Noname manuscript No.
(will be inserted by the editor)

A Global Interior-Point Method for Non-convex Geometric Programming

Roberto Quirino do Nascimento
· Rubia Mara de Oliveira Santos ·
Nelson Maculan

Received: date / Accepted: date

Abstract In this paper we solve the non-convex or signomial geometric programming problem. The strategies found in the literature to solve this problem are basically branch and bound or condensation methods that locally transform the problem into a convex problem. The presented strategy differs substantially from the existing ones, since we formulate the problem as an optimization problem of the difference of convex functions in its standard form. The necessary and sufficient conditions for the existence of global solutions have also been developed. The existing challenge in the standard form is due to a constraint $g(t) \geq 1$ with g convex. Such difficulty is circumvented by using the classical inequality between the weighted arithmetic and harmonic means, which allows us to write the DC optimality conditions as a convex geometric programming problem, and to use a primal dual predictor-corrector interior-point method to solve it, using the predictor phase to update weights. The interior-point method solves the dual geometric programming problem and the exponential transformation finds the primal solution. We developed the algorithm on the `Fortran 90` language and applied a set of test problems from the literature for validation. The proposed method solved all evaluated problems and the computational results are presented along with the solutions

Keywords Global Optimization · Difference of Convex Functions · Geometric Programming · Interior Point Method

Mathematics Subject Classification (2010) MSC 90C26 · MSC 90C30 · 90C99

1 Introduction

Geometric Programming (GP) is a generally non-convex optimization technique well known and applied in real-world problems. Such problems can be written as an optimization problem where the objective function and each of its constraints are the difference between two convex functions. Problems with

Depto. of Scientific Computing
Federal University of Paraiba (UFPB) - Brasil
E-mail: roberto@ci.ufpb.br · Institute of Mathematics
Federal University of Mato Grosso do Sul (UFMS) - Brasil
E-mail: rubia.oliveira@ufms.br · Coppe Sistemas
Federal University of Rio de Janeiro (UFRJ) - Brasil
E-mail: maculan@cos.ufrj.br

such objective function and constraints are known in the literature as DC Optimization. Under this class of problems, it is possible to establish necessary and sufficient conditions to obtain global minima under certain assumptions. The goal of this paper is to present a strategy to solve GP problems using the DC optimization theory. Firstly, the problem is converted into a DC problem in canonical form, using the inequality between the arithmetic mean and the weighted harmonic mean. We locally transform the problem into a convex GP problem and use a primal dual predictor-corrector interior-point method to find the optimality conditions for a DC problem in the canonical form. In the correcting phase, besides updating the barrier parameter, we also update the peaks and consequently the dual residuals. This paper is organized as follows: section 2 introduces the GP problem, the DC optimization problem, the canonical form, and the global optimality conditions; we also write the GP problem as a DC optimization problem in canonical form and the global optimality conditions for the GP problem; section 3 presents a canonical form similar to the canonical DC form for GP in the "posynomial" form and we transform the reverse constraint into a posynomial constraint; section 4 presents the dual GP problem; sections 5 and 6 present the convex linearly constrained differentiable programming, which is a particular case of the dual PG problem and its optimality conditions, respectively; section 7 describes the primal dual predictor-corrector interior-point method, where the corrector step is fundamental for the update of the weights and the requirement of the global optimality conditions; a proof of weak convergence is also presented in the section; in Section 8 we present the computational results from some problems found in the literature; the concluding remarks are presented in section 9; all the evaluated problems and the solutions are detailed in the Appendix (section 10).

2 Geometric programming and DC optimization

Geometric programming is an optimization technique to solve algebraically nonlinear optimization problems. It was developed in the early 1960s by Clarence Melvin Zener (1905 - 1993), Richard J. Duffin (1909 - 1996) and Elmor L. Peterson(1938). In 1961 Zener, using the inequality between the arithmetic and geometric mean for positive numbers, discovered a simple way to minimize a special class of functions called posynomials that were defined as follows:

$$g(t) = \sum_{i=1}^{n} c_i \prod_{j=1}^{m} t_j^{a_{ij}}$$

In a posynomial function $c_i > 0,\ t_j > 0,\ a_{ij}\ \in \mathbf{R}$ and the existing parts of function g are called monomials. By transforming $t = e^x,\ x \in \mathbf{R}$ we make function g convex. When at least one c_i in function g is negative, we lose convexity so that many authors call this case a *signomial function.* Optimization problems characterized by objective functions or constraints involving signomial functions are called *Signomial Geometric Problems (SGP).* For some cases of those problems, it is only possible to determine a local minimum, hence the need to investigate methodologies that allow finding global solutions. Formally a SGP is defined as follows:

$$\textbf{SGP} \qquad V_{SGP} := \textit{Minimize } g_0(t) \tag{1}$$

$$\textit{Subject to}:$$

$$g_k(t) \leq 1, \quad k = 1, \ldots, p \tag{2}$$

$$\textit{where } g_k(t) = \sum_{i \in J[k]} \sigma_i c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \qquad k = 0, \ldots, p \tag{3}$$

$$J[k] = \{m_k, m_{k+1}, \ldots, n_k\} \qquad k = 0, \ldots, p \tag{4}$$

$m_0 = 1, \quad m_1 = n_0 + 1, m_2 = n_1 + 1, \ldots, m_p = n_{p-1} + 1,\ n_p = n$, exponents a_{ij} are arbitrary constants, coefficients c_i are positive constants, functions g_k are named signomial functions and terms $c_i \prod_{j=1}^{m} t_j^{a_{ij}}$ in the problem are called monomials, variables t_j are primal variables, when $\sigma_i = 1$ for all i there is a posynomial GP problem.

Posynomial GP problem was the initial problem developed by Duffin (1967) which can be transformed into a convex problem by changing variables $t_j = e^{z_j}$, $z_j \in \mathbf{R}$, $j = 1, \ldots, m$ and for which several methods have been developed, see [2], [9],[10]. For the signomial (non-convex) case strategies for determining global solutions can be found in [7], [11], [15], [19] and [29]. In [28] although a proper global optimization technique is not used, the authors adopted an interior-points method making several setups of the algorithm with different starting points. In [12] SGP problems are used as tests to validate a methodology called Continuous General Variable Neighborhood Search (CGVNS). A well-structured review is performed in [7], which recounts strategies for solving signomial problems. Among all the references studied, no approach was found exploring the optimality conditions for global minima, existing in DC optimization, and this became the motivation to develop this work. In the following, we present the fundamentals of DC optimization needed to develop this work.

Definition 1 *A function f defined on a convex set $X \subseteq \boldsymbol{R}^n$ is a difference of convex functions (DC) on X if, for every $x \in X$, there exist convex functions $g, h : X \to \boldsymbol{R}$ such that:*

$$f(x) = g(x) - h(x)$$

Definition 2 *An optimization problem is called a DC optimization problem if it is written in the form:*

$$\begin{array}{lll} \textbf{DC} & \textit{Minimize} & f_0(x) \\ & \textit{Subject to:} & f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & & x \in X \end{array}$$

where $f_0, \ldots, f_m$ are DC functions.

Definition 3 *A DC optimization problem is written in canonical DC (CDC) if it is written in the form:*

$$\textbf{CDC} \qquad \text{Minimize} \quad c^T x$$
$$\text{Subject to: } g(x) \geq 0, \quad x \in D \tag{5}$$

where $c \in \boldsymbol{R}^n, g : \boldsymbol{R}^n \to \boldsymbol{R}$ *is convex, and* D *is a compact convex subset of* $\boldsymbol{R}^n$*. The constraint* $g(x) \geq 0$ *given in (5) is called reverse convex constraint or simply reverse constraint.*

Proposition 1 *Let* f*,* $f_i(i = 1, \ldots, m)$ *DC functions. Then the following are also DC functions*

1. $\sum_{i=1}^{m} \lambda_i f_i(x)$ *for some real number* λ_i*;*
2. $max_{i=1,\ldots,m} f_i(x)$ *and* $min_{i=1,\ldots,m} f_i(x)$ *;*
3. $|f(x)|$
4. $f^+ := max\{0, f(x)\}, f^- := min\{0, f(x)\}$*;*
5. *product* $\prod_{i=1}^{m} f_i(x)$*.*

Proof See [8].

Consider the canonical DC problem and let F be the feasible solutions set given by:

$$F = \{x \in D : g(x) \geq 0\},$$

it is assumed in this paper that the following assumptions are satisfied:

(**A1**) **Assumption** A_1 The inner set D given in (5) is non-empty, $intD \neq \emptyset$.
(**A2**) **Assumption** A_2 There exists $x^0 \in D$ such that:

$$g(x^0) < 0 \quad \text{e} \quad c^T x^0 < min\left\{c^T x : x \in D, g(x) \geq 0\right\}. \tag{6}$$

(**A3**) **Assumption** A_3 $F = cl\,(intF)$.

Definition 4 *Constraint* $g(x) \geq 0$ *given in (5) is said to be essential if* **Assumption** A_2 *is satisfied.*

Based on those assumptions one has an initial result on global minima for DC problems.

Proposition 2 *Suppose that set* $F \neq \emptyset$*, under* **Assumption** A_2*, there exists an optimal solution* x^* *to the CDC problem satisfying:*

$$x^* \in \partial\{x \in \boldsymbol{R}^n; g(x) \geq 0\} \cap \partial D$$

where ∂S *stands for the boundary of set* S*.*

Proof See [8].

The necessary and sufficient global optimality conditions for problems in the CDC form are presented in the following.

Theorem 1 *Necessary Optimality Condition for the CDC problem.*
Under **Assumption** A_1 *and* **Assumption** A_2*. If* x^* *is an optimal solution to the CDC problem, then*

$$max\left\{g(x); x \in D, c^T x \leq c^T x^*\right\} = 0$$

Proof See [8].

Theorem 2 *Sufficient Optimality Condition for the CDC problem.*
Under **Assumption** A_1 *and* **Assumption** A_2*. If* $S \supseteq F$ *and* $x^* \in F$ *is such that*

$$max\left\{g(x) : x \in S, c^T x \leq c^T x^*\right\} = 0.$$

Then x *is an optimal solution for the CDC problem.*

Proof See [8].

Proposition 3 *Every SGP Problem can be transformed into a CDC optimization problem.*

Proof For each $k = 0, \ldots, p$ consider set $J[k]$ in (4) and define the following sets:

$$J_+[k] = \{i \in J[k]; \sigma_i = 1\}, \tag{7}$$

$$J_s[k] = \{i \in J[k]; \sigma_i = -1\} \tag{8}$$

by changing variables $t_j = e^{z_j}$, where $f(z) = e^z$ is the exponential function, $z_j \in \mathbf{R}$, and $j = 1, \ldots, m$, one can write functions $g_k(t)$ $k = 0, \ldots, p$ defined in (3) as follows:

$$g_k(z) = \sum_{i \in J_+[k]} c_i e^{\sum_{j=1}^m a_{ij} z_j} - \sum_{i \in J_s[k]} c_i e^{\sum_{j=1}^m a_{ij} z_j}$$

since $c_i > 0$, one can observe that $g_k(z)$ is a difference of convex functions.

■

It has been proved that the SGP problem is an optimization DC problem. The next step is to prove that problem SGP can be transformed into a CDC problem. We initially have the following results:

Theorem 3 *Given the SGP problem, there are posynomial functions* $f_0, f_1, \ldots, f_p$ *and* h *that make it equivalent to the following problem:*

$$\begin{array}{lll} \textbf{CDCG} & \textit{Minimize} & t_0 \\ & \textit{Subject to}: & h(t_0, t) \geq 1 \quad t_0, t \in D \end{array}$$

$$where\ D \subset \boldsymbol{R}^{m+1}\ is\ defined\quad by: \quad D = D_1 \cap D_2\ and,$$

$$D_1 = \left\{(t_0, t) \in \boldsymbol{R}^{m+1}_{++}; \quad Max\{f_k(t_0, t), k = 1, \ldots, p\} \leq 1\right\} \tag{9}$$

$$D_2 = \left\{t \in \boldsymbol{R}^m_{++}; 0 < l_j \leq t_j \leq U_j\right\} \tag{10}$$

Proof Let $\sigma = \min\{\sigma_i; i \in J[0]\}$, as we do not know the sign of function g_0 and we want the objective function to be linear, we can rewrite the SGP problem as follows:

$$\begin{array}{llrll} \textbf{SGP1} & \textit{Minimize} & t_0 & & \\ & \textit{Subject to:} & & & \\ & & g_0(t) & \leq & t_0^{\frac{3+\sigma}{4}} - \dfrac{1-\sigma}{2} t_0^{-\frac{3+\sigma}{4}}, \qquad (11) \\ & & g_k(t) & \leq & 1, \qquad k = 1, \ldots, p. \\ & & t_0 > 0, & & 0 < l_j \leq t_j \leq U_j, \quad j = 1, \ldots, m \end{array}$$

$g_k(t)$ being defined in (3).

When $\sigma = 1$ the objective function is posynomial, so $\min\{g_0(t), t > 0\}$ is equivalent to the problem: $\min\{t_0; g_0(t) \leq t_0, \;\; t_0, t > 0\}$. When $\sigma = -1$ the objective function is signomial, then problem $\min\{g_0(t), t > 0\}$ is equivalent to the problem $\min\left\{t_0; g_0(t) \leq t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}}, t_0, t > 0\right\}$. One may notice that function $f : \mathbf{R}_{++} \to \mathbf{R}$ defined by $f(x) = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$ is strictly increasing, $\lim_{x \to 0} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$.

Given the sets $J_+[k], J_s[k],\; k = 0, \ldots, p$, defined in (7) and (8) respectively, the following sets are defined:

$$J_{sig} = \cup_{k=0}^{p} J_s[k]; \;\; J_s^c[k] = J_{sig} \backslash J_s[k]; \;\; J_{s0}^c[k] = J_{sig} \backslash \left(J_s[0] \cup J_s[k]\right)$$

where $J_{sig} \backslash J_s[k] = \{\sigma_i; \sigma_i \in J_{sig},\; \sigma_i \notin J_s[k]\}$. Now any constraint of the SGP problem is written as follows:

$$g_k(t) = \sum_{i \in J_+[k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} + \sum_{i \in J_s^c[k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} - \sum_{i \in J_{sig}} c_i \prod_{j=1}^{m} t_j^{a_{ij}},$$

constraint (11) is equivalent to:

$$\overline{g}_0(t_0, t) = t_0^{-\frac{3+\sigma}{4}} \sum_{i \in J_+[0]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} + \frac{1-\sigma}{2} t_0^{-\frac{3+\sigma}{2}} + \sum_{i \in J_s^c[0]} c_i \prod_{j=1}^{m} t_j^{a_{ij}}$$

$$-t_0^{-\frac{3+\sigma}{4}} \sum_{i \in J_s[0]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} - \sum_{i \in J_s^c[0]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} \leq 1.$$

Doing

$$\begin{aligned} \overline{f}_0(t_0, t) = \;& t_0^{-\frac{3+\sigma}{4}} \sum_{i \in J_+[0]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} + \sum_{i \in J_s^c[0]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} \\ & + \tfrac{1-\sigma}{2} t_0^{-\frac{3+\sigma}{2}} \end{aligned} \qquad (12)$$

$$\overline{f}_k(t_0, t) = \sum_{i \in J_+[k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} + t_0^{-\frac{3+\sigma}{4}} \sum_{i \in J_s[0]} c_i \prod_{j=1}^{m} t_j^{a_{ij}}$$

$$+ \sum_{i\in J_{s0}^c[k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} \qquad k = 1,\dots,p \tag{13}$$

$$\overline{h}(t_0,t) = t_0^{-\frac{3+\sigma}{4}} \sum_{i\in J_s[0]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} + \sum_{i\in J_s^c[0]} c_i \prod_{j=1}^{m} t_j^{a_{ij}}$$

it can be stated that

$$\overline{g}_0(t_0,t) = \overline{f}_0(t_0,t) - \overline{h}(t_0,t) \tag{14}$$

$$g_k(t) = \overline{f}_k(t_0,t) - \overline{h}(t_0,t) \quad k = 0,\dots,p. \tag{15}$$

One may now write the constraints given in (2) and (11) as a single constraint:

$$max\,\{\overline{g}_0(t_0,t), g_k(t_0,t), k = 0,\dots,p\} \le 1, \tag{16}$$

applying (14) and (15) in (16) one has,

$$max\,\{\overline{g}_0(t_0,t), g_k(t_0,t)\} = max\left\{\overline{f}_k(t_0,t), k = 0,\dots,p\right\} - \overline{h}(t_0,t) \le 1. \tag{17}$$

now the inequality given in (17) will be written as:

$$max\left\{\overline{f}_k(t_0,t), k = 0,\dots,p\right\} \le 1 + \overline{h}(t_0,t).$$

which will be separated into two inequalities, using the auxiliary variable z resulting in:

$$\left(\overline{h}(t_0,t) + 1\right) z^{-1} \ge 1,$$

$$\overline{f}_k(t_0,t) z^{-1} \le 1, \qquad k = 0,\dots,p,$$

eventually, by making $z = t_{m+1}$, one writes the SGP1 problem as follows:

$$\textit{Minimize} \quad t_0$$

$$\textit{Subject to}: h(t_0,t) \ge 1 \quad t_0, t \in D_1 \cap D_2.$$

where D_1 and D_2, are defined in (9) and (10) respectively, and

$$f_k(t_0,t) = \overline{f}_k(t_0,t) t_{m+1}^{-1}, \qquad k = 0,\dots,p$$

$$h(t_0,t) = \left(\overline{h}(t_0,t) + 1\right) t_{m+1}^{-1}.$$

where $\overline{f}_0$ and $\overline{f}_k$ are given in (12) and (13).

