Instituto Alberto Luiz Coimbra de
Pós-Graduação e Pesquisa de Engenharia
COPPE UFRJ

IMMERSIONS OF CLIQUES IN GRAPHS WITH INDEPENDENCE NUMBER 2 AND MAXIMUM DEGREE AT MOST 19N/29

Rui Aldé Lopes

Dissertação de Mestrado apresentada ao Programa de Pós-graduação em Engenharia de Sistemas e Computação, COPPE, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Mestre em Engenharia de Sistemas e Computação.

Orientadores: Fábio Happ Botler
Laura Silvia Bahiense da Silva Leite

Rio de Janeiro
Fevereiro de 2025

IMMERSIONS OF CLIQUES IN GRAPHS WITH INDEPENDENCE NUMBER 2 AND MAXIMUM DEGREE AT MOST 19N/29

Rui Aldé Lopes

DISSERTAÇÃO SUBMETIDA AO CORPO DOCENTE DO INSTITUTO ALBERTO LUIZ COIMBRA DE PÓS-GRADUAÇÃO E PESQUISA DE ENGENHARIA DA UNIVERSIDADE FEDERAL DO RIO DE JANEIRO COMO PARTE DOS REQUISITOS NECESSÁRIOS PARA A OBTENÇÃO DO GRAU DE MESTRE EM CIÊNCIAS EM ENGENHARIA DE SISTEMAS E COMPUTAÇÃO.

Orientadores: Fábio Happ Botler
Laura Silvia Bahiense da Silva Leite

Aprovada por: Prof. Fábio Happ Botler
Profa. Cristina Gomes Fernandes
Prof. Franklin de Lima Marquezino

RIO DE JANEIRO, RJ – BRASIL
FEVEREIRO DE 2025

Lopes, Rui Aldé

Immersions of cliques in graphs with independence number 2 and maximum degree at most 19n/29/Rui Aldé Lopes. – Rio de Janeiro: UFRJ/COPPE, 2025.

VII, 37 p. 29, 7cm.

Orientadores: Fábio Happ Botler

Laura Silvia Bahiense da Silva Leite

Dissertação (mestrado) – UFRJ/COPPE/Programa de Engenharia de Sistemas e Computação, 2025.

Referências Bibliográficas: p. 34 – 37.

1. Graph Theory. 2. Immersion. I. Botler, Fábio Happ *et al.* II. Universidade Federal do Rio de Janeiro, COPPE, Programa de Engenharia de Sistemas e Computação. III. Título.

Acknowledgments

I would like to thank my research partners Fábio, Bruno, Maycon, Carla, Cris and Suchi.

I would like to thank Professor Laura for the help with the internal workings of departmental rules and bureaucracies.

I would like to thank my mother and my father for the support in the first and second half of this journey, respectively.

Resumo da Dissertação apresentada à COPPE/UFRJ como parte dos requisitos necessários para a obtenção do grau de Mestre em Ciências (M.Sc.)

IMERSÕES DE CLIQUES EM GRAFOS COM NÚMERO DE INDEPENDÊNCIA 2 E GRAU MÁXIMO NO MÁXIMO 19N/29

Rui Aldé Lopes

Fevereiro/2025

Orientadores: Fábio Happ Botler
Laura Silvia Bahiense da Silva Leite

Programa: Engenharia de Sistemas e Computação

Em 1943, Hadwiger conjecturou que todo grafo G com número cromático $\chi(G) = t$ tem um clique com t vértices como um minor. Este trabalho explora uma questão análoga para imersões, com foco no caso em que o número de independência de G é no máximo 2, conforme proposto por Vergara em 2017. Um grafo $G = (V, E)$ contém uma imersão de um grafo $H = (V', E')$ se existe uma função injetiva $f\colon V' \to V$ e um conjunto de caminhos aresta-disjuntos $\mathcal{P} = \{P_e : e \in E'\}$ em G onde o caminho P_{uv} tem precisamente $f(u)$ e $f(v)$ como vértices finais.

