tomescu [14]). *A graph G satisfies the same neighbors condition if, for every set $X \subseteq V(G)$, the set $Y = \{y \in V : N(y) = X\}$ has at most $|X| + 1$ elements.*

Lemma 22 (M. Milanič, A. Tomescu [10]). *Every set graph satisfies the same neighbors condition.*

Proof. Let D be an eao of G. Let $X \subseteq V(G)$, and let $Y = \{y \in V(G) : N(y) = X\}$. By Corolary 20, the out-neighborhoods of vertices in Y have pairwise distinct cardinalities. But these out-neighborhoods are subsets of X, whose cardinalities range over $\{0, \ldots, |X|\}$. Thus, $|Y| \leq |X| + 1$. □

The converse of Lemma 22 does not hold:

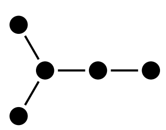

Figure 2.6: Counter-example to the converse of Lemma 22

The graph in Figure 2.6 satisfies the *same neighbors condition*, but is not a set graph: Since this graph is a tree, for each vertex v, there is a unique acyclic orientation such that v is the only sink. To prove that this graph is not a set graph we verify that, for each vertex v, the acyclic orientation that has v as the only sink is not extensional. In Figure 2.7, we present, for every vertex v of the graph in

Figure 2.6, the only acyclic orientation that has v as the only sink. In each case, we represent v by a square node and we paint in black a pair of vertices that have the same out-neighborhood.

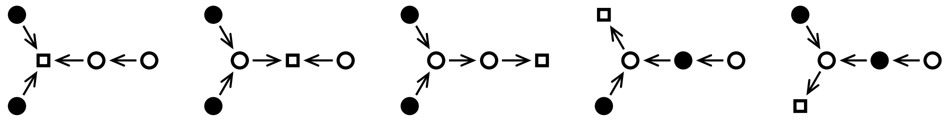

Figure 2.7: Every acyclic orientation of the graph in Figure 2.6 that only has one sink

Since the *same neighbors condition* (of Lemma 22) can be verified in polynomial time, and the recognition of set graphs is NP-complete, the recognition of set graphs restricted to the class of graphs satisfying the *same neighbors condition* is NP-complete. Thus, it is likely that there is no simple characterization of which graphs satisfying the *same neighbors condition* are set graphs.

From Lemmas 17 and 22, one obtains the following generalization of Theorem 13, characterizing the complete bipartite graphs that are set graphs.

Corollary 23 (M. Milanič, A. Tomescu [10])**.** *For any $m, n \in \mathbb{N}$, $K_{m,n}$ is a set graph if and only if $|m - n| \leq 1$.*

Proof. Suppose $K_{m,n} = (X \cup Y, E)$ is a set graph. Since G is bipartite complete, $N(x) = Y$ and $N(y) = X$ for all $x \in X$ and $y \in Y$. Therefore, $X = \{x \in V(G) : N(x) = Y\}$, and consequently, by Lemma 22, $|X| \leq |Y| + 1$. Similarly, $Y = \{y \in V(G) : N(y) = X\}$, and consequently, by Lemma 22, $|Y| \leq |X| + 1$. So, $|X| - 1 \leq |Y| \leq |X| + 1$. Therefore, $|m - n| \leq 1$. Conversely, suppose $|m - n| \leq 1$. Assume, $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. Suppose w.l.o.g. $m \leq n$. Then, $m = n - 1$ or $m = n$. If $m = n - 1$, then $y_1 x_1 \ldots y_m x_m y_n$ is a Hamiltonian path; and if $m = n$, then $y_1 x_1 \ldots y_m x_m$ is a Hamiltonian path. Thus, by Lemma 17, $K_{m,n}$ is a set graph. □

The proof given for Lemma 13, that $K_{1,n}$ is not a set graph (for $n \geq 3$), hints at another important necessary condition for being a set graph, known as *the cut-set condition*:

Definition 24 (A. Tomescu [14])**.** *A graph G satisfies the cut-set condition if, for every $X \subseteq V(G)$, the induced subgraph $G \setminus X$ has at most $2^{|X|}$ connected components.*

Lemma 25 (M. Milanič, A. Tomescu [10])**.** *Every set graph satisfies the cut-set condition.*

Proof. Let D be an eao of the set graph G. Let k be the number of connected components of G. Let $G_1, \ldots, G_k$ be the connected components of $G \setminus X$. Since D is acyclic, the restricted orientation $D[V \setminus X]$ has a sink in each connected component of $G \setminus X$. Let $y_1, \ldots, y_k$ be sinks of each of these connected components. Then, in D, the out-neighborhoods of $y_1, \ldots, y_k$ must be subsets of X, and must be pairwise distinct because D is extensional. Thus, X has at least k subsets, i.e., $k \leq 2^{|X|}$. $\square$

The converse of Lemma 25 does not hold. For instance, $K_{2,4}$ satisfies the condition that, for every $X \subseteq V(K_{2,4})$, the graph $K_{2,4} \setminus X$ has at most $2^{|X|}$ connected components (the *cut-set condition*), but by Corollary 23, $K_{2,4}$ is not a set graph.

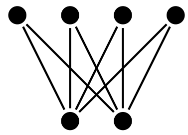

Figure 2.8: $K_{2,4}$

Moreover, a graph may satisfy both the *cut-set condition* and the *same neighbors condition* and still not be a set graph. For instance, consider the graph in Figure 2.9. It is easy to verify that the graph in Figure 2.9 satisfies both the *cut-set* and *same*

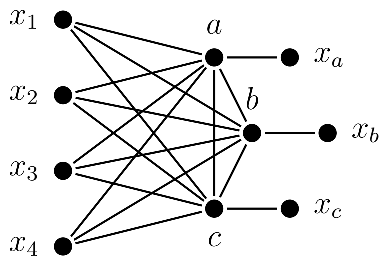

Figure 2.9: Graph satisfying the cut-set and the same-neighbors conditions that is not a set graph

neighbors conditions. But it is not a set graph, for suppose, for a contradiction, the graph in Figure 2.9 has an eao D. Since x_1, x_2, x_3, x_4 have the same neighborhood $\{a, b, c\}$, by Corollary 20, their out-neighborhoods have pairwise distinct cardinalities. Thus, the out-neighborhoods of x_1, x_2, x_3, x_4 are subsets of $\{a, b, c\}$ of every possible cardinality, from 0 to 3. Assume, w.l.o.g. that $|N_D^+(x_1)| = 0$ and $|N_D^+(x_2)| = 1$. Since x_1 is a sink and D is extensional, the vertices x_a, x_b, x_c are not sinks, and thus are sources. Then, since $|N_D^+(x_2)| = 1$, x_2 must collide with one of the vertices x_a, x_b, x_c, contradicting the extensionality of D.

It would be interesting to find a characterization of the graphs satisfying the *cut-set condition* that are set graphs, or a characterization of the graphs satisfying both the *same neighbors condition* and the *cut set condition* that are set graphs. But, to the best of our knowledge, no progress has been made on this matter.

As a corollary of Lemma 17 and Lemma 25, one obtains a characterization of the trees that are set graphs.

Corollary 26 (M. Milanič, A. Tomescu [10]). *A tree T is a set graph if and only if $\Delta(T) \leq 2$, i.e., T is a path.*

Proof. Let T be a tree. Suppose T is a set graph. Suppose, for a contradiction, T has a vertex $v \in V(T)$ such that $d(v) \geq 3$. Since T is a tree, $T \setminus \{v\}$ has $d(v)$ connected components. Thus, $T \setminus \{v\}$ has at least 3 connected components, while $2^{|\{v\}|} = 2$, contradicting Lemma 25. Thus, $\Delta(T) \leq 2$, i.e., T is a path. Conversely, suppose $\Delta(T) \leq 2$. Then, T is a path, and consequently has a Hamiltonian path. Thus, by Lemma 17, T is a set graph. □

The characterization in Corollary 26 gives us a direct proof that the graph in Figure 2.6 is not a set graph.

Let G and H be graphs. We say that G is H-free if no induced subgraph of G is isomorphic to H. A. Tomescu proved, in [14], the following theorem.

Theorem 27 (M. Milanič, A. Tomescu [14]). *Every connected $K_{1,3}$-free graph is a set graph.*

Proof. Let G be a connected $K_{1,3}$-free graph. The proof follows by induction on $|V(G)| = n$.
Base: If $n = 1$, since the orientation $D = \emptyset$ is an eao, G is a set graph.
Induction Hypothesis: Suppose that every connected $K_{1,3}$-free graph on n vertices is a set graph.
Step: Suppose G is a connected $K_{1,3}$-free graph with $n + 1$ vertices. By taking, for instance, a leaf of a spanning tree of G, let $v \in V(G)$ be such that $G \setminus \{v\}$ is connected. Then, $G \setminus \{v\}$ is a connected $K_{1,3}$-free graph with n vertices. By IH, $G \setminus \{v\}$ is a set graph. Let D' be an eao of $G \setminus \{v\}$. Let s be the unique sink of D'. We consider two cases, either $s \in N(v)$ or not.

Case 1: Suppose $s \in N(v)$. Extend D' to an orientation D of G by setting v as a sink, i.e., $D = D' \cup \{xv : x \in N(v)\}$. We prove that D is acyclic and extensional. First, we prove that v is the only sink of D. Let $w \in V(G) \setminus \{v, s\}$. Since s is the only sink of D' and $w \neq s$, w is not a sink of D'. And since $D' \subseteq D$, $\emptyset \neq N^+_{D'}(w) \subseteq N^+_D(w)$, so w is not a sink of D. Moreover, s is not a sink of D because, since $s \in N(v)$, by the definition of D, $sv \in D$. Thus, v is the only sink of D. Suppose, for a contradiction, that D has a cicle C. Since D' is acyclic, C must contain v, contradicting that v is a sink. Thus, D is acyclic. Secondly, we prove that D is extensional. Let $x, y \in V(G)$ and suppose $x \neq y$. We consider four subcases.

Subcase 1.1: Suppose $x, y \in V(G) \setminus \{v\}$. Assume, for a contradiction, $N^+_D(x) = N^+_D(y)$. Then, $N^+_{D'}(x) = N^+_D(x) \setminus \{v\} = N^+_D(y) \setminus \{v\} = N^+_{D'}(y)$, contradicting that D' is extensional.

Subcase 1.2: Suppose $x \in V(G) \setminus \{v\}$ and $y = v$. We have already proved that v is the only sink of D. So $N^+_D(x) \neq \emptyset = N^+_D(v)$.

Subcase 1.3: Suppose $x = v$ and $y \in V(G) \setminus \{v\}$. This case is analogous to Subcase 1.2.

Subcase 1.4: Suppose $x = v$ and $y = v$. Then, $x = y$, contradicting our assumption that $x \neq y$.

Thus, D is extensional.

Case 2: If $s \notin N(v)$, extend D' to an orientation D of G by making v a source, i.e., $D = D' \cup \{(v, x) : x \in N(v)\}$. First, we prove that s is the unique sink of D. Let $w \in V(G) \setminus \{v, s\}$. Since s is the only sink of D' and $w \neq s$, w is not a sink of D'. And since $D' \subseteq D$, $\emptyset \neq N^+_{D'}(w) \subseteq N^+_D(w)$, so w is not a sink of D. Moreover, v is not a sink because, since G is connected, v is not an isolated vertex, and since v is a source, $vw \in D$ for some $w \in N(v)$. Then, s is the unique sink of D. Since D' is acyclic, any cycle of D must contain v. But v is a source. Thus, D is acyclic. It remains to prove that D is extensional. Let $x, y \in V(G)$ and suppose $x \neq y$. We consider four subcases:

Subcase 2.1: Suppose $x, y \in V(G) \setminus \{v\}$. Since v is a source, $v \notin N^+_D(x)$. So $N^+_D(x) = N^+_{D'}$. Analogously, $N^+_{D'}(y) = N^+_D(y)$. Since D' is extensional, $N^+_D(x) = N^+_{D'}(x) \neq N^+_{D'}(y) = N^+_D(y)$.

Subcase 2.2: Suppose w.l.o.g, $x \in V(G) \setminus \{v\}$ and $y = v$. Assume, for a contradiction, that $N^+_D(x) = N^+_D(v)$. Let w be a sink of $D[N(v)]$. Since v is a source and $w \in N(v)$, $vw \in D$. And since $N^+_D(x) = N^+_D(v)$, $xw \in D$. Since s is the only sink of D and $s \notin N(v)$, w is not a sink of D. Then, let $z \in N^+_D(w)$. We will prove that $G[\{x, v, z, w\}]$ is a $K_{1,3}$. We already know that $vw \in D$, $xw \in D$ and $wz \in D$. So w is adjacent to x, v and z. So, it suffices to prove that x, v, z are pairwise non-adjacent. Firstly, since $N^+_D(v) = N^+_D(x)$, v and x are not adjacent because otherwise, $vx \in D$ or $xv \in D$ and consequently $xx \in D$ or $vv \in D$, contradicting that graphs have no loops. Secondly, $z \notin N(v)$ because otherwise, since $wz \in D$, w would not be a sink of $D[N(v)]$. Lastly, suppose, for a contradiction, that $z \in N(x)$. Then $zx \in D$ because otherwise, $z \in N^+_D(x) = N^+_D(v) \subseteq N(v)$, contradicting that $z \notin N(v)$. But, if $zx \in D$, we have the cycle $zxwz$ in D', a contradiction. Thus, $z \notin N(x)$. Therefore, $G[\{x, v, w, z\}]$ is an induced $K_{1,3}$ in G, contradicting that G is $K_{1,3}$-free.

Subcase 2.3: Suppose $x = v$ and $y \in V(G) \setminus \{v\}$. This case is analogous to Subcase 2.2.

Subcase 2.4: Suppose $x = v$ and $y = v$. Then, $x = y$, contradicting our assumption that $x \neq y$.

This concludes the proof. $\square$

It follows from Theorem 27 that $K_{1,3}$ is the smallest connected graph that does not admit an eao, as every other connected graph on at most 4 vertices is $K_{1,3}$-free, and, consequently, is a set graph.

Note, however, that the property of being a set graph is not *hereditary*, in the sense that some connected induced subgraph of a set graph may not be a set graph. Accordingly, some set graphs are not $K_{1,3}$-free. In fact, by Lemma 17, every graph admitting a Hamiltonian path is a set graph. And, as a corollary of the following lemma, every graph is an induced subgraph of a graph admitting a Hamiltonian path. Thus, every graph is an induced subgraph of some set graph.

Definition 28. *A vertex $v \in V(G)$ is a universal vertex if it is adjacent to every other vertex in G.*

Given a graph $G = (V, E)$, to add a universal vertex to G is to construct the graph $G' = (V \cup \{u\}, E \cup \{\{u, v\} : v \in V\})$, with some $u \notin V$.

Lemma 29. *Let G be a graph with n vertices and H be the graph obtained by adding successively $n - 1$ universal vertices to G. Then, H admits a Hamiltonian path.*

Proof. Let $x_1, \ldots, x_n$ be the n vertices of G, and let $y_1, \ldots, y_{n-1}$ be the $n - 1$ additional vertices of H. Then, $x_1 y_1 \ldots x_{n-1} y_{n-1} x_n$ is a Hamiltonian path of H. $\square$

Corollary 30. *Every graph G is an induced subgraph of a set graph H.*

Proof. Let H be the graph obtained by successively adding $|V(G)| - 1$ universal vertices to G. Then, by Lemma 29, H admits a Hamiltonian path. By Lemma 17, H is a set graph. Thus, G is an induced subgraph of the set graph H. $\square$

For example, by adding one universal vertex to $K_{1,3}$ one obtains the graph in Figure 2.10, that admits a Hamiltonian path.

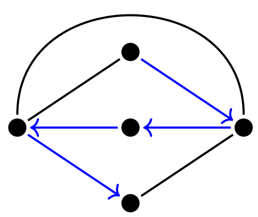

Figure 2.10: Adding a universal vertex to $K_{1,3}$ yields a graph with a Hamiltonian path.

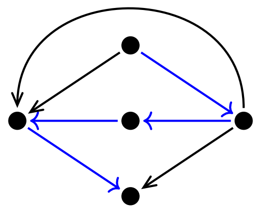

Figure 2.11: An eao can be obtained from a Hamiltonian path

Following the idea of the proof of Lemma 17, we define an eao, in Figure 2.11, from the Hamiltonian path in Figure 2.10.

As we saw in Figure 2.5, not every set graph admits a Hamiltonian path. By successively adding universal vertices to a given graph, we may obtain a set graph before obtaining a graph that admits a Hamiltonian path. For instance, if we add one universal vertex to the graph in Figure 2.12, consisting of a $K_{1,3}$ with an added isolated vertex, we obtain the graph in Figure 2.13 which does not admit a Hamiltonian path. However, it does already admit an extensional acyclic orientation (see Figure 2.14). Hence, we raise one of the main questions addressed in this text:

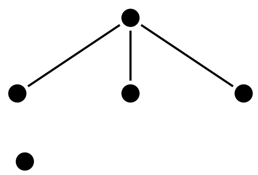

Figure 2.12: $K_{1,3} \cup K_1$

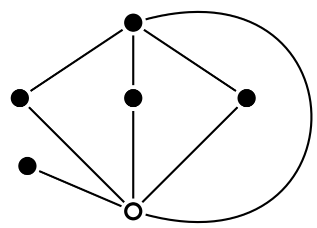

Figure 2.13: $(K_{1,3} \cup K_1) \wedge K_1$

> Given a graph G, what is the minimum number k such that we obtain a set graph by successively adding k universal vertices to G?

Clearly, this number is 0 if and only if the given graph is a set graph. Thus, by computing this number, we are able to decide whether or not a graph is a set graph. One of the concepts defined in Chapter 3 is the *set-deficicency* of a graph, which measures, from another point of view, how far a graph is from being a set graph.

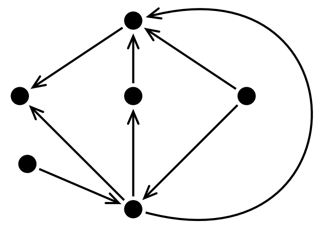

Figure 2.14: Eao of the graph in Figure 2.13

Later, in Chapter 4, we prove that the set-deficiency coincides with the answer to the question above. As we shall see, by combining the results of Chapters 3, 4 and 5 we are able to compute this number for any given cograph in polynomial time.

Chapter 3

Layered Extensional Acyclic Orientations

In this chapter, we define a generalization of extensional acyclic orientations that we call *layered extensional acyclic orientations* (leaos). These are obtained by relaxing the extensionality condition allowing the graph to be separated into layers and only requiring the given orientation to be *extensional-by-layers* (i.e., out-neighborhoods only need to be distinct between pairs of vertices in the same layer) and *downwards* (i.e., directed edges cannot go from a lower to a higher layer). Formally, the layers of a leao will be given by a labeling function on the vertices. Then, we prove that a graph admits an eao if and only if it admits a leao with a single layer (Lemma 34). In this manner, the problem of recognizing set graphs can be reduced to the problem of finding, for a given graph, the minimum number of layers a leao can have. This minimum number of layers will be called the *set-deficiency* of the graph and can be seen as a measure of how far the graph is from being a set graph.

Definition 31. *Given a function $\ell : V(G) \to \mathbb{N}$ and an orientation D of a graph G, we say that the pair $\mathfrak{L} = (\ell, D)$ is:*
(a) *a layered orientation of G. We call the number $\ell(x)$ the layer of the vertex x, and we say that x is in the layer $\ell(x)$. A number $n \in \mathbb{N}$ is a layer of $\mathfrak{L}$ if there is a vertex $x \in V(G)$ such that $\ell(x) = n$.*
(b) *extensional-by-layers when for all $x, y \in V(G)$, if $x \neq y$ and $\ell(x) = \ell(y)$, then $N^+_D(x) \neq N^+_D(y)$, i.e., pairs of vertices on the same layer have distinct out-neighborhoods.*
(c) *downwards if, for all $x, y \in V(G)$, if $xy \in D$, then $\ell(x) \geq \ell(y)$, i.e., directed edges can only connect vertices if they are in the same layer or the first is in a higher layer than the second.*
(d) *a layered extensional acyclic orientation (leao) if D is acyclic and $\mathfrak{L}$ is extensional-by-layers and downwards.*

In Figure 3.1 we present a few leaos of $K_{1,4}$, a graph that does not admit an eao, by Corollary 23. The number labeling a vertex v in the figure corresponds to the layer $\ell(v)$ of v.

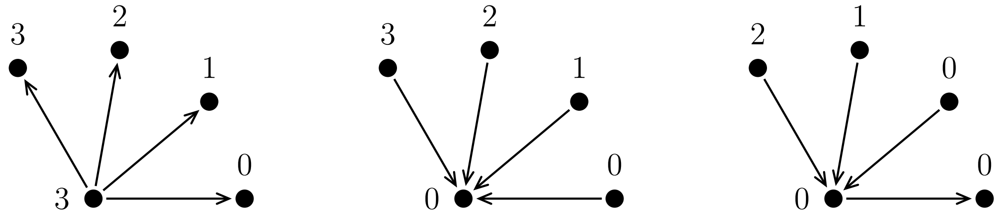

Figure 3.1: Three different leaos of $K_{1,4}$

In Figure 3.2, we present examples of layered orientations that are not leaos. The first is not a leao because it is not acyclic; the second is not extensional-by-layers (there are two sinks in the layer 0); and the third is not downwards.