■

Lemma 1 *Hölder Inequality*
Given $a_i, b_i > 0, p > 1, q > 0,\ i = 1,\dots,n$ *with* $\frac{1}{p} + \frac{1}{q} = 1$

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}. \tag{18}$$

Lemma 2 *Function* $f(z) = log\left(\sum_{i\in J_+[k]} c_i e^{\sum_{j=1}^{m} a_{ij}z_j}\right)$ *is convex.*

Proof Given $z, w \in \mathbf{R}^m$ and $\lambda \in (0,1)$,

$$f(\lambda z + (1-\lambda)w) = log\left(\sum_{i\in J_+[k]} c_i^{(1-\lambda)} c_i^{\lambda} e^{\lambda \sum_{j=1}^{m} a_{ij}z_j} e^{(1-\lambda)\sum_{j=1}^{m} a_{ij}w_j}\right)$$

Making $a_i = c_i^{\lambda} e^{\lambda \sum_{j=1}^{m} a_{ij}z_j}$, $b_i = c_i^{(1-\lambda)} e^{(1-\lambda)\sum_{j=1}^{m} a_{ij}w_j}$, $p = \frac{1}{\lambda}, q = \frac{1}{1-\lambda}$ and applying Hölder (18) inequality, we have:

$$f(\lambda z + (1-\lambda)w) \le \lambda log\left(\sum_{i\in J_+[k]} a_i^{1/\lambda}\right) + (1-\lambda) log\left(\sum_{i\in J_+[k]} b_i^{1/(1-\lambda)}\right)$$

From the definition of a_i and b_i we conclude the result.

■

Proposition 3, Theorem 3, and Lemma 2 enable us to write the SGP problem as an optimization DC problem in canonical form:

$$\textbf{SGPDC} \quad \begin{array}{ll} \textit{Minimize} & c^t x \\ \textit{Subject to}: & h_0(x) \ge 0. \\ & g_k(x) \le 0, \qquad k = 1, \ldots, p. \\ & l_j \le x_j \le U_j, \ j = 1, \ldots, m \end{array}$$

3 Posynomials functions maximization and reverse constraints

Let us now consider a GP problem in the form:

$$\textbf{SGP1} \quad \begin{array}{ll} \textit{Minimize} & t_0 \\ \textit{Subject to}: & g_0(t) \le t_0. \\ & g_k(t) \le 1, \qquad k = 1, \ldots, p-1. \\ & h(t) \ge 1. \end{array}$$

$$t_0 > 0, t \in D \subset \mathbf{R}_{++}^m \ \textit{compact}.$$

where

$$g_k(t) = \sum_{i\in J[k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}} \quad k = 0, 1, \ldots, p-1, \tag{19}$$

$$h(t) = \sum_{i\in J[p]} c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \ c_i > 0 \ a_{ij} \in \mathbf{R}. \tag{20}$$

Let us now consider simultaneously the following problems:

PMIN		**PMAX**	
	Minimize $g_0(t)$		*Minimize* $\frac{1}{h(t)}$
	Subject to :		*Subject to* :
	$g_k(t) \le 1,$		$g_k(t) \le 1,$
	$t \in D \subset \mathbf{R}^m_{++}$ *compact.*		$\frac{1}{h(t)} \le 1.$

Our efforts will now be targeted to transform $\frac{1}{h(t)}$ into a posynomial function, for which we will use the inequality between arithmetic mean and harmonic mean, establishing the following result:

Proposition 4 *Let* $x_1, x_2, \ldots, x_n \in \boldsymbol{R}^n_{++}$ *then,*

$$\frac{1}{\sum_{i=1}^{n} x_i} \le \sum_{i=1}^{n} \frac{\omega_i^2}{x_i}$$

where $\sum_{i=1}^{n} \omega_i = 1, \omega_i > 0$*, the equality being valid if and only if*

$$\omega_i = \frac{x_i}{\sum_{i=1}^{n} x_i}$$

.

Proof Consider function $H : \mathbf{R}^n_{++} \to \mathbf{R}$ given by

$$H(\omega) = \sum_{i=1}^{n} \frac{\omega_i^2}{x_i} - \frac{1}{\sum_{i=1}^{n} x_i}$$

and the following optimization problem:

$$\begin{array}{ll} \textit{Minimize} & H(\omega) \\ \textit{Subject to}: & \sum_{i=1}^{n} \omega_i = 1, \omega_i > 0. \end{array}$$

This problem has a unique solution given by $\omega_i^* = \dfrac{x_i}{\sum_{i=1}^{n} x_i}$, and $H(\omega^*) = 0$. Then $H(\omega) \ge 0$ for all ω such that $\sum_{i=1}^{n} \omega_i = 1, \omega_i > 0$ thus proving the result.

■

From proposition 4 and from the expression of the reverse function h given in (20), we can now write the following inequality:

$$\frac{1}{h(t)} \le \sum_{i \in J[p]} \frac{\omega_i^2}{c_i \prod_{j=1}^{m} t_j^{a_{ij}}} = \sum_{i \in J[p]} \frac{\omega_i^2}{c_i} \prod_{j=1}^{m} t_j^{-a_{ij}} = H_\omega(t)$$

$\sum_{i=1}^{n_p} \omega_i = 1,\ \omega_i > 0$ where n_p is the cardinality of the set $J[p]$, and the equality being valid if and only if:

$$\omega_i = \frac{c_i \prod_{j=1}^{m} t_j^{a_{ij}}}{h(t)} \tag{21}$$

We can now write problems PMIN and PMAX as a single geometric programming problem:

$$\textbf{PMINMAX} \qquad \textit{Minimize} \quad \alpha t_0 \tag{22}$$

$$\begin{aligned} \textit{Subject to}: \; & g_0(t) \leq t_0 \\ & g_k(t) \leq 1, \qquad k = 1, \ldots, p-1 \\ & H_\omega(t) \leq 1 \qquad (23) \\ & H_\omega(t) \leq \alpha \qquad (24) \end{aligned}$$

where $g_k(t)$ *is given in* (*19*),

$$H_\omega(t) = \sum_{i \in J[p]} \frac{\omega_i^2}{c_i} \prod_{j=1}^{m} t_j^{-a_{ij}} \tag{25}$$

$$J[k] = \{m_k, m_{k+1}, \ldots, n_k\} \qquad k = 0, 1, \ldots, p+1 \tag{26}$$

The main result of this work can be stated:

Theorem 4 *Suppose that there exists* $\hat{t}$ *the optimal solution to the PMIN problem with* $h(\hat{t}) < 1$*, the PMINMAX problem has an interior feasible solution and its dual problem has a solution with positive coordinates. If* (t_0^*, t^*) *is an optimal solution to the PMINMAX problem,* ω^* *is obtained according to (21), and* $\alpha^* \geq 1$*, then this solution is a global optimal solution to the SGP1 problem.*

Proof Using the optimality conditions for GP (see [6], page 117), if x^* is the optimal solution of the dual problem of the PMINMAX problem, then:

1. $\displaystyle\sum_{i \in J[0]} x_i^* = 1,$
2. $\displaystyle\prod_{j=1}^{m} t_j^{*a_{ij}} = \frac{u(x^*)x_i^*}{c_i}, i \in i \in J[0],$ where $u(x)$ is the dual function,
3. $\displaystyle\prod_{j=1}^{m} t_j^{*a_{ij}} = \frac{x_i^*}{\lambda_k^* c_i}, i \in i \in J[k],$ where $\displaystyle\lambda_k = \sum_{i \in J[k]} x_i$
4. $\displaystyle\frac{1}{\alpha^*} \prod_{j=1}^{m} t_j^{*-a_{ij}} = \frac{x_i^* c_i}{\omega_i^{*2}}, i \in i \in J[p]$

adding each parcel of item (4) with respect to i, we have:

$$\sum_{i \in J[p]} \frac{\omega_i^{*2}}{c_i} \prod_{j=1}^{m} t_j^{*-a_{ij}} = \alpha^* \cdot \sum_{i \in J[p]} x_i^* = \alpha^*,$$

$\omega_i^* = \dfrac{c_i \prod_{j=1}^{m} t_j^{*a_{ij}}}{h(t^*)}$, $H_\omega(t^*) = \alpha^*$, as $H_\omega(t^*) \leq 1$ we conclude that $h(t^*) \geq 1$, $\alpha^* = 1$ and $g_k(t^*) \leq 1$. By the GP optimality conditions, we have: $v(x^*) = t_0^* = g_0(t^*) = \text{Min} g_0(t), \text{Max} h(t) = h(t^*) = 1$. Therefore, (t_0^*, t^*) satisfies the following condition:

$$max\left\{h(t) : \{t \in D, g_0(t) \leq t_0, g_k(t) \leq 1, h(t) \geq 1\}, t_0 \leq t_0^*\right\} = 1.$$

Since PMINMAX has an inner feasible solution, $\hat{t}$ is such that $h(\hat{t}) < 1$, and $\hat{t}$ is an optimal solution for PMIN in the set $\{t \in D, g_k(t) \leq 1, h(t) < 1\}$, since assumptions **A1, A2** (given in (6)) are satisfied, by Theorem 2 we may conclude that (t_0^*, t^*) is a global solution to the SGP1 problem.

■

4 The primal dual pair of the Geometric Programming

According to [6] and [10] the Primal problem of the posynomial geometric problem is defined as follows:

$$\begin{aligned}
\mathbf{GP} \qquad V_{GP} := \textit{Minimize} \quad & g_0(t) \\
\textit{Subject to}: \quad & g_k(t) \leq 1 \quad k = 1, \ldots, p \\
& t > 0. \\
\textit{where} \quad & g_k(t) = \sum_{i \in J[k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \quad j = 1, \ldots, m \qquad (27)\\
J[k] = \{m_k, m_{k+1}, \ldots, n_k\}, \quad & k = 0, \ldots, p
\end{aligned}$$

$m_0 = 1, \quad m_1 = n_0 + 1, m_2 = n_1 + 1, \ldots, m_p = n_{p-1} + 1, \quad n_p = n.$
Exponents a_{ij} are arbitrary constants, coefficients c_i are positive.

The Dual Geometric (GD) Programming problem is defined as follows:

$$\begin{aligned}
\mathbf{GD} \qquad V_{GD} := \textit{Maximize}\ & u(x) \\
\textit{Subject to}: \ & \sum_{i \in J[0]} x_i = 1, \\
& \sum_{i=1}^{n} a_{ij} x_i = 0, \quad x_i \geq 0 \quad j = 1, 2, \ldots, m \\
\textit{where}, \quad & u(x) = \prod_{i=1}^{n} \left(\frac{c_i}{x_i}\right)^{x_i} \prod_{k=1}^{p} \lambda_k^{\lambda_k}, \qquad (28)\\
& \lambda_k = \sum_{i \in J[k]} x_i.
\end{aligned}$$

Consider function $f(x) = \log(u(x))$ where u is given in (28) then:

1) $$f(x) = \sum_{i=1}^{n} x_i \log\left(\frac{c_i}{x_i}\right) + \sum_{k=1}^{p} \lambda_k \log(\lambda_k); \qquad (29)$$

2) f *is concave* ;

3) *If* $\lambda_0 = 1 \qquad f(\alpha x) = \alpha f(x) \quad \alpha > 0$;

4) *If* $x \in \mathbf{R}^{\mathbf{n}}_{++}$

$$\frac{\partial f}{\partial x_i}(x) = \begin{cases} \log\left(\frac{c_i}{x_i}\right) + 1 & \text{if} \quad i \in J[0] \\ \log\left(\frac{c_i \lambda_k}{x_i}\right) & \text{if} \quad i \in J[k] \quad k = 1, \dots, p \end{cases}$$

5) $f(x) = x^t \nabla f(x) + \lambda_0$

6) *The Hessian matrix de* f *is of the block diagonal form* :

$$\nabla^2 f(x) = \begin{bmatrix} H_0 & & & \\ & H_1 & & \\ & & \ddots & \\ & & & H_p \end{bmatrix} \tag{30}$$

where

$$H_0 = \begin{bmatrix} -\frac{1}{x_0} & & & \\ & -\frac{1}{x_1} & & \\ & & \ddots & \\ & & & -\frac{1}{x_{n_0}} \end{bmatrix} \text{ e } H_k = \begin{bmatrix} \frac{1}{\lambda_k} - \frac{1}{x_{m_k}} & \frac{1}{\lambda_k} & \cdots & \frac{1}{\lambda_k} \\ \frac{1}{\lambda_k} & \frac{1}{\lambda_k} - \frac{1}{x_{m_k+1}} & \cdots & \frac{1}{\lambda_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\lambda_k} & \frac{1}{\lambda_k} & \cdots & \frac{1}{\lambda_k} - \frac{1}{x_{n_k}} \end{bmatrix}$$

Definition 5 *The GP problem is said to be consistent if there exists a solution* $\tilde{t} \in \mathbf{R}^m_{++}$ *such that* $g_k(t) \leq 1,\ k = 1, \dots, p,$ *if* $\theta g_k(t) \leq 1,\ k = 1, \dots, p$ *for each* $\theta \in (0, 1)$ *GP is said subconsistent. The GD problem is said Canonical if there exists a solution* $\tilde{x} \in \mathbf{R}^n_{++}$ *such that* $A\tilde{x} = 0, \sum_{i \in J[0]} \tilde{x}_i = 1.$

Remark 1 *The GD problem belongs to the class of linearly constrained convex problems, for which there is an established theory, so solving the dual problem and converting it to the primal problem GP has been a recurrent strategy, see [6] page. 81 and [10], although many works adopt a primal strategy to solve non-convex problems, in this work we will also use this methodology.*

5 Linearly constrained interior-differentiable convex programming

In order to solve the GP problem we use the duality theory in GP by solving the dual problem GD, from which we find the solution of the primal problem GP, the dual problem can be written as the following problem:

$$\begin{aligned} \mathbf{P} \quad V_p &:= \mathit{Minimize} F(x) \\ &\mathit{Subject\ to} : Ax = b \quad x \in \mathbf{R}^n_+ \end{aligned}$$

Under the following assumptions:

(A4) $A \in \mathbf{R^{m\times n}}, \mathbf{b} \in \mathbf{R^m}, \mathbf{x} \in \mathbf{R^n}$, rank(A) = m.
(A5) $F : \mathbf{R}^n_+ \to \mathbf{R}$ is a convex function.
(A6) F has continuous partial derivatives in $\mathbf{R}^n_{++}$
(A7) $-\infty < V_P < \infty$, where we adopt the convention that $V_P = \infty$ f and only if (P) inconsistent.