Neste trabalho revisamos os resultados conhecidos e fornecemos uma prova alternativa de que todo grafo G com no máximo n vértices e número de independência 2 possui uma imersão de um clique com $2\lfloor n/5 \rfloor$ vértices, conforme estabelecido por Gauthier, Le e Wollan em 2017. Além disso, provamos que, se o grau máximo de G é no máximo $19n/29$, então G contém uma imersão de um clique com $\lfloor n/2 \rfloor$ vértices.

Por fim, apresentamos uma abordagem baseada em programação inteira para identificar grandes imersões de cliques, juntamente com sua implementação no SageMath.

Abstract of Dissertation presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Master of Science (M.Sc.)

IMMERSIONS OF CLIQUES IN GRAPHS WITH INDEPENDENCE NUMBER 2 AND MAXIMUM DEGREE AT MOST 19N/29

Rui Aldé Lopes

February/2025

Advisors: Fábio Happ Botler
Laura Silvia Bahiense da Silva Leite

Department: Systems Engineering and Computer Science

In 1943, Hadwiger posed the question of whether every graph G with chromatic number $\chi(G) = t$ has a clique of t vertices as a minor. This work explores an analogous question for immersions, focusing on the case where the independence number of G is at most 2, as proposed by Vergara in 2017. A graph $G = (V, E)$ contains an immersion of a graph $H = (V', E')$ if there is an injective function $f\colon V' \to V$ and a set of edge-disjoint paths $\mathcal{P} = \{P_e : e \in E'\}$ in G where the path P_{uv} has precisely $f(u)$ and $f(v)$ as endpoints.

We review known results and provide an alternative proof that every graph G with at most n vertices and independence number 2 has an immersion of a clique with $2\lfloor n/5 \rfloor$ vertices, as established by Gauthier, Le, and Wollan in 2017. Furthermore, we prove that if the maximum degree of G is at most $19n/29$, then G contains an immersion of a clique with $\lceil n/2 \rceil$ vertices.

Finally, we present an integer programming approach for identifying large clique immersions, along with its implementation in SageMath.

Contents

Chapter 1

Introduction

When studying graphs, some of the natural questions that quickly appear concern the existence of special substructures. For example, consider a graph G. Does G have a triangle? Does G have a long path, that is, a path above a certain length? Does G contain a sequence of vertices that traverses all of its edges? Does G have a path that begins and ends at given vertices? Such questions lead to the definition of certain just as natural structures such as, for example, subgraphs and induced subgraphs. For a more complete introduction to basic concepts in Graph Theory, we refer the reader to BONDY and MURTY (2008) and BOTLER *et al.* (2021).

One of the simplest substructures studied in graph theory is the *subgraph*, which can then be generalized in several ways, such as *subdivisions*, *immersions*, and *minors*, which we define next. All of these structures are generalizations of subgraphs in the sense that every subgraph is also (trivially) a minor, a subdivision, and an immersion, while the converses do not hold.

For any graph $G = (V, E)$, a *subgraph* of G is any graph $G' = (V^*, E^*)$ such that $V^* \subseteq V$ and $E^* \subseteq E$. We say that a graph $G = (V, E)$ contains a subgraph isomorphic to a graph $H = (V', E')$, if there exists an injective function $f: V' \to V$ such that if $uv \in E'$, then $f(u)f(v) \in E$. If G contains a subgraph isomorphic to H, we write $H \subseteq G$. If, moreover, it also holds that for every $uv \notin E'$ we have $f(u)f(v) \notin E$, then we say G contains an *induced subgraph* isomorphic to H, which we write as $H \subseteq_{IND} G$. When a graph G does not contain a subgraph isomorphic to H, we say that G *is* H*-free.*

There are many sufficient conditions for the existence of certain subgraphs in any given graph. One particularly influential such condition was presented by MANTEL (1907).

Theorem 1 (Mantel, 1907)**.** *For $n \in \mathbb{N}$, if G is a triangle-free graph on n vertices, then*

$$|E(G)| \leq \frac{n^2}{4}.$$

This result was later generalized for every clique by TURÁN (1941).