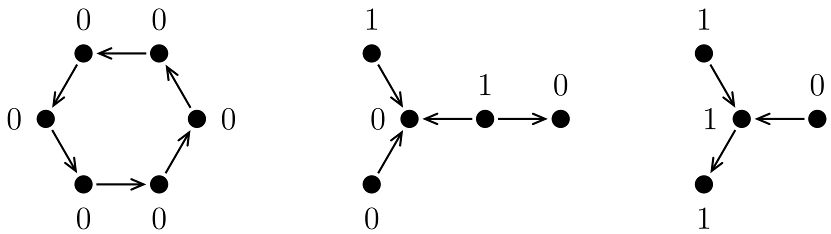

Figure 3.2: Not leaos

Theorem 32. *Every graph admits a leao.*

Proof. Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_n\}$. Define $\ell : V \to \mathbb{N}$ such that, for all $i \in [n]$, $\ell(v_i) = i - 1$ (one vertex per layer). For every edge $\{x, y\} \in E$, define $xy \in D$ if and only if $\ell(x) > \ell(y)$. By definition, $\mathfrak{L} = (\ell, D)$ is a leao of G. $\square$

Intuitively, the fewer layers a leao has, the closer it is to being an eao.

Definition 33. *Given a leao $\mathfrak{L} = (\ell, D)$ of a graph G, the height of $\mathfrak{L}$, denoted $|\mathfrak{L}|$, is the highest layer of $\mathfrak{L}$, i.e., $|\mathfrak{L}| = \max \ell = \max\{\ell(v) : v \in V(G)\}$.*

Lemma 34. *A graph G is a set graph if and only if G admits a leao $\mathfrak{L}$ such that $|\mathfrak{L}| = 0$.*

Proof. Suppose G is a set graph. Then, G has an eao D. Take $\ell : V(G) \to \mathbb{N}$ such that $\ell(v) = 0$ for all $v \in V(G)$. Then, $\mathfrak{L} := (\ell, D)$ is extensional-by-layers because D is extensional; is downwards because $\ell(u) = 0 \geq 0 = \ell(v)$ for all $uv \in D$; and D is acyclic because it is an eao. Moreover, $|\mathfrak{L}| = 0$ by defintion. Conversely, suppose G has a leao $\mathfrak{L} = (\ell, D)$ such that $|\mathfrak{L}| = 0$. D is extensional because (ℓ, D) is extensional-by-layers and all vertices have the same layer 0. D is acyclic because (ℓ, D) is a leao. Thus, D is an eao of G. $\square$

We will be mostly interested in leaos that have minimum height, as the height of a minimum leao can be seen as a measure of how far a graph is from being a set graph.

Definition 35. *A leao $\mathfrak{L}$ of a graph G is a minimum leao of G if, for every leao $\mathfrak{L}'$ of G, $|\mathfrak{L}| \leq |\mathfrak{L}'|$.*

Definition 36. *Let G be a graph. The set-deficiency of G, denoted by $\mathcal{S}_\Delta(G)$, is the height of a minimum leao of G, i.e., $\mathcal{S}_\Delta(G)$ is the minimum number k such that G has a leao $\mathfrak{L}$ with $k = |\mathfrak{L}|$.*

Of course, the number of vertices of a graph G provides a natural upper bound for the set-deficiency of G:

Lemma 37. *For any graph G, $\mathcal{S}_\Delta(G) \leq |V(G)| - 1$.*

Proof. The leao constructed, for an arbitrary graph G, in the proof of Theorem 32, has height $|V(G)| - 1$. Thus, $\mathcal{S}_\Delta(G) \leq |V(G)| - 1$. □

Example 38. We may determine, for instance, the set-deficiency of $K_{1,3}$. Since $K_{1,3}$ is not a set graph (by Theorem 13), $\mathcal{S}_\Delta(K_{1,3}) \geq 1$. And since $K_{1,3}$ has a leao of height 1 (in Figure 3.3), we have $\mathcal{S}_\Delta(K_{1,3}) = 1$.

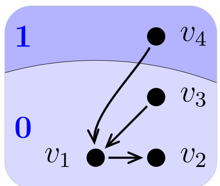

Figure 3.3: A 2-layers leao of $K_{1,3}$

The number of connected components of a graph provides a lower bound for its set-deficiency:

Lemma 39. *If a graph G has p connected components, then $\mathcal{S}_\Delta(G) \geq p - 1$.*

Proof. Let (ℓ, D) be a minimum leao of G. Since D is acyclic, each connected component of G has a sink. Then, D has at least p sinks. Since (ℓ, D) is extensional-by-layers, the layers of the sinks must be pairwise distinct. Then, we have $|\{\ell(v) : v \in V(G)\}| \geq p$. And consequently, $\mathcal{S}_\Delta(G) = \max \ell \geq p - 1$. □

For another example, consider the graph $\overline{K_n}$ having n vertices and no edges.

Lemma 40. $\mathcal{S}_\Delta(\overline{K_n}) = n - 1$.

Proof. By Lemma 37, $\mathcal{S}_\Delta(\overline{K_n}) \leq n-1$. And, since $\overline{K_n}$ has n connected components, by Lemma 39, $\mathcal{S}_\Delta(\overline{K_n}) \geq n - 1$. Therefore, $\mathcal{S}_\Delta(\overline{K_n}) = n - 1$. □

Now, we may connect this concept of set-deficiency to the motivation presented in the end of Chapter 1. It turns out that the set-deficiency of a graph G corresponds to the minimum number k such that we can obtain a set graph by adding k universal vertices to G. We will not give a proof of this claim at this point of the text, because a proof will follow directly from Theorem 58, which will be presented in Chapter 4.

3.1 Leaos of induced subgraphs

Given a leao $\mathfrak{L} = (\ell, D)$ of a graph G, we may want to produce a leao for an induced subgraph $G[X]$. This may not be as simple as taking $(\ell|_X, D \cap X^2)$, the restriction of $\mathfrak{L}$ to $G[X]$, as the result of this operation will rarely be extensional-by-layers. For instance, the layered orientation in Figure 3.4 is a leao, but if we restrict it to the induced $K_{1,3}$ (in white), it loses the property of being extensional-by-layers.

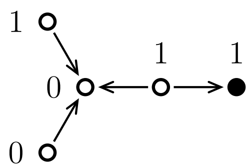

Figure 3.4: A restricted leao may not be a leao

As we have proved in Corollary 30, every graph is an induced subgraph of a set graph. Thus, an induced subgraph $G[X]$, of a graph G, may have a higher set-deficiency than G, i.e., it may happen that $\mathcal{S}_\Delta(G) < \mathcal{S}_\Delta(G[X])$.

In fact, as the next example shows, the difference $\mathcal{S}_\Delta(G[X]) - \mathcal{S}_\Delta(G)$ may be exponential with respect to $|V(G) \setminus X|$. For instance, in Example 41, we present a graph G with an induced subgraph $G[X]$ such that $\mathcal{S}_\Delta(G[X]) = \mathcal{S}_\Delta(G) + 2^{|V \setminus X|} - 1$.

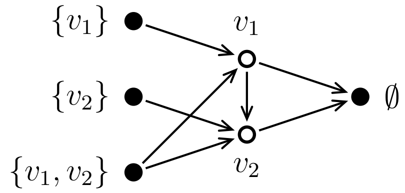

Figure 3.5: Induced subgraphs with high set-deficiency ($n = 2$)

Example 41. Let $X = \{v_1, \ldots, v_n\}$ be a set. Let $G = (V, E)$ be the graph such that $V(G) = X \cup \mathcal{P}(X)$ and $E(G) = \binom{X \cup \{\emptyset\}}{2} \cup \{\{v, Y\} : v \in X, Y \in \mathcal{P}(X),\text{ and } v \in Y\}$, i.e., all pairs of vertices in X are adjacent, every vertex $v \in X$ is adjacent to the vertex $\emptyset \in \mathcal{P}(X)$, no pair of vertices in $\mathcal{P}(X)$ are adjacent, and $N(Y) = Y$ for every $Y \in \mathcal{P}(X) \setminus \{\emptyset\}$. Define the orientation D such that, for all $v_i, v_j \in X$, $v_i \to v_j$ if and only if $i < j$; and $N^+_D(Y) = Y$ for every $Y \in \mathcal{P}(X)$. For illustrations of

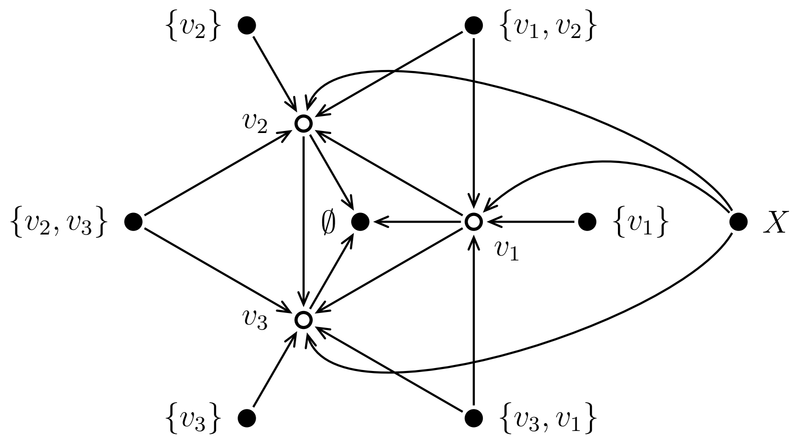

Figure 3.6: Induced subgraphs with high set-deficiency ($n = 3$)

the constructed graph and orientation, see Figure 3.5,for $n = 2$, and Figure 3.6, for $n = 3$. The vertices in X are painted in white, and the vertices of $\mathcal{P}(X)$ are painted in black.

We prove that D is an eao. D is acyclic because directed edges between vertices of X can only go from v_i to v_j if $i < j$ and every vertex in $\mathcal{P}(X)$ is either a source (the non-empty sets) or a sink (the empty set). D is extensional because if $v_i, v_j \in X$ and $i < j$, then $v_j \in N^+(v_i) \setminus N^+(v_j)$; if $v_i \in X$ and $Y \in \mathcal{P}(X)$, then $\emptyset \in N^+_D(v_i) \setminus N^+_D(Y)$; and if $Y_1, Y_2 \in \mathcal{P}(X)$ with $Y_1 \neq Y_2$, then $N^+(Y_1) = Y_1 \neq Y_2 = N^+(Y_2)$. Thus, $\mathcal{S}_\Delta(G) = 0$. Still, the induced subgraph $G[\mathcal{P}(X)]$ consists of 2^n isolated vertices. Thus, by Lemma 40, $\mathcal{S}_\Delta(G[\mathcal{P}(X)]) = 2^n - 1$. As such, we have $\mathcal{S}_\Delta(G[\mathcal{P}(X)]) = \mathcal{S}_\Delta(G) + 2^n - 1$.

However, if a set of vertices X is a *module* (cf. Section 3.2) of a graph G, then we can obtain a leao for $G[X]$, from a given leao of G, using at most $|N^+_D(X) \setminus X|$ additional layers (cf. Lemma 44 in Section 3.3). In the next Section 3.2, we review the concept of a *module*, and in Section 3.3 we present some results regarding the leaos of subgraphs induced by modules.

3.2 Modules

In this section, we review the concept of *module* of a graph as well as its main properties. This fundamental concept plays a key role in all of the forthcoming chapters.

Definition 42. *Let $G = (V, E)$ be a graph and let $X \subseteq V$. We say that X is a* <u>module</u> *of G if $N(x) \setminus X = N(y) \setminus X$ for all $x, y \in X$.*

The concept of a module generalizes the concept of a set of vertices such that, for any two vertices in the set, both vertices have the same neighborhood. The key difference is that the concept of a module only takes into account the parts of the neighborhoods outside the module.

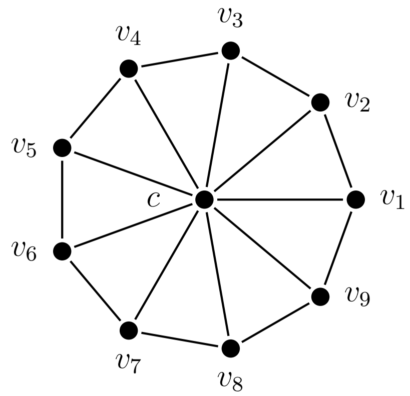

Figure 3.7: W_9

For instance, consider the *wheel graph* in Figure 3.7. The set $X_1 = \{v_i : i \in [9]\}$ is a module because, for every $i \in [9]$, $N(v_i) \setminus X_1 = \{c\}$. However, the subset $X_2 = \{v_1, v_2, v_3, v_4\}$ is not a module, because $v_9 \in N(v_1) \setminus X_2$ but $v_9 \notin N(v_2) \setminus X_2$. Notice that a subset of a module may not be a module. In contrast, it is the case that if every vertex in a set X has the same neighborhood and $Y \subseteq X$, then every vertex in Y has the same neighborhood.

Given a graph G, the sets $\emptyset$, $V(G)$, and $\{v\}$, for any $v \in V(G)$, are modules. These modules are called the trivial modules of G.

Aiming to apply these concepts to the study of set graphs, we generalize the key Lemma 19 from Chapter 1.

Lemma 43. *Let D be an acyclic orientation of a graph $G = (V, E)$ and let $X \subseteq V$ be a module. Then, the set $\Gamma_X = \{N_D^+(v) \setminus X : v \in X\}$ is nested.*

Proof. Let $x, y \in X$. Suppose, for a contradiction, that there exist $z \in (N_D^+(x) \setminus X) \setminus (N_D^+(y) \setminus X)$ and $w \in (N_D^+(y) \setminus X) \setminus (N_D^+(x) \setminus X)$. Since $z \in N(x) \setminus X$, $w \in N(y) \setminus X$ and X is a module, $z \in N(y)$ and $w \in N(x)$. But, since $z \notin (N_D^+(y) \setminus X)$ and $z \notin X$, we have $z \in N_D^-(y)$. Analogously, $w \in N_D^-(x)$. As in Figure 3.8, we have the cycle $xzywx$, contradicting that D is acyclic. Therefore, $N_D^+(x) \setminus X \subseteq N_D^+(y) \setminus X$ or $N_D^+(y) \setminus X \subseteq N_D^+(x) \setminus X$. □

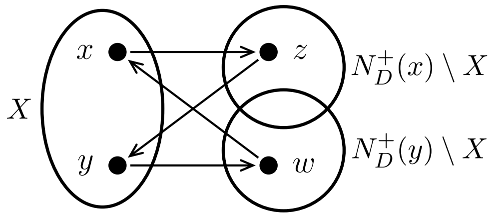

Figure 3.8: Illustration for the proof of Lemma 43

3.3 Leaos of subgraphs induced by modules

In this section, we present two results concerning the leaos of subgraphs induced by modules that we will use repeatedly for proving, for instance, that certain leaos are minimum.

Lemma 44. *Let G be a graph with a leao $\mathfrak{L} = (\ell, D)$ and a module $X \subseteq V(G)$. Let $\ell_X : X \to \mathbb{N}$ be such that $\ell_X(x) = \ell(x) + |N_D^+(x) \setminus X|$ for every $x \in X$; and let $D_X = D \cap X^2$. Then, $\mathfrak{L}_X = (\ell_X, D_X)$ is a leao of $G[X]$.*

Proof. D_X is acyclic because $D_X \subseteq D$ and D is acyclic.

To prove that $\mathfrak{L}_X$ is extensional-by-layers, let $x, y \in X$ be such that $x \neq y$ and $\ell_X(x) = \ell_X(y)$. Then, $\ell(x) + |N_D^+(x) \setminus X| = \ell(y) + |N_D^+(y) \setminus X|$. We consider three cases.

Case 1: Suppose $\ell(x) = \ell(y)$. Then, $|N_D^+(x) \setminus X| = |N_D^+(y) \setminus X|$. By Lemma 43, $\{N_D^+(z) \setminus X : z \in X\}$ is nested. Thus, $N_D^+(x) \setminus X = N_D^+(y) \setminus X$. Suppose, for a contradiction, that $N_{D_X}^+(x) = N_{D_X}^+(y)$. Since $N_{D_X}^+(x) = N_{D_X}^+(y)$ and $N_D^+(x) \setminus X = N_D^+(y) \setminus X$, $N_D^+(x) = N_{D_X}^+(x) \cup (N_D^+(x) \setminus X) = N_{D_X}^+(y) \cup (N_D^+(y) \setminus X) = N_D^+(y)$, contradicting that D is extensional-by-layers. Thus, $N_{D_X}^+(x) \neq N_{D_X}^+(y)$.

Case 2: Suppose $\ell(x) > \ell(y)$. Then, $|N_D^+(x) \setminus X| < |N_D^+(y) \setminus X|$. Thus, there is a vertex $z \in (N^+(y) \setminus X) \setminus (N^+(x) \setminus X)$. Since X is a module and $z \in N(y) \setminus X$, then $z \in N(x) \setminus X$. But since $z \notin N^+(x) \setminus X$, we have $zx \in D$. Since $\mathfrak{L}$ is downwards and yzx is a path in D, then $\ell(y) \geq \ell(z) \geq \ell(x)$, contradicting the assumption that $\ell(x) > \ell(y)$. Therefore, this case is not possible.

Case 3: Suppose $\ell(x) < \ell(y)$. This case is analogous to Case 2.

$\mathfrak{L}_X$ is downwards: Let $xy \in D_X$. Since $D_X \subseteq D$, $xy \in D$, and since $\mathfrak{L}$ is downwards, $\ell(x) \geq \ell(y)$. Suppose, for a contradiction, that $|N_D^+(x) \setminus X| < |N_D^+(y) \setminus X|$. Then, there is a vertex $z \in (N_D^+(y) \setminus X) \setminus (N_D^+(x) \setminus X)$. Since X is a module and $z \in N(y) \setminus X$, then $z \in N(x) \setminus X$. But $z \notin N_D^+(x)$, so $zx \in D$. Then, the acyclic orientation D has the cycle $xyzx$, a contradiction. Thus, we have $|N_D^+(x) \setminus X| \geq |N_D^+(y) \setminus X|$, and consequently, $\ell_X(x) = \ell(x) + |N_D^+(x) \setminus X| \geq \ell(y) + |N_D^+(y) \setminus X| = \ell_X(y)$. □

From Lemma 44, we obtain an upper-bound for the set deficiency of subgraphs induced by modules.

Theorem 45. *Let $G = (V, E)$ be a graph and $X \subseteq V(G)$ be a module of G. Then, $\mathcal{S}_\Delta(G[X]) \leq \mathcal{S}_\Delta(G) + |N(X) \setminus X|$.*

Proof. Let $\mathfrak{L} = (\ell, D)$ be a minimum leao of G. Let $\ell_X : X \to \mathbb{N}$ be such that $\ell_X(x) = \ell(x) + |N_D^+(x) \setminus X|$ for every $x \in X$, and $D_X = D \cap X^2$. By Lemma 44, $\mathfrak{L}_X = (\ell_X, D_X)$ is a leao of $G[X]$. By definition, $|\mathfrak{L}_X| \leq |\mathfrak{L}| + |N(X) \setminus X|$. Since $\mathfrak{L}$ is a minimum leao, $\mathcal{S}_\Delta(G) = |\mathfrak{L}|$. Hence, $\mathcal{S}_\Delta(G[X]) \leq |\mathfrak{L}_X| \leq \mathcal{S}_\Delta(G) + |N(X) \setminus X|$, which concludes the proof. $\square$

Chapter 4

Leaos of Co-Disconnected Graphs

In this chapter we prove some results about the minimum leaos of graphs that have a disconnected complement. We show how a minimum leao can be constructed, for a graph with a disconnected complement, from minimum leaos of some of its subgraphs. This method is one of the two components for the polynomial-time algorithm we present in Section 5.4 for constructing minimum leaos for cographs.

A graph G is said to be co-disconnected if its complement, $\overline{G}$, is disconnected.

Given two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the join of G_1 and G_2, denoted $G_1 \wedge G_2$, is the graph obtained by taking the union of G_1 and G_2 and making every vertex of G_1 adjacent to every vertex of G_2, i.e.,

$$G_1 \wedge G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{\{u, v\} : u \in V_1, v \in V_2\})$$

Lemma 46. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs. Then, $\overline{G_1} \wedge \overline{G_2} = \overline{G_1 \cup G_2}$.*

Proof. Both $\overline{G_1} \wedge \overline{G_2}$ and $\overline{G_1 \cup G_2}$ have $V_1 \cup V_2$ as set of vertices. It remains to prove that u and v are adjacent in $\overline{G_1} \wedge \overline{G_2}$ if and only if they are adjacent in $\overline{G_1 \cup G_2}$. Let $u, v \in V_1 \cup V_2$. We consider four cases:

Case 1: Suppose $u, v \in V_1$. Then, $\{u, v\} \in E(\overline{G_1} \wedge \overline{G_2})$ if and only if $\{u, v\} \in E(\overline{G_1})$ if and only if $\{u, v\} \notin E(G_1)$ if and only if $\{u, v\} \notin E(G_1 \cup G_2)$ if and only if $\{u, v\} \in E(\overline{G_1 \cup G_2})$.

Case 2: Suppose $u, v \in V_2$. Then, $\{u, v\} \in E(\overline{G_1} \wedge \overline{G_2})$ if and only if $\{u, v\} \in E(\overline{G_2})$ if and only if $\{u, v\} \notin E(G_2)$ if and only if $\{u, v\} \notin E(G_1 \cup G_2)$ if and only if $\{u, v\} \in E(\overline{G_1 \cup G_2})$.

Case 3: Suppose $u \in V_1$ and $v \in V_2$. Then, $\{u, v\} \in E(\overline{G_1} \wedge \overline{G_2})$ by definition. Moreover, by definition, $\{u, v\} \notin E(G_1 \cup G_2)$ and, consequently, $\{u, v\} \in E(\overline{G_1 \cup G_2})$.

Case 4: Suppose $u \in V_2$ and $v \in V_1$. This case is analogous to Case 3.