This problem is a linearly constrained convex problem, in [10], a very efficient interior-point method was developed using the dual Problem due to Wolfe given by:

$$\begin{aligned} \mathbf{D} \qquad V_D := \mathit{Maximize} \qquad & b^t y - x^t \nabla F(x) + F(x) \\ \mathit{Subject\ to:} & \\ A^t y + z - \nabla F(x) = 0, \quad & y, z \geq 0, x > 0. \end{aligned}$$

Definition 6 *Let* $(x, y, z) \in \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_+$
We define the primal and dual residuals respectively as follows:

$$\begin{aligned} r_P(x) &:= b - Ax \\ r_D(x, y, z) &:= -\nabla F(x) + A^T y + z \end{aligned}$$

and the complementarity residue is defined by:

$$\mu(u, v, z, w) := \frac{x^T z}{n} \tag{31}$$

The Duality gap between (P) and (D) as:

$$x^T \nabla F(x) - b^T y \tag{32}$$

Definition 7 *We say that (P) is subconsistent if and only if there exists a sequence* $x^k \in bfR^n_+$ *called subfeasible solution such that:*

$$lim_{k\to\infty} r_P(x^k) = 0$$

The set of all subfeasible solutions is denoted by $AF(P)$*, and the subvalue of (P) is*

$$\bar{V}_P := inf_{\{x^k\}\in AF(P)} \liminf_k f\left(x^k\right)$$

When (P) is not subconsistent, $\bar{V}_P := \infty$*. A subfeasible solution is optimal if its value is* $\bar{V}_P$*. Similarly, the set of feasible solutions for (D), denoted by* $AF(D)$ *is formed by the sequences:*

$$\left(x^k, y^k, z^k\right) \in \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_+$$

such that

$$lim_{k\to\infty} r_D\left(x^k, y^k, z^k\right) = 0$$

with subvalor dual

$$\bar{V}_D := inf_{\{(x^k,y^k,z^k)\}\in AF(D)} \liminf_k f\left(b^t y^k - x^t \nabla F(x^k) + F(x^k)\right)$$

When (D) is not subconsistent, $\bar{V}_D := -\infty$*. A subfeasible solution is optimal if its value is* $\bar{V}_D$*.*

6 Solving the KKT system

We now consider the system:

$$\textbf{KKT}-\mu \qquad \begin{aligned} -\nabla F(x) + A^T y + z &= 0 \qquad (33)\\ \mathbf{Ax} - \mathbf{b} &= 0 \qquad (34)\\ Xz &= \mu e \qquad (35)\end{aligned}$$

To solve the system (KKT) we will use Newton's damped method due to the requirement of the variables x and z to be positive, in this case the decomposition LDL^t is used since the extended system is undefined, Newton's iterates consist of solving the following system of linear equations:

$$\begin{bmatrix} -(\nabla^2 F(x) + ZX^{-1}) & A^t \\ A & \mathbf{0} \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = - \begin{pmatrix} r_D - X^{-1}(Xz - \tau\mu e) \\ r_P \end{pmatrix} \qquad (36)$$

being

$$\Delta z = -ZX^{-1}\Delta x - X^{-1}(Xz - \mu e))$$

where X and Z are diagonal matrices formed with the coordinates of x and z respectively, and α satisfies the condition:

$$\alpha = \min\left\{\min\left\{-\tfrac{x_i}{\Delta x_i}, \Delta x < 0\right\}, \min\left\{-\tfrac{z_i}{\Delta z_i}, \Delta z < 0\right\}, 1\right\}$$

If $(\Delta x, \Delta y, \Delta z)$ is (36) solution and

$$\begin{aligned} x(\alpha) &= x + \alpha\Delta x,\\ y(\alpha) &= y + \alpha\Delta y,\\ z(\alpha) &= z + \alpha\Delta z. \end{aligned}$$

then:

$$\begin{aligned} (x(\alpha))^T z(\alpha) &= (1 - \alpha(1-\tau))x^T z + \alpha^2 \Delta x^T \Delta z,\\ Ax(\alpha) - b &= (1-\alpha) r_P,\\ -\nabla F(x(\alpha)) + A^T y(\alpha) + z(\alpha) &= (1-\alpha) r_D - \nabla F(x(\alpha)) + \nabla F(x)\\ &\quad -\alpha \nabla^2 F(x) \Delta x \end{aligned}$$

Remark 2 *In the case of the problem GD the expression:*

$$R(\alpha) = -\nabla F(x(\alpha)) + \nabla F(x) - \alpha \nabla^2 F(x) \Delta x$$

does not depend on the coefficients $\mathbf{c}$, *moreover if there is a perturbation in the coefficients say* $\mathbf{c}^+$ *then we will have:*

$$-\nabla F(x^+) + A^T y^+ z^+ = (1-\alpha) r_D + R(\alpha) + log(\mathbf{c}^+./\mathbf{c})$$

where ./ stands for coordinate-to-coordinate division.

Given $(x^0, z^0) \in \mathbf{R}^n_{++} \times \mathbf{R}^n_{++}$ and constants $\beta \in [1, \infty), \sigma \in (0, 1), C_p > 1$ we will define the following region $R \subset \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{++}$.

$$R = \left\{ \begin{array}{l} (u, v, w) \in \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{++}; \\ \qquad \|u\|_2 \leq C_P \min\left\{\|w\|_1, (x^0)^T z^0\right\}, \\ \qquad \|v\|_2 \leq C_P \min\left\{\|w\|_1, (x^0)^T z^0\right\}, \\ \qquad \|w\|_1 \leq (x^0)^T z^0, \\ \qquad \left\| w - \frac{\|w\|_1}{n} \right\|_2 \leq \beta \|w\|_1 \\ \qquad w \geq \sigma \max\left\{\frac{\|w\|_1}{n}, \mu_0\right\}. \end{array} \right\} \tag{37}$$

Proposition 5 *Given* $(x^0, z^0) \in \mathbf{R}^n_{++} \times \mathbf{R}^n_{++}$ *and constants* $\beta \in [1, \infty), \sigma \in (0, 1)$. *If* $x^0 = e$, $y^0 = 0$, $z_i^0 = max(1.0001max(0, -\frac{\partial}{\partial x_i} F(x^0) + a_i^t y^0), 1)$ *where* a_i *is the ith columm of the matrix* A, $i = 1, \ldots, n$ *and* $C_P \geq \frac{\max\left\{\|r_P(x^0)\|_2, \|r_D(x^0, y^0, z^0)\|_2\right\}}{(x^0)^T z^0}$, *then* $R \neq \emptyset$.

Proof If $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} r_D(x^0, y^0, z^0) \\ r_P(x^0) \\ X^0 z^0 \end{pmatrix}$, from the definition of the constants it is easy to see $R \neq \emptyset$.

■

Given $(x, y, z) \in R$ and $(\Delta x, \Delta y, \Delta z)$ a solution of (36), we will determine by linear search (backtracking) the largest α such that:

$$\begin{array}{l} \bar{\alpha} = \min\left\{\min\left\{-\frac{x_i}{\Delta x_i}, \Delta x < 0\right\}, \min\left\{-\frac{z_i}{\Delta z_i}, \Delta z < 0\right\}\right\} \\ x(\alpha) = x + \alpha \Delta x, \\ y(\alpha) = y + \alpha \Delta y, \\ z(\alpha) = z + \alpha \Delta z, \\ (r_D(x(\alpha), y(\alpha), z(\alpha)), r_P(x(\alpha)), X(\alpha) z(\alpha)) \in R, \ 0 < \alpha_{min} \leq \alpha \leq \bar{\alpha} \end{array}$$

This is a characteristic step of a primal-dual interior point method, what differentiates some methods is the way of updating the parameter μ or the way of solving the linear system etc, here we will have some updates that are different from the traditional ones, since we will update besides the parameter μ the parameter ω given in (21).

Given $(x^0, z^0) \in \mathbf{R}^n_{++} \times \mathbf{R}^n_{++}$ consider the set $S \subset \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{++}$ defined by:

$$S = \left\{ \begin{array}{l} (x, y, z) \in \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{+}; \\ \qquad \|r_P(x)\| \leq C_P \min\left\{x^T z, (x^0)^T z^0\right\}, \\ \qquad \|r_D(x, y, z)\| \leq C_P \min\left\{x^T z, (x^0)^T z^0\right\} \\ \qquad x^T z \leq 2(x^0)^T z^0 \\ \qquad \left\| Xz - \frac{x^T x}{n} \right\|_2 \leq \beta x^T z \\ \qquad Xz \geq \sigma \max\left\{\frac{x^T z}{n}, \mu_0\right\}. \end{array} \right\} \tag{38}$$

The S region is a neighborhood of the central trajectory, by generating a sequence with these characteristics we are generating a limitedly subfeasible.

We now define the application:

$$G: \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{++} \to \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n_{+}$$

defined by:

$$G(x,y,z) = \begin{pmatrix} -\nabla F(x) + A^T y + z \\ b - Ax \\ Xz \end{pmatrix} \tag{39}$$

7 Convergence analysis

In the sense of proving the existence of an accumulation point for the sequence generated by the 1 algorithm we will from now on create conditions to prove that the set S given in (38) is a compact set. First, We state a lemma that will be used later

Lemma 3 *Suppose (x,z) and $(\widehat{x},\widehat{z})$ are two vectors in $\mathbf{R}^n \times \mathbf{R}^n$ such that $(x-\widehat{x})^T(z-\widehat{z}) \geq 0$. Then the following statements are true:*

a) If $x+z = \widehat{x}+\widehat{z}$ then $(x,z) = (\widehat{x},\widehat{z})$
b) If $(x,z) > 0, (\widehat{x},\widehat{z}) > 0$ and $Xz = \widehat{X}\widehat{z}$ then $(x,z) = (\widehat{x},\widehat{z})$.

where X and Z are diagonal matrices formed with the coordinates of x and z respectively.

Proof

a) If $x-\widehat{x} = -(z-\widehat{z})$ then $\|x-\widehat{x}\|^2 = (x-\widehat{x})^T(x-\widehat{x}) = -(x-\widehat{x})^T(z-\widehat{z}) \geq 0$, as $(x-\widehat{x})^T(z-\widehat{z}) \geq 0$, we conclude that $\|x-\widehat{x}\|^2 = 0$, then $x = \widehat{x}$ by substituting the value of x for $\widehat{x}$ in (a) we find $z = \widehat{z}$.
b) If $(x,z) > 0, (\widehat{x},\widehat{z}) > 0$ and $X\widehat{x} = Z\widehat{z}$ we can write $z-\widehat{z} = -X^{-1}\widehat{Z}(x-\widehat{x})$ e $\|x-\widehat{x}\|^2_{X^{-1}\widehat{Z}} = (x-\widehat{x})^T X^{-1}\widehat{Z}(x-\widehat{x}) = -(x-\widehat{x})^T(z-\widehat{z}) \geq 0$ therefore $x = \widehat{x}$, writing $x-\widehat{x} = -Z^{-1}\widehat{X}(z-\widehat{z})$ we have $\|z-\widehat{z}\|^2_{Z^{-1}\widehat{X}} = (z-\widehat{z})^T Z^{-1}\widehat{X}(z-\widehat{z}) = -(z-\widehat{z})^T(x-\widehat{x}) \geq 0$ therefore $z = \widehat{z}$.

■

Theorem 5 *If $F(x) = -ln(u(x))$ where $u(x)$ is given in (28), then:*

a) The application G given in (39) is injetive,
b) $\nabla G(x,y,z)$ is nonsingular for all $(x,y,z) \in \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{++}$, where $\nabla G(x,y,z)$ is a Jacobian matrix of G in (x,y,z).

Proof If $G(x,y,z) = G(\widehat{x},\ \widehat{y},\widehat{z})$ then $\nabla F(\widehat{x}) - \nabla F(x) + A^T(y-\widehat{y}) + z - \widehat{z} = 0$, $A(x-\widehat{x}) = 0$, $Xz = \widehat{X}\widehat{z}$, as F is convex, we have:

$$(x - \widehat{x})^T(z - \widehat{z}) = (x - \widehat{x})^T(\nabla F(x) - F(\widehat{x})) \geq 0,$$

by lemma 3 (b) $(x, z) = (\widehat{x}, \widehat{z})$ consequently $A^T(y - \widehat{y}) = 0$ by assumption **A1** the matrix A has full rank, therefore the linear equation $(AA^T)(y - \widehat{y}) = 0$ has unique solution, therefore $y = \widehat{y}$, proving that G is injective. Let's now prove that $\nabla G(x, y, z)$ is nonsingular, given $(x, y, z) \in \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{++}$ and $\bar{w} = (u, v, w) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n$ such that $\nabla G(x, y, z)\bar{w} = \mathbf{0}$, that is:

$$\begin{bmatrix} -\nabla^2 F(x) & A^t & \mathbf{I}_{n\times n} \\ A & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times n} \\ Z & \mathbf{0}_{n\times m} & X \end{bmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{40}$$

Solving (40) we obtain $w = -Z^{-1}Xu$, $-(\nabla^2 F(x) + Z^{-1}X)u + A^t v = 0$, $Au = 0$, where $\nabla^2 F(x)$ is given in (30), concluding that $-u^T(\nabla^2 F(x) + Z^{-1}X)u = 0$ so $u = 0$ due to the convexity of F, substituting $u = 0$ in $-(\nabla^2 F(x) + Z^{-1}X)u + A^t v = 0$ we obtain $A^t v = 0$, as the matrix A has full rank, $v = 0$, therefore $\bar{w} = 0$, consequently $\nabla G(x, y, z)$ is nonsingular.

■

Theorem 6 *Suppose the problems* **(P)** *and* **(D)** *are subconsistent, if* $F(x) = -ln(u(x))$ *where* $u(x)$ *is given in (28). Then the sequence* $\{(x^k, y^k, z^k)\}$ *generated by the algorithm (1) has an accumulation point, which is the solution to the KKT-*μ *(33) - (35) system.*

Proof The sequence$\{(x^k, y^k, z^k)\} \in S = G^{-1}(R)$ where R and S are given by (37) and (38) respectively, by the theorem 5 the application G given in (39) is injective and $\nabla G(x, y, z)$ is non-singular for all $(x, y, z) \in \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{++}$, by the inverse function theorem G is a local injective diffeomorphism, hence a global diffeomorphism over $G(\mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{++})$ as R is compact, the set $S = G^{-1}(R)$ is also compact, so $\{(x^k, y^k, z^k)\}$ has an accumulation point (x^*, y^*, z^*) such that $G(x^*, y^*, z^*) \in R$ as $(x^*)^T z^* = \lim_{k\to\infty} (x^k)^T z^k = n\mu_0$, since $\alpha_k \geq \alpha_{min} > 0$,

$$\|r_P(x^*, y^*, z^*)\| = \lim_{k\to\infty} \left\|r_P(x^k, y^k, z^k)\right\| \leq P \lim_{k\to\infty} (x^k)^T z^k$$
$$\|r_D(x^*, y^*, z^*)\| = \lim_{k\to\infty} \left\|r_D(x^k, y^k, z^k)\right\| \leq P \lim_{k\to\infty} (x^k)^T z^k.$$

Proving that (x^*, y^*, z^*) is a solution of the KKT system, by the injectivity of G this solution is unique, the fact that G is a diffeomorphism tells us that there exists an open ball of center $G(x^*, y^*, z^*) \in \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{++}$ and radius ϵ where the application G is a bijection of this ball into an open ball of center $(x^*, y^*, z^*) \subset \times \mathbf{R}^m \times \mathbf{R}^n_{++}$, if $\|r_P(x^*, y^*, z^*)\| \leq nP\mu_0$, $\|Ax^* - b\| \leq nP\mu_0$, $(x^*)^T z^* < n\mu_0$ e $nP\mu_0$ is small enough there is $0 \leq \theta < 1$ e $(\tilde{x}, \tilde{y}, \tilde{z})$ such that $\|(\tilde{x}, \tilde{y}, \tilde{z}) - (x^*, y^*, z^*)\| < \epsilon$, $\|r_D(\tilde{x}, \tilde{y}, \tilde{z}) - \theta\mu_0 e\| = 0$, $\|r_P(\tilde{x}) - \theta\mu_0 e_m\| = 0$, $(\tilde{x})^T \tilde{z} = n\mu^0$.