Theorem 2 (Turan, 1941)**.** *For $n, k \in \mathbb{N}$, if G is a K_{k+1}-free graph on n vertices, then*

$$|E(G)| \leq \left(1 - \frac{1}{k}\right)\frac{n^2}{2}.$$

In other words, Theorem 2 says that if a graph has sufficiently many edges, it contains a large clique as a subgraph.

The first generalization of the definition of subgraph we introduce are subdivisions. We say that a graph $G = (V, E)$ contains a *subdivision* of a graph $H = (V', E')$ if there exists an injective function $f\colon V' \to V$ and a set of internally-disjoint paths $\mathcal{P} = \{P_e : e \in E'\}$ in G for which the path P_{uv} has precisely $f(u)$ and $f(v)$ as endpoints. A set of paths is internally-disjoint if the internal vertices of any path in the set do not occur in the other paths in the set, but the paths may share their endpoints. We say that the vertices in the set $\{f(v) : v \in V'\}$ are the *branch vertices* of the subdivision. When a graph G contains a subdivision of a graph H, we write $H \subseteq_{SUB} G$.

Subdivisions have an equivalent definition given in terms of the operation of edge subdivision. An *edge subdivision* is the operation in which, given a graph G and an edge $uv \in E(G)$, we obtain a new graph G' with $V(G') = V(G) \cup \{w\}$ and $E(G') = E(G) \cup \{uw, vw\} \setminus \{uv\}$. In other words, an edge subdivision is the operation in which we replace an edge by a path of length 2 whose internal vertex is a new vertex. If G contains a subgraph H' that can be obtained from H through a sequence of edge subdivisions, then $H \subseteq_{SUB} G$.

The next structure we define is the minor, or the minor relation, for which we must first introduce edge contractions. An *edge contraction* is the operation in a graph G and an edge $uv \in E(G)$ and obtain a new graph G' with $V(G') = V(G) \cup \{w\} \setminus \{u, v\}$, where $N_{G'}(w) = N_G(u) \cup N_G(v) \setminus \{u, v\}$. If we can obtain a graph H from a graph G through a sequence of operations of edge removals, vertex removals and edge contractions, then H is a minor of G. If H is a minor of G, we write $H \subseteq_{MIN} G$.

The next proposition shows that whenever a graph G contains a subdivision of a graph H, H is a minor of G.

Proposition 1. *For any graphs G and H, if $H \subseteq_{SUB} G$ then $H \subseteq_{MIN} G$.*

Proof. Let us take H' to be the subdivision of H in G. To find H as a minor of G, we start by removing from G everything outside of H'. Then, since H' can be obtained from H by edge subdivisions, we simply contract the edges that were previously subdivided, getting back to H, thus showing $H \subseteq_{MIN} G$. □

To properly understand the conjectures explored in this text, it is important to set the historical context of the problem being studied In the following section, we discuss some of the main theorems and conjectures connected to the existence of cliques as minors and subdivisions in graphs.

1.1 The Four Color Theorem and Hadwiger's Conjecture

First published as a conjecture in 1854[1] by F.G. (June 10th, 1854) (believed to be Francis or Frederick Guthrie), the Four Color Theorem is one of the most well known theorems in combinatorics, for several reasons.

Theorem 3 (Four Color Theorem)**.** *Every planar graph can be colored with 4 colors.*

One of the reasons for its importance is the discrepancy between the ease of stating it and the complexity of proving its validity. As a consequence of that challenge, it also made history by having the first computer-assisted mathematical proof, presented by APPEL and HAKEN (1977).

However, graphs are not always planar. Minors and subdivisions are fundamentally connected to the planarity of graphs, as characterized by KURATOWSKI (1930) and WAGNER (1937).