This concludes the proof. $\square$

Lemma 47. *A graph G is co-disconnected if and only if there are two disjoint graphs G_1 and G_2 such that $G = G_1 \wedge G_2$.*

Proof. Suppose G is co-disconnected. Then, $\overline{G}$ is disconnected. Let G_1 be a connected component of $\overline{G}$ and $G_2 = \overline{G} \setminus V(G_1)$. Then, G_1 and G_2 are two disjoint graphs such that $G_1 \cup G_2 = \overline{G}$. Thus, $G = \overline{G_1 \cup G_2}$. By Lemma 46, $G = \overline{G_1} \wedge \overline{G_2}$.

Conversely, suppose there are two disjoint graphs G_1 and G_2 such that $G = G_1 \wedge G_2$. Then, by Lemma 46, $G = \overline{\overline{G_1}} \wedge \overline{\overline{G_2}} = \overline{\overline{G_1} \cup \overline{G_2}}$, so $\overline{G} = \overline{G_1} \cup \overline{G_2}$. Since $\overline{G_1}$ and $\overline{G_2}$ are disjoint graphs, $\overline{G}$ is disconnected. Therefore, G is co-disconnected. $\square$

4.1 Stretching leaos

Before constructing leaos for joins of graphs, we need to introduce the *stretching of leaos*. This method will be useful for increasing the amount of (non-empty) layers of a leao on-demand.

Definition 48. *We say that a leao $\mathfrak{L}$ of a graph G is gapless if, for all $k \leq |\mathfrak{L}|$, there is a vertex $v \in V(G)$ such that $\ell(v) = k$.*

Lemma 49. *If a leao $\mathfrak{L}$ is minimum, then it is gapless.*

Proof. Let $\mathfrak{L} = (\ell, D)$ be a minimum leao of a graph G. Assume, for a contradiction, that $\mathfrak{L}$ is not gapless. Then, there is a number $k \leq |\mathfrak{L}|$ such that, for all $v \in V(G)$, $\ell(v) \neq k$. Define $\ell' : V \to \mathbb{N}$ such that

$$\ell'(v) = \begin{cases} \ell(v) & \text{if } \ell(v) < k \\ \ell(v) - 1 & \text{if } \ell(v) \geq k \end{cases}$$

Take $\mathfrak{L}' := (\ell', D)$. D is acyclic by hypothesis. To prove that $\mathfrak{L}'$ is extensional-by-layers, let $x, y \in V(G)$ be such that $x \neq y$ and $\ell'(x) = \ell'(y)$. We consider four cases.

Case 1: Suppose $\ell(x) < k$ and $\ell(y) < k$. Then, by definition, $\ell(x) = \ell'(x) = \ell'(y) = \ell(y)$. Since $\mathfrak{L}$ is extensional-by-layers, $N_D^+(x) \neq N_D^+(y)$.

Case 2: Suppose $\ell(x) \geq k$ and $\ell(y) < k$. Since k is such that $\ell(x) \neq k$, we have $\ell(x) > k$. Then, by definition, $\ell'(x) = \ell(x) - 1 \geq k$. Moreover, by definition, $\ell'(y) = \ell(y)$. Hence, $\ell'(x) \geq k > \ell(y) = \ell'(y)$, contradicting the assumption that $\ell'(x) = \ell'(y)$.

Case 3: Suppose $\ell(x) < k$ and $\ell(y) \geq k$. Since k is such that $\ell(y) \neq k$, we have $\ell(y) > k$. Then, by definition, $\ell'(y) = \ell(y) - 1 \geq k$. Moreover, by definition, $\ell'(x) = \ell(x)$. Hence, $\ell'(y) \geq k > \ell(x) = \ell'(x)$, contradicting the assumption that $\ell'(x) = \ell'(y)$.

Case 4: Suppose $\ell(x) \geq k$ and $\ell(y) \geq k$. Then, by definition, $\ell(x) = \ell'(x) + 1 = \ell'(y) + 1 = \ell(y)$. Since $\mathfrak{L}$ is extensional-by-layers, $N_D^+(x) \neq N_D^+(y)$.

In any case, $N_D^+(x) \neq N_D^+(y)$. So, $\mathfrak{L}'$ is extensional-by-layers. To prove that $\mathfrak{L}'$ is downwards, let $x, y \in V(G)$ be such that $xy \in D$. Since $\mathfrak{L}$ is downwards, $\ell(x) \geq \ell(y)$. We consider four cases.

Case 1: Suppose $\ell(x) < k$ and $\ell(y) < k$. Then, by definition, $\ell'(x) = \ell(x) \geq \ell(y) = \ell'(y)$.

Case 2: Suppose $\ell(x) \geq k$ and $k > \ell(y)$. Then, by definition, $\ell'(x) = \ell(x) - 1 \geq k - 1 \geq \ell(y) = \ell'(y)$.

Case 3: Suppose $k > \ell(x)$ and $\ell(y) \geq k$. Then, $\ell(y) > \ell(x)$, a contradiction.

Case 4: Suppose $\ell(x) \geq k$ and $\ell(y) \geq k$. Then, $\ell'(x) = \ell(x) - 1 \geq \ell(y) - 1 = \ell'(y)$.

In any case, $\ell'(x) \geq \ell'(y)$. Thus, $\mathfrak{L}'$ is a leao of G. But $|\mathfrak{L}'| = |\mathfrak{L}| - 1$, contradicting the minimality of $\mathfrak{L}$. Hence, $\mathfrak{L}$ is gapless. □

Lemma 50. *Let $\mathfrak{L}$ be a leao of a graph $G = (V, E)$, and let $k \in \mathbb{N}$. If $\mathfrak{L}$ is gapless and $|\mathfrak{L}| \leq k < |V|$, then there is a gapless leao $\mathfrak{L}^*$ with height $|\mathfrak{L}^*| = k$.*

The general procedure to obtain such a larger leao is straightforward. As long as we have a gapless leao $\mathfrak{L}$ such that $|\mathfrak{L}| < |V| - 1$, we are guaranteed, by the pigeonhole principle, to have a layer being shared by more than one vertex. So by carefully partitioning this layer into two layers, we obtain a new gapless leao with one extra layer. We iterate this process until we obtain a gapless leao with the particular desired height. This procedure is more precisely described in Construction 51.

Construction 51. [Stretching a leao]
INPUT: A triple $(\mathfrak{L}, G, k)$, where $\mathfrak{L}$ is a gapless leao of the graph $G = (V, E)$ and $k \in \mathbb{N}$ is such that $|\mathfrak{L}| \leq k < |V|$.
CONSTRUCTION: The procedure applies recursion on k.
Base: If $k = |\mathfrak{L}|$, take $\mathfrak{L}^* := \mathfrak{L}$.
Recursive step: Suppose $|\mathfrak{L}| < k < |V|$. Apply Construction 51 on $(\mathfrak{L}, G, k - 1)$ to obtain a gapless leao $\mathfrak{L}^*_{rec} = (\ell^*_{rec}, D)$ such that $|\mathfrak{L}^*_{rec}| = k - 1$.

Since $\mathfrak{L}^*_{rec}$ is gapless, $|\{\ell^*_{rec}(v) : v \in V(G)\}| = |\mathfrak{L}^*_{rec}| + 1 = k < |V|$. Thus, by the pigeonhole principle, there is a layer $p \leq |\mathfrak{L}^*_{rec}|$ that has at least two vertices, i.e.,

$|\ell^{*-1}_{rec}(p)| \geq 2$. Let s be a sink of the acyclic orientation $D[\ell^{*-1}_{rec}(p)]$. Let $\ell^* : V \to \mathbb{N}$ be defined by:

$$\ell^*(x) = \begin{cases} \ell^*_{rec}(x) & \text{if } x \in \ell^{*-1}_{rec}([0, p-1]) \cup \{s\}, \\ \ell^*_{rec}(x) + 1 & \text{if } x \in \ell^{*-1}_{rec}([p, k-1]) \setminus \{s\}, \end{cases}$$

for every $x \in V$.

OUTPUT: Take $\mathfrak{L}^* := (\ell^*, D)$. In Lemma 52, we prove that $\mathfrak{L}^*$ is a gapless leao with height $|\mathfrak{L}^*| = k$. △

Lemma 52. *The output $\mathfrak{L}^*$ of Construction 51 is a gapless leao with height $|\mathfrak{L}^*| = k$.*

Proof. If $|\mathfrak{L}| = k$, then $\mathfrak{L}^* = \mathfrak{L}$ ia a gapless leao with height k.

Suppose $|\mathfrak{L}| < k$. We will prove that $\mathfrak{L}^* = (\ell^*, D)$ is a gapless leao of G with height k.

By hypothesis, D is acyclic.

To prove that $\mathfrak{L}^*$ is extensional-by-layers, let $x, y \in V(G)$ be such that $x \neq y$ and $\ell^*(x) = \ell^*(y)$. We consider three cases.

Case 1: Suppose $\ell^*(x) = \ell^*(y) < p$. Then, $\ell^*_{rec}(x) = \ell^*(x) = \ell^*(y) = \ell^*_{rec}(y)$. And since $\mathfrak{L}^*_{rec}$ is extensional-by-layers, $N^+(x) \neq N^+(y)$.

Case 2: Suppose $\ell^*(x) = \ell^*(y) = p$. Assume, for a contradiction that $x \neq s$. We consider two cases.

Subcase 2.1: If $\ell(x) < p$, by definition, $\ell'(x) = \ell(x) < p$, contradicting our hypothesis that $\ell(x) = p$.

Subcase 2.2: If $\ell(x) \geq p$, since $x \neq s$, by definition, $\ell'(x) = \ell(x) + 1 > p$, contradicting our hypothesis that $\ell(x) = p$.

Thus, $x = s$. Analogously, $y = s$. Then, $x = s = y$, contradicting that $x \neq y$.

Case 3: Suppose $\ell^*(x) = \ell^*(y) > p$. Then, $\ell^*_{rec}(x) = \ell^*(x) - 1 = \ell^*(y) - 1 = \ell^*_{rec}(y)$. And since $\mathfrak{L}^*_{rec}$ is extensional-by-layers, $N^+(x) \neq N^+(y)$.

To prove that $\mathfrak{L}^*$ is downwards, let $x, y \in V(G)$ be such that $xy \in D$. Then, since $\mathfrak{L}^*_{rec}$ is downwards, $\ell^*_{rec}(x) \geq \ell^*_{rec}(y)$. We consider four cases:

Case 1: If $x, y \in \ell^{*-1}_{rec}([0, p-1]) \cup \{s\}$, then $\ell^*(x) = \ell^*_{rec}(x) \geq \ell^*_{rec}(y) = \ell^*(y)$.

Case 2: If $x, y \in \ell^{*-1}_{rec}([p, k-1]) \setminus \{s\}$, then $\ell^*(x) = \ell^*_{rec}(x) + 1 \geq \ell^*_{rec}(y) + 1 = \ell^*(y)$.

Case 3: Suppose $x \in \ell^{*-1}_{rec}([0, p-1]) \cup \{s\}$ and $y \in \ell^{*-1}_{rec}([p, k-1]) \setminus \{s\}$. Since $\ell^*_{rec}(x) \geq \ell^*_{rec}(y) \geq p$, $x \notin \ell^{*-1}_{rec}([0, p-1])$, and consequently, $x = s$. Moreover, since $p = \ell^*_{rec}(s) = \ell^*_{rec}(x) \geq \ell^*_{rec}(y) \geq p$, we have $\ell^*_{rec}(x) = p = \ell^*_{rec}(y)$. But $xy \in D$ and $x = s$, contradicting that s is a sink of $D[\ell^{*-1}_{rec}(p)]$.

Case 4: If $x \in \ell_{rec}^{*-1}([p, k-1]) \setminus \{s\}$ and $y \in \ell_{rec}^{*-1}([0, p-1]) \cup \{s\}$, then $\ell^*(x) = \ell_{rec}^*(x) + 1 > \ell_{rec}^*(y) = \ell^*(y)$.

In every possible case, $\ell^*(x) \geq \ell^*(x)$. Thus $\mathfrak{L}^*$ is a leao of G. Additionally, to prove that $\mathfrak{L}^*$ is gapless, let $i \in [0, k]$. We consider four cases.

Case 1: Suppose $i < p$. Since $\mathfrak{L}_{rec}^*$ is gapless, there exists a vertex x such that $\ell_{rec}^*(x) = i$. Then, since $\ell_{rec}^*(x) < p$, by definition, $\ell^*(x) = \ell_{rec}^*(x) = i$.

Case 2: Suppose $i = p$. Then, $\ell^*(s) = p = i$.

Case 3: Suppose $i = p + 1$. Since p was taken such that $|\ell_{rec}^{*-1}(p)| \geq 2$, there is a vertex $x \in \ell_{rec}^{*-1}(p)$ such that $x \neq s$. Then, by definition, $\ell^*(x) = \ell_{rec}^*(x) + 1 = p + 1 = i$.

Case 4: Suppose $i > p+1$. Since $i - 1 \leq k - 1 = |\mathfrak{L}_{rec}^*|$ and $\mathfrak{L}_{rec}^*$ is gapless, there is a vertex x such that $\ell_{rec}^*(x) = i - 1$. Since $\ell_{rec}^*(x) = i - 1 > p = \ell_{rec}^*(s)$, $x \neq s$. Thus, $x \in \ell_{rec}^{*-1}([p, k-1]) \setminus \{s\}$. So, by definition, $\ell^*(x) = \ell_{rec}^*(x) + 1 = i$.

Thus, $\mathfrak{L}^*$ is a gapless leao of G. Moreover, by definition, $|\mathfrak{L}^*| = |\mathfrak{L}_{rec}^*| + 1 = k$. □

4.2 Leaos of joins

In this section, we show how a minimum leao of a join $G_1 \wedge G_2$ can be constructed from minimum leaos of G_1 and G_2.

Given two disjoint graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, we always have $N_{G_1 \wedge G_2}(v_1) \setminus V_1 = V_2$ and $N_{G_1 \wedge G_2}(v_2) \setminus V_2 = V_1$ for all $v_1 \in V_1$ and all $v_2 \in V_2$. So, V_1 and V_2 are always modules of $G_1 \wedge G_2$. Thus, we can apply our results from Chapter 3 to obtain a lower-bound for $\mathcal{S}_\Delta(G_1 \wedge G_2)$ in terms of $\mathcal{S}_\Delta(G_1)$ and $\mathcal{S}_\Delta(G_2)$.

Lemma 53. *Let $G = G_1 \wedge G_2$, where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then, $\mathcal{S}_\Delta(G_1) - |V_2| \leq \mathcal{S}_\Delta(G)$.*

Proof. Since V_1 is a module of G, by Theorem 45, $\mathcal{S}_\Delta(G_1) \leq \mathcal{S}_\Delta(G) + |N(V_1) \setminus V_1|$. But $N(V_1) \setminus V_1 = V_2$. Thus, $\mathcal{S}_\Delta(G_1) - |V_2| \leq \mathcal{S}_\Delta(G)$. □

Now, we construct a leao of a co-disconnected graph $G_1 \wedge G_2$. Following the construction, we prove, in Lemma 55, that it is correct, i.e., we prove that the resulting layered orientation is indeed a leao.

Construction 54. [Leaos for joins]
INPUT: A tuple $(G_1, \mathfrak{L}_1, G_2, \mathfrak{L}_2)$, where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are disjoint graphs, and $\mathfrak{L}_1 = (\ell_1, D_1)$ and $\mathfrak{L}_2 = (\ell_2, D_2)$ are gapless leaos of G_1 and G_2, respectively, such that $|\mathfrak{L}_1| \geq |\mathfrak{L}_2|$.

CONSTRUCTION: Let $k = \min\{|V_2| - 1, |\mathfrak{L}_1|\}$. Then, $k < |V_2|$. Since, by Lemma 37, $|\mathfrak{L}_2| \le |V_2| - 1$, and $|\mathfrak{L}_2| \le |\mathfrak{L}_1|$, we have $|\mathfrak{L}_2| \le k$. Use Construction 51 on $(\mathfrak{L}_2, G_2, k)$ to obtain a gapless leao $\mathfrak{L}_2^*$ of G_2 with height $|\mathfrak{L}_2^*| = k$.

Let $D = D_1 \cup D_2 \cup D_{1,2} \cup D_{2,1}$ be the orientation of G where

$$D_{1,2} = \{xy : x \in V_1,\ y \in V_2,\ \ell_1(x) > \ell_2^*(y)\},$$

$$D_{2,1} = \{yx : x \in V_1,\ y \in V_2,\ \ell_1(x) \le \ell_2^*(y)\}.$$

Let $\ell : V(G) \to \mathbb{N}$ defined by:

$$\ell(x) = \begin{cases} \max\{\ell_1(x) - |V_2|, 0\} & \text{if } x \in V_1 \\ \ell(x) = 0 & \text{if } x \in V_2 \end{cases}$$

OUTPUT: Take $\mathfrak{L} = (\ell, D)$. In Lemma 55, we prove that $\mathfrak{L}$ is a leao of $G = G_1 \wedge G_2$ with height $|\mathfrak{L}| = \max\{|\mathfrak{L}_1| - |V_2|, 0\}$ and that D has exactly one sink. △

Lemma 55 (Correctness of Construction 54)**.** *In Construction 54, the output $\mathfrak{L} = (\ell, D)$ is a leao of G with height $|\mathfrak{L}| = \max\{|\mathfrak{L}_1| - |V_2|, 0\}$. Moreover, D has exactly one sink.*

Proof. To prove D is acyclic, suppose, for a contradiction, that there is a cycle $C = x_1 x_2 \dots x_p x_1$ in D. Let $g : V_1 \cup V_2 \to \mathbb{N}$ be the *union* of ℓ_1 and ℓ_2^*, that is,

$$g(x) = \begin{cases} \ell_1(x) & \text{if } x \in V_1 \\ \ell_2^*(x) & \text{if } x \in V_2 \end{cases}$$

Since $V_1 \cap V_2 = \emptyset$, g is well-defined. Let xy be an edge of the cycle C. We consider four cases:

Case 1: If $\{x, y\} \subseteq V_1$, then $\ell_1(x) \ge \ell_1(y)$ because $\mathfrak{L}_1$ is downwards.

Case 2: If $\{x, y\} \subseteq V_2$, then $\ell_2^*(x) \ge \ell_2^*(y)$ because $\mathfrak{L}_2^*$ is downwards.

Case 3: If $x \in V_1$ and $y \in V_2$, then $\ell_1(x) > \ell_2^*(y)$ by the definition of D.

Case 4: If $x \in V_2$ and $y \in V_1$, then $\ell_2^*(x) \ge \ell_1(y)$ by the definition of D.

In any case, $g(x) \ge g(y)$ and, in the special *Case 3*, $g(x) > g(y)$. But C is a cycle, so we must have $g(x_i) = g(x_j)$ for all $i, j \in [p]$. Since D_1 and D_2 are acyclic orientations of G_1 and G_2, any cycle in D must have vertices of both V_1 and V_2. Thus, the cycle C must have an edge xy such that $x \in V_1$ and $y \in V_2$ (*Case 3*), but this implies that $g(x) > g(y)$, a contradiction. Therefore, D is acyclic.

To prove $\mathfrak{L}$ is extensional-by-layers, let $x, y \in V(G)$ be such that $x \neq y$ and $\ell(x) = \ell(y)$. We consider four cases.

Case 1: Suppose $\ell(x) = \ell(y) > 0$. Then, $\{x, y\} \subseteq V_1$ because, by definition, $\ell(z) = 0$ for every $z \in V_2$. Since $x \in V_1$, $\ell(x) = \max\{\ell_1(x) - |V_2|, 0\}$. And since $\ell(x) > 0$, $\ell(x) = \ell_1(x) - |V_2|$. Then, $\ell(x) + |V_2| = \ell_1(x)$. Analogously, $\ell(y) + |V_2| = \ell_1(y)$. Thus, since $\ell(x) = \ell(y)$, we have $\ell_1(x) = \ell_1(y)$. Since $\mathfrak{L}_1$ is extensional-by-layers, $N^+_{D_1}(x) \neq N^+_{D_1}(y)$. Suppose, for a contradiction, that $N^+_D(x) = N^+_D(y)$. Then, $N^+_{D_1}(x) = N^+_D(x) \cap V_1 = N^+_D(y) \cap V_1 = N^+_{D_1}(y)$, a contradiction. Thus, $N^+_D(x) \neq N^+_D(y)$.

Case 2: Suppose $\ell(x) = \ell(y) = 0$ and $\{x, y\} \subseteq V_1$. Suppose w.l.o.g. $\ell_1(x) \geq \ell_1(y)$. We consider two subcases:

Subcase 2.1: Suppose $\ell_1(x) = \ell_1(y)$. Then, since $\mathfrak{L}_1$ is extensional-by-layers, $N^+_{D_1}(x) \neq N^+_{D_1}(y)$. Assume, for a contradiction, $N^+_D(x) = N^+_D(y)$. Then, $N^+_{D_1}(x) = N^+_D(x) \cap V_1 = N^+_D(y) \cap V_1 = N^+_{D_1}(y)$, a contradiction. Thus, $N^+_D(x) \neq N^+_D(y)$.

Subcase 2.2: Suppose $\ell_1(x) > \ell_1(y)$. Since $\ell(x) = 0$, $\ell_1(x) - |V_2| \leq 0$. And since $\ell_1(x) \leq |\mathfrak{L}_1|$, we have $\ell_1(x) - 1 \leq \min\{|V_2| - 1, |\mathfrak{L}_1|\} = k = |\mathfrak{L}^*_2|$. Since $0 \leq \ell_1(y) < \ell_1(x)$, we have $0 \leq \ell_1(x) - 1$. Thus, $0 \leq \ell_1(x) - 1 \leq |\mathfrak{L}^*_2|$. Since $\mathfrak{L}^*_2$ is a gapless leao, there is a vertex $z \in V_2$ such that $\ell^*_2(z) = \ell_1(x) - 1$. Then, $\ell_1(x) > \ell^*_2(z)$, and consequently, by the definition of D, $xz \in D$. And since $\ell^*_2(z) = \ell_1(x) - 1 \geq \ell_1(y)$, by the definition og D, we have $zy \in D$. Therefore, $z \in N^+_D(x) \setminus N^+_D(y)$.