■

7.1 The Primal Dual Predictor-Corrector Method

In this section we present a feasible interior-point algorithm for the Posynomial Geometric programming problem, the algorithm is of the type presented in [10] but in the problem to be discussed here some c_i coefficients are perturbed, this is the case when we approximate signomial by posynomial problems using the harmonic mean. The algorithm 1 describes the developed method in detail.

Algorithm 1 Predictor-Corrector Interior-Point Method

Input $\epsilon = 10^{-10}$, $\mu_0 = 10^{-16}$, $epsp = 10^{-6}$,$\alpha_{min} = 10^{-6}$, $fac = 100$,
$Stoping\_Criterio = 1$, ω^0 *(Initials weigths)*, $e = (1, \dots, 1)^T \in \mathbf{R}^n$,
$(x^0, y^0, z^0) \in \mathbf{R}^n_{++} \times \mathbf{R}^m \times \mathbf{R}^n_{++}$, $x^0_i = 1$, $y^0_j = 0$, n_- *(number of negative coef.)*,
$z^0_i = max(1.0001max(0, -\frac{\partial}{\partial x_i}F(x^0) + a^t_i y^0), 1)$ $J_\omega = \{i, c_i < 0, i = 1, \dots, n_-\}$.
numiter = 0 (iteratitons number)
Compute the initial residues:

$$\begin{aligned} r_D &= -\nabla F_\omega(x^0) + A^t y^0 + z^0, \\ r_P &= Ax^0 - b, \\ M^0 &= 2(x^0)^T z^0 \end{aligned}$$

while Stoping_Criterio $> \epsilon$

Step 1 Predicted Phase

- *Solve the linear system given in (36) with* $\tau = 0, x = x^0, z = z^0, r_D, r_P$
- *Do*

$$\begin{aligned} x_{pred} &\leftarrow x^0 + 0.999\alpha\Delta x, \\ y_{pred} &\leftarrow y^0 + 0.999\alpha\Delta y, \\ z_{pred} &\leftarrow z^0 + 0.999\alpha\Delta z. \end{aligned}$$

where
$\alpha = min\left\{min\left\{-\frac{x^0_i}{(\Delta x)_i}, \Delta x < 0\right\}, min\left\{-\frac{z^0_i}{(\Delta z)_i}, \Delta z < 0\right\}\right\}$,
$\alpha = min\left\{\alpha, \frac{(x^0)^T z^0}{\Delta x^T \Delta x}\right\}$, *se* $\Delta x^T \Delta x > 0$

Step 2 Corrective Phase
If $n_- > 0$

1. *Weights update.*
   Calculate: $h = c^t_h e^{A_h y^0_{pred}}$, $\omega = \frac{diag(c_h)e^{A_h y^0_{pred}}}{h}$.
   If $\|\omega - \omega^0\| > epsp$
   Update the coeficients $c_h(\omega)$.
   $\omega^0 \leftarrow \omega$,
   Calculate $r_D = -\nabla F_\omega(x^0) + A^t y^0 + z^0$.
   End If
   End If
2. *Solve the Linear System (36) with*
   $\tau = .5\frac{x^T_{pred} z_{pred}}{(x^0)^T z^0}, x = x^0, z = z^0, \mu = \frac{(x^0)^T z^0}{n}, r_D, r_P$ *being* $(\Delta x, \Delta y)$ *its solution and* $\Delta z = -ZX^{-1}\Delta x - X^{-1}(Xz - \mu e)$
3. *Determine* $\alpha > 0$ *such that:* $x = x^0 + \alpha\Delta x$, $y = y^0 + \alpha\Delta y$ *e* $z = z^0 + \alpha\Delta z$ *that satisfy:* $\bar{\alpha} = min\left\{min\left\{-x^0_i/\Delta x, \Delta x < 0\right\}, min\left\{-z^0_i/\Delta z, \Delta z < 0\right\}\right\}$,

$$Xz \geq \sigma \frac{x^T z}{n}$$

$$\left\| Xz - \frac{x^T z}{n} e \right\| \leq \beta \frac{x^T z}{n}$$

$$\| r_P(x,y,z) \| \leq P x^T z, \qquad \textit{where } \sigma \in (0,1] \ e \ \beta \in (0,\infty]$$

$$\| r_D(x,y,z) \| \leq P x^T z,$$

$$x^T z < \min\left\{ (x^0)^T z^0, M^0 \right\}, \quad \alpha_{min} \leq \alpha \leq \bar{\alpha}.$$

4. Update

$$x^0 \leftarrow x^0 + 0.9995\alpha \Delta x$$
$$y^0 \leftarrow y^0 + 0.9995\alpha \Delta y$$
$$z^0 \leftarrow z^0 + 0.9995\alpha \Delta z$$
$$r_P \leftarrow Ax^0 - b,$$
$$\mu \leftarrow \max\left\{ \frac{(x^0)^T z^0}{n}, \mu_0 \right\}$$

Step 3 Resetting dual slacks

If $\left\| X^0 z^0 \right\|_\infty < epsp$
Do
$s = \nabla F_\omega(x) - A^T y$
for $i = 1, \ldots, n$ *do*
If $s_i \geq 0$ *then*

$$z_i^0 = \begin{cases} s_i & se \quad s_i \in (z_i/fac, z_i \times fac) \\ z_i^0/fac & se \quad s_i \leq z_i/fac \\ z_i^0 \times fac & se \quad s_i \geq z_i \times fac. \end{cases}$$

end If
end for
end If
Update :

$$r_D = -\nabla F_\omega(x^0) + A^t y^0 + z^0, \quad M^0 = 2(x^0)^T z^0$$

$numiter = numiter + 1$

Stoping_Criterio $\leftarrow \max\left(\frac{10^2 (x0)^T z0}{1+\left\|x^0\right\|_1 + \left\|z^0\right\|_1}, \frac{\|r_P\|_1}{1+\left\|x^0\right\|_1}, \frac{\|r_D\|_1}{1+\left\|z^0\right\|_1} \right)$

end while

7.2 The Global Method

In this section we present the algorithm that determines a global solution to the GP problem. Since the corrector-predictor method can be used for both convex and nonconvex cases of GP it is sufficient to distinguish both cases conveniently.

Notation: We denote by $\#(\Omega)$ the cardinal, or the number of elements of the set $\#(\Omega)$

Algorithm 2 Global Method for GP - DCGP

START
Input *A (exponent matrix), c coefficients,*
J (delimiter vector), σ *(signs vector of the terms) given in (4),*
Step 1 Characterization
Build the following sets:
$J_+[k] = \{i \in J[k]; \sigma_i = 1\}, \quad J_s[k] = \{i \in J[k]; \sigma_i = -1\}$
$J_{sig} = \bigcup_{k=0}^{p} J_s[k] = \{i; \sigma_i = -1, i = 1, \ldots, n\}.$
If $J_{sig} = \emptyset$ *then* $\#(J_{sig}) = 0,$

this is a posynomial GP problem, in which case the predictor-corrector method will be used directly with A, c and J as data
If $J_{sig} \neq \emptyset$ the problem is a Signomial GP problem and therefore non-convex.
$n_{rev} = \#(J_{sig})$, n_{rev} will be the number of terms of the reverse constraint given in (20).

Step 2 Standard DC form
For each $k = 0, \ldots, p$ use theorem 3 to build the exponent matrix,the coefficients vector of each function f_k and also of the reverse function h.

Step 3 PMINMAX Problem
Take $\omega_j^0 = \frac{1}{n_{rev}}$ as initial weight and use the harmonic mean to locally transform the reverse constraint $h(t) \geq 1$ into a constraint $H_{\omega^0}(t) \leq 1$ where H_{ω^0} is of type (27),
by doing:

$$A_{DC} = \begin{bmatrix} 1 & 0 & 1 \\ -e_0 & A_0 & 0 \\ 0 & A_g & 0 \\ 0 & -A_h & 0 \\ 0 & -A_h & -e_h \end{bmatrix} \quad c_{DC_\omega} = \begin{pmatrix} 1 \\ c_0 \\ c_g \\ \frac{\omega^2}{c_h} \\ \frac{\omega^2}{c_h} \end{pmatrix} \quad J_{DC_\omega} = \begin{pmatrix} 1 \\ J_0 \\ J_g \\ J_\omega \\ J_\omega \end{pmatrix}$$

where A_0, A_g e A_h are the exponent matrices of the objective function, constraints and the reverse constraint, respectively,
$J_0 = \{1,\ J_+[0],\ J_{sig} \backslash J_s[0]\}$,
$J_g\,[k] = \{J_s[0],\ J_+[k],\ J_{sig} \backslash \left(J_s[0] \cup J_s[k]\right)\}$,
$J_\omega = \{J_{sig}, n\}$

Step 4
Use the correct-predictor method with $n_{rev} > 0$ to solve the problem whose data are $A_{DC}, c_{DC_{omega}}, J_{DC}$.

Step 5 Determining the solution
If (x^*, y^*, z^*) is the solution by the corrector predictor method, make $t^* = e^{y^*}$ then substitute in the SGP problem, in order to determine the active constraints, optimal value, etc.

END

8 Computational implementation, Weights control

8.1 Computational implementation

The computational implementation of the proposed strategy was performed using GNU Fortran 90 Compiler version 2018. The code implementation has been divided into three parts:

- Routines for transforming the original SGP problem (1) to (4) into the PMINMAX problem (22),(26);
- Routines for solving the linear system given in (36);
- Implementation of the Interior-Points Corrector-Predictor Method

The linear system given in (36) was solved using the decomposition LDL^t for indefinite symmetric systems, from the approach presented in [26], with scale, without permutations, but with partial iterative refinement. If $\mathbf{x}^k$ is the solution of the $Ax^k = b^k$ system, and $\left\|\mathbf{r}^k\right\| = \left\|Ax^k - \mathbf{b}^k\right\| \geq 10^{-10}$, we solve the system $A\Delta x^k = \mathbf{r}^k$ and do $\mathbf{x}^{k+1} = \mathbf{x}^k - \Delta x^k$.

Table 1 Solving SGP by a primal-dual infeasible predictor-corrector algorithm

Problem Summary					Efficiency		Accuracy	
Problem		Number of					Objective value of	
N^o	Name	vars	cons	terms	Iter	Time	Primal GP	Dual GP
1	Demb7603	8	28	58	3285	43.45	0.122722612095E+04	0.122721666943E+04
2	Demb764a	8	4	16	136	0.02	0.395116344078E+01	0.395116344073E+01
3	Demb7606	13	39	62	303	3.77	0.976071987758E+02	0.976071984641E+02
4	Demb7607	16	51	93	3253	184.97	0.174790706173E+03	0.174790703280E+03
5	RM197809	2	1	5	58	0.00	0.119643371198E+02	0.119643371198E+02
6	RM197810	3	4	9	54	0.00	-.832497284048E+02	-.832497284060E+02
7	RM197811	4	2	7	28	0.00	-.573982030359E+01	-.573982030359E+01
8	RM197812	8	4	15	123	0.03	-.604823288886E+01	-.604823289416E+01
9	RM197813	8	6	19	364	0.22	0.704924891369E+04	0.704924890161E+04
10	RM197814	10	29	36	32	0.02	0.114362316109E+01	0.114362316027E+01
11	RM197815	10	9	15	34	0.00	0.205653413173E+00	0.205653413173E+00
12	RM197816	10	5	16	122	0.03	0.196631321203E+00	0.196631321203E+00
13	RM197817	11	33	41	886	0.64	0.140606724804E+00	0.140606672182E+00
14	RM197818	13	9	21	574	0.28	0.186162725391E+01	0.186162723732E+01
15	RM197819	8	5	28	134	0.03	0.174859882958E+05	0.174859882957E+05
16	RM197820	13	11	30	228	0.19	-0.126449860706E+01	-.126449865211E+01
17	RM197821	10	22	38	2363	3.28	-0.125092953064E+01	-.125090865224E+01
18	RM197822	9	10	57	3963	67.77	-0.390536810985E+01	-.390536813436E+01
19	RM197823	5	16	31	24	0.02	0.101224305490E+05	0.101224305493E+05
20	Ċ B2005001	2	6	13	37	0.09	0.122264777590E+02	0.122264777589E+02
21	ĊB2005002	3	16	30	4506	19.88	0.345236507913E+04	0.345236507913E+04
22	ĊTC200501	4	8	31	31	0.02	0.700678063085E+04	0.700678063085E+04

8.2 Weight control

For some problems, the computational tests revealed that for the weights strategy ω given in (21) was resulting in one of the following phenomena: $\omega_i^k \to 0$ or started the process with very small coordinates requiring many iterations, and in some cases not succeeding. We added the following constraint to the problem to avoid such a situation:

$$(2\tau^k)^{\frac{1}{4}} C^k \left(\prod_{j=1}^{m} t_j^{a_{ij}} \right)^{-\frac{1}{4}} \leq 1$$

where $\tau^k = \frac{1}{2} \frac{(x^k_{pred})^t (z^k_{pred})}{(x^k)^t z^k}$, $C^k = \left(\prod_{j=1}^m (t^k_{pred})_j^{a_{ij}} \right)^{\frac{1}{4}}$, $t^k_{pred} = \exp(y_{pred})$

This constraint had identical behavior to the logarithmic barrier, allowing a uniformity of the peaks in the initial iterations ($\tau^k \approx 1$) and allowing small peak values when the gap is small ($\tau^k \approx 1$). The constraint enabled solutions to problems 09, 16, 17, and 18. The computational tests were performed from a set of existing instances in the literature in [9],[5],[12] and [18], In most cases the results were similar to those in the literature or were better. The method solves both posynomial and signomial problems and it is clear the difference in iterations and processing time in both cases. We adopted a continuous approach to solve some discrete problems successfully, showing that the methodology can also be applied in exceptional cases for this purpose. The processing runtime and the number of iterations were competitive because global methods are generally considered slow. Even so we still consider the number of iterations in some problems to be high.