Theorem 4 (Kuratowski, 1930)**.** *A graph G is planar if and only if $K_{3,3} \not\subseteq_{SUB} G$ and $K_5 \not\subseteq_{SUB} G$.*

Theorem 5 (Wagner, 1937)**.** *A graph G is planar if and only if $K_{3,3} \not\subseteq_{MIN} G$ and $K_5 \not\subseteq_{MIN} G$.*

From the Four Color Theorem, we can see that every graph with no K_5 or $K_{3,3}$ as a minor is 4-colorable. Naturally, any K_5 subgraph always needs 5 different colors, and the largest clique of a graph is a lower bound for its chromatic number. It is odd, however, that while K_5 itself is not 4-colorable and is an intuitive structure to forbid in planar graphs, $K_{3,3}$ is 2-colorable. A reasonable question, then, is if all graphs that do not have K_5 as a substructure in some way can be coloured with 4 colors. This approach discards the planarity hypothesis and ignores any $K_{3,3}$ substructures.

In 1943, HADWIGER (1943) formalized this idea in an attempt to generalize the Four Color Theorem to non-planar graphs. We denote by $\chi(G)$ the *chromatic number* of G, which is the smallest amount of colors necessary to color G with no pair of adjacent vertices receiving the same color.

[1]In 1852, Augustus De Morgan wrote about the problem in a letter to William Rowan Hamilton, crediting the problem to one of his students, likely Frederick Guthrie (see MCKAY (2012)). For a deeper look at the history of the problem, we recommend WILSON and NASH (2003).

Conjecture 1 (Hadwiger's Conjecture)**.** *If K_t is not a minor of G, then $\chi(G) \leq t-1$.*

Hadwiger's Conjecture is one of the most important, long-standing open conjectures in graph theory. In his paper, Hadwiger verified the case where $K_4 \nsubseteq_{MIN} G$. The case where $K_5 \nsubseteq_{MIN} G$ was proved to be equivalent to the Four Color Theorem by WAGNER (1937), and the case when G is free from K_6 minors was verified by ROBERTSON *et al.* (1993); however, their proof assumes the Four Color Theorem as a hypothesis. Hadwiger's Conjecture remains open for the cases where $K_t \nsubseteq_{MIN} G$ with $t \geq 7$. There have been two major surveys on this conjecture, by TOFT (1996) and SEYMOUR (2016).

Given the importance of Hadwiger's Conjecture, and the fact that it remains open, it is interesting to study its variations and restrictions. One such restriction of Conjecture 1 that has been fruitful is when we restrict the independence number of the graph, which directly constrains the chromatic number, since $\chi(G) \geq \lceil n/\alpha(G) \rceil$. In particular, BALOGH and KOSTOCHKA (2011) showed that every graph G with n vertices has a K_t minor with $t \geq 0.513n/\alpha(G)$. We will see this hypothesis is also useful even when we replace minors by immersions.

Another historically interesting variation is the analogue to Hadwiger's Conjecture for subdivisions, credited to Hajós by ERDŐS and FAJTLOWICZ (1981).

Conjecture 2 (Hajós Conjecture)**.** *Every graph G contains a subdivisions of $K_{\chi(G)}$.*

Unlike Hadwiger's Conjecture, Conjecture 2 has been proven not to hold for all G with $\chi(G) \geq 6$ by CATLIN (1979), and ERDŐS and FAJTLOWICZ (1981) proved that almost every graph is a counterexample to Conjecture 2.

In this dissertation, the main type of substructure studied is the *immersion*. We say a graph $G = (V, E)$ contains an immersion of a graph $H = (V', E')$ if there is an injective function $f\colon V' \to V$ and a set of edge-disjoint paths $\mathcal{P} = \{P_e : e \in E'\}$ in G for which the path P_{uv} has precisely $f(u)$ and $f(v)$ as endpoints. We say that the vertices in the set $\{f(v) : v \in V'\}$ are the *branch vertices* of the immersion. If G contains an immersion of H, we write $H \subseteq_{IM} G$. In the case where the branch vertices occur only as the endpoints of the paths in $\mathcal{P}$, i.e., that the branch vertices are not internal vertices of the paths in $\mathcal{P}$, we say that $\mathcal{I}$ is a *strong immersion* (see Figure 1.1).