Case 3: Suppose $\ell(x) = \ell(y) = 0$ and $\{x, y\} \subseteq V_2$. Suppose w.l.o.g. $\ell^*_2(x) \geq \ell^*_2(y)$. We consider two subcases:

Subcase 3.1: Suppose $\ell^*_2(x) = \ell^*_2(y)$. Then, since $\mathfrak{L}^*_2$ is extensional-by-layers, $N^+_{D_2}(x) \neq N^+_{D_2}(y)$. Assume, for a contradiction, $N^+_D(x) = N^+_D(y)$. Then, $N^+_D(x) \cap V_2 = N^+_{D_2}(x) \neq N^+_{D_2}(y) = N^+_D(y) \cap V_2$, a contradiction. Thus, $N^+_D(x) \neq N^+_D(y)$.

Subcase 3.2: Suppose $\ell^*_2(x) > \ell^*_2(y)$. Since $\mathfrak{L}_1$ is a gapless leao and, by definition, $\ell^*_2(x) \leq |\mathfrak{L}^*_2| = k \leq |\mathfrak{L}_1|$, there is a vertex $z \in V_1$ such that $\ell_1(z) = \ell^*_2(x)$. Then, $xz \in D$ by definition. And since $\ell_1(z) = \ell^*_2(x) > \ell^*_2(y)$, we have $zy \in D$. Therefore, $z \in N^+_D(x) \setminus N^+_D(y)$.

Case 4: Suppose $\ell(x) = \ell(y) = 0$ and $|\{x, y\} \cap V_1| = 1$. Then $N^+_D(x) \neq N^+_D(y)$ because x and y are adjacent in $G_1 \wedge G_2$.

To prove that $\mathfrak{L}$ is downwards, let $x, y \in V(G)$ and suppose $xy \in D$. We consider two cases.

Case 1: Suppose $\ell(y) = 0$. Then, $\ell(x) \geq \ell(y)$.

Case 2: Suppose $\ell(y) > 0$. Then, since $\ell(z) = 0$ for every $z \in V_2$, we have $y \in V_1$. We consider two subcases.

Subcase 2.1: Suppose $x \in V_1$. Since $\{x, y\} \subseteq V_1$, we have $xy \in D_1$. Since $\mathfrak{L}_1$ is downwards, $\ell_1(x) \geq \ell_1(y)$. Then, $\ell_1(x) - |V_2| \geq \ell_1(y) - |V_2|$, and consequently, $\ell(x) = \max\{\ell_1(x) - |V_2|, 0\} \geq \max\{\ell_1(y) - |V_2|, 0\} = \ell(y)$.

Subcase 2.2: Suppose $x \in V_2$. Since $0 < \ell(y) = \max\{\ell_1(y) - |V_2|, 0\}$, we have $|V_2| < \ell_1(y)$. Moreover, by definition, $\ell_2^*(x) \leq |\mathfrak{L}_2^*| = k < |V_2|$. Thus, $\ell_2^*(x) < \ell_1(y)$, and consequently, by the definition of D, $yx \in D$, contradicting our assumption that $xy \in D$.

Therefore, $\mathfrak{L}$ is downwards.

Thus, $\mathfrak{L}$ is a leao of G, and by definition, $|\mathfrak{L}| = \max\{|\mathfrak{L}_1| - |V_2|, 0\}$.

Finally, we prove that D has exactly one sink. Suppose, for a contradiction, that D has a sink s such that $\ell(s) > 0$. Since $\ell(s) > 0$ and $\ell(z) = 0$ for every $z \in V_2$, $s \in V_1$. Then, $\ell(s) = \max\{\ell_1(s) - |V_2|, 0\}$, and since $\ell(s) > 0$, we have $\ell_1(s) > |V_2|$. Since $|\mathfrak{L}_2^*| = k < |V_2| < \ell_1(s)$, we have $\ell_2^*(x) < \ell_1(s)$ for every $x \in V_2$. Then, by definition of D, $V_2 \subseteq N_D^+(s)$, contradicting that s is a sink. Thus, for every sink s of D, $\ell(s) = 0$. Therefore, since we have proved that $\mathfrak{L}$ is extensional-by-layers, D has at most one sink. But, since D is acyclic, D has a sink. Therefore, D has exactly one sink. $\square$

Finally, using our lower-bound for the set deficiency of a co-disconnected graph (Lemma 53), we prove that if Construction 54 receives minimum leaos as input, then it outputs a minimum leao.

Lemma 56. *Let G_1 and G_2 be two graphs with leaos $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively, such that $|\mathfrak{L}_1| \geq |\mathfrak{L}_2|$. If $\mathfrak{L}_1$ is minimum and $\mathfrak{L}_2$ is gapless, then Construction 54 on $(G_1, \mathfrak{L}_1, G_2, \mathfrak{L}_2)$ yields a minimum leao $\mathfrak{L}$ of $G = G_1 \wedge G_2$.*

Proof. By Lemma 55, $\mathfrak{L}$ is a leao of G with height $|\mathfrak{L}| = \max\{|\mathfrak{L}_1| - |V(G_2)|, 0\}$. Since $\mathfrak{L}_1$ is minimum, $|\mathfrak{L}_1| = \mathcal{S}_\Delta(G_1)$. So $|\mathfrak{L}| = \max\{\mathcal{S}_\Delta(G_1) - |V(G_2)|, 0\}$. On the other hand, by Lemma 53, $\mathcal{S}_\Delta(G_1) - |V(G_2)| \leq \mathcal{S}_\Delta(G)$. And since, by definition, $0 \leq \mathcal{S}_\Delta(G)$, we have $|\mathfrak{L}| = \max\{\mathcal{S}_\Delta(G_1) - |V(G_2)|, 0\} \leq \mathcal{S}_\Delta(G)$. Therefore, $\mathfrak{L}$ is a minimum leao of G. $\square$

Theorem 57. *Let G_1 and G_2 be two disjoint graphs. If $\mathcal{S}_\Delta(G_1) \geq \mathcal{S}_\Delta(G_2)$, then $\mathcal{S}_\Delta(G_1 \wedge G_2) = \max\{\mathcal{S}_\Delta(G_1) - |V(G_2)|, 0\}$.*

Proof. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be minimum leaos of G_1 and G_2, respectively. Let $\mathfrak{L}$ be the leao of $G_1 \wedge G_2$ given by Construction 54 on $(\mathfrak{L}_1, G_1, \mathfrak{L}_2, G_2)$. Then, by Lemma 55, $|\mathfrak{L}| = \max\{|\mathfrak{L}_1| - |V(G_2)|, 0\}$. Since $\mathfrak{L}_1$ is minimum, $|\mathfrak{L}_1| = \mathcal{S}_\Delta(G_1)$.

So $|\mathfrak{L}| = \max\{\mathcal{S}_\Delta(G_1) - |V(G_2)|, 0\}$. And, since $\mathfrak{L}_1$ is minimum and $\mathfrak{L}_2$ is gapless, by Lemma 56, $\mathfrak{L}$ is a minimum leao of $G_1 \wedge G_2$. Thus, $\mathcal{S}_\Delta(G_1 \wedge G_2) = \max\{\mathcal{S}_\Delta(G_1) - |V(G_2)|, 0\}$. □

In particular, applying Theorem 57 we obtain the following characterization of co-disconnected set graphs.

Theorem 58. *Let G_1 and G_2 be graphs such that $\mathcal{S}_\Delta(G_1) \geq \mathcal{S}_\Delta(G_2)$. Then, the join $G = G_1 \wedge G_2$ is a set graph if and only if $\mathcal{S}_\Delta(G_1) \leq |V(G_2)|$.*

Proof. G is a set graph if and only if $\mathcal{S}_\Delta(G) = 0$ if and only if (by Theorem 57) $\max\{\mathcal{S}_\Delta(G_1) - |V(G_2)|, 0\} = 0$ if and only if $\mathcal{S}_\Delta(G_1) \leq |V(G_2)|$. □

With Theorem 58, it becomes clear that $\mathcal{S}_\Delta(G)$ corresponds to the minimum number n such that we can obtain a set graph by successively adding n universal vertices to G. Note that adding n universal vertices to G is equivalent to taking the join $G \wedge K_n$.

Corollary 59. *For every graph G, $\mathcal{S}_\Delta(G)$ is the minimum number n such that $G \wedge K_n$ is a set graph, i.e., $\mathcal{S}_\Delta(G)$ is the minimum number n such that we obtain a set graph by adding n universal vertices to G.*

Proof. Since K_n admits a hamiltonian path, by Lemma 17, it is a set graph. Then, $\mathcal{S}_\Delta(G) \geq 0 = \mathcal{S}_\Delta(K_n)$. Hence, by Theorem 58, $G \wedge K_n$ is a set graph if and only if $\mathcal{S}_\Delta(G) \leq |V(K_n)| = n$. This concludes the proof. □

Example 60. We have already proved, in Lemma 40, that $\mathcal{S}_\Delta(\overline{K_n}) = n - 1$. Now, we may also determine the set-deficiency of complete bipartite graphs, generalizing the characterization of Corollary 23. Note that $K_{m,n}$ is the join $\overline{K_m} \wedge \overline{K_n}$. Since $K_{m,n}$ is isomorphic to $K_{n,m}$, let us assume w.l.o.g. that $m \geq n$. Then, $\mathcal{S}_\Delta(\overline{K_m}) = m-1 \geq n-1 = \mathcal{S}_\Delta(\overline{K_n})$. Hence, by Theorem 57, $\mathcal{S}_\Delta(K_{m,n}) = \max\{m-1-n, 0\}$. In particular, if $m > n$, $\mathcal{S}_\Delta(K_{m,n}) = m - n - 1$. Thus, if one is looking for a connected graph with an arbitrarily large set-deficiency, one may use, for instance, $K_{1,n}$ that, for any $n \geq 2$, has set-deficiency $\mathcal{S}_\Delta(K_{1,n}) = n - 2$.

Corollary 61. *If $m \geq n$, then $\mathcal{S}_\Delta(K_{m,n}) = \max\{m - n - 1, 0\}$.*

Chapter 5

Leaos of Disconnected Graphs

In this chapter, we prove some results about minimum leaos of disconnected graphs. The construction presented here is the second component of our algorithm for constructing minimum leaos for cographs. Here, we present a method for constructing minimum leaos for a disconnected graph whenever minimum leaos are given for every connected component of the given graph, and the orientations of these minimum leaos have exactly one sink each. Not every connected graph has a minimum leao with only one sink (cf. Section 5.3). However, all the methods presented in Chapters 4 and 5 for constructing minimum leaos yield such leaos with one sink per connected component. This will be sufficient for constructing minimum leaos for every cograph.

5.1 Leaos of disconnected graphs

In this section, we present how a leao of a disconnected graph G can be obtained from leaos of the connected components of G, whenever certain conditions are met.

Definition 62. *We say that a leao $\mathfrak{L} = (\ell, D)$ of a connected graph G is a* <u>*single-sink leao*</u> *if D has exactly one sink.*

Construction 63. [Leaos for disconnected graphs]
INPUT: A pair $(G, \{(G_i, \mathfrak{L}_i) : i \in [0, k]\})$, where G is a graph with $k+1$ connected components $G_0, \ldots, G_k$ and, for each $i \in [0, k]$, $\mathfrak{L}_i = (\ell_i, D_i)$ is a gapless single-sink leao of G_i.
CONSTRUCTION: Let $\ell : V(G) \to \mathbb{N}$ be such that, for every $i \in [0, k]$ and $v \in V(G_i)$, $\ell(v) = \ell_i(v) + i$; and let $D = \bigcup\{D_i : i \in [0, k]\}$. Let $\mathfrak{L} := (\ell, D)$.
OUTPUT: Take $\mathfrak{L} = (\ell, D)$. In Lemma 64, we prove that $\mathfrak{L}$ is a leao of G, with height $|\mathfrak{L}| = \max\{|\mathfrak{L}_i| + i : i \in [0, k]\}$, having exactly one sink per connected component. $\triangle$

Lemma 64 (Correctness of Construction 63)**.** *The output $\mathfrak{L} = (\ell, D)$ of Construction 63 is a leao of G with height $|\mathfrak{L}| = \max\{|\mathfrak{L}_i| + i : i \in [0, k]\}$. Moreover, D has exactly one sink per connected component of G.*

Proof. D is acyclic because in each connected component G_i of G, $D[V(G_i)] = D_i$ is acyclic.

To prove that $\mathfrak{L}$ is extensional-by-layers, let $x, y \in V(G)$ be such that $x \neq y$ and $\ell(x) = \ell(y)$. We consider two cases:

Case 1: Suppose $x, y \in V(G_i)$ for some $i \in [0, k]$. Then, by definition, $\ell_i(x) = \ell(x) - i = \ell(y) - i = \ell_i(y)$. Since $\mathfrak{L}_i$ is extensional-by-layers, $N^+_{D_i}(x) \neq N^+_{D_i}(y)$. But, since G_i is a connected component of G, $N^+_{D_i}(x) = N^+_D(x)$ and $N^+_{D_i}(y) = N^+_D(y)$. Therefore, $N^+_D(x) \neq N^+_D(y)$.

Case 2: Suppose $x \in V(G_i)$ and $y \in V(G_j)$ for $i, j \in [0, k]$ with $i \neq j$. Assume, for a contradiction, that $N^+_D(x) = N^+_D(y)$. Since x and y are in different connected components, we have $N^+_D(x) = N^+_D(y) \subseteq V(G_i) \cap V(G_j) = \emptyset$. Since, by hypothesis, D_i and D_j each have unique sinks, x and y are the only sinks of D_i and D_j, respectively. Since $\mathfrak{L}_i$ is gapless, there is a vertex $z \in V(G_i)$ such that $\ell_i(z) = 0$. Since D_i is acyclic and x is the only sink of D_i, there is a directed path in D_i from z to x. And since $\mathfrak{L}_i$ is downwards, $0 = \ell_i(z) \geq \ell_i(x)$. Hence, $\ell_i(x) = 0$. Similarly, $\ell_j(y) = 0$. But, by definition, $\ell(x) = \ell_i(x) + i = i$ and $\ell(y) = \ell_j(y) + j = j$, and consequently, $i = \ell(x) = \ell(y) = j$, contradicting the assumption that $i \neq j$. Thus, $N^+_D(x) \neq N^+_D(y)$.

To prove that $\mathfrak{L}$ is downwards, let $x, y \in V(G)$, and suppose $xy \in D$. By definition of D, we have $xy \in D_i$ for some $i \in [0, k]$. Since $\mathfrak{L}_i$ is downwards, $\ell_i(x) \geq \ell_i(y)$. Then, $\ell(x) = \ell_i(x) + i \geq \ell_i(y) + i = \ell(y)$. Therefore, $\mathfrak{L}$ is downwards.

Thus, $\mathfrak{L}$ is a leao of G.

By definition, for any $i \in [0, k]$ and $v \in V(G_i)$, $\ell(v) = \ell_i(v) + i$. Hence, $|\mathfrak{L}| = \max\{|\mathfrak{L}_i| + i : i \in [0, k]\}$.

Moreover, by hypothesis, for every connected component G_i, there is only one sink in $D[V(G_i)] = D_i$. □

Corollary 65. *Let G be a graph with k connected components $G_0, \ldots, G_k$ such that, for every $i \in [0, k]$, G_i admits a single-sink minimum leao. Then, $\mathcal{S}_\Delta(G) \leq \max\{\mathcal{S}_\Delta(G_i) + i : i \in [0, k]\}$.*

Proof. For every $i \in [0, k]$, let $\mathfrak{L}_i$ be a single-sink minimum leao of G_i. Then, using Construction 63 on $(G, \{(\mathfrak{L}_i, G_i) : i \in [0, k]\})$, we obtain a leao $\mathfrak{L}$ of G with height $\max\{|\mathfrak{L}_i| + i : i \in [0, k]\}$. But, since $\mathfrak{L}_i$ is minimum for every $i \in [0, k]$, we have $|\mathfrak{L}| = \max\{\mathcal{S}_\Delta(G_i) + i : i \in [0, k]\}$. Hence, $\mathcal{S}_\Delta(G) \leq \max\{\mathcal{S}_\Delta(G_i) + i : i \in [0, k]\}$. □

Now, we provide an example which illustrates Construction 63.

Example 66. Let G be a graph with 3 connected components G_0, G_1 and G_2, as in Figure 5.1.

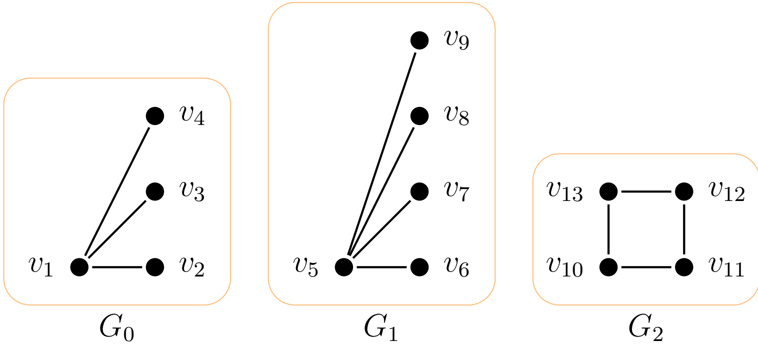

Figure 5.1: A disconnected graph G

In Figure 5.2, we provide leaos $\mathfrak{L}_0, \mathfrak{L}_1, \mathfrak{L}_2$ for each of the connected components of G.

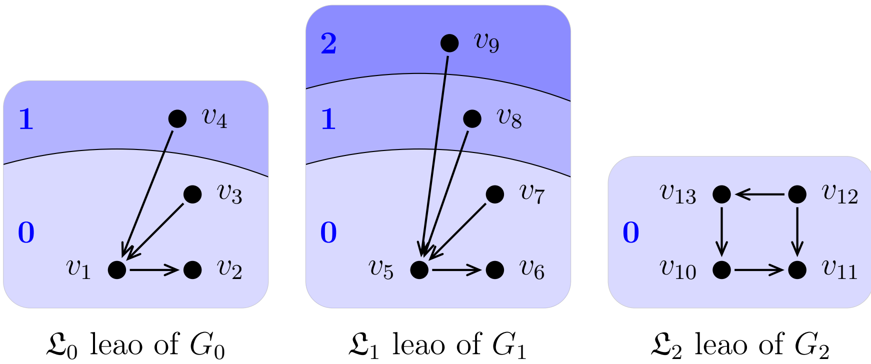

Figure 5.2: Leaos of each connected component of G

Construction 63 on $(G, \{(G_i, \mathfrak{L}_i) : i \in [0, 2]\})$ yields the leao represented in Figure 5.3, with height 3.

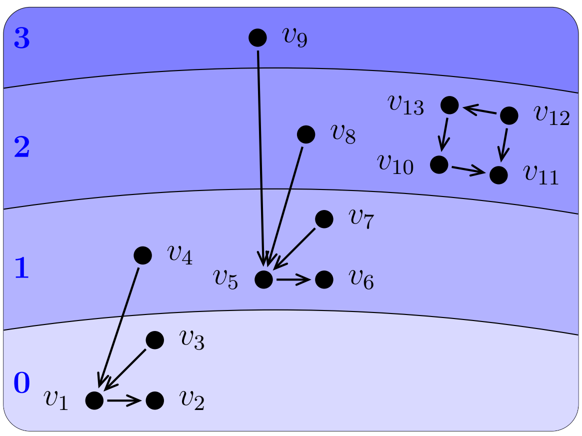

Figure 5.3: Leao of G, from Construction 5.3 on G_0, G_1, G_2

However, by reindexing the connected components in the input of Construction 63, we may obtain a leao of a lower height. Let π be the permutation of $[0, 2]$ such that $\pi(0) = 1$, $\pi(1) = 0$ and $\pi(2) = 2$. Then, Construction 63 on $(G, \{(G_{\pi(i)}, \mathfrak{L}_{\pi(i)}) : i \in [0, 2]\}$ yields the leao represented in Figure 5.4, with height 2.

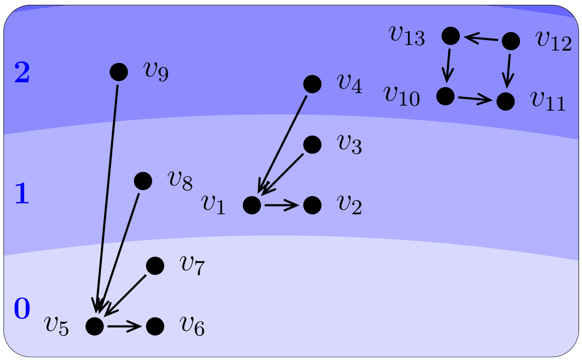

Figure 5.4: Leao of G, from Construction 5.3 on $G_{\pi(0)}, G_{\pi(1)}, G_{\pi(2)}$

5.2 Reordering connected components

The height of the leao obtained from Construction 63 depends on the ordering of the connected components of the given graph, as shown in Example 66. In this section, we show how to choose an ordering of the connected components that minimizes the height of the leao produced by Construction 63.

Definition 67. *Let $X, Y \subseteq \mathbb{N}$. We say that a function $f : X \to Y$ is decreasing if, for all $x, y \in X$, if $x \leq y$ then $f(x) \geq f(y)$; and is increasing if, for all $x, y \in X$, if $x \leq y$ then $f(x) \leq f(y)$.*

Theorem 70 shows that to minimize the height of the leao produced by Construction 63, it suffices to index the connected components in order of decreasing set-deficiency. To prove this, we need the following lemmas.