Table 2 Accuracy statistics

Problem Summary					Efficiency		Accuracy		
Problem		Number of							
N^o	Name	vars	cons	terms	Iter	Time	Gap	Inf_D	Inf_D
1	Demb76003	8	28	58	3285	43.45	.419E-14	.460E-07	.169E-12
2	Demb7604a	8	4	16	136	0.02	.199E-14	.324E-11	.264E-15
3	Demb76006	13	39	62	303	3.77	.525E-09	.801E-08	.339E-13
4	Demb76007	16	53	93	3253	184.97	.254E-14	.256E-07	.591E-13
5	RM1978009	2	1	5	58	0.00	.116E-14	.314E-07	.212E-14
6	RM1978010	3	4	9	54	0.00	.191E-12	.341E-14	.226E-15
7	RM1978011	4	2	7	28	0.00	.294E-11	.681E-10	.461E-15
8	RM1978012	8	4	15	123	0.03	.300E-14	.561E-07	.117E-13
9	RM1978013	8	6	19	364	0.22	.276E-14	.340E-07	.114E-13
10	RM1978014	10	29	36	32	0.02	.307E-14	.652E-07	.322E-13
11	RM1978015	10	9	15	34	0.00	.252E-14	.298E-11	.229E-15
12	RM1978016	10	9	18	122	0.03	.109E-14	.372E-11	.251E-15
13	RM1978017	10	5	16	886	0.64	.341E-14	.761E-07	.572E-13
14	RM1978018	13	9	21	574	0.28	.310E-14	.887E-10	.121E-13
15	RM1978019	8	5	28	134	0.03	.157E-14	.415E-11	.561E-15
16	RM1978020	13	11	30	228	0.19	.218E-14	.262E-09	.406E-13
17	RM1978021	10	22	38	2363	3.28	.571E-14	.212E-07	.212E-12
18	RM1978022	9	10	57	3963	69.89	.259E-08	.518E-09	.105E-13
19	RM1978023	5	16	31	24	0.02	.461E-10	.240E-10	.490E-15
20	CB2005001	2	6	13	37	0.09	.216E-10	.870E-14	.472E-13
21	CB2005002	3	16	30	4506	19.88	.822E-12	.135E-09	.183E-09
22	CTC200501	4	8	31	31	0.100	.140E-15	.130E-14	.100E-15

Table 3 Result comparisons

Problem Summary					Efficiency			Accuracy
Problem		Number of						Objective value of
N^o	Name	vars	cons	terms	Iter	Time	ref.	Primal GP
1	Demb7603	8	28	58	3285	43.45	DCGP	1227.226120946139
							[5]	1227.1831610
							[12]	1227.33
2	Demb764a	8	4	16	136	0.02	DCGP	3.951163440777
							[5]	3.9516982
							[11]	3.9511
							[28]	3.95116
							[12]	3.9511
3	Demb7606	13	39	62	303	3.77	DCGP	97.607198775822
							[5]	97.591034
							[18]	97.591034
							[28]	97.59237
4	Demb7607	16	51	93	3253	184.97	DCGP	174.790706172889
							[5]	174.788807
5	RM197809	2	1	5	58	0.00	DCGP	11.964337119829
							[18]	11.91
							[28]	11.96438
6	RM197810	3	4	9	54	0.00	DCGP	-83.249728404775
							[18]	-83.21
							[11]	-83.254
							[28]	-83.24973
							[12]	-83.2535
7	RM197811	4	2	7	28	0.00	DCGP	-5.739820303591
							[18]	-5.7398.
							[11]	-5.7398.
							[28]	-5.73982
							[12]	-5.7398
8	RM197812	8	4	15	123	0.03	DCGP	-6.048232888864
							[18]	-6.0482
							[11]	-6.0482
							[28]	-6.04823
							[12]	-6.0482
9	RM197813	8	6	19	364	0.22	DCGP	7049.248913688329
							[5]	7049.324305
							[18]	7049.247
							[11]	7049.24
							[28]	7049.2477
							[12]	7049.25
10	RM197814	10	5	16	32	0.02	DCGP	1.143623161094
							[18]	1.1975
							[11]	1.1436
							[28]	1.14362
							[12]	1.1437
11	RM197815	10	7	15	34	0.00	DCGP	0.205653413173
							[18]	0.2015
							[28]	0.20565

Table 4 Result comparisons (cont.)

Problem Summary					Efficiency			Accuracy
Problem		Number of						Objective value of
N^o	Name	vars	cons	terms	Iter	Time	ref.	Primal GP
12	RM197816	10	7	18	122	0.03	DCGP	0.196631321203
							[18]	0.1966.
							[28]	0.19663
13	RM197817	11	9	19	122	0.03	DCGP	0.140606724804
							[18]	0.1406
							[11]	0.1406
							[28]	0.14061
							[12]	0.1406
14	RM197818	13	9	21	574	0.28	DCGP	1.861627253912
							[18]	1.81830
							[28]	1.86163
15	RM197819	8	5	28	134	0.03	DCGP	17485.988295811050
							[18]	17486.0
							[28]	17485.988
16	RM197820	13	11	30	228	0.19	DCGP	-116.4498607061
							[18]	-121.54
							[28]	-116.4515
17	RM197821	10	22	38	2363	3.28	DCGP	-1250.929530643
							[18]	-1237.55
18	RM197822	9	10	57	3963	67.77	DCGP	-390.5368109851
							[18]	-375.784
							[28]	-376.3109
19	RM197823	5	16	31	24	0.02	DCGP	10122.430548961020
							[5]	10126.64252
							[18]	10127.13.
							[28]	10121.773
							[12]	10122.6964
20	CB2005001	2	6	13	37	0.09	DCGP	12.2264777590620
							[3]	12.2263
21	CB2005002	3	16	30	4506	19.88	DCGP	3452.36507912689
							[3]	3452.365079365079
22	CTC200501	4	8		31	0.02	DCGP	7006.78063084608
							[4]	7006.367

9 Concluding remarks

This work generalizes the work of Kortaneck et al [10] since it also solves the cases of non-convex GP.

We present in this work results that guarantee a weak convergence of the generated sequence by the proposed algorithm. Some concepts and procedures from [10] had a fundamental role in the computational performance. We highlight the backlash update routine, which here was used only when the max $\left\{x_i^k z_i^k < 10^{-4}\right\}$ greatly decreased the number of iterations. The ω weight update in the predictor phase was an initiative that we believe contributed to the good performance of the method. The weights control constraint presented in the previous section, consolidated the work.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Beightler,C. S.,Phillps, D. T. Applied Geometric Programming. (John Wiley & Sons), United States (1976).
2. Boyd S.,Kim S. J.,Vandenberghe L.,Hassibi and A. A tutorial on geometric programming. Optimization and Engineering, 8:67–127, 2007.
3. Choi J. C. and Bricker D. L., *A Heuristic Procedure for Rounding Posynomiial Geometric Programming Solutions to Discrete Values*, Computers Ind. Engineering Vol. 30, No. 4, pp. 623-629, 1996.

4. Ching-Ter Chang *On the mixed integer signomial programming problems*,Applied Mathematics and Computation 170 (2005) 1436-1451
5. Dembo R.S., (1976), A Set of Geometric Programming test problems and their Solutions, Mathematical Programming 10, 192-213.
6. Duffin, R.J., Peterson, E.L. and Zener, C., "Geometric Programming – Theory and Applications", John Wiley & Sons, New York (1967).
7. Gongxian Xu, *Global optimization of signomial geometric programming problems* European Journal of Operational Research 233,500-510 (2014)
8. Horst R., P. N. Pardalos,Thoai N. V., Introduction to Global Optimization, 349 Kluwer Academic Publishers,(2000)
9. Kortanek, K.O. and No, H., "A second order affine scaling Algorithm for geometric programming dual with logarithmic barrier", Optimization (23), 303–322 (1992).
10. Kortanek, K.O., Xu, X. and Ye, Y., "An Infeasible Interior Point Method for Solving Primal and Dual Geometric Programs", Mathematical Programming 76, 155-181 (1995)
11. Maranas C.D., Floudas C.A., Global Optimization in Generalized Geometric Programming, Computers chem. Engineering Vol. 21, No. 4, pp. 351-369, (1997)
12. Mladenovic N., et al. General Variable Neighborhood Search for the Continuous Optimization, European Journal of Operational Research 191, 753-770 (2008).
13. Monteiro,R.D.C. and Pang,J.S. *Properties of an Interior-Point Mapping for Mixed Complementary Problems*,Mathematics of Operations Research, Vol 21,Num 3, pag 629-654 August 1996.
14. Mutapcic A. et al *GGPLAB: a simple matlab toolbox for geometric programming* A Mutapcic, K Koh, S Kim, L Vandenberghe, S Boyd Web page and software: http://stanford.edu/boyd/ggplab. (2006)
15. Peiping Shen and Hongwei Jiao, *A new rectangle branch-and-pruning approach for generalized geometric programming*, Applied Mathematics and Computation, Volume 183, Issue 2, 15 December 2006, Pages 1027-1038
16. Pörn, R., Björk, K. M., & Westerlund, T. Global solution of optimization problems with signomial parts. Discrete Optimization, 5(1), 108-120 (2008).
17. Qu, S., Zhang, K. C., & Ji, Y. A new global optimization algorithm for signomial geometric programming via Lagrangian relaxation. Applied Mathematics and Computation, 184(2), 886-894 (2007).
18. Rijckaert, M. J., & Martens, X. M. Comparison of generalized geometric programming algorithms. Journal of Optimization Theory and Applications, 26(2), 205-242. (1978).
19. Shen, P. Linearization method of global optimization for generalized geometric programming. Applied Mathematics and Computation, 162(1), 353-370 (2005).
20. Toscano, R., & Amouri, S. B. Some heuristic approaches for solving extended geometric programming problems. Engineering Optimization, 44(12), 1425-1446- (2012).
21. Tsai, J. F., & Lin, M. H. Global optimization of signomial mixed-integer nonlinear programming problems with free variables. Journal of Global Optimization, 42(1), 39-49 . (2008).
22. Tsai, J. F., & Lin, M. H. An efficient global approach for posynomial geometric programming problems. INFORMS Journal on Computing, 23(3), 483-492. (2011).
23. Tsai, J. F., Lin, M. H., & Hu, Y. C. On generalized geometric programming problems with non-positive variables. European Journal of Operational Research, 178(1), 10-19. (2007).
24. Tuy H.; ,F.Al-Khayyal; F.Zhou; A DC Optimization Method for Single Facility Location Problems J. Global Optim., 7: 209-227 (1995).
25. H.Tuy; Canonical DC programming problem: Outer approximation methods revisited, Operations Research Letters 18, 99-106 (1995).
26. Vanderbei, R.J. , "Linear Programming – Foundations and Extensions", Kluwer Academic Publishers, Boston (1997).
27. Wang, Y., Zhang, K. C., & Gao, Y. Global optimization of generalized geometric programming. Computers & Mathematics with Applications, 48(10-11) 1505-1516. (2004).
28. Yang H.,Bricker D. L. Investigation of Path-Following Algoritms for Signomial Geometric Programming Problems, European Journal of Operational Research, 103:230-241 (1996)
29. Yanjun Wang and Zhian Liang, *A deterministic global optimization algorithm for generalized geometric programming*, Applied Mathematics and Computation, Volume 168, Issue 1, 1 September, Pages 722-737 (2005)

10 Apendix: Problems Tests and the solutions obtained

01 D76003 Ref. [5] Prob. 3, [12] Prob. GGP1

Minimize $1.715t_1 + 0.035t_1t_6 + 4.0565t_3 + 10.0t_4 + 3000.0 - 0.063t_3t_5$,
Subject to :
$0.0059553571t_6^2 + 0.88392857t_1^{-1}t_3 - 0.11756250t_6 \leq 1$,
$1.10880000t_1t_3^{-1} + 0.13035330t_1t_3^{-1}t_6 - 0.00660330t_1t_3^{-1}t_6^2 \leq 1$,
$0.00066173269t_6^2 + 0.017239878t_5 - 0.0056595559t_4 - 0.019120592t_6 \leq 1$,
$56.850750t_5^{-1} + 1.08702000t_5^{-1}t_6 + 0.32175000t_4t_5^{-1} - 0.03762000t_5^{-1}t_6^2 \leq 1$,
$0.00619800t_7 + 2462.3121t_2t_3^{-1}t_4^{-1} - 25.125634t_2t_3^{-1} \leq 1$,
$161.18996t_7^{-1} + 5000t_2t_3^{-1}t_7^{-1} - 489510.00t_2t_3^{-1}t_4^{-1}t_7^{-1} \leq 1$,
$44.333333t_5^{-1} + 0.33000000t_5^{-1}t_7 \leq 1$,
$0.02255600t_5 - 0.00759500t_7 \leq 1$
$0.00061000t_3 - 0.0005t_1 \leq 1$
$0.81967200t_1t_3^{-1} + 0.81967200t_2t_3^{-1} \leq 1$
$24500.0t_2t_3^{-1}t_4^{-1} - 250.0t_2t_3^{-1} \leq 1$,
$0.010204082t_4 + 0.000012244898t_2^{-1}t_3t_4 \leq 1$,
$0.00006250t_1t_6 + 0.00006250t_1 - 0.00007625t_3 \leq 1$,
$1.22t_1t_3^{-1} + 1.0t_1^{-1} - 1.0t_6 \leq 1$
$1500 \leq t_1 \leq 2000$,
$1 \leq t_2 \leq 120$,
$3000 \leq t_3 \leq 3500$,
$85 \leq t_4 \leq 93$,
$90 \leq t_5 \leq 95$,
$3 \leq t_6 \leq 12$,
$145 \leq t_7 \leq 162$.

solution Demb76003

$g_0^* =$	1227.226120946139
$F^* =$	1227.216669426738
$\mathbf{max\{g_k\}} =$	1.000000366132
$h^* = 0.999999993532$	
$x_0^{*T} z_0^* =$	$.9800137224E-08$
$\lVert r_D \rVert_1 =$	$.2127201385E-12$
$\lVert r_P \rVert_1 =$	$.2167984349E-07$
T(seg.) =	42.625000000000
Iterations =	3285
$\mathbf{t}^*$	**weight**
$t_1 = 1698.1834899721$	$\omega_1 = 0.4058834305$
$t_2 = 53.6651301434$	$\omega_2 = 0.0338889562$
$t_3 = 3031.2983403019$	$\omega_3 = 0.0111960799$
$t_4 = 90.1098268814$	$\omega_4 = 0.0140018120$
$t_5 = 95.0000000073$	$\omega_5 = 0.0055117653$
$t_6 = 10.4992831507$	$\omega_6 = 0.0011985170$
$t_7 = 153.5353546394$	$\omega_7 = 0.0122126600$
$t_8 = 1227.2166694267$	$\omega_8 = 0.0171978538$
	$\omega_9 = 0.0320159141$
	$\omega_{10} = 0.0233122585$
	$\omega_{11} = 0.1215159385$
	$\omega_{12} = 0.0063459736$
	$\omega_{13} = 0.2882633170$
	$\omega_{14} = 0.0274555237$

02 D7604a Ref. [5] Prob. 4A]

Minimize $0.4t_1^{0.67}t_7^{-0.67} + 0.4t_2^{0.67}t_8^{-0.67} + 10.0 - 1.0t_1 - 1.0t_2$,
Subject to :
$0.0588t_5t_7 + 0.1t_1 \leq 1$,
$0.0588t_6t_8 + 0.1t_1 + 0.1t_2 \leq 1$,
$4t_3t_5^{-1} + 2t_3^{-0.71}t_5^{-1} + 0.0588t_3^{-1.3}t_7 \leq 1$,
$4t_4t_6^{-1} + 2t_4^{-0.71}t_6^{-1} + 0.0588t_4^{-1.3}t_8 \leq 1$,

Solution Demb7604a

$g_0^* =$	3.951163440777
$F^* =$	3.951163440728
$\mathbf{max\{g_k\}} =$	1.000000000000
$h^* = 1.000000000004$	
$x_0^{*T} z_0^* =$	$.1993309065E-14$
$\lVert r_D \rVert_1 =$	$.2640429207E-15$
$\lVert r_P \rVert_1 =$	$.3241598468E-11$
T(seg.) =	0.015625000000
Iterations =	136
$\mathbf{t}^*$	**weight**
$t_1 = 6.4649938883$	$\omega_1 = 0.5016424892$
$t_2 = 2.2328219767$	$\omega_2 = 0.1732528126$
$t_3 = 0.6674010012$	$\omega_3 = 0.3251046982$
$t_4 = 0.5957566651$	
$t_5 = 5.9326960009$	
$t_6 = 5.5272359005$	
$t_7 = 1.0133529773$	
$t_8 = 0.4006702281$	

03 D76006 Ref. [5] Prob. 6
Minimize $1.0t_{11} + 1.0t_{12} + 1.0t_{13}$
Subject to :
$1.262626t_8t_{11}^{-1} - 1.231059t_1t_8t_{11}^{-1} \leq 1$,
$1.262626t_9t_{12}^{-1} - 1.231059t_2t_9t_{12}^{-1} \leq 1$,
$1.262626t_9t_{12}^{-1} - 1.231059t_2t_9t_{12}^{-1} \leq 1$,
$1.262626t_{10}t_{13}^{-1} - 1.231059t_3t_{10}t_{13}^{-1} \leq 1$,
$0.034750t_2t_5^{-1} + 0.975000t_2 - 0.009750t_2^2t_5^{-1} \leq 1$,
$0.034750t_3t_6^{-1} + 0.975000t_3 - 0.009750t_3^2t_6^{-1} \leq 1$,
$1.0t_1t_5^{-1}t_7^{-1}t_8 + 1.0t_4t_5^{-1} - 1.0t_4t_5^{-1}t_7^{-1}t_8 \leq 1$,
$0.002t_2t_9 + 0.002t_5t_8 + 1.0t_6 + 1.0t_5 - 0.002t_1t_8 - 0.002t_6t_9 \leq 1$,
$1.0t_2^{-1}t_3t_9^{-1}t_{10} + 1.0t_2^{-1}t_6 + 500.0t_9^{-1} - 1.0t_9^{-1}t_{10} - 500.0t_2^{-1}t_6t_9^{-1} \leq 1$,
$0.9t_2^{-1} + 0.002t_{10} - 0.002t_2^{-1}t_3t_{10} \leq 1$,
$1.0t_2t_3^{-1} \leq 1$,
$0.002t_7 - 0.002t_8 \leq 1$,
$0.034750t_1t_4^{-1} + 0.975000t_1 - 0.009750t_1^2t_4^{-1} \leq 1$,
$0.1 \leq t_1 \leq 1$,
$0.1 \leq t_2 \leq 1$,
$0.9 \leq t_3 \leq 1$,
$0.0001 \leq t_4 \leq 0.1$,
$0.1 \leq t_5 \leq 0.9$,
$0.1 \leq t_6 \leq 0.9$,
$0.1 \leq t_7 \leq 1000$,
$0.1 \leq t_8 \leq 1000$,
$500 \leq t_9 \leq 1000$,
$0.1 \leq t_{10} \leq 500$,
$1 \leq t_{11} \leq 150$,
$0.0001 \leq t_{12} \leq 150$,
$0.0001 \leq t_{13} \leq 150$.