Analogously to subdivisions, immersions also have an equivalent definition in terms of the operation of lifting. A *lifting* is the operation in which we choose two edges $uv, vw \in E(G)$ incident to the same vertex v [2] and obtain a graph G' for which $V(G') = V(G)$ and $E(G') = E(G) \cup \{uw\} \setminus \{uv, vw\}$. If we can obtain a

[2] In some cases it is additionally required that $uw \notin E(G)$ to avoid parallel edges.

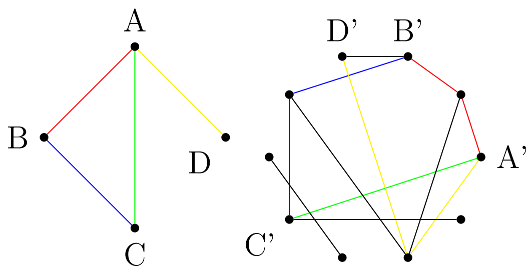

Figure 1.1: The graph on the right contains an immersion of the graph on the left, in which each colored path on the right pairs with the edge of the same color on the left.

graph H from a graph G through a sequence of operations of edge deletions, vertex deletions and liftings, then $H \subseteq_{IM} G$.

It is interesting to note that, unlike subdivisions and minors, for any graph H, it is possible to find a planar graph G such that $H \subseteq_{IM} G$.

Theorem 6. *For any graph G, there exists a planar graph G' such that $G \subseteq_{IM} G'$.*

Proof. For any G, take any representation P of G in the plane. In Figure 1.2, we chose K_6. Then, add a vertex for every edge intersection in P, effectively turning each intersection into the common vertex of multiple paths in an immersion. The obtained graph G' is a planar graph that has G as an immersion. □

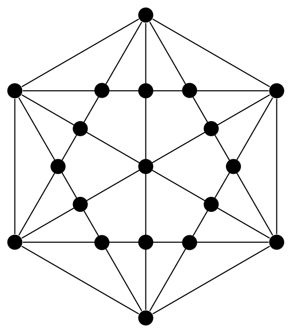

Figure 1.2: Planar embedding of a graph with K_6 as an immersion.

The next proposition shows that, whenever a graph G contains a subdivision of a graph H, G contains an immersion H.

Proposition 2. *For any graphs G and H, if $H \subseteq_{SUB} G$ then $H \subseteq_{IM} G$.*

Proof. Since a set of vertex-disjoint paths is also a set of edge-disjoint paths, any subdivision of H in G is itself also an immersion of H in G. □

The converse, however, does not hold. To show that the existence of immersions does not imply on the existence of subdivisions, let G be the star with four leaves whose center is a vertex u, and let H be the graph with four vertices and two nonadjacent edges. Observe that G contains an immersion of H (see Figure 1.3), but not as a subdivision since any path with at least two vertices must contain u.

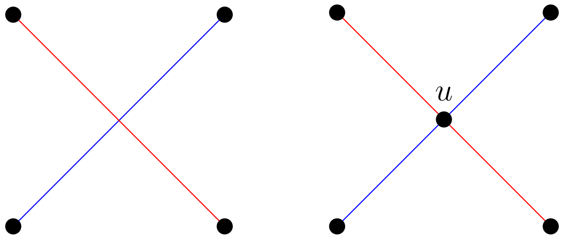

Figure 1.3: On the left the graph H, and on the right the graph G, with immersion paths shown in the color of the corresponding edge of H.

Propositions 1 and 2 show us that both minors and immersions are relaxations of the definition of subdivision, but it remains to be seen if the converse holds, that is, if any pair of the definitions above are equivalent.

Let us begin by checking that the existence of minors does not imply in the existence of subdivisions and immersions. For that, we may simply consider the Petersen graph, P (see Figure 1.4).