Lemma 68. *Let $f : [0, k] \to \mathbb{N}$ be a decreasing function and let $g : [0, k] \to \mathbb{N}$ be an injective function. Then,* $\max\{f(x)+g(x) : x \in [0, k]\} \geq \max\{f(x)+x : x \in [0, k]\}$.

Proof. The proof goes by induction on k.
Basis: Suppose $k = 0$. Then, $\max\{f(x) + g(x) : x \in [0, 0]\} = f(0) + g(0) \geq f(0) + 0 = \max\{f(x) + x : x \in [0, 0]\}$.
Induction Hypothesis: Suppose the statement in Lemma 68 holds for some $k \in \mathbb{N}$.
Step: Let $f : [0, k + 1] \to \mathbb{N}$ be a decreasing function, and $g : [0, k + 1] \to \mathbb{N}$ be an injective function. Now, we prove that, for all $i \in [0, k + 1]$, $\max\{f(x) + g(x) : x \in [0, k + 1]\} \geq f(i) + i$. Let $i \in [0, k + 1]$. We consider two cases.

Case 1: Suppose $i \in [0,k]$. Since $f|_{[0,k]}$ is decreasing and $g|_{[0,k]}$ is injective, by IH, we have $\max\{f(x)+g(x) : x \in [0,k]\} \geq \max\{f(x)+x : x \in [0,k]\}$. Thus, since $i \in [0,k]$, $\max\{f(x)+g(x) : x \in [0,k+1]\} \geq f(i)+i$.

Case 2: Suppose $i = k+1$. Since g is injective, $g([0,k+1]) \not\subseteq [0,k]$. Then, there exists $j \in [0,k+1]$ such that $g(j) \notin [0,k]$. So, $g(j) \geq k+1$. Since f is decreasing and $j \leq k+1$, we have $f(j) \geq f(k+1)$. Hence, $f(j)+g(j) \geq f(k+1)+k+1$. Therefore, $\max\{f(x)+g(x) : x \in [0,k+1]\} \geq f(k+1)+k+1 = f(i)+i$.

Thus, $\max\{f(x)+g(x) : x \in [0,k+1]\} \geq \max\{f(x)+x : x \in [0,k+1]\}$. $\square$

Lemma 69. *Let G be a disconnected graph, $\mathfrak{L} = (\ell, D)$ be a leao of G, and H be a connected component of G. Besides, let $m := \min\{\ell(x) : x \in V(H)\}$ and $\ell_H : V(H) \to \mathbb{N}$ be defined by $\ell_H(x) = \ell(x) - m$, for every $x \in V(H)$. Let $D_H := D \cap V(H)^2$. Then, $\mathfrak{L}_H := (\ell_H, D_H)$ is a leao of H.*

Proof. D_H is acyclic because $D_H \subseteq D$ and D is acyclic.

To prove that $\mathfrak{L}_H$ is extensional-by-layers, let $x, y \in V(H)$ be such that $x \neq y$ and $\ell_H(x) = \ell_H(y)$. Then, $\ell(x) = \ell_H(x) + m = \ell_H(y) + m = \ell(y)$. Since $\mathfrak{L}$ is extensional-by-layers, $N^+_D(x) \neq N^+_D(y)$. Since H is the connected component of x and y, $N^+_D(x) = N^+_{D_H}(x)$ and $N^+_D(y) = N^+_{D_H}(y)$. Hence, $N^+_{D_H}(x) \neq N^+_{D_H}(y)$.

To prove that $\mathfrak{L}_H$ is downwards, let $x, y \in V(H)$ and suppose $xy \in D_H$. Then $xy \in D$. And since $\mathfrak{L}$ is downwards, $\ell(x) \geq \ell(y)$. Thus, $\ell_H(x) = \ell(x) - m \geq \ell(y) - m = \ell_H(y)$. $\square$

Now, we may prove that, in order to minimize the height of the output of Construction 63, it suffices to index the leaos of the connected components from highest to lowest height.

Theorem 70. *Let G be a graph with connected components $G_0, \ldots, G_k$. For every $i \in [0,k]$, let $\mathfrak{L}_i$ be a single-sink minimum leao of G_i. If $|\mathfrak{L}_1| \geq \cdots \geq |\mathfrak{L}_k|$, then Construction 63 on $(G, \{(G_i, \mathfrak{L}_i) : i \in [0,k]\})$ yields a minimum leao $\mathfrak{L}$ of G.*

Proof. Let $\mathfrak{L}' = (\ell', D')$ be a leao of G. Let $g : [0,k] \to \mathbb{N}$ be defined by $g(i) = \min\{\ell'(x) : x \in V(G_i)\}$, for every $i \in [0,k]$.

First, we prove that g is injective. Let $i, j \in [0,k]$ be such that $g(i) = g(j)$. Then, by definition, there are vertices $x \in V(G_i)$ and $y \in V(G_j)$ such that $\ell'(x) = g(i)$ and $\ell'(y) = g(j)$. Since D' is acyclic, there are paths, in D', from x to a sink $s_i \in V(G_i)$ of D', and from y to a sink $s_j \in V(G_j)$ of D'. Since $\mathfrak{L}'$ is downwards, $g(i) = \ell'(x) \geq \ell'(s_i)$ and $g(j) = \ell'(y) \geq \ell'(s_j)$. But, by definition of g, $\ell'(s_i) \geq g(i)$ and $\ell'(s_j) \geq g(j)$. Thus, $\ell'(s_i) = g(i)$ and $\ell'(s_j) = g(j)$. Suppose, for a contradiction, $s_i \neq s_j$. Since $\ell'(s_i) = g(i) = g(j) = \ell'(s_j)$ and $\mathfrak{L}'$ is extensional-by-layers, $N^+_{D'}(s_i) \neq$

$N^+_{D'}(s_j)$, contradicting that s_i and s_j are sinks of D'. Then, $s_i = s_j$. Hence, G_i and G_j are the same connected component. So, $i = j$. Thus, g is injective.

Since $|\mathfrak{L}_i|$ is decreasing as a function of $i \in [0, k]$ and g is injective, by Lemma 68, $\max\{|\mathfrak{L}_i| + g(i) : i \in [0, k]\} \geq \max\{|\mathfrak{L}_i| + i : i \in [0, k]\}$. Besides, for every $i \in [0, k]$, $|\mathfrak{L}'_i| \geq |\mathfrak{L}_i|$ because, by hypothesis, $\mathfrak{L}_i$ is minimum. Thus, $\max\{|\mathfrak{L}'_i| + g(i) : i \in [0, k]\} \geq \max\{|\mathfrak{L}_i| + g(i) : i \in [0, k]\} \geq \max\{|\mathfrak{L}_i| + i : i \in [0, k]\}$. Moreover, by definition, $\max\{|\mathfrak{L}'_i| + g(i) : i \in [0, k]\} = |\mathfrak{L}'|$ and $\max\{|\mathfrak{L}_i| + i : i \in [0, k]\} = |\mathfrak{L}|$. Hence, $|\mathfrak{L}'| \geq |\mathfrak{L}|$. We conclude that $\mathfrak{L}$ is a minimum leao of G. □

As a corollary, Theorem 71 simplifies the statement of Theorem 70 in terms of set-deficiencies.

Theorem 71. *Let G be a graph with connected components $G_0, \ldots, G_k$ such that, for every $i \in [0, k]$, G_i admits a single-sink minimum leao. If $\mathcal{S}_\Delta(G_0) \geq \cdots \geq \mathcal{S}_\Delta(G_k)$, then $\mathcal{S}_\Delta(G) = \max\{\mathcal{S}_\Delta(G_i) + i : i \in [0, k]\}$.*

Proof. Suppose $\mathcal{S}_\Delta(G_0) \geq \cdots \geq \mathcal{S}_\Delta(G_k)$. For every $i \in [0, k]$, let $\mathfrak{L}_i = (\ell_i, D_i)$ be a single-sink minimum leao of G_i. Use Construction 63 on $(G, \{(G_i, \mathfrak{L}_i) : i \in [0, k]\})$ to obtain a leao $\mathfrak{L}$ of G. By definition, $|\mathfrak{L}| = \max\{|\mathfrak{L}_i| + i : i \in [0, k]\}$. Moreover, $|\mathfrak{L}| = \mathcal{S}_\Delta(G)$ because, by Lemma 70, $\mathfrak{L}$ is a minimum leao of G. And, for every $i \in [0, k]$, $|\mathfrak{L}_i| = \mathcal{S}_\Delta(G_i)$ because, by hypothesis, $\mathfrak{L}_i$ is minimum. Therefore, $\mathcal{S}_\Delta(G) = \max\{\mathcal{S}_\Delta(G_i) + i : i \in [0, k]\}$. □

5.3 On the unicity of sinks of minimum leaos

Construction 63, presented in this chapter, requires that the given leaos of the connected components be single-sink leaos. One may wonder if every connected graph has a single-sink minimum leao. In Example 72, we will show that there are connected graphs for which no minimum leao is a single-sink leao.

Let $\mathcal{C}$ be the class of all graphs that admit minimum leaos with one sink per connected component. Since both Constructions 54 and 63 yield minimum leaos with one sink per connected components, $\mathcal{C}$ is closed under the operations of join and disjoint union. The smallest class of graphs that contains K_1 and is closed under joins and disjoint unions is called the class of *cographs* (see Section 5.4 and [5]). Thus, the cographs are a subclass of $\mathcal{C}$.

Note, however, that not every graph is in $\mathcal{C}$. In Example 72, we present a graph that is not in $\mathcal{C}$, i.e., a graph G such that, for every minimum leao $\mathfrak{L} = (\ell, D)$ of G, there is a connected component of G with more than one sink of D.

Example 72. Consider the graph G in Figure 5.5.

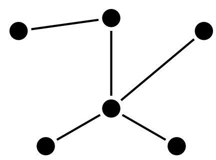

Figure 5.5: A connected graph that does not admit a single-sink minimum leao

To verify that G does not admit a single-sink minimum leao, we list all of the possible acyclic orientations of G that have only one sink. Since G is a tree, for every vertex $v \in V(G)$, there is a unique acyclic orientation such that v is the only sink. In Figure 5.6, we present every acyclic orientation of G that has only one sink. In each case, we represent, by a square node, the only sink of the orientation and, by black nodes, a triple of vertices with the same out-neighborhood. Thus proving that any single-sink leao of G has at least 3 layers.

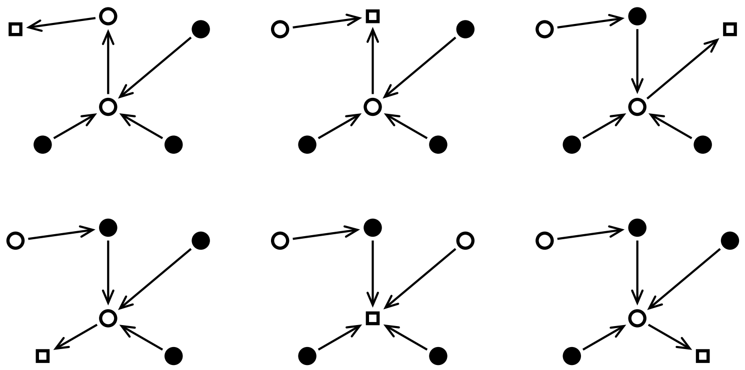

Figure 5.6: Every acyclic orientation of G having only one sink

Thus, every single-sink leao $\mathfrak{L}$ of G has height $|\mathfrak{L}| \geq 2$. However, the leao of G in Figure 5.7 has height 1.

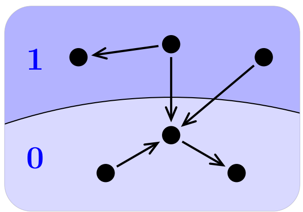

Figure 5.7: A minimum leao of G with height 1 and two sinks

Therefore, the graph in Figure 5.5 does not admit a single-sink minumum leao.

5.4 Minimum leaos of cographs

With the results we have proved so far, we can already produce minimum leaos for any graph in the class of cographs (or complement-reducible graphs).

The class of cographs has appeared naturally in many areas of mathematics and, in fact, was rediscovered independently by many authors (cf. [5]).

Definition 73. *The class of cographs is defined inductively as the smallest class of graphs such that*

(i) K_1 *is a cograph.*

(ii) If $G_1, \ldots, G_k$ *are pairwise disjoint cographs, then the union* $G_1 \cup \cdots \cup G_k$ *is a cograph, i.e., cographs are closed under disjoint union.*

(iii) If G *is a cograph, then the complement* $\overline{G}$ *is a cograph, i.e., cographs are closed under complementation.*

With many rediscoveries came many characterizations of the cographs. For instance, a graph is a cograph if and only if it is P_4-free (cf. [5]), where P_4 is the path with 4 vertices. Another well-known characterization of cographs can be stated in terms of disconnected and co-disconnected graphs:

Lemma 74 (Corneil et al. [5])**.** *For every graph* G *the following conditions are equivalent.*

(a) G *is a cograph.*

(b) For every $X \subseteq V(G)$ *such that* $|X| > 1$*,* $G[X]$ *is either disconnected or co-disconnected, i.e., every non-trivial induced subgraph of* G *is either disconnected or co-disconnected.*

Proof. *((a)⇒(b))*

First, we prove, by induction on the structure of the class of cographs, that every cograph satisfies *(b)*.

Let G be a cograph.

Basis: Suppose $G = K_1$. Since K_1 does not have non-trivial induced subgraphs, it trivially satisfies *(b)*.

Induction Hipothesis (Union): Suppose $G_1, \ldots, G_k$ are pairwise disjoint cographs that satisfy *(b)*.

Step (Union): Suppose $G = G_1 \cup \cdots \cup G_k$. Let $X \subseteq V(G)$ be such that $|X| > 1$. Let $v \in X$ and let $i \in [k]$ be such that $v \in V(G_i)$. We consider two cases.

Case 1: Suppose $X \subseteq V(G_i)$. Then, $G[X] = G_i[X]$ is disconnected or co-disconnected because, by IH, G_i satisfies *(b)*.

Case 2: Suppose $X \not\subseteq V(G_i)$. Then, X has vertices of G_i and G_j for some $j \neq i$, and thus, $G[X]$ is disconnected.

In any case, $G[X]$ is either disconnected or co-disconnected. Therefore, G satisfies *(b)*.

Induction Hypothesis (Complement): Suppose H is a cograph satisfying *(b)*.

Step (Complement): Suppose $G = \overline{H}$. Let $X \subseteq V(G)$ be such that $|X| > 1$. Since $G = \overline{H}$, we have $G[X] = \overline{H}[X] = \overline{H[X]}$. Since, by IH, H satisfies *(b)*, $H[X]$ is disconnected or co-disconnected. If $H[X]$ is disconnected, then $G[X] = \overline{H[X]}$ is co-disconnected; and if $H[X]$ is co-disconnected, then $G[X] = \overline{H[X]}$ is disconnected. In any case, $G[X]$ is disconnected or co-disconnected. Therefore, G satisfies *(b)*.

Thus, every cograph satisfies *(b)*.

(*(b)⇒(a)*)

Conversely, we prove by induction on n that, for every graph G such that $|V(G)| = n$, if G satisfies *(b)*, then G is a cograph.

Let G be a graph satisfying *(b)*.

Basis: Suppose $|V(G)| = 1$. Then, $G = K_1$. Thus, G is a cograph.

Induction Hypothesis: Let $n \geq 1$. Suppose that, for every graph H, if H satisfies *(b)* and $|V(H)| \leq n$, then H is a cograph.

Step: Suppose $|V(G)| = n + 1$. Since G satisfies *(b)* and $|V(G)| > 1$, G is either disconnected or co-disconnected. We consider both cases.

Case 1: Suppose G is disconnected. Let $G_1 = (V_1, E_1), \ldots, G_k = (V_k, E_k)$ be the connected components of G. Let $i \in [k]$. Let $X \subseteq V_i$ be such that $|X| > 1$. Then, $G_i[X] = G[X]$ is either disconnected or co-disconnected because G satisfies *(b)*. Therefore, G_i satisfies *(b)*. Since G has more than one connected component, $|V_i| < |V(G)| = n + 1$. Then, $|V_i| \leq n$, and since G_1 satisfies *(b)*, by IH, G_i is a cograph. Since i is arbitrary, G_i is a cograph for every $i \in [k]$. Since, $G = G_1 \cup \cdots \cup G_k$ is a disjoint union of cographs, G is a cograph.

Case 2: Suppose G is co-disconnected. Then, $\overline{G}$ is a disconnected graph with $|V(\overline{G})| = n+1$. Since G satisfies *(b)*, $\overline{G}$ also satisfies *(b)*. Analogously to Case 1, we prove that $\overline{G}$ is a cograph. Let $G_1 = (V_1, E_1), \ldots, G_k = (V_k, E_k)$ be the connected components of $\overline{G}$. Let $i \in [k]$. Let $X \subseteq V_i$ be such that $|X| > 1$. Then, $G_i[X] = \overline{G}[X]$ is either disconnected or co-disconnected because $\overline{G}$ satisfies *(b)*. Therefore, G_i satisfies *(b)*. Since $\overline{G}$ has more than one connected component, $|V_i| < |V(\overline{G})| = n + 1$. Then, $|V_i| \leq n$, and since G_1 satisfies *(b)*, by IH, G_i is a cograph. Since i is arbitrary, G_i is a cograph for every $i \in [k]$. Since $\overline{G} = G_1 \cup \cdots \cup G_k$ is a disjoint union of cographs, $\overline{G}$ is a cograph. Since $\overline{G}$ is a cograph, G is a cograph.

In any case G is a cograph. Thus, every graph satisfying *(b)* is a cograph. This completes the proof. □

Now, we implement Constructions 54 and 63 to produce an algorithm for constructing minimum leaos for cographs.

Construction 75. [Minimum Leaos for cographs]
INPUT: A cograph G.
CONSTRUCTION: The construction applies recursion on $n = |V(G)|$.
Base: Suppose $|V(G)| = 1$. Let $\ell : V(G) \to \mathbb{N}$ be such that $\ell(v) = 0$. Let $D := \emptyset$. Take $\mathfrak{L} := (\ell, D)$.
Recursive Step: Suppose $|V(G)| > 1$. Then, by Lemma 74, the induced subgraph $G[V(G)] = G$ is disconnected or co-disconnected. We consider both cases.

Case 1: Suppose G is disconnected. Let $G_0, \dots, G_k$ be the connected components of G. For each $i \in [0, k]$, apply Construction 75 on G_i to obtain a minimum leao $\mathfrak{L}_i$ with one sink per connected component. Let π be a permutation of $[0, k]$ such that $|\mathfrak{L}_{\pi(0)}| \geq \cdots \geq |\mathfrak{L}_{\pi(k)}|$. Then, apply Construction 63 on $(G, \{(G_{\pi(i)}, \mathfrak{L}_{\pi(i)}) : i \in [0, k]\})$ to obtain a minimum leao $\mathfrak{L}$ of G with one sink per connected component.

Case 2: Suppose G is co-disconnected. Let H be a connected component of $\overline{G}$. Let $G_1 := \overline{H}$ and let $G_2 := G \setminus V(H)$. Then, $G = G_1 \wedge G_2$. For each $i \in [2]$, apply Construction 75 on G_i to obtain a minimum leao $\mathfrak{L}_i$ of G_i with one sink per connected component. If $|\mathfrak{L}_1| \geq |\mathfrak{L}_2|$, then apply Construction 54 on $(G_1, \mathfrak{L}_1, G_2, \mathfrak{L}_2)$ to obtain a minimum leao $\mathfrak{L}$ of G with one sink per connected component. If $|\mathfrak{L}_1| < |\mathfrak{L}_2|$, then apply Construction 54 on $(G_2, \mathfrak{L}_2, G_1, \mathfrak{L}_1)$ to obtain a minimum leao $\mathfrak{L}$ of G with one sink per connected component.

OUTPUT: The minimum leao $\mathfrak{L}$, of G, with exactly one sink per connected component. $\triangle$

Finally, we have:

Theorem 76. EAO *restricted to the class of cographs can be solved in polynomial time.*

Proof. Applying Construction 75, we obtain a minimum leao for any given cograph in polynomial time. Then, a cograph is a set graph if and only if the leao produced by Construction 75 on that cograph has height 0. $\square$

Chapter 6

NP-Completeness of EAO on Split Graphs

In this chapter we prove that the recognition problem of set graphs, EAO, restricted to *split* graphs is NP-complete. More specifically, we present a polynomial-time reduction of the TOTAL ORDERING PROBLEM (TOP), known to be NP-Complete [12], to the recognition problem of set graphs restricted to split graphs. We also provide a detailed proof of the NP-completeness of TOP, based on a sketch, given in [12], of a polynomial-time reduction from the 3-SATISFIABILITY problem (3-SAT).

To define TOP, we consider that an *order*, $\prec$, on a set X is a binary relation on X such that, for all $x, y, z \in X$, $x \not\prec x$ (irreflexive), and if $x \prec y$ and $y \prec z$ then $x \prec z$ (transitive). Moreover, an order $\prec$ on a set X is total if, for all $x, y \in X$, if $x \neq y$, then either $x \prec y$ or $y \prec x$.

Definition 77. *Let $\prec$ be an order on a set X and let $T \subseteq X^3$ be a set of triples. The elements of T are called betweenness restrictions. The order $\prec$ satisfies a betweenness restriction $(a, b, c) \in T$ if either $a \prec b \prec c$ or $c \prec b \prec a$, i.e., b is between a and c in the order $\prec$. Additionally, $\prec$ satisfies T if, for every $t \in T$, $\prec$ satisfies t.*

Note that if an order $\prec$ satisfies a set of betweenness restrictions T, then the reverse of $\prec$ is another order that also satisfies T.