Solution Demb76006

$g_0^* =$	97.607198775822
$F^* =$	97.607198464072
$\mathbf{max\{g_k\}} =$	1.000000000463
$h^* = 0.999999999996$	
$x_0^{*T} z_0^* =$	$.5246095218E-09$
$\lVert r_D \rVert_1 =$	$.3394249288E-13$
$\lVert r_P \rVert_1 =$	$.8012316884E-08$
T(seg.) =	4.671875000000
Iterations =	303

$\mathbf{t}^*$	**weight**
$t_1 = 0.8037731576$	$\omega_1 = 0.2069738974$
$t_2 = 0.9000000000$	$\omega_2 = 0.4092473157$
$t_3 = 0.9000000006$	$\omega_3 = 0.2001164818$
$t_4 = 0.1000000000$	$\omega_4 = 0.0023643205$
$t_5 = 0.1908367340$	$\omega_5 = 0.0005395137$
$t_6 = 0.8363072815$	$\omega_6 = 0.0038640748$
$t_7 = 574.0996378242$	$\omega_7 = 0.0068054723$
$t_8 = 74.0996377969$	$\omega_8 = 0.0477798266$
$t_9 = 500.0000000012$	$\omega_9 = 0.0000114264$
$t_{10} = 0.1000000000$	$\omega_{10} = 0.0530886962$
$t_{11} = 20.2391171575$	$\omega_{11} = 0.0000114264$
$t_{12} = 77.3364499943$	$\omega_{12} = 0.0084669067$
$t_{13} = 0.0316313122$	$\omega_{13} = 0.0035987386$
	$\omega_{14} = 0.0571319031$

04 D76007 Ref. [5] Prob. 7

Minimize $1.262626t_{12} + 1.262626t_{13} + 1.262626t_{14} + 1.262626t_{15} + 1.262626t_{16} - 1.231060t_1t_{12} - 1.231060t_2t_{13} - 1.231060t_3t_{14} - 1.231060t_4t_{15} - 1.231060t_5t_{16}$

Subject to :

$0.03475t_1t_6^{-1} + 0.975t_1 - 0.0097t_1^2t_6^{-1} \leq 1,$
$0.03475t_2t_7^{-1} + 0.975t_2 - 0.00975t_2^2t_7^{-1} \leq 1,$
$0.03475t_3t_8^{-1} + 0.975t_3 - 0.00975t_3^2t_8^{-1} \leq 1,$
$0.03475t_4t_9^{-1} + 0.975t_4 - 0.00975t_4^2t_9^{-1} \leq 1,$
$0.03475t_5t_{10}^{-1} + 0.975t_5 - 0.00975t_5^2t_{10}^{-1} \leq 1,$
$t_6t_7^{-1} + t_1t_7^{-1}t_{11}^{-1}t_{12} - t_6t_7^{-1}t_{11}^{-1}t_{12} \leq 1,$
$t_7t_8^{-1} + 0.002t_7t_8^{-1}t_{12} + 0.002t_2t_8^{-1}t_{13} - 0.002t_{13} - 0.002t_1t_8^{-1}t_{12} \leq 1,$
$t_8 + 0.002t_8t_{13} + 0.002t_3t_{14} + t_9 - 0.002t_2t_{13} - 0.002t_9t_{14} \leq 1,$
$t_3^{-1}t_9 + t_3^{-1}t_4t_{14}^{-1}t_{15} + 500.0t_3^{-1}t_{10}t_{14}^{-1} - 500.0t_3^{-1}t_9t_{14}^{-1} - t_3^{-1}t_8t_{14}^{-1}t_{15} \leq 1,$
$t_4^{-1}t_5t_{15}^{-1}t_{16} + t_4^{-1}t_{10} + 500.0t_{15}^{-1} - t_{15}^{-1}t_{16} - 500.0t_4^{-1}t_{10}t_{15}^{-1} \leq 1,$
$0.9t_4^{-1} + 0.002t_{16} - 0.002t_4^{-1}t_5t_{16} \leq 1,$
$0.002t_{11} - 0.002t_{12} \leq 1,$
$t_{11}^{-1}t_{12} \leq 1,$
$t_4t_5^{-1} \leq 1,$
$t_3t_4^{-1} \leq 1,$
$t_2t_3^{-1} \leq 1,$
$t_1t_2^{-1} \leq 1,$
$t_9t_{10}^{-1} \leq 1,$
$t_8t_9^{-1} \leq 1,$
$0.1 \leq t_1 \leq 0.9,$
$0.1 \leq t_2 \leq 0.9,$
$0.1 \leq t_3 \leq 0.9,$
$0.1 \leq t_4 \leq 0.9,$
$0.9 \leq t_5 \leq 1.0,$
$0.0001 \leq t_6 \leq 0.1,$
$0.1 \leq t_7 \leq 0.9,$
$0.1 \leq t_8 \leq 0.9,$
$0.1 \leq t_9 \leq 0.9,$
$0.1 \leq t_{10} \leq 0.9,$
$1.0 \leq t_{11} \leq 1000.0,$
$0.000001 \leq t_{12} \leq 500.0,$
$1.0 \leq t_{13} \leq 500.0,$
$500.0 \leq t_{14} \leq 1000.0$
$500.0 \leq t_{15} \leq 1000.0$
$0.000001 \leq t_{16} \leq 500.0.$

Solution Demb76007

$g_0^* =$	174.790706172889
$F^* =$	174.790703279747
$\mathbf{max\{g_k\}} =$	1.000000000061
$h^* = 1.000000000026$	
$x_0^{*T} z_0^* =$	$.2537237200E-14$
$\lVert r_D \rVert_1 =$	$.5909268298E-13$
$\lVert r_P \rVert_1 =$	$.2560164865E-07$
T(seg.) =	184.968750000000
Iterations =	3253
$\mathbf{t}^*$	**weigtht**
$t_1 = 0.8037731572$	$\omega_1 = 0.0036591705$
$t_2 = 0.8181458208$	$\omega_2 = 0.0453465953$
$t_3 = 0.9000000000$	$\omega_3 = 0.3444899373$
$t_4 = 0.9000000000$	$\omega_4 = 0.3444899373$
$t_5 = 0.9000000000$	$\omega_5 = 0.0000005870$
$t_6 = 0.1000000000$	$\omega_6 = 0.0068468208$
$t_7 = 0.1082720460$	$\omega_7 = 0.0065518969$
$t_8 = 0.1908367350$	$\omega_8 = 0.0044982647$
$t_9 = 0.1908367350$	$\omega_9 = 0.0044982647$
$t_{10} = 0.1908367350$	$\omega_{10} = 0.0044982647$
$t_{11} = 505.9468243023$	$\omega_{11} = 0.0011799979$
$t_{12} = 5.9468244219$	$\omega_{12} = 0.0157397304$
$t_{13} = 72.4018964201$	$\omega_{13} = 0.0054450765$
$t_{14} = 499.9999999995$	$\omega_{14} = 0.0128773946$
$t_{15} = 500.0000000000$	$\omega_{15} = 0.0207433707$
$t_{16} = 0.0008519936$	$\omega_{16} = 0.0230481896$
	$\omega_{17} = 0.0230481896$
	$\omega_{18} = 0.0000001852$
	$\omega_{19} = 0.0230481896$
	$\omega_{20} = 0.0000001852$
	$\omega_{21} = 0.0012928033$
	$\omega_{22} = 0.1086969481$

05 RM7809 Ref. [18] Prob. 9

Minimize $3.7t_1^{0.85} + 1.985t_1 + 700.3t_2^{-0.75},$
Subject to :
$0.7673t_2^{0.05} - 0.05t_1 \leq 1,$
$t_j > 0, i = 1, 2.$

Solution RM1978009

$g_0^* =$	11.964337119829
$F^* =$	11.964337119824
$\mathbf{max\{g_k\}} =$	0.999999999989
$h^* = 1.000000000011$	
$x_0^{*T} z_0^* =$	$.4967839200E-11$
$\lVert r_D \rVert_1 =$	$.4654170454E-15$
$\lVert r_P \rVert_1 =$	$.5331494966E-12$
T(seg.) =	0.000000000000
Iterations =	58
$\mathbf{t}^*$	**weigtht**
$t_1 = 0.8113379810$	$\omega_1 = 0.0389853830$
$t_2 = 442.6863888235$	$\omega_2 = 0.9610146170$

06 RM7810 Ref. [18] Prob. 10

Minimize $0.5t_1t_2^{-1} - t_1 - 5t_2^{-1},$
Subject to :
$0.01t_2t_3^{-1} + 0.01t_1 + 0.0005t_1t_3 \leq 1,$
$1 \leq t_i \leq 100, i = 1, 2, 3.$

Solution RM1978010

$g_0^* =$	−83.249728404775
$F^* =$	−83.249728406002
$\mathbf{max\{g_k\}} =$	1.000000000000
$h^* = 1.000000000014$	
$x_0^{*T} z_0^* =$	$.1906137824E-12$
$\lVert r_D \rVert_1 =$	$.2256881884E-15$
$\lVert r_P \rVert_1 =$	$.3406664635E-14$
T(seg.) =	0.000000000000
Iterations =	54
$\mathbf{t}^*$	**weight**
$t_1 = 88.3559452410$	$\omega_1 = 0.9925445690$
$t_2 = 7.6726026111$	$\omega_2 = 0.0073205134$
$t_3 = 1.3178575281$	$\omega_3 = 0.0001349176$

07 RM7811 Ref. [18] Prob. 11

Minimize $-t_1 + 0.4t_1^{0.67}t_3^{-0.67},$
Subject to :
$0.05882t_3t_4 + 0.1t_1 \leq 1,$
$4t_2t_4^{-1} + 2t_2^{-0.71}t_4^{-1} + 0.05882t_2^{-1.3}t_3 \leq 1,$

Solution RM1978011

$g_0^* =$	-5.739820303591
$F^* =$	-5.739820303586
$\max\{\mathbf{g_k}\} =$	1.000000000000
$h^* = 0.999999999999$	
$x_0^{*T} z_0^* =$	$.2938712383E-11$
$\lVert r_D \rVert_1 =$	$.4611924825E-15$
$\lVert r_P \rVert_1 =$	$.6811043555E-10$
T(seg.) =	0.000000000000
Iterations =	28
$\mathbf{t}^*$	**weigtht**
$t_1 = 8.1300721655$	$\omega_1 = 0.9796089073$
$t_2 = 0.6153662462$	$\omega_2 = 0.0203910927$
$t_3 = 0.5640437585$	
$t_4 = 5.6362082107$	

08 RM7812 Ref. [18] Prob. 12

Minimize $-t_1 - t_5 + 0.4t_1^{0.67}t_3^{-0.67} + 0.4t_5^{0.67}t_7^{-0.67}$,
Subject to :
$0.05882t_3t_4 + 0.1t_1 \le 1$,
$0.05882t_7t_8 + 0.1t_1 + 0.1t_5 \le 1$,
$4t_2t_4^{-1} + 2t_2^{-0.71}t_4^{-1} + 0.05882t_2^{-1.3}t_3 \le 1$,
$4t_6t_8^{-1} + 2t_6^{-0.71}t_8^{-1} + 0.05882t_6^{-1.3}t_7 \le 1$,

Solution RM1978012

$g_0^* =$	-6.048232888864
$F^* =$	-6.048232894162
$\max\{\mathbf{g_k}\} =$	1.000000000180
$h^* = 1.000000000131$	
$x_0^{*T} z_0^* =$	$.2996346052E-14$
$\lVert r_D \rVert_1 =$	$.1172030589E-13$
$\lVert r_P \rVert_1 =$	$.5613654368E-07$
T(seg.) =	0.031250000000
Iterations =	123
$\mathbf{t}^*$	**weight**
$t_1 = 6.4637375629$	$\omega_1 = 0.7296591759$
$t_2 = 0.6674377031$	$\omega_2 = 0.2521607796$
$t_3 = 1.0133321377$	$\omega_3 = 0.0181800445$
$t_4 = 5.9329085131$	
$t_5 = 2.2337841512$	
$t_6 = 0.5957671238$	
$t_7 = 0.4006202891$	
$t_8 = 5.5272935743$	

09 RM7813 Ref. [5] Prob. 5,[18] Prob. 13

Minimize $t_1 + t_2 + t_3$,
Subject to :
$833.332352t_1^{-1}t_4t_6^{-1} + 100t_6^{-1} - 83333.333t_1^{-1}t_6^{-1} \le 1$,
$1250t_2^{-1}t_5t_7^{-1} + t_4t_7^{-1} - 1250t_2^{-1}t_4t_7^{-1} \le 1$,
$1250000t_3^{-1}t_8^{-1} + t_5t_8^{-1} - 2500t_3^{-1}t_5t_8^{-1} \le 1$,
$0.0025t_4 + 0.0025t_6 \le 1$,
$0.0025t_5 + 0.0025t_7 - 0.0025t_4 \le 1$,
$0.01t_8 - 0.01t_5 \le 1$,

Solution RM1978013

$g_0^* =$	7049.248913688329
$F^* =$	7049.248901613300
$\max\{\mathbf{g_k}\} =$	1.000000000023
$h^* = 1.000000000031$	
$x_0^{*T} z_0^* =$	$.2760766285E-14$
$\lVert r_D \rVert_1 =$	$.1144528951E-13$
$\lVert r_P \rVert_1 =$	$.3398148279E-07$
T(seg.) =	0.203125000000
Iterations =	364
$\mathbf{t}^*$	**weight**
$t_1 = 577.9663698134$	$\omega_1 = 0.1098036771$
$t_2 = 1360.9612542577$	$\omega_2 = 0.0969183737$
$t_3 = 5110.3212775423$	$\omega_3 = 0.0607127979$
$t_4 = 181.9056731085$	$\omega_4 = 0.0755320587$
$t_5 = 295.5871489094$	$\omega_5 = 0.4909424867$
$t_6 = 218.0943269008$	$\omega_6 = 0.1660906059$
$t_7 = 286.3185241828$	
$t_8 = 395.5871489081$	