Total Ordering Problem (TOP)
INSTANCE: A pair (X, T), where X is a finite set and $T \subseteq X^3$ is a set of triples.
QUESTION: Is there a total order $\prec$ on X that satisfies T?

Example 78. Let $X := \{a, b, c, d, e\}$ and $T := \{(a, b, d), (b, c, a), (d, e, c), (c, e, b)\}$. The order given by $a \prec c \prec e \prec b \prec d$ satisfies T, whereas the order given by $c \prec e \prec d \prec b \prec a$ does not, since in the latter, $(b, c, a) \in T$ is not satisfied.

Let TOP* denote the restriction of TOP to the instances (X, T) such that

(i) every element of X is involved in some restriction of T, i.e., for every $x \in X$, there exists $t = (a, b, c) \in T$ such that $x \in \{a, b, c\}$; and

(ii) every triple $t \in X^3$ involves three pairwise distinct elements, i.e., for every $(a, b, c) \in T$, we have $|\{a, b, c\}| = 3$.

According to Opatrny [12], TOP appears naturally in the design of electronic circuits, as one may desire to arrange pins in a linear fashion having some pins be placed between certain pairs of other pins. The time complexity of TOP was posed as an open problem by R. Karp (cf. [12]), and was solved by Opatrny in [12], where TOP is proved to be NP-complete. In the following section, we present a complete proof that TOP is NP-complete.

6.1 NP-completeness of TOP (and TOP*)

To prove that TOP is a NP-complete problem, we will present a polynomial reduction from the 3-SATISFIABILITY problem (the canonical example of NP-completeness) to TOP. This reduction was sketched by Opatrny in [12], alongside an alternative reduction from the problem of 2-COLORABILITY OF HYPERGRAPHS OF RANK 3, which was given in detail in the same paper. The reason we insist on presenting a detailed development of the reduction from 3-SAT to TOP is the lack of references for the NP-completeness of the problem of 2-COLORABILITY OF HYPERGRAPHS OF RANK 3. Unfortunately, the reference [9], given by Opatrny, only applies to hypergraphs of rank strictly greater than 3. Thus, we decide to develop the alternative sketch presented by Opatrny into a complete proof.

Definition 79. *Let Var be a set of variables. We define the set of literals on Var by* $Lit := \{x, \neg x : x \in Var\}$. *We say that a function* $v : Lit \to \{0, 1\}$ *is a valuation if* $v(\neg x) = 1 - v(x)$ *for all* $x \in Var$. *We say that a finite set of literals is a clause. We say that a valuation* $v : Lit \to \{0, 1\}$ *satisfies a clause* C *if there is a literal* $\alpha \in C$ *such that* $v(\alpha) = 1$. *We say that a valuation* v *satisfies a set of clauses* B *if* v *satisfies every clause in* B.

3-SATISFIABILITY (3-SAT)
INSTANCE: A pair (Var, B) such that Var is a finite set of variables with $|\text{Var}| \geq 2$; B is a finite set of clauses such that, for every $C \in B$, $|C| = 3$.
QUESTION: Is there a valuation v that satisfies the set of clauses B?

Theorem 80 (S. Cook [4]). *3-SAT is NP-complete.*

Construction 81. Reducing 3-SAT to TOP*.
INPUT: An instance (Var, B) of 3-SAT.

PROCEDURE: Let $\text{Var} = \{x_i : i \in [m]\}$, $B = \{C_i : i \in [n]\}$ and, for every $i \in [n]$, let $C_i = \{a_i, b_i, c_i\}$.

We define the sets X and $T \subseteq X^3$ as follows. Let F, M and, for each $i \in [n]$, V_i, Y_i, Z_i, U_i be new elements not in Lit. Then,

$$X := \text{Lit} \cup \{V_i, Y_i, Z_i, U_i : i \in [n]\} \cup \{F, M\}$$

and

$$\begin{aligned} T := \{&(a_i, V_i, b_i), (V_i, M, Z_i), (U_i, Y_i, c_i), (Y_i, M, F), (Z_i, M, U_i) : i \in [n]\} \cup \\ &\{(x, M, \neg x) : x \in \text{Var}\}. \end{aligned}$$

OUTPUT: (X, T), an instance of TOP* such that T can be satisfied by a total order if and only if there is a valuation $v : \text{Lit} \to \{0, 1\}$ that satisfies B. △

We prove the correctness of this construction in Lemma 83.

Example 82. Let (Var, B) be an instance of 3-SAT, where $\text{Var} = \{x_1, x_2, x_3\}$ and $B = \{\{x_1, \neg x_1, x_2\}, \{x_1, \neg x_2, \neg x_3\}\}$.

Then, Construction 81 on (Var, B) yields (X, T), where

$$X = \{x_1, \neg x_1, x_2, \neg x_2, x_3, \neg x_3, V_1, V_2, Y_1, Y_2, Z_1, Z_2, U_1, U_2, F, M\}\}$$

and

$$T = \left\{ \begin{array}{l} (x_1, V_1, \neg x_1), (V_1, M, Z_1), (U_1, Y_1, x_2), (Y_1, M, F), (Z_1, M, U_1), \\ (x_1, V_2, \neg x_2), (V_2, M, Z_2), (U_2, Y_2, \neg x_3), (Y_2, M, F), (Z_2, M, U_2), \\ (x_1, M, \neg x_1), (x_2, M, \neg x_2), (x_3, M, \neg x_3) \end{array} \right\}.$$

Note that B is satisfied by any valuation v such that $v(x_1) = 1$. Moreover, the total order of X given by

$$F \prec \neg x_3 \prec \neg x_2 \prec \neg x_1 \prec Z_2 \prec Z_1 \prec M \prec U_1 \prec Y_2 \prec V_1 \prec U_2 \prec V_2 \prec x_1 \prec Y_1 \prec x_2 \prec x_3$$

satisfies each betweenness restriction of T. In the following Lemma 83, we show how a total order on X that satisfies T can be constructed from a valuation of v satisfying B.

Lemma 83 (Correctness of Construction 81)**.** *Given an instance* (Var, B) *of 3-SAT, the output* (X, T) *of Construction 81 on* (Var, B) *is an instance of* TOP* *such that* T *can be satisfied by a total order if and only if there is a valuation* $v : \mathit{Lit} \to \{0, 1\}$ *that satisfies* B*.*

Proof. First, note that, by definition, (i) every element of X is in some triple of T, and (ii) every triple of T has three pairwise distinct elements of X. Hence, (X, T) is indeed an instance of TOP*.

We shall prove that there is a total order on X that satisfies T if and only if every clause of B is satisfied by v.

To prove the implication from left to right, suppose there is a total order $\prec$ on X satisfying T. As we have noted, the reverse of $\prec$ also satisfies T. Hence, we assume, without loss of generality, that $F \prec M$.

Define the function $v : \mathrm{Lit} \to \{0, 1\}$, such that, for all $\alpha \in \mathrm{Lit}$,

$$v(\alpha) = \begin{cases} 1 & \text{if } M \prec \alpha \\ 0 & \text{if } \alpha \prec M \end{cases}.$$

Let $x \in \mathrm{Var}$. Since $(x, M, \neg x) \in T$ and $\prec$ satisfies T, we have $x \prec M \prec \neg x$ or $\neg x \prec M \prec x$. In any case, $v(\neg x) = 1 - v(x)$. Thus, v is a valuation.

We will prove that v satisfies every clause in B. Let $C_i \in B$. We consider two cases.

Case 1: Suppose $v(a_i) = 1$ or $v(b_i) = 1$. Then, v satisfies C_i.

Case 2: Suppose $v(a_i) = 0$ and $v(b_i) = 0$. Then, $a_i \prec M$ and $b_i \prec M$. Since $(a_i, V_i, b_i) \in T$, we have $a_i \prec V_i \prec b_i \prec M$ or $b_i \prec V_i \prec a_i \prec M$. In either case, $V_i \prec M$. Since $(V_i, M, Z_i) \in T$, we have $M \prec Z_i$. Since $(Z_i, M, U_i) \in T$, we have $U_i \prec M$. Moreover, since $(Y_i, M, F) \in T$ and $F \prec M$, we have $M \prec Y_i$. Thus, $U_i \prec M \prec Y_i$. Then, since $(U_i, Y_i, c_i) \in T$, we have $U_i \prec M \prec Y_i \prec c_i$. Thus, $v(c_i) = 1$ and, consequently, v satisfies C_i.

In any case, v satisfies C_i. Therefore, v satisfies every clause of B.

To prove the converse, suppose there is a valuation v satisfying B. We will define an injective function from X to $\mathbb{R}$ to induce a total order on X from the natural order of $\mathbb{R}$. Recall that $\mathrm{Var} = \{x_i : i \in [m]\}$, $B = \{C_i : i \in [n]\}$ and, for every $i \in [n]$, $C_i = \{a_i, b_i, c_i\}$. Define the function $f : X \to \mathbb{R}$ such that

- $f(M) = 0$,
- $f(F) = -(m + 1)$;

for every $i \in [m]$,

- $f(x_i) = \begin{cases} i & \text{if } v(x_i) = 1, \\ -i & \text{if } v(x_i) = 0, \end{cases}$
- $f(\neg x_i) = -f(x_i)$;

and, for every $i \in [n]$,

- $f(Y_i) = \max\{f(a_i), f(b_i), f(c_i)\} - 2^{-i}$;
- $f(U_i) = \max\{f(a_i), f(b_i)\} - 3^{-i}$;
- $f(V_i) = \max\{f(a_i), f(b_i)\} - 5^{-i}$;
- $f(Z_i) = -f(V_i)$.

By definition, f is an injective function. Thus, we can define the relation $\prec$ on X such that $x \prec y$ if and only if $f(x) < f(y)$, for all $x, y \in X$ such that $x \neq y$. By definition, $\prec$ is a total order.

It remains to check if $\prec$ satisfies T. Let $i \in [n]$.

- (a_i, V_i, b_i) is satisfied. Indeed, since $f(a_i)$ and $f(b_i)$ are distinct integers, we have $\min\{f(a_i), f(b_i)\} \leq \max\{f(a_i), f(b_i)\} - 1$. And since, $f(V_i) = \max\{f(a_i), f(b_i)\} - 5^{-i}$, we have $\min\{f(a_i), f(b_i)\} < f(V_i) < \max\{f(a_i), f(b_i)\}$. Thus, either $f(a_i) < f(V_i) < f(b_i)$ or $f(b_i) < f(V_i) < f(a_i)$.
- (V_i, M, Z_i) is satisfied. Indeed, since, by definition, $f(Z_i) = -f(V_i)$ and $f(V_i) \neq 0$, the signs of $f(V_i)$ and $f(Z_i)$ are opposite. And since $f(M) = 0$, we have either $f(Z_i) < f(M) < f(V_i)$ or $f(V_i) < f(M) < f(Z_i)$.
- To show (U_i, Y_i, c_i) is satisfied, we consider two cases:

  Case 1: Suppose $\max\{f(a_i), f(b_i)\} \leq f(c_i)$. Since f is injective, $\max\{f(a_i), f(b_i)\} < f(c_i)$. Since $\max\{f(a_i), f(b_i)\}$ and $f(c_i)$ are integers, $\max\{f(a_i), f(b_i)\} < f(c_i) - 2^{-i}$. Besides, since $f(Y_i) = \max\{f(a_i), f(b_i), f(c_i)\} - 2^{-i}$ and $\max\{f(a_i), f(b_i)\} \leq f(c_i)$, we have $f(Y_i) = f(c_i) - 2^{-i}$. Thus,

  $$\begin{aligned} f(U_i) &= \max\{f(a_i), f(b_i)\} - 3^{-i} \\ &< \max\{f(a_i), f(b_i)\} \\ &< f(c_i) - 2^{-i} = f(Y_i) \\ &< f(c_i). \end{aligned}$$

  Case 2: Suppose $f(c_i) < \max\{f(a_i), f(b_i)\}$. Since $f(c_i)$ and $\max\{f(a_i), f(b_i)\}$ are integers, $f(c_i) < \max\{f(a_i), f(b_i)\} - 2^{-i}$. Thus,

  $$\begin{aligned} f(c_i) &< \max\{f(a_i), f(b_i)\} - 2^{-i} \\ &= \max\{f(a_i), f(b_i), f(c_i)\} - 2^{-i} = f(Y_i) \\ &< \max\{f(a_i), f(b_i), f(c_i)\} - 3^{-i} = f(U_i). \end{aligned}$$

- (Y_i, M, F) is satisfied. Indeed, since C_i is satisfied by v, we have $v(\alpha) = 1$ for some $\alpha \in \{a_i, b_i, c_i\}$. Then, $0 < \max\{f(a_i), f(b_i), f(c_i)\}$. Since $\max\{f(a_i), f(b_i), f(c_i)\}$ is an integer, $0 < \max\{f(a_i), f(b_i), f(c_i)\} - 2^{-i} = f(Y_i)$. Besides, by definition, $f(F) = -(m+1) < 0 = f(M)$. Therefore, we have $f(F) < f(M) < f(Y_i)$.

- (Z_i, M, U_i) is satisfied. Indeed, since $f(U_i) = \max\{f(a_i), f(b_i)\} - 3^{-i}$ and $f(V_i) = \max\{f(a_i), f(b_i)\} - 5^{-i}$, $f(U_i)$ and $f(V_i)$ have the same sign as $\max\{f(a_i), f(b_i)\}$. Then, $f(U_i)$ and $f(V_i)$ have the same sign. Besides, $f(Z_i) = -f(V_i)$. So $f(Z_i)$ and $f(V_i)$ have opposite signs. Thus, $f(Z_i)$ and $f(U_i)$ have opposite signs. And since $f(M) = 0$, either $f(Z_i) < f(M) < f(U_i)$ or $f(U_i) < f(M) < f(Z_i)$.

- $(x_i, M, \neg x_i)$ is satisfied. Indeed, since, $f(\neg x_i) = -f(x_i)$ by definition, $f(\neg x_i)$ and $f(x_i)$ have opposite signs. Moreover, $f(M) = 0$. Thus, either $f(\neg x_i) < f(M) < f(x_i)$ or $f(x_i) < f(M) < f(\neg x_i)$.

Therefore, $\prec$ satisfies T. This concludes the proof. □

Theorem 84 (J. Opatrny [12]). TOP* *is NP-complete.*

Proof. Note that, given an instance (X, T) of TOP* and a total order $\prec$ of X, we can verify whether all of the betweenness restrictions of T are satisfied by $\prec$ in time $\mathcal{O}(|X| \cdot |T|)$. Thus, TOP* is in NP.

Moreover, Construction 81 is a polynomial-time reduction of 3-SAT to TOP*. And since, by Theorem 80, 3-SAT is NP-Complete, we conclude that TOP* is NP-complete. □

6.2 Reducing TOP* to EAO on Split Graphs

Finally, we arrive at the main point of this chapter. We prove that EAO restricted to *split* graphs is NP-complete.

First, let us briefly recall what a *split* graph is.

Definition 85. *Given a graph $G = (V, E)$, a set $X \subseteq V$ is a stable set if the vertices of X are pairwise non-adjacent; and is a clique if the vertices of X are pairwise adjacent.*

Definition 86. *A graph $G = (V, E)$ is a split graph if there is a partition $\{S, Q\}$ of V such that S is a stable set and Q is a clique of G.*

The strategy for reducing TOP* to EAO is the following. Note that any acyclic orientation of a graph induces a total order on each of its cliques. Given the sets X

and T of an instance of TOP*, we will construct a split graph $G_T = (S \cup Q, E)$ by putting the elements of X in the clique Q, and defining the vertices of S and the edges between S and Q in such a way as to force any total order of Q contained in an eao of G to satisfy every betweenness restriction of T.

We begin by constructing a split graph G_t that contains, in its clique, the elements involved in a single betweenness restriction $t = (a, b, c)$, and is such that, every eao of G_t satisfying certain conditions will contain a total order of $\{a, b, c\}$ satisfying t (see Lemma 88).

Construction 87. INPUT: A triple $t = (a, b, c)$ of pairwise distinct elements.
PROCEDURE: Let $Q_t := \{a, b, c, t\}$ and $S_t := \{s_t^i : i \in [4]\} \cup \{q_t, r_t\}$ be a set of new elements. Define the split graph $G_t := (S_t \cup Q_t, E_t)$, in which S_t is a stable set, Q_t is a clique, and is such that $N(s_t^i) = Q_t$ for each $i \in [4]$, $N(q_t) = \{b, t\}$ and $N(r_t) = \{a, c, t\}$ (see Figure 6.1).
OUTPUT: The split graph G_t. △

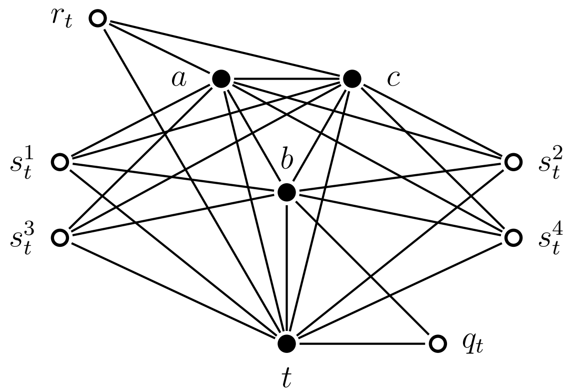

Figure 6.1: G_t given by Construction 87

Lemma 88. *Let $t = (a, b, c)$ be a triple such that a, b and c are pairwise distinct, and let G_t be given by Construction 87 on t. If D is an eao of G_t such that t is a sink and q_t, r_t are sources, then either $\{ab, bc\} \subseteq D$ or $\{cb, ba\} \subseteq D$, i.e., the order of Q_t contained in D satisfies the betweenness restriction t.*

Proof. Let D be an eao of G_t such that t is a sink and q_t, r_t are sources.

Since $N(s_t^i) = Q_t$ for every $i \in [4]$, by Lemma 19, $\{N_D^+(s_t^i) : i \in [4]\}$ is nested. Then, there is a permutation π of $[4]$ such that $N_D^+(s_t^{\pi(1)}) \subseteq N_D^+(s_t^{\pi(2)}) \subseteq N_D^+(s_t^{\pi(3)}) \subseteq N_D^+(s_t^{\pi(4)})$. Since t is a sink and $t \in Q_t = N(s_t^{\pi(1)})$, we have $t \in N_D^+(s_t^{\pi(1)})$. Additionally, for all $i, j \in [4]$ if $i \neq j$, since $N(s_t^i) = Q_t = N(s_t^j)$, by Corollary 20, $|N_D^+(s_t^i)| \neq |N_D^+(s_t^j)|$. So $1 \leq |N_D^+(s_t^{\pi(1)})| < |N_D^+(s_t^{\pi(2)})| < |N_D^+(s_t^{\pi(3)})| < |N_D^+(s_t^{\pi(4)})| \leq 4$. Hence,

$|N_D^+(s_t^{\pi(i)})| = i$ for each $i \in [4]$. Then, for some permutation τ of $\{a, b, c\}$, we have

$$\begin{aligned}
N_D^+(s_t^{\pi(1)}) &= \{t\} \\
N_D^+(s_t^{\pi(2)}) &= \{t, \tau(a)\} \\
N_D^+(s_t^{\pi(3)}) &= \{t, \tau(a), \tau(b)\} \\
N_D^+(s_t^{\pi(4)}) &= \{t, \tau(a), \tau(b), \tau(c)\}.
\end{aligned}$$

Next, we prove that $\tau(b) = b$. The roles of the vertices q_t and r_t are to eliminate the alternative possibilities of having $\tau(b) = a$ or $\tau(b) = c$. Suppose, for a contradiction, that $\tau(b) \in \{a, c\}$. Since τ is a permutation and $\tau(b) \neq b$, either $\tau(a) = b$ or $\tau(c) = b$. We consider two cases.

Case 1: Suppose $\tau(a) = b$. Since, by hypothesis, q_t is a source, $N_D^+(q_t) = N(q_t) = \{b, t\}$. Then, $N_D^+(q_t) = \{b, t\} = \{t, \tau(a)\} = N_D^+(s_t^{\pi(2)})$, contradicting the extensionality of D.

Case 2: Suppose $\tau(c) = b$. Since τ is a permutation of $\{a, b, c\}$, we have $\{\tau(a), \tau(b)\} = \{\tau(a), \tau(b), \tau(c)\} \setminus \{\tau(c)\} = \{a, b, c\} \setminus \{b\} = \{a, c\}$. So, since r_t is a source, $N_D^+(r_t) = \{a, c, t\} = \{t, \tau(a), \tau(b)\} = N_D^+(s_t^{\pi(3)})$, contradicting the extensionality of D.

In both cases we have a contradiction. Thus, $\tau(b) = b$.

Since $\tau(c)s_t^{\pi(3)}\tau(b)s_t^{\pi(2)}\tau(a)$ is a path in D and D is acyclic, $\tau(c)\tau(b) \in D$ and $\tau(b)\tau(a) \in D$. Since $\tau(b) = b$, we have $\tau(c)b \in D$ and $b\tau(a) \in D$. If $\tau(a) = a$, then $\tau(c) = c$ and we have $\{cb, ba\} \subseteq D$. And if $\tau(a) = c$, then $\tau(c) = a$ and we have $\{ab, bc\} \subseteq D$. In any case, either $\{ab, bc\} \subseteq D$ or $\{cb, ba\} \subseteq D$. □

So far, we have established a method for encoding a single betweenness restriction into the structure of a split graph. To enforce all of the restrictions of T at once, we will define a split graph G_T with the following properties: (i) G_t is an induced subgraph of G_T for every $t \in T$; (ii) any eao D of G_T, when restricted to G_t, is an eao of G_t such that t is a sink and q_t, r_t are sources (the conditions for applying Lemma 88 are met); (iii) if T can be satisfied by some total order of X, then G_T admits an eao.