10 RM7814 Ref. [18] Prob. 14

Minimize $t_6 + 0.4t_4^{0.67} + 0.4t_9^{0.67}$,
Subject to :
$t_1^{-1}t_2^{-1.5}t_3t_4^{-1}t_5^{-1} + 5t_1^{-1}t_2^{-1}t_3t_5^{1.2} \le 1$,
$0.05t_3 + 0.05t_2 \le 1$,
$10t_3^{-1} - t_1t_3^{-1} \le 1$,
$t_6^{-1}t_7^{-1.5}t_8t_9^{-1}t_{10}^{-1} + 5t_6^{-1}t_7^{-1}t_8t_{10}^{1.2} \le 1$,
$t_2^{-1}t_7 + t_2^{-1}t_8 \le 1$,
$t_1t_8^{-1} - t_6t_8^{-1} \le 1$,
$10t_{10} \le 1$,

Solution RM1978014

$g_0^* =$	1.143623161094
$F^* =$	1.143623160274
$\max\{\mathbf{g_k}\} =$	1.000000000302
$h^* = 0.999999999869$	
$x_0^{*T} z_0^* =$	$.3067277405E-14$
$\lVert r_D \rVert_1 =$	$.3216969076E-13$
$\lVert r_P \rVert_1 =$	$.6519037594E-07$
T(seg.) =	0.000000000000
Iterations =	32
$\mathbf{t}^*$	**weight**
$t_1 = 2.0951912914$	$\omega_1 = 0.1718589867$
$t_2 = 12.0951912939$	$\omega_2 = 0.1797456043$
$t_3 = 7.9048087063$	$\omega_3 = 0.6483954090$
$t_4 = 0.4594056835$	
$t_5 = 0.3579263266$	
$t_6 = 0.4547551911$	
$t_7 = 10.4547551945$	
$t_8 = 1.6404360998$	
$t_9 = 1.1974604599$	
$t_{10} = 0.1000000000$	

11 RM7815 Ref. [18] Prob. 15

Minimize $0.05t_1 + 0.05t_2 + 0.05t_3 + t_9$,
Subject to :
$0.5t_9t_{10}^{-1} + 0.25t_{10}^{-1} \le 1$,
$t_7^{-1}t_{10} - 0.5t_1t_4t_7^{-1} \le 1$,
$t_7t_8^{-1} - 0.5t_2t_5t_8^{-1} \le 1$,
$t_8t_9^{-1} - 0.5t_3t_6t_9^{-1} \le 1$,
$0.79681t_4t_7^{-1} \le 1$,
$0.79681t_5t_8^{-1} \le 1$,
$0.79681t_6t_9^{-1} \le 1,\ t_j > 0, j = 1 \cdots, 10.$

Solution RM1978015

$g_0^* =$	0.205653413173
$F^* =$	0.205653413173
$\max\{\mathbf{g_k}\} =$	1.000000000000
$h^* = 1.000000000002$	
$x_0^{*T} z_0^* =$	$.2516764773E-14$
$\lVert r_D \rVert_1 =$	$.2288362244E-15$
$\lVert r_P \rVert_1 =$	$.2978168587E-11$
T(seg.) =	0.015625000000
Iterations =	34
$\mathbf{t}^*$	**weight**
$t_1 = 0.7240462847$	$\omega_1 = 0.1922710170$
$t_2 = 0.7240462847$	$\omega_2 = 0.1922710170$
$t_3 = 0.7240462847$	$\omega_3 = 0.1922710170$
$t_4 = 0.2576067462$	$\omega_4 = 0.4231869490$
$t_5 = 0.1771295832$	
$t_6 = 0.1217937406$	
$t_7 = 0.2052636315$	
$t_8 = 0.1411386232$	
$t_9 = 0.0970464705$	
$t_{10} = 0.2985232352$	

12 RM7816 Ref. [18] Prob. 16

Minimize $0.0t_1 + 0.05t_2 + 0.05t_3 + t_9$
Subject to :
$0.5t_9t_{10}^{-1} + 0.25t_{10}^{-1} \le 1$,
$t_7^{-1}t_{10} - 0.5t_1t_4t_7^{-1} \le 1$,
$t_7t_8^{-1} - 0.5t_2t_5t_8^{-1} \le 1$,
$t_8t_9^{-1} - 0.5t_3t_6t_9^{-1} \le 1$,
$0.700329t_4t_7^{-1} + 0.307795t_7 \le 1$,
$0.700329t_5t_8^{-1} + 0.307795t_8 \le 1$,
$0.700329t_6t_9^{-1} + 0.307795t_9 \le 1$.

Solution RM1978016

$g_0^* =$	0.196631321203
$F^* =$	0.196631321203
$\max\{\mathbf{g_k}\} =$	1.000000000000
$h^* = 1.000000000005$	
$x_0^{*T} z_0^* =$	$.1092329201E-14$
$\lVert r_D \rVert_1 =$	$.2513431779E-15$
$\lVert r_P \rVert_1 =$	$.3719809349E-11$
T(seg.) =	0.015625000000
Iterations =	122
$\mathbf{t}^*$	**weights**
$t_1 = 0.7295974658$	$\omega_1 = 0.1983906924$
$t_2 = 0.7133051592$	$\omega_2 = 0.1980883855$
$t_3 = 0.7029907344$	$\omega_3 = 0.1979559977$
$t_4 = 0.2653361383$	$\omega_4 = 0.4055649244$
$t_5 = 0.1820601417$	
$t_6 = 0.1240561647$	
$t_7 = 0.1978740396$	
$t_8 = 0.1329418204$	
$t_9 = 0.0893366532$	
$t_{10} = 0.2946683266$	

13 RM7817 Ref. [18] Prob. 17

Minimize t_3^{-1},
Subject to :
$0.1t_{10} + t_7 t_{10} \le 1$,
$10t_1 t_4 + 10t_1 t_4 t_7^2 \le 1$
$t_4^{-1} - 100t_7 t_{10} \le 1$
$t_{10} t_{11}^{-1} - 10t_8 \le 1$
$t_1^{-1} t_2 t_5 + t_1^{-1} t_2 t_5 t_8^2 \le 1$
$t_5^{-1} - 10t_1^{-1} t_8 t_{11} \le 1$
$10t_{11} - 10t_9 \le 1$
$t_2^{-1} t_3 t_6 + t_2^{-1} t_3 t_6 t_9^2 \le 1$
$t_6^{-1} - t_2^{-1} t_9 \le 1$
$t_j > 0, j = 1, \cdots, 11.$

Solution RM1978017

$g_0^* =$	0.140606724804
$F^* =$	0.140606672182
$\max\{\mathbf{g_k}\} =$	1.000000215669
$h^* = 0.999999998313$	
$x_0^{*T} z_0^* =$	$.3412153120E-14$
$\lVert r_D \rVert_1 =$	$.5722177464E-13$
$\lVert r_P \rVert_1 =$	$.7606740006E-07$
T(seg.) =	0.703125000000
Iterations =	886
$\mathbf{t}^*$	**weights**
$t_1 = 7.0040534693$	$\omega_1 = 0.9041784774$
$t_2 = 7.6461595143$	$\omega_2 = 0.0409176134$
$t_3 = 7.1120380312$	$\omega_3 = 0.0026477274$
$t_4 = 0.0124657665$	$\omega_4 = 0.0403153625$
$t_5 = 0.8117007374$	$\omega_5 = 0.0005272629$
$t_6 = 0.9558434800$	$\omega_6 = 0.0114135565$
$t_7 = 0.3812249178$	
$t_8 = 0.3585001170$	
$t_9 = 0.3532234894$	
$t_{10} = 2.0780303827$	
$t_{11} = 0.4532235060$	

14 RM7818 Ref. [18] Prob. 18

Minimize t_9^{-1}
Subject to :
$t_1 + t_1 t_{10} + t_1 t_{10} t_{12} \le 1$,
$t_1^{-1} t_4 t_{10}^{-1} + 0.01t_1^{-1} t_4 t_{12}^{-1} + 0.01t_1^{-1} t_4 \le 1$,
$100t_4^{-1} t_7 t_{10}^{-1} \le 1$,
$t_1^{-1} t_2 + t_1^{-1} t_2 t_{11} + t_1^{-1} t_2 t_{11} t_{13} \le 1$,
$-t_2 t_4^{-1} t_{11} + t_4^{-1} t_5 + 0.001t_4^{-1} t_5 t_{11} t_{13}^{-1} +$
$0.01t_4^{-1} t_5 t_{11} \le 1$,
$-0.01t_5 t_7 t_{11} + t_7^{-1} t_8 \le 1$,
$12601t_2^{-1} t_3 \le 1$,
$-2100t_3 t_5^{-1} + 26.5t_5^{-1} t_6 \le 1$,
$-21t_6 t_8^{-1} + t_8^{-1} t_9 \le 1.$

Solution RM1978018

$g_0^* =$	1.861627253912
$F^* =$	1.861627237317
$\max\{\mathbf{g_k}\} =$	1.000000003637
$h^* = 0.999999999988$	
$x_0^{*T} z_0^* =$	$.3095118067E-14$
$\lVert r_D \rVert_1 =$	$.1207180737E-13$
$\lVert r_P \rVert_1 =$	$.8871196266E-10$
T(seg.) =	0.484375000000
Iterations =	574
$\mathbf{t}^*$	**weights**
$t_1 = 0.3608700233$	$\omega_1 = 0.0054759706$
$t_2 = 0.1125420373$	$\omega_2 = 0.0000012111$
$t_3 = 0.0000089312$	$\omega_3 = 0.0004214871$
$t_4 = 0.4978063866$	$\omega_4 = 0.9796806166$
$t_5 = 0.6416993276$	$\omega_5 = 0.0144207147$
$t_6 = 0.0252082003$	
$t_7 = 0.0077916051$	
$t_8 = 0.0077922595$	
$t_9 = 0.5371644656$	
$t_{10} = 1.5651878566$	
$t_{11} = 1.6796547291$	
$t_{12} = 0.1315452088$	
$t_{13} = 0.3136838972$	

15 RM7819 Ref. [18] Prob. 19

Minimize $2.0425t_1^{0.782} + 52.25t_2 + 192.85t_2^{0.9} + 5.25t_2^3 + 61.465t_6^{0.467} + 0.01748t_3^{1.33} t_4^{-0.8} + 100.7t_4^{0.546} + (3.66 \times 10^{-10}) t_3^{2.85} t_4^{-1.7} + 0.00945t_5 + (1.06 \times 10^{-10}) t_4^{-1.8} t_5^{2.8} + 116t_6 - 205t_6 t_7 - 278t_2^3 t_7$,
Subject to :
$129.4t_2^{-3} + 105t_6^{-1} \le 1$,
$(1.03 \times 10^5) t_2^3 t_3^{-1} t_7 t_8^{-1} + (1.2 \times 10^6) t_3^{-1} t_8^{-1} \le 1$,
$4.68t_1^{-1} t_2^3 + 61.3t_1^{-1} t_2^2 + 160.5t_1^{-1} t_2 \le 1$,
$1.79t_7 + 3.02t_2^3 t_6^{-1} t_7 + 35.7t_6^{-1} \le 1$,
$(1.22 \times 10^{-3}) t_3 t_4^{-0.2} t_5^{-0.8} t_8 + (1.67 \times 10^{-3}) t_3^{0.4} t_4^{-0.43} t_8 + (3.6 \times 10^{-5}) t_3 t_4^{-1} t_8 + (2 \times 10^{-3}) t_3 t_5^{-1} t_8 + (4 \times 10^{-3}) t_8 \le 1.$

Solution RM1978019

$g_0^* =$	17485.988295811050
$F^* =$	17485.988295670832
$\max\{\mathbf{g_k}\} =$	1.000000000000
$h^* = 1.000000000004$	
$x_0^{*T} z_0^* =$	$.1567956644E-14$
$\lVert r_D \rVert_1 =$	$.5607639536E-15$
$\lVert r_P \rVert_1 =$	$.4148890565E-11$
T(seg.) =	0.031250000000
Iterations =	134
$\mathbf{t}^*$	**weights**
$t_1 = 5153.5320817306$	$\omega_1 = 0.1472567781$
$t_2 = 6.6494207551$	$\omega_2 = 0.3130502884$
$t_3 = 169413.5996007385$	$\omega_3 = 0.5396929334$
$t_4 = 743.3944009036$	
$t_5 = 87999.0451805067$	
$t_6 = 187.5441685127$	
$t_7 = 0.1240970111$	
$t_8 = 29.2653086647$	

16 RM7820 Ref. [18] Prob. 20

Minimize $-0.28t_1 t_6^{-1} + 0.6732t_2 t_6^{-1} + 1.12t_3 t_6^{-1} - 3104.139t_5^{-1} + 0.0074t_5 t_6^{-1} + 10$,
Subject to :
$0.73398t_3^{-1} t_4^{1.67} t_7 t_{10} t_{11} \le 1$,
$0.639926t_4^{-0.25} t_8 t_{10}^{-1} - 0.156564t_4^{0.42} t_9^{-1} t_{11} - 0.1t_{10} t_{13}^{-1} \le 1$,
$3809.973t_4^{-1.25} t_7^{-1} t_9^{-1} t_{10}^{-1} + 0.195706t^{0.42} t_9^{-1} t_{11} \le 1$,
$0.31254t_2^{-1} t_4^{1.25} t_7 t_9 t_{10} t_{12} t_{13}^{-1} \le 1$,
$t_1 t_4^{-1} t_7^{-1} t_8^{-1} t_9^{-1} - 0.31254t_4^{0.25} t_{10} t_{13}^{-1} \le 1$,
$0.02t_5^2 t_6^{-1} t_7 \le 1$,
$t_{11}^{-1} t_{13} + 1.25014t_4^{1.25} t_7 t_9 t_{10} t_{11}^{-1} - .24466t_4^{1.67} t_7 t_{10} \le 1$,
$t_5^{-1} t_{12} + 0.73398t_4^{1.67} t_5^{-1} t_7 t_{10} t_{11} + t_5^{-1} t_{11} - t_5^{-1} t_{12} \le 1$,

$t_{10}t_{12}^{-1} + 0.24466t_4^{0.67}t_9^{-1}t_{10}t_{11}t_{12}^{-1} +$
$0.15627t_4^{.25}t_{10}^{2}t_{12}^{-1}t_{13}^{-1} + t_9t_{12}^{-1} + 11.0t_{12}^{-1}t_{13} +$
$1.5628t_4^{.25}t_{10}t_{12}^{-1} \le 1,$
$6.14 \le t_4 \le 129.53$

Solution RM1978020

$g_0^* =$	-116.4498607061
$F^* =$	-116.4498652112
$\mathbf{max\{g_k\}} =$	1.000000005249
$h^* = 0.999999999819$	
$x_0^{*T}z_0^* =$	$.2179870824E-14$
$\lVert r_D \rVert_1 =$	$.4060664775E-13$
$\lVert r_P \rVert_1 =$	$.2616307774E-09$
T(seg.) =	0.265625000000
Iterations =	228
$\mathbf{t}^*$	**weights**
$t_1 = 13511.0987675454$	$\omega_1 = 0.0993270635$
$t_2 = 36299.7373348968$	$\omega_2 = 0.8151555355$
$t_3 = 3204.6385324366$	$\omega_3 = 0.0068144500$
$t_4 = 106.9843087033$	$\omega_4 = 0.0025158170$
$t_5 = 370087.7405901506$	$\omega_5 = 0.0252880115$
$t_6 = 32.1504418971$	$\omega_6 = 0.0025728180$
$t_7 = 0.0000000117$	$\omega_7 = 0.0018272354$
$t_8 = 47479.7267090123$	$\omega_8 = 0.0464990691$
$t_9 = 146795.5146999597$	
$t_{10} = 7868.4599482367$	
$t_{11} = 19306.0306494466$	
$t_{12} = 362120.1048956879$	
$t_{13} = 14543.0315447451$	