Note that, by Lemma 88, properties (i) and (ii) ensure that, if G_T admits an eao D, then T is satisfied by a total order of X; while property (iii) ensures the converse. G_T will be constructed by taking the union $\bigcup_{t \in T} G_t$ and adding some auxiliary structure to ensure that the resulting graph is in fact a split graph, and that the properties (i), (ii) and (iii) hold.

Construction 89. INPUT: An instance of TOP*, (X, T), where $X = \{x_i : i \in [n]\}$ and $T = \{t_i : i \in [p]\}$ for some $n, p \in \mathbb{N}$.

PROCEDURE: For each $t \in T$, let $G_t = (S_t \cup Q_t, E_t)$ be the split graph given by Construction 87 on t. Recall that, for every $t = (a_t, b_t, c_t) \in T$, $S_t = \{s_t^i : i \in [4]\} \cup \{q_t, r_t\}$ is a stable set, $Q_t = \{a_t, b_t, c_t, t\}$ is a clique, $N(s_t^i) = Q_t$ for each $i \in [4]$, $N(q_t) = \{b_t, t\}$ and $N(r_t) = \{a_t, c_t, t\}$.

Let $X' = \{x_i' : i \in [n]\}$ be a copy of X; $T' = \{t_i' : i \in [p]\}$ be a copy of T; and d_-, d_+ and d be three additional elements. Define the split graph A_T with stable set $X' \cup T' \cup \{d_-, d_+\}$, clique $X \cup T \cup \{d\}$, such that $N(x_i') = \{x_i\}$ for every $x_i' \in X'$, and, for every $v \in T' \cup \{d_-, d_+\}$, $N(v) = T \cup \{d\}$ (see Figure 6.2).
OUTPUT: The graph $G_T := \bigcup\{G_t : t \in T\} \cup A_T$. △

Note that the vertices of G_T can be partitioned into the stable set $S_T := \bigcup\{S_t : t \in T\} \cup X' \cup T' \cup \{d_+, d_-\}$ and the clique $Q_T := X \cup T \cup \{d\}$. Thus, G_T is a split graph. Moreover, since $|S_t| = 6$ for each $t \in T$, $|X| = |X'| = n$ and $|T| = |T'| = p$, the graph G_T has $2n + 8p + 3$ vertices. By Theorem 95, we will conclude that G_T is a set graph if and only if X admits a total order satisfying T.

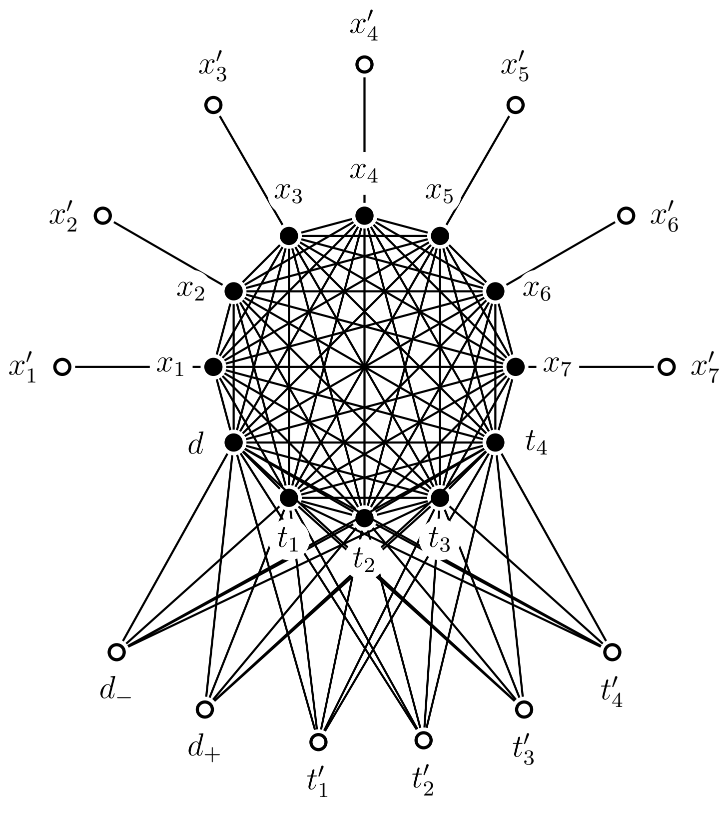

Figure 6.2: A_T (for $|X| = 7$ and $|T| = 4$)

Lemma 90. *Let G_T be given by Construction 89 and let D be an eao of G_T. Then, the sink of D is in $T' \cup \{d_-, d_+\}$.*

Proof. Since $N_D^+(v) = T \cup \{d\}$ for all $v \in T' \cup \{d_-, d_+\}$, $\{|N_D^+(v)| : v \in T' \cup \{d_-, d_+\}\} \subseteq [0, |T \cup \{d\}|] = [0, p+1]$. Besides, for all $u, v \in T' \cup \{d_-, d_+\}$, since

$N(u) = T \cup \{d\} = N(v)$, by Corollary 20, if $u \neq v$, then $|N_D^+(v)| \neq |N_D^+(w)|$. So, since $|T' \cup \{d_-, d_+\}| = p+2$, we have $|\{|N_D^+(v)| : v \in T' \cup \{d_-, d_+\}\}| = p+2$. Thus, $\{|N_D^+(v)| : v \in T' \cup \{d_-, d_+\}\} = [0, p+1]$. Hence, there is a vertex $v \in T' \cup \{d_-, d_+\}$ such that $|N_D^+(v)| = 0$, i.e., v is a sink. Since D is extensional, the only sink of D is $v \in T' \cup \{d_-, d_+\}$. □

Lemma 91. *Let G_T be given by Construction 89, let D be an eao of G_T and $t = (a, b, c) \in T$. Then, there are permutations π and τ, respectively, of $[4]$ and $\{a, b, c\}$, such that*

$$\begin{aligned}
N_D^+(s_t^{\pi(1)}) &= \{t\} \\
N_D^+(s_t^{\pi(2)}) &= \{t, \tau(a)\} \\
N_D^+(s_t^{\pi(3)}) &= \{t, \tau(a), \tau(b)\} \\
N_D^+(s_t^{\pi(4)}) &= \{t, \tau(a), \tau(b), \tau(c)\}.
\end{aligned}$$

Proof. By Lemma 90, S_t does not have sinks. So $1 \leq |N_D^+(s_t^i)| \leq |Q_t| = 4$ for all $i \in [4]$. Since $N(s_t^i) = Q_t$ for all $i \in [4]$, by Lemma 19, $\{N_D^+(s_t^i) : i \in [4]\}$ is nested. Thus, there is a permutation $\pi : [4] \to [4]$ such that $N(s_t^{\pi(1)}) \subseteq N(s_t^{\pi(2)}) \subseteq N(s_t^{\pi(3)}) \subseteq N(s_t^{\pi(4)})$. By Corollary 20, for all $i, j \in [4]$, if $i \neq j$, $|N_D^+(s_t^i)| \neq |N_D^+(s_t^j)|$. So we have $1 < |N_D^+(s_t^{\pi(1)})| < |N_D^+(s_t^{\pi(2)})| < |N_D^+(s_t^{\pi(3)})| < |N_D^+(s_t^{\pi(4)})| < |Q_t| = 4$. Therefore, $|N_D^+(s_t^{\pi(i)})| = i$ for all $i \in [4]$. Thus, there is a permutation $\tau : Q_t \to Q_t$ such that

$$\begin{aligned}
N_D^+(s_t^{\pi(1)}) &= \{\tau(t)\} \\
N_D^+(s_t^{\pi(2)}) &= \{\tau(t), \tau(a)\} \\
N_D^+(s_t^{\pi(3)}) &= \{\tau(t), \tau(a), \tau(b)\} \\
N_D^+(s_t^{\pi(4)}) &= \{\tau(t), \tau(a), \tau(b), \tau(c)\}.
\end{aligned}$$

Suppose, for a contradiction, that $t \notin N_D^+(s_t^{\pi(1)})$. Then, $N_D^+(s_t^{\pi(1)}) = \{x\}$ for some $x \in \{a, b, c\}$. But, by Lemma 90, there are no sinks in X', so $N_D^+(x') = N(x') = \{x\} = N_D^+(s_t^{\pi(1)})$, contradicting the extensionality of D. Therefore, $t \in N_D^+(s_t^{\pi(1)})$, and so $\tau(t) = t$. Thus, $\tau|_{Q_t \setminus \{t\}}$ is a permutation of $\{a, b, c\}$ satisfying the statement in Lemma 91. □

In the following Lemmas 92 and 93, we prove that if D_T is an eao of G_T and $t \in T$, then $D_t = D_T \cap (S_t \cup Q_t)^2$ is an eao of G_t in which t is a sink and q_t and r_t are sources.

Lemma 92. *Let G_T be given by Construction 89 and let $t = (a, b, c) \in T$. If D is an eao of G_T, then $D_t = D \cap (S_t \cup Q_t)^2$ is an eao of G_t.*

Proof. First, we prove that D_t is extensional. Let $x, y \in S_t \cup Q_t$ be such that $x \neq y$. We consider four cases.

Case 1: Suppose $x, y \in S_t$. Since $N_D^+(x) \subseteq N(x) = Q_t \subseteq S_t \cup Q_t$, we have $N_{D_t}^+(x) = N_D^+(x)$. Analogously, $N_{D_t}^+(y) = N_D^+(y)$. Since D is extensional, $N_D^+(x) \neq N_D^+(y)$. Therefore, $N_{D_t}^+(x) \neq N_{D_t}^+(y)$.

Case 2: Suppose $x \in S_t$ and $y \in Q_t$. We consider two subcases.

Subcase 2.1: Suppose $y \neq t$. Then, $y \in Q_t \setminus \{t\} = \{a, b, c\}$. By Lemma 91, there is a permutation $\pi : [4] \to [4]$ such that, for all $i \in [4]$, $|N_D^+(s_t^{\pi(i)})| = i$. Moreover, $N_D^+(s_t^{\pi(1)}) = \{t\}$. Since $y \notin \{t\} = N_{D_t}^+(s_t^{\pi(1)})$ and, since $y \in Q_t = N(s_t^{\pi(1)})$, we have $ys_t^{\pi(1)} \in D \cap (S_t \cup Q_t)^2 = D_t$. Since S_t is a stable set, $s_t^{\pi(1)}$ and x are not adjacent in G_t. So $s_t^{\pi(1)} \notin N_{D_t}^+(x)$. Then, $s_t^{\pi(1)} \in N_{D_t}^+(y) \setminus N_{D_t}^+(x)$.

Subcase 2.2: Suppose $y = t$. Then, $N_{D_t}^+(x) \neq N_{D_t}^+(y)$ because x and y are adjacent in G_t.

Case 3: Suppose $x \in Q_t$ and $y \in S_t$. Analogously to *Case 2*, $N_{D_t}^+(x) \neq N_{D_t}^+(y)$.

Case 4: If $x, y \in Q_t$, then $N_{D_t}^+(x) \neq N_{D_t}^+(y)$ because x and y are adjacent in G_t.

Therefore, D_t is extensional.

D_t is acyclic because D is acyclic and $D_t \subseteq D$. ☐

Lemma 93. *Let G_T be given by Construction 89, let D be an eao of G_T and $t \in T$. Let $D_t = D \cap (S_t \cup Q_t)^2$ be the restriction of D to G_t. Then, t is a sink of D_t and q_t, r_t are sources of D_t.*

Proof. Let $t = (a, b, c)$. By Lemma 91, there are permutations π and τ of $[4]$ and $\{a, b, c\}$, respectively, such that

$$\begin{aligned}
N_D^+(s_t^{\pi(1)}) &= \{t\} \\
N_D^+(s_t^{\pi(2)}) &= \{t, \tau(a)\} \\
N_D^+(s_t^{\pi(3)}) &= \{t, \tau(a), \tau(b)\} \\
N_D^+(s_t^{\pi(4)}) &= \{t, \tau(a), \tau(b), \tau(c)\}.
\end{aligned}$$

First, we show that t is a sink of D_t. Suppose, for a contradiction, that $t \notin N_D^+(r_t)$. By Lemma 90, r_t is not a sink, so there exists $x \in N(r_t) \setminus \{t\} = \{a, c\}$ such that $r_t x \in D$. Since $t \notin N_D^+(r_t)$, we have $tr_t \in D$. Then, $xs_t^{\pi(1)}tr_tx$ is a cycle in D, a contradiction. Therefore, $t \in N_D^+(r_t)$. Analogously, $t \in N_D^+(q_t)$. Moreover, suppose for a contradiction, that, for some $x \in \{a, b, c\}$ we have $tx \in D$. Then, we have the cycle $xs_t^{\pi(1)}tx$ in D, a contradiction. Hence, for every $x \in \{a, b, c\}$, since x is adjacent to t, we must have $xt \in D$. Then, for every $v \in S_t \cup Q_t \setminus \{t\}$, we have $t \in N_D^+(v)$. Hence, t is a sink of D_t.

Secondly, we show that q_t and r_t are sources of D. We have already shown that $t \in N_D^+(q_t)$ and $t \in N_D^+(r_t)$. If $b \notin N_D^+(q_t)$, then $N_D^+(q_t) = \{t\} = N_D^+(s_t^{\pi(1)})$, contradicting that D is extensional. Therefore, $N_D^+(q_t) = \{t, b\}$, i.e., q_t is a source. Suppose, for a contradiction, that r_t is not a source. Since $t \in N_D^+(r_t)$, we consider three cases, one for each proper subset of $N(r_t) = \{t, a, c\}$ that contains t:

Case 1: Suppose $N_D^+(r_t) = \{t\}$. Then, $N_D^+(r_t) = \{t\} = N_D^+(s_t^{\pi(1)})$, contradicting that D is extensional.

Case 2: Suppose $N_D^+(r_t) = \{t, a\}$. Since D is extensional, $\{t, a\} = N_D^+(r_t) \neq N_D^+(s_t^{\pi(2)}) = \{t, \tau(a)\}$, so $\tau(a) \neq a$; and, again, since D is extensional and we have already established that q_t is a source, $\{t, b\} = N_D^+(q_t) \neq N_D^+(s_t^{\pi(2)}) = \{t, \tau(a)\}$, so $\tau(a) \neq b$. Hence, $\tau(a) = c$. Since $c \in N(r_t)$ but $c \notin N_D^+(r_t)$, we have $cr_t \in D$; $r_t a \in D$ by hypothesis; since $a \in Q_t = N(s_t^{\pi(2)})$ but $N(s_t^{\pi(2)}) = \{t, \tau(a)\}$ and $\tau(a) = c$, we have $as_t^{\pi(2)} \in D$; since $N(s_t^{\pi(2)}) = \{t, \tau(a)\}$ and $\tau(a) = c$, we have $s_t^{\pi(2)}c \in D$. Thus, $cr_t a s_t^{\pi(2)} c$ is a cycle in D, a contradiction.

Case 3: Suppose $N_D^+(r_t) = \{t, c\}$. Since D is extensional, $\{t, c\} = N_D^+(r_t) \neq N_D^+(s_t^{\pi(2)}) = \{t, \tau(a)\}$, so $\tau(a) \neq c$; and, again, since D is extensional and we have already established that q_t is a source, $\{t, b\} = N_D^+(q_t) \neq N_D^+(s_t^{\pi(2)}) = \{t, \tau(a)\}$, so $\tau(a) \neq b$. Hence, $\tau(a) = a$. Since $a \in N(r_t)$ but $a \notin N_D^+(r_t)$, we have $ar_t \in D$; $r_t c \in D$ by hypothesis; since $c \in Q_t = N(s_t^{\pi(2)})$ but $N(s_t^{\pi(2)}) = \{t, \tau(a)\}$ and $\tau(a) = a$, we have $cs_t^{\pi(2)} \in D$; since $N(s_t^{\pi(2)}) = \{t, \tau(a)\}$ and $\tau(a) = a$, we have $s_t^{\pi(2)}a \in D$. Thus, $ar_t c s_t^{\pi(2)} a$ is a cycle in D, a contradiction.

In any case, we reach a contradiction. Thus, r_t is a source of D. Since q_t and r_t are sources of D and $D_t \subseteq D$, q_t and r_t are sources of D_t. □

Lemmas 88, 92 and 93 tell us that if the split graph G_T given by Construction 89 on an instance (X, T) of TOP* admits an eao D, then D contains a total order of X satisfying T. It remains to prove that, conversely, X only admits a total order satisfying T if G_T admits an eao.

Lemma 94. *Let (X, T) be an instance of* TOP* *and let G_T be the split graph given by Construction 89 on (X, T). If T is satisfied by some total order on X, then G_T admits an eao.*

Proof. Suppose T is satisfied by a total order $\prec$ on X. Define the orientation D of G_T as follows:

- The edges between vertices of the clique Q are oriented as follows:
  - For each $x_i, x_j \in X$, $x_i \to x_j$ if and only if $x_i \prec x_j$.
  - For each $x_i \in X$ and $v \in T \cup \{d\}$, $x_i \to v$.
  - For each $t_i, t_j \in T$, $t_i \to t_j$ if and only if $i < j$.
  - For each $t_i \in T$, $t_i \to d$.
- For every $t \in T$ with $t = (a_t, b_t, c_t)$, let τ_t be the permutation of $\{a_t, c_t\}$ such that $\tau_t(a_t) \prec b_t \prec \tau_t(c_t)$. Then, the edges between the stable set S_t and the clique Q_t are oriented as such:
  - q_t and r_t are sources, i.e., $N^+(q_t) = \{t, b_t\}$ and $N^+(r_t) = \{t, a_t, c_t\}$;
  - $N^+(s_t^1) = \{t\}$;
  - $N^+(s_t^2) = \{t, \tau_t(c_t)\}$;
  - $N^+(s_t^3) = \{t, b_t, \tau_t(c_t)\}$;
  - $N^+(s_t^4) = \{t, \tau_t(a_t), b_t, \tau_t(c_t)\}$.
- For each $i \in [n]$, $N^+(x'_i) = \{x_i\}$, i.e., every vertex in X' is a source.
- The edges between the stable set $T' \cup \{d_-, d_+\}$ and the clique $T \cup \{d\}$ are defined such that, for all $i, j \in [p]$,
  - $t'_i \to t_j$ if $i < j$, and $t_j \to t'_i$ if $j \leq i$;
  - $t'_i \to d$;
  - $d_+ \to t_j \to d_-$;
  - $d_+ \to d \to d_-$.

First, we prove that D is acyclic. Note that the vertices in $X' \cup \{q_t, r_t : t \in T\} \cup \{d_+\}$ are sources and d_- is the sink of D. Thus, no directed cycle contains vertices of $X' \cup \{q_t, r_t : t \in T\} \cup \{d_-, d_+\}$. Since $N^+(d) = \{d_-\}$ and d_- is a sink, d cannot be in a cycle of D.

Suppose, for a contradiction, that D has a cycle C containing a vertex in T. Let $i \in [p]$ be maximal such that t_i is in the cycle C. Let u_1 be the vertex after t_i in C. Then $u_1 \in N_D^+(t_i) \subseteq T \cup T' \cup \{d_-, d_+, d\}$. But, as we have proved above, no vertex in $\{d_-, d_+, d\}$ can be in a cycle of D. So $u_1 \in T \cup T'$. Then, we consider two cases.

Case 1: Suppose $u_1 \in T$. Then, $u_1 = t_j$ for some $j \in [p]$. Since $t_i t_j \in D$, we have $j > i$, contradicting the maximality of i.

Case 2: Suppose $u_1 \in T'$. Then, $u_1 = t'_j$ for some $j \in [p]$. Since $t_i t'_j \in D$, we have $j \geq i$. Let u_2 be the vertex after u_1 in C. Then, $u_2 \in N_D^+(t'_j) \subseteq T \cup \{d\}$. Since d cannot be in a cycle of D, we have $u_2 \in T$. Then, $u_2 = t_k$ for some $k \in [p]$. Since $t'_j t_k \in D$, we have $k > j$. Then, $u_2 = t_k$ is a vertex in C with $k > j \geq i$, contradicting the maximality of i.

In both cases, we reach a contradiction. Therefore, no cycle of D contains vertices of T. Since $N^+(t'_i) \subseteq T \cup \{d\}$ for all $t'_i \in T'$, no cycle of D contains vertices of T'.

Suppose, for a contradiction, that D has a cycle C containing a vertex in X. Let $x \in X$ be maximal, with respect to $\prec$, such that x is in C, i.e., for all $y \in X$ such that $x \neq y$, if C contains y, then $y \prec x$. Let u_1 be the vertex after x in C. Then, $u_1 \in N_D^+(x) \subseteq X \cup (\bigcup\{S_t : t \in T\}) \cup T \cup \{d\}$. Since we have already proved that the vertices of $T \cup \{d\}$ are not in any cycle of D, we have $u_1 \in X \cup (\bigcup\{S_t : t \in T\}$. We consider two cases.

Case 1: Suppose $u_1 \in X$. Then, $u_1 = y$ for some $y \in X$. Since $xy \in D$, $x \prec y$, contradicting the maximality of x.

Case 2: Suppose $u_1 \in S_t$ for some $t \in T$. Since q_t and r_t are sources, $u_1 \notin \{q_t, r_t\}$. Then, $u_1 = s_t^i$ for some $i \in [4]$. Let u_2 be the following vertex, after u_1 in C. Since $u_2 \in N_D^+(s_t^i) \subseteq Q_t$ and t cannot be in a cycle, $u_2 \in Q_t \setminus \{t\}$. Since $xu_1 = xs_t^i \in D$ and $N(s_t^i) = Q_t$, we have $x \in Q_t$. We consider four cases.

Case 1: Suppose $u_1 = s_t^1$. Then, $N_D^+(u_1) = N_D^+(s_t^1) = \{t\}$. But $u_1u_2 \in D$. So $u_2 = t$, contradicting that C does not have vertices in T.

Case 2: Suppose $u_1 = s_t^2$. Then, $N_D^+(u_1) = N_D^+(s_t^2) = \{t, \tau_t(c_t)\}$. Since $u_1u_2 \in D$ and $u_2 \neq t$, we have $u_2 = \tau_t(c_t)$. Since $xs_t^2 \in D$, we have $x \in N_D^-(s_t^2) = N(s_t^2) \setminus N_D^+(s_t^2) = Q_t \setminus N_D^+(s_t^2) = \{\tau_t(a_t), b_t\}$. And since τ_t is such that $\tau_t(a_t) \prec b_t \prec \tau_t(c_t)$, we have $x \prec \tau_t(c_t) = u_2$, contradicting the maximality of x in C.

Case 3: Suppose $u_1 = s_t^3$. Then, $N_D^+(u_1) = N_D^+(s_t^3) = \{t, b_t, \tau_t(c_t)\}$. Since $u_1u_2 \in D$ and $u_2 \neq t$, we have $u_2 \in \{b_t, \tau_t(c_t)\}$. Since $xs_t^3 \in D$, we have $x \in N_D^-(s_t^3) = N(s_t^3) \setminus N_D^+(s_t^3) = Q_t \setminus N_D^+(s_t^3) = \{\tau_t(a_t)\}$. So $x = \tau_t(a_t)$. And since τ_t is such that $\tau_t(a_t) \prec b_t \prec \tau_t(c_t)$ and we have $u_2 \in \{b_t, \tau_t(c_t)\}$, we have $x = \tau_t(a_t) \prec u_2$ contradicting the maximality of x in C.

Case 4: Suppose $u_1 = s_t^4$. Then, $N_D^+(u_1) = N_D^+(s_t^4) = \{t, \tau_t(a_t), b_t, \tau_t(c_t)\}$. Since $xu_1 \in D$ and, by definition, $u_1 = s_t^4$ is a source, we have a contradiction.

In any case, we reach a contradiction. Therefore, no cycle of D contains vertices of X. Since $N^+(s_t^i) \subseteq X \cup T$, no cycle of D contains s_t^i, for any $t \in T$ and $i \in [4]$.

Since no vertex of $V(G_T) = T \cup T' \cup \{d_-, d_+, d\} \cup X \cup X' \cup \bigcup\{S_t : t \in T\}$ can be in a cycle of D, we conclude that D is acyclic.

Now, we prove that D is extensional. We will prove that (i) if $x, y \in S$ are distinct, then $N_D^+(x) \neq N_D^+(y)$, (ii) if $x \in S$ and $y \in Q$ are distinct, then $N_D^+(x) \neq N_D^+(y)$ and (iii) if $x, y \in Q$ are distinct, then $N_D^+(x) \neq N_D^+(y)$.
(i) Recall that $S = \bigcup\{S_t : t \in T\} \cup X' \cup T' \cup \{d_+, d_-\}$. Let $u, v \in S$ be such that $u \neq v$. We consider ten cases, organized according to the following table.

	$u \in \bigcup\{S_t : t \in T\}$	$u \in X'$	$u \in T'$	$u = d_+$	$u = d_-$
$v \in \bigcup\{S_t : t \in T\}$	1	2*	3*	3*	4*
$v \in X'$	2	5	6*	6*	7*
$v \in T'$	3	6	8	9*	10*
$v = d_+$	3	6	9	-	10*
$v = d_-$	4	7	10	10	-

In the above table we consider all of the 25 possible configurations for the vertices $u, v \in S$. Cells with a number $k \in [10]$ indicate that *Case* k, in the list below, proves that $N_D^+(u) \neq N_D^+(v)$ for the corresponding configuration of u and v. Certain cells contain k* for some $k \in [10]$, indicating that *Case* k proves $N_D^+(u) \neq N_D^+(v)$ in the analogous configuration in which u and v are swapped, one for the other. Cells containing "-" indicate that the corresponding configuration is not consistent with the hypothesis that $u \neq v$.

Case 1: Suppose $u, v \in \bigcup\{S_t : t \in T\}$.

Subcase 1.1: Suppose $u, v \in S_t$ for some $t \in T$.

Subcase 1.1.1: Suppose $u = s_t^i$ and $v = s_t^j$ for some $i, j \in [4]$. Since $u \neq v$, we have $i \neq j$. Then, $N_D^+(u) \neq N_D^+(v)$ because, by definition, $|N_D^+(s_t^i)| = i \neq j = |N_D^+(s_t^j)|$.

Subcase 1.1.2: Suppose $u = s_t^i$ and $v = r_t$ for some $i \in [4]$. We consider two cases.

Subcase 1.1.2.1: Suppose $i \neq 3$. Then $N_D^+(u) \neq N_D^+(v)$ because, by definition, $|N_D^+(s_t^i)| = i \neq 3 = |N_D^+(r_t)|$.

Subcase 1.1.2.2: Suppose $i = 3$. Then, by definition, $N_D^+(s_t^3) = \{t, b_t, \tau_t(c_t)\}$ and $N_D^+(r_t) = \{t, a_t, c_t\}$. Therefore, $b_t \in N_D^+(u) \setminus N_D^+(v)$.

Subcase 1.1.3: Suppose $u = s_t^i$ and $v = q_t$. We consider two cases.

Subcase 1.1.3.1: Suppose $i \neq 2$. Then, $N_D^+(u) \neq N_D^+(v)$ because, by definition, $|N_D^+(s_t^i)| = i \neq 2 = |N_D^+(q_t)|$.

Subcase 1.1.3.2: Suppose $i = 2$. Then, by definition, $N_D^+(s_t^2) = \{t, \tau_t(c_t)\}$ and $N_D^+(q_t) = \{t, b_t\}$. Since, by definition, τ_t is a permutation of $\{a_t, c_t\}$, $\tau_t(c_t) \in \{a_t, c_t\}$. So $\tau_t(c_t) \neq b_t$. Therefore, $b_t \in N_D^+(q_t) \setminus N_D^+(r_t) = N_D^+(v) \setminus N_D^+(u)$.

Subcase 1.1.4: Suppose $u = r_t$ and $v = q_t$. Then, $N_D^+(u) \neq N_D^+(v)$ because, by definition $|N_D^+(r_t)| = 3 \neq 2 = |N_D^+(q_t)|$.

Subcase 1.2: Suppose $u \in S_{t_i}$ and $v \in S_{t_j}$ for some $t_i, t_j \in T$ such that $t_i \neq t_j$. Then, $t_i \in N_D^+(u) \setminus N_D^+(v)$.

Case 2: Suppose $u \in (\bigcup\{S_t : t \in T\})$ and $v \in X'$. Let $t \in T$ be such that $u \in S_t$. Then, $t \in N_D^+(u) \setminus N_D^+(v)$.

Case 3: Suppose $u \in (\bigcup\{S_t : t \in T\})$ and $v \in T' \cup \{d_+\}$. Then, $d \in N_D^+(v) \setminus N_D^+(u)$.

Case 4: Suppose $u \in (\bigcup\{S_t : t \in T\})$ and $v = d_-$. Let $t \in T$ be such that $u \in S_t$. Then, $t \in N_D^+(u) \setminus N_D^+(v)$.

Case 5: Suppose $u, v \in X'$. Then, $u = x'_i$ and $v = x'_j$ for some $i, j \in [n]$. Since $u \neq v$, we have $i \neq j$. Then, $N_D^+(x'_i) = \{x_i\} \neq \{x_j\} = N_D^+(x'_j)$. So $N_D^+(u) \neq N_D^+(v)$

Case 6: Suppose $u \in X'$ and $v \in T' \cup \{d_+\}$. Then, $d \in N_D^+(v) \setminus N_D^+(u)$.

Case 7: Suppose $u \in X'$ and $v = d_-$. Then, $N_D^+(u) \neq N_D^+(v)$ because $|N_D^+(u)| = 1$ and $|N_D^+(v)| = 0$.

Case 8: Suppose $u, v \in T'$. Then, $u = t'_i$ and $v = t'_j$ for some $i, j \in [p]$. Since $u \neq v$, $i \neq j$. Recall that, by definition, for all $r, s \in [p]$, $t'_r t_s \in D$ if and only if $r < s$. So, if $i < j$, then $t_j \in N_D^+(t'_i) \setminus N_D^+(t'_j)$; and if $j < i$, then $t_i \in N_D^+(t'_j) \setminus N_D^+(t'_i)$. In any case, $N_D^+(u) \neq N_D^+(v)$.

Case 9: Suppose $u \in T'$ and $v = d_+$. Let $i \in [p]$ be such that $u = t'_i$. By definition, $t_i t'_i \in D$. So, $t_i \in N_D^+(d_+) \setminus N_D^+(t'_i)$.

Case 10: Suppose $u \in T' \cup \{d_+\}$ and $v = d_-$. Then, $d \in N_D^+(u) \setminus N_D^+(v)$.

Thus, if $u, v \in S$ are distinct, then $N_D^+(u) \neq N_D^+(v)$.

(ii) Let $u \in S$ and $v \in Q$. Recall that $S = \bigcup\{S_t : t \in T\} \cup X' \cup T' \cup \{d_+, d_-\}$ and $Q = X \cup T \cup \{d\}$. We consider two cases: $u \in \bigcup\{S_t : t \in T\} \cup X'$ or $u \in T' \cup \{d_+, d_-\}$.

Case 1: Suppose $u \in \bigcup\{S_t : t \in T\} \cup X'$.

Subcase 1.1: Suppose $v \in X \cup T$. Then, $d \in N_D^+(v) \setminus N_D^+(u)$.

Subcase 1.2: Suppose $v = d$. Then, $d_- \in N_D^+(v) \setminus N_D^+(u)$.

Case 2: Suppose $u \in T' \cup \{d_+, d_-\}$.

Subcase 2.1: Suppose $v \in X$. Since (X, T) is an instance of TOP*, there is a restriction $t = (a_t, b_t, c_t) \in T$ such that $v \in \{a_t, b_t, c_t\}$. Since $N(s_t^1) = Q_t = \{t, a_t, b_t, c_t\}$ and $N_D^+(s_t^1) = \{t\}$, we have $vs_t^1 \in D$. And since $N_D^+(u) \subseteq T \cup \{d\}$, $s_t^1 \in N_D^+(v) \setminus N_D^+(u)$.

Subcase 2.2: Suppose $v \in T \cup \{d\}$. Then, $N_D^+(u) \neq N_D^+(v)$ because u and v are adjacent.

Thus, if $u \in S$ and $v \in Q$, then $N_D^+(u) \neq N_D^+(v)$.

(iii) Let $u, v \in Q$. Then, $N_D^+(u) \neq N_D^+(v)$ because u and v are adjacent.

Thus, D is extensional.

We conclude that D is an eao of G_T. □

Finally, we apply all of the previous lemmas to prove the main theorem of this section.

Theorem 95. *Let (X, T) be an instance of* TOP**. Then, the split graph G_T, given by Construction 89, is a set graph if and only if there is a total order of X satisfying T.*

Proof. Suppose G_T is a set graph. Let D be an eao of G_T. Let $\prec$ be the relation on X such that, for all $x, y \in X$, $x \prec y$ if and only if $xy \in D$.

First, we prove that $\prec$ is a total order on X. Since D is an orientation, for all $x \in X$, $xx \notin D$, so $\prec$ is irreflexive. Suppose $x, y, z \in X$ are such that $x \prec y$ and $y \prec z$. Then, $xy \in D$ and $yz \in D$. Since X is a clique, x and z are adjacent. So, $xz \in D$ or $zx \in D$. But, if $zx \in D$, we would have the cycle $xyzx$ in D, a contradiction. Thus, $xz \in D$, and so $x \prec z$. Therefore, $\prec$ is transitive. Since $\prec$ is irreflexive and transitive, $\prec$ is an order on X. Since X is a clique, for every $x, y \in X$, either $xy \in D$ or $yx \in D$. Thus, $\prec$ is a total order on X.

Next, we prove that $\prec$ satisfies every betweenness restriction of T. Let $t = (a, b, c) \in T$. Since D is an eao of G_T, then, by Lemma 92, $D_t := D \cap (S_t \cup Q_t)^2$ is an eao of G_t. By Lemma 93, t is a sink of D_t and q_t, r_t are sources of D_t. Then, by Lemma 88, either $\{ab, bc\} \subseteq D_t \subseteq D$ or $\{cb, ba\} \subseteq D_t \subseteq D$. Then, either $a \prec b \prec c$ or $c \prec b \prec a$. So $\prec$ satisfies t. Since t is arbitrary, $\prec$ satisfies all of the restrictions in T.

Conversely, suppose (X, T) is satisfied by some total order $\prec$. Then, by Lemma 94, G_T is a set graph. This completes the proof. □

Theorem 96. EAO *restricted to split graphs is NP-complete.*

Proof. First we prove that EAO is in NP. Let D be an orientation of a graph $G = (V, E)$. If D is acyclic, then it has a sink $s \in V$ and $D[V \setminus \{s\}]$ is an acyclic orientation of $G \setminus \{s\}$. Conversely, if $s \in V$ is a sink of D and $D[V \setminus \{s\}]$ is an acyclic orientation of $G \setminus \{s\}$, then D must be acyclic. Therefore, acyclicity can be verified recursively in linear time. Extensionality can be verified in quadratic time by comparing all pairs of out-neighborhoods in the orientation. Hence, EAO is in NP.

Secondly, we prove that EAO restricted to split graphs is NP-hard. Let (X, T) be an instance of TOP*, and let G_T be the split graph given by Construction 89. By Theorem 95, G_T is a set graph if and only if there is a total order of X satisfying T. Moreover, by definition, G_T is a split graph with $2|X| + 8|T| + 3$ vertices. Thus, Construction 89 is a polynomial-time reduction of TOP* to EAO restricted to split graphs. Since, by Lemma 84, TOP* is NP-complete, we conclude that EAO restricted to split graphs is NP-complete. □

Chapter 7

Conclusion

In this dissertation, we have studied the time complexity of the SET GRAPH RECOGNITION problem (EAO). We have developed an approach for recognizing set graphs in the class of cographs, and have proved that EAO restricted to split graphs is NP-complete. In this chapter, we discuss how these results have been and can be extended further.

In Chapter 2 we have described the class of set graphs, as defined by A. Tomescu, and have discussed some of its basic properties. In particular, we have shown the necessity of the cut-set condition and the same neighbors condition for being a set graph. A question that remains open from this discussion is that of finding possible characterizations of the graphs satisfying both the cut-set condition and the same neighbors condition that are not set graphs.

In Chapter 3, we have introduced the two main concepts on which we base our approach for recognizing set graphs in the class of cographs: the layered extensional acyclic orientation and the set-deficiency. Then, by generalizing some lemmas from Chapter 2 to the context of leaos, we have proved some basic results concerning set-deficiencies. In particular, we have shown that if X is a module of a graph G, then $\mathcal{S}_\Delta(G[X])$ is bounded above by $\mathcal{S}_\Delta(G)+|N(X)\setminus X|$. If, however, X is not a module, $\mathcal{S}_\Delta(G[X])$ is not similarly bounded. In fact, we have presented an example of a graph G and a subset $X \subset V(G)$ such that $\mathcal{S}_\Delta(G[X]) = \mathcal{S}_\Delta(G)+2^{|N(X)\setminus X|}$ in Example 41. It remains an open problem to find a tight upper bound for $\mathcal{S}_\Delta(G[X])$ in terms of $\mathcal{S}_\Delta(G)$ and $|N(X) \setminus X|$, i.e., to find a minimal function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for every graph G and every subset $X \subseteq V(G)$, $\mathcal{S}_\Delta(G[X]) \leq f(\mathcal{S}_\Delta(G), |N(X) \setminus X|)$. We conjecture that $(\mathcal{S}_\Delta(G) + 1)2^{|N(X)\setminus X|} - 1$ is a tight upper bound for $\mathcal{S}_\Delta(G[X])$, but further thought on this is needed.

In Chapter 4, we have presented how a minimum leao can be constructed for the join of two disjoint graphs from minimum leaos of the two given graphs. By applying Theorem 57 of Chapter 4, one is able to solve the following problem in linear time:

SET-DEFICIENCY OF JOINS (SDJ)
INSTANCE: $(G_1, G_2, \mathcal{S}_\Delta(G_1), \mathcal{S}_\Delta(G_2))$, where G_1 and G_2 are disjoint graphs.
TASK: Compute $\mathcal{S}_\Delta(G_1 \wedge G_2)$.

One may wonder if similar results can be obtained for other graph operations such as, for instance, the cartesian product of graphs. It is left as a future work to determine whether the following problem can be solved in polynomial time:

SET-DEFICIENCY OF CARTESIAN PRODUCTS
INSTANCE: $(G_1, G_2, \mathcal{S}_\Delta(G_1), \mathcal{S}_\Delta(G_2))$, where G_1 and G_2 are disjoint graphs.
TASK: Compute $\mathcal{S}_\Delta(G_1 \times G_2)$.

The more problems of this type we can solve in polynomial time, the larger the class of graphs in which set graphs can be recognized in polynomial time.

Another operation of interest is that of vertex substitution. Given two disjoint graphs G_1 and G_2, and a vertex $v \in V(G)$, the substitution of v for G_2 in G_1 is obtained by removing v from G_1 and joining every vertex of G_2 to every vertex of $N_{G_1}(v)$, and is denoted by $G_1(v \to G_2)$. A. Tomescu has proved, in [14], that the class of set graphs is closed under vertex substitutions, i.e., if G_1 and G_2 are set graphs and $v \in V(G_1)$, then $G_1(v \to G_2)$ is a set graph. In the context of set-deficiencies, we have the following problem.

SET-DEFICIENCY OF VERTEX SUBSTITUTIONS (SDS)
INSTANCE: $(G, H, v, \mathcal{S}_\Delta(G), \mathcal{S}_\Delta(H))$, where G and H are disjoint graphs and $v \in V(G)$.
TASK: Compute $\mathcal{S}_\Delta(G(v \to H))$.

It can be shown that, if there is a polynomial-time algorithm that solves SDS, then set graphs can be recognized in polynomial time in the class of split graphs. However, by Theorem 96, EAO restricted to split graphs is NP-complete. Hence, SDS is NP-hard. More details and the complete proofs of these claims will be presented in a subsequent publication, currently in preparation.

In the first two sections of Chapter 5, we have presented how a minimum leao can be constructed for a given graph from single-sink minimum leaos of each connected component of the graph. In Section 5.3, we have briefly shown that not every graph admits a single-sink minimum leao. Hence, it is left for a future work to find a polynomial-time algorithm for computing, in general, the set-deficiency of a graph from the given set-deficiency of each connected component, i.e., an algorithm that solves the following general problem:

SET-DEFICIENCY OF UNIONS (SDU)
INSTANCE: $(G, CC(G), \mathcal{S}_\Delta|_{CC(G)})$, where G is a graph, $CC(G)$ is the set of connected components of G and $\mathcal{S}_\Delta|_{CC(G)} : CC(G) \to \mathbb{N}$ is the function such that, for

every $H \in CC(G)$, $\mathcal{S}_\Delta|_{CC(G)}(H) = \mathcal{S}_\Delta(H)$.
TASK: Compute $\mathcal{S}_\Delta(G)$.

In Section 5.4, we have applied our polynomial-time algorithms, for solving SDJ and partially solving SDU, to obtain a polynomial-time algorithm for recognizing set graphs in the class of cographs. This solution relies on a well-known structural characterization of cographs in terms of disconnected and co-disconnected graphs. A larger class of graphs, that of P_4-sparse graphs, admits a similar characterization (see [8]), in terms of disconnected, co-disconnected graphs and *spiders*. Using this characterization, we were able to extend our methods, for recognizing set graphs in polynomial time, to the class of P_4-sparse graphs. The details of this additional result are currently being written, and will be presented in a subsequent publication.

In Chapter 6, we have proved that the recognition of set graphs in the class of split graphs is an NP-complete problem. Unresolved questions tangential to this have appeared, but of a general interest and not particular to the study of set graphs. The main proof, given by Opatrny in [12], that TOP is NP-complete uses a reduction from the problem of 2-COLORABILITY OF HYPERGRAPHS OF RANK 3. Opatrny, then, points to a paper by L. Lovász [9] as a reference for the NP-completeness of 2-COLORABILITY OF HYPERGRAPHS OF RANK 3. The book [7], by Garey and Johnson, reaffirms the NP-completeness of 2-COLORABILITY OF HYPERGRAPHS OF RANK 3 (see the problem named SET SPLITTING, in [7]), and also points to [9] for a proof. However, the proof by L. Lovász, in [9], only applies to hypergraphs of rank strictly greater than 3. Hence, the time complexity of the 2-COLORABILITY OF HYPERGRAPHS OF RANK 3 remains elusive.

In some classes of graphs, in which the problem of SET GRAPH RECOGNITION can be solved in polynomial time, the more general problem, of determining the set-deficiency of the graph, remains of unknown complexity. For instance, Corollary 26 states that a tree is a set graph if and only if it is a path. So, SET GRAPH RECOGNITION in the class of trees is trivial. But it remains an interesting open problem to find an efficient method for computing the set-deficiency, or a minimum leao, of an arbitrary tree.

The class of set graphs has been an interesting and rich object of study. As we have pointed in this final chapter, the theory of layered extensional acyclic orientations can be extended much further, and has already allowed us to better understand the SET GRAPH RECOGNITION problem.

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