17 RM7821 Ref. [18] Prob.21

Minimize $-0.063t_4t_7 + 5.04t_1 + 0.035t_2 + 10t_3 + 3.35t_5,$
Subject to :
$0.89286t_1^{-1}t_4 - 0.11756t_8 + 0.005955t_8^2 \le 1,$
$0.01741t_7 - 0.01912t_8 + 0.0006617t_8^2 - 0.0056596t_6 \le 1,$
$35.82t_9^{-1} - 0.222t_9^{-1}t_{10} \le 1,$
$0.333t_7^{-1}t_{10} + 44.3333t_7^{-1} \le 1,$
$(1.0204 \times 10^{-5})t_3^{-1}t_4t_6t_9 + (1.0204 \times 10^{-2})t_6 \le 1,$
$1.22t_4t_5^{-1} - t_1t_5^{-1} \le 1,$
$t_1t_2^{-1}t_8 - 1.22t_2^{-1}t_4 + t_1t_2^{-1} \le 1,$
$t_1 \le 2000, t_2 \le 19200, t_3 \le 120,$
$t_4 \le 5000, t_5 \le 2000,$
$85 \le t_6 \le 93,$
$90 \le t_7 \le 95,$
$3 \le t_8 \le 12,$
$1.2 \le t_9 \le 4,$
$145 \le t_{10} \le 162,$

Solution RM1978021

$g_0^* =$	-1250.929530643
$F^* =$	-1250.908652238
$\mathbf{max\{g_k\}} =$	1.000000000035
$h^* = 1.000000005115$	
$x_0^{*T}z_0^* =$	$.5709268285E-14$
$\lVert r_D \rVert_1 =$	$.2122779514E-12$
$\lVert r_P \rVert_1 =$	$.2122852059E-07$
T(seg.) =	4.312500000000
Iterations =	2363
$\mathbf{t}^*$	**weights**
$t_1 = 1770.7222474328$	$\omega_1 = 0.6211336082$
$t_2 = 18860.9598055321$	$\omega_2 = 0.0257849936$
$t_3 = 94.8599683528$	$\omega_3 = 0.0041936805$
$t_4 = 3090.7557179242$	$\omega_4 = 0.0096676834$
$t_5 = 2000.0000000699$	$\omega_5 = 0.3003970638$
$t_6 = 91.7514558192$	$\omega_6 = 0.0164833002$
$t_7 = 94.9276340979$	$\omega_7 = 0.0037220692$
$t_8 = 11.7810427184$	$\omega_8 = 0.0186176011$
$t_9 = 2.0904445938$	
$t_{10} = 151.9349371205$	

18 RM7822Ref. [18] Prob. 22

Minimize $2.8485t_1 - 22.499t_1t_2 + 2.8952t_1t_3 + 0.3057t_1t_4 - 4.4318t_1t_5 + 0.14t_1t_5^2 + 3.5974t_1t_6 + 0.05t_1t_7,$
Subject to :
$0.025616t_1^2t_7^{-1} + 0.293164t_1^2t_6t_7^{-1} + 0.83877t_1^2t_6^2t_7^{-1} \le 1,$
$100t_3t_9^{-1} - 100t_3t_8^{0.01}t_9^{-1.01} + t_8t_9^{-1} \le 1,$
$0.4744t_1^{-1}t_4t_8^{-1} + 0.87564t_1^{-1}t_4t_6t_8^{-1} + 0.012152t_1t_8^{-1} + 0.1391t_1t_6t_8^{-1} + 0.3979t_1t_6^2t_8^{-1} - 5.7222t_6 \le 1,$
$10.4351t_1^{-1}t_4t_5^{-1}t_9^{-1} + 72.5476t_5^{-1} + 5.6303t_3t_5^{-1} + 0.1279t_4t_5^{-1} - 1.8459t_6 - 133.9131t_5^{-1}t_6 + 10.3930t_3t_5^{-1}t_6 + 0.2362t_4t_5^{-1}t_6 + 19.2611t_1^{-1}t_4t_5^{-1}t_6t_9^{-1} \le 1,$
$-4.44t_5^{-1} + 41.04t_2t_5^{-1} + 5.63t_3t_5^{-1} + 0.1228t_4t_5^{-1} \le 1,$
$3.309 \times 10^{-3}t_1 - 6.61 \times 10^{-3}t_1t_3 - 4.858 \times 10^{-4}t_1t_4 + 1.009 \times 10^{-2}t_1t_5 - 1.294 \times 10^{-6}t_1^3t_3 - 1.49 \times 10^{-5}t_1^3t_6 - 4.237 \times 10^{-5}t_1^3t_6^2 - 2.5322 \times 10^{-4}t_1t_5^2 \le 1,$
$0.4t_6^{-1} \le 1,$
$21.3351t_4^{-1} - 1.8458t_6 \le 1,$
$0.002017t_1 + 0.004878t_1t_2 + 0.005735t_1t_5 - 0.000744t_1t_3 - 0.000063t_1t_4 - 0.000019t_1t_7 \le 1,$
$0.001817t_1 + 0.011287t_1t_2 + 0.010795t_1t_5 + 0.000013t_1t_7 - 0.003304t_1t_3 - 0.000471t_1t_4 - 0.000363t_1t_5^2 \le 1.$

Solution RM1978022

$g_0^* =$	-390.5368109851
$F^* =$	-390.5368134356
$\mathbf{max\{g_k\}} =$	0.999999999995
$h^* = 1.000000002344$	
$x_0^{*T}z_0^* =$	$.2591878712E-08$
$\lVert r_D \rVert_1 =$	$.1047606283E-13$
$\lVert r_P \rVert_1 =$	$.5178857702E-09$
T(seg.) =	69.937500000000
Iterations =	3963
$\mathbf{t}^*$	**weights**
$t_1 = 11.6001176608$	$\omega_1 = 0.0994168794$
$t_2 = 0.4055502437$	$\omega_2 = 0.6622390929$
$t_3 = 0.0006429340$	$\omega_3 = 0.0001610880$
$t_4 = 12.2734026369$	$\omega_4 = 0.0518474862$
$t_5 = 13.7145769831$	$\omega_5 = 0.0167252586$
$t_6 = 0.4000000000$	$\omega_6 = 0.0884718928$
$t_7 = 37.2852870594$	$\omega_7 = 0.0073333976$
$t_8 = 9.0407374733$	$\omega_8 = 0.0000011167$
$t_9 = 9.0408201302$	$\omega_9 = 0.0015667134$
	$\omega_{10} = 0.0000457537$
	$\omega_{11} = 0.0002107357$
	$\omega_{12} = 0.0002397013$
	$\omega_{13} = 0.0125149729$
	$\omega_{14} = 0.0167243525$
	$\omega_{15} = 0.0000001257$
	$\omega_{16} = 0.0002031761$
	$\omega_{17} = 0.0001861479$
	$\omega_{18} = 0.0000005582$
	$\omega_{19} = 0.0015189832$
	$\omega_{20} = 0.0179406648$
	$\omega_{21} = 0.0226519023$

19 RM7823 Ref. [18] Prob.23, [18],[5] Prob. 2, [12] Prob. GGP5]

Minimize $5.3578t_3^2 + 0.8357t_1t_5 + 37.2392t_1,$
Subject to :
$0.00002584t_3t_5 - 0.00006663t_2t_5 - 0.0000734t_1t_4 \le 1,$
$0.000853007t_2t_5 + 0.00009395t_1t_4 - 0.00033085t_3t_5 \le 1,$
$1330.3294t_2^{-1}t_5^{-1} - 0.42t_1t_5^{-1} - 0.30586t_2^{-1}t_3^2t_5^{-1} \le 1,$
$0.00024186t_2t_5 + 0.00010159t_1t_2 + 0.00007379t_3^2 \le 1,$
$2275.1327t_3^{-1}t_5^{-1} - 0.2668t_1t_5^{-1} - 0.40584t_4t_5^{-1} \le 1,$
$0.00029955t_3t_5 + 0.00007992t_1t_3 + 0.00012157t_3t_4 \le 1,$
$78 \le t_1 \le 102,$
$33 \le t_2 \le 45,$
$27 \le t_3 \le 45,$
$27 \le t_4 \le 45,$
$27 \le t_5 \le 45.$

Solution RM1978023

$g_0^* =$	10122.430548961020
$F^* =$	10122.430549309804
$\max\{\mathbf{g_k}\} =$	0.999999999999
$h^* = 1.000000000004$	
$x_0^{*T} z_0^* =$	$.4613817076E-10$
$\lVert r_D \rVert_1 =$	$.4904649473E-15$
$\lVert r_P \rVert_1 =$	$.2401908574E-10$
T(seg.) =	0.015625000000
Iterations =	24
$\mathbf{t}^*$	**weights**
$t_1 = 78.0000000001$	$\omega_1 = 0.0208211454$
$t_2 = 33.0000000001$	$\omega_2 = 0.0663390586$
$t_3 = 29.9955107270$	$\omega_3 = 0.0939741193$
$t_4 = 45.0000000003$	$\omega_4 = 0.2293941151$
$t_5 = 36.7751740123$	$\omega_5 = 0.0583892179$
	$\omega_6 = 0.1457161780$
	$\omega_7 = 0.1278727502$
	$\omega_8 = 0.2574934155$

20. CB2005001 - Machining economics [3]

Minimize $\mathbf{g(V, F)} = 452V^{-1}F^{-1} + 10^{-5}V^{2.33}F^{0.4}$,
Subject to :
$1.9273 \times 10^{-2} V F^{0.83} \leq 1$,
$1.1 \times 10^4 V^{-1.52} F \leq 1$,
$F > 0, V \in \{150, 160, 170, 180, 190, 200\}$.
Continuous Solution MachEc

$g_0^* =$	12.0976375861661
$F^* =$	0.94558537294763
$\max\{\mathbf{g_k}\} =$	1.00000000000000
$x_0^{*T} z_0^* =$	0.00000000000000
$\lVert r_D \rVert_1 =$	0.00000000000000
$\lVert r_P \rVert_1 =$	0.00000000000000
T(seg.) =	0.10667560000000
Iterations =	18
$\mathbf{t}^*$	
$V = 174.386698874543$	
$F = 0.23211735701777$	

SGP equivalent

Minimize $\mathbf{g(V, F)} = 452V^{-1}F^{-1} + 10^{-5}V^{2.33}F^{0.4}$,
Subject to :
$1.9273 \times 10^{-2} V F^{0.83} \leq 1$,
$1.1 \times 10^4 V^{-1.52} F \leq 1$,
$\frac{170e-180}{e-1} V^{-1} + \frac{10u}{e-1} V^{-1} = 1$
$\frac{9}{4} u_i^{\ln(\frac{4}{9})+2\ln(1+\frac{e^\rho}{2})} - u^\rho - \frac{u^{2\rho}}{4} \leq 1.$
$u \in [1, e], \quad \rho = 1, \ e = exp(1)$

Solution MachEc

$g_0^* =$	12.2264777590620
$F^* =$	12.2264777589867
$\max\{\mathbf{g_k}\} =$	1.00000000000029
$h^* =$	1.00000000000000
$x_0^{*T} z_0^* =$	0.00000000216918
$\lVert r_D \rVert_1 =$	0.00000000000087
$\lVert r_P \rVert_1 =$	0.00000000000472
T(seg.) =	0.08895550000000
Iterations =	37
$\mathbf{t}^*$	**weights**
$V = 180.000000000571$	$\omega_1 = 0.4884124397$
$F = 0.22342426067835$	$\omega_2 = 0.3319106649$
$u = 2.71828182845985$	$\omega_3 = 0.1796768954$

21 . CB2005002 Economic Order Quantity [3]

Minimize $\mathbf{K(Q)} = 50000Q_1^{-l} + 50000Q_2^{-1} + 100000Q_3^{-1} + 5Q_1 + 2Q_2 + 8Q_3$,
Subject to :
$50Q_1 + 20Q_2 + 80Q_3 \leq 15000$,
$Q_1 = 6I, \ Q_2 = 5I, \ Q_3 = 6I$,
I Positive and Integer .

Solution EOQ (Solução Contínua)

$g_0^* =$	3450.893587977819
$F^* =$	3450.893587977818
$\max\{\mathbf{g_k}\} =$	0.99999999999891
$x_0^{*T} z_0^* =$	0.00000000000026
$\lVert r_D \rVert_1 =$	0.00000000000000
$\lVert r_P \rVert_1 =$	0.00000000000000
T(seg.) =	0.00824210000000
Iterations =	12
$\mathbf{t}^*$	
$t_1 = 87.68571323197986$	
$t_2 = 138.6432860346997$	
$t_3 = 98.03560772113349$	

SGP equivalent

Minimize $\mathbf{K(Q)} = 50000Q_1^{-l} + 50000Q_2^{-1} + 100000Q_3^{-1} + 5Q_1 + 2Q_2 + 8Q_3$,
Subject to :
$50Q_1 + 20Q_2 + 80Q_3 \leq 15000$,
$\frac{84e-90}{e-1} Q_1^{-1} + \frac{6u_1}{e-1} Q_1^{-1} = 1$
$\frac{135e-140}{e-1} Q_2^{-1} + \frac{5u_2}{e-1} Q_2^{-1} = 1$
$\frac{96e-102}{e-1} Q_3^{-1} + \frac{6u_3}{e-1} Q_3^{-1} = 1$
$\frac{9}{4} u_i^{\ln(\frac{4}{9})+2\ln(1+\frac{e^\rho}{2})} - u_i^\rho - \frac{u_i^{2\rho}}{4} \leq 1.$
$u_i \in [1, e] \quad i = 1, 2, 3, \ \rho = 1, \ e = exp(1)$

Solution EOQ

$g_0^* =$	3452.36507912689
$F^* =$	3451.659382393322
$\max\{\mathbf{g_k}\} =$	1.00000009477064
$h^* =$	1.00000000000454
$x_0^{*T} z_0^* =$	0.00000000000822
$\lVert r_D \rVert_1 =$	0.00000013463365
$\lVert r_P \rVert_1 =$	0.00000000182779
T(seg.) =	19.8893183000000
Iterations =	4506
$\mathbf{t}^*$	**weights**
$Q_1 = 90.0000003328490$	$\omega_1 = 0.2278004433$
$Q_2 = 139.999999738614$	$\omega_2 = 0.1987754269$
$Q_3 = 95.9999999971371$	$\omega_3 = 0.2278004239$
$u_1 = 2.71828192378011$	$\omega_4 = 0.1987753931$
$u_2 = 2.71828173863058$	$\omega_5 = 0.0652659168$
$u_3 = 0.99999999918012$	$\omega_6 = 0.0163164792$
	$\omega_7 = 0.0652659169$

22. CC2005001 [4] Pressure Vessel Design

Minimize $0.6224x_1x_3x_4 + 1.7781x_2x_3^2 + 3.1661x_1^2x_4 + 19.84x_1^2x_3$,
Subject to: $0.0193x_1^{-1}x_3 \leq 1$,
$0.00954x_2^{-1}x_3 \leq 1$
$x_5 - \frac{\pi x_3^2 x_4}{750\times1728} - \frac{4\pi x_3^3}{3\times750\times1728} \leq 1$,
$45 \leq x_3 \leq 55$,
$80 \leq x_4 \leq 110$,
$x_5 \geq 2$,
$x_1 \in \{0.9375 + 0.0625i; \ i = 1, \ldots 7\}$,
$x_2 \in \{0.5625 + 0.0625i; \ i = 1, \ldots 7\}$.
$1 \leq x_1 \leq 1.375$,
$0.625 \leq x_2 \leq 1$.
Continuous Solution PVD

$g_0^* =$	7006.78063084608
$F^* =$	7006.78063084525
$\max\{\mathbf{g_k}\} =$	1.00000000000000
$h^* =$	1.00000000000003
$x_0^{*T} z_0^* =$	0.00000000000014
$\lVert r_D \rVert_1 =$	0.00000000000013
$\lVert r_P \rVert_1 =$	0.00000000000000
T(seg.) =	0.16040170000000
Iterations =	31
$\mathbf{t}^*$	**weights**
$x_1 = 1.00000000000002$	$\omega_1 = 0.2752075684$
$x_2 = 0.62500000000001$	$\omega_2 = 0.2247924316$
$x_3 = 51.8134715025912$	$\omega_3 = 0.5000000000$
$x_4 = 84.5785266878473$	
$x_5 = 2.00000000000001$	

In this problem, continuous and the discrete solution are equal.