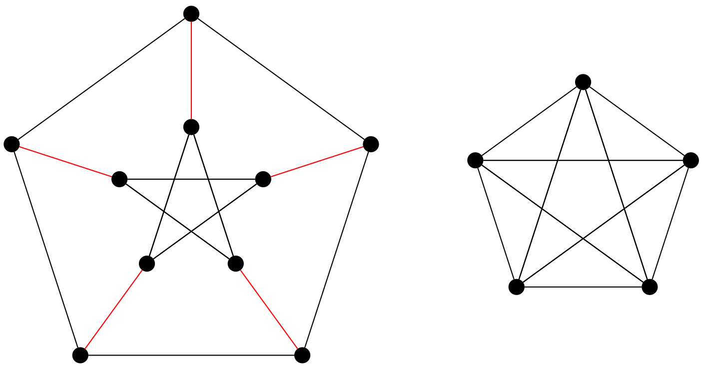

Figure 1.4: On the left, the Petersen graph; On the right, the complete graph K_5, obtained as a minor of by contracting the red edges.

Clearly, $K_5 \subseteq_{MIN} P$. Next, let graphs G and H be such that $H \subseteq_{IM} G$. If $v' \in V(G)$ is the branch vertex associated with some vertex $v \in V(H)$, there must be at least $d_H(v)$ edge-disjoint paths in G ending in v' and connecting it to the other branch vertices. This implies $d_G(v') \geq d_H(v)$, and $\Delta(G) \geq \Delta(H)$. Since $\Delta(P) = 3$, then P does not contain an immersion of K_5. The same argument shows that P does not contain a subdivision of K_5.

It remains to show that immersions and minors are not relaxations of one another but fundamentally different concepts. We have already shown that the existence of minors does not imply the existence of immersions, and to show the converse, it is sufficient to present a planar graph with an immersion of the $K_{3,3}$ graph (see Figure 1.5).

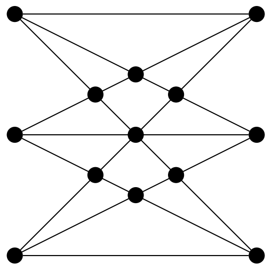

Figure 1.5: A planar graph with an immersion of the complete bipartite graph $K_{3,3}$, where each straight line is a different path.

Now that the structures and their relations with each other are well understood, we may introduce the main conjecture explored in this dissertation.

1.2 Vergara's conjecture

LESCURE and MEYNIEL (1988) proposed the analogue of Hadwiger's Conjecture for strong immersions, and ABU-KHZAM and LANGSTON (2003) proposed it for any immersion.

Conjecture 3 (Lescure and Meyniel, 1988)**.** *Every graph G has $K_{\chi(G)}$ as a strong immersion.*

Conjecture 4 (Abu-Khzam and Langston, 2003)**.** *Every graph G has $K_{\chi(G)}$ as an immersion.*

Since $\chi(G) \geq \lceil n/\alpha(G) \rceil$, a consequence of the conjectures above is that G contains an immersion of $K_{\lceil n/\alpha(G) \rceil}$. In that direction of research, GAUTHIER *et al.* (2019) showed that every graph G has an immersion with $\lceil n/3.54\alpha - 1.13 \rceil$ vertices, and BUSTAMANTE *et al.* (2022) improved that when $\alpha(G) \geq 3$ to $\lfloor \frac{n}{2.25\alpha(G) - f(\alpha(G))} \rfloor$ where f is a nonnegative function. When $\alpha(G) = 3$, BUSTAMANTE *et al.* (2019) showed that G has an immersion on at least $\lceil 2n/9 \rceil - 1$ vertices, and that increases to $\lceil 4n/27 \rceil - 1$ vertices when $\alpha(G) = 4$.

When the independence number is small, VERGARA (2017) proved that the case $\alpha(G) \leq 2$ of Abu-Khzam and Langston's Conjecture is equivalent to the following statement.

Conjecture 5 (Vergara, 2017)**.** *Every graph G with n vertices and $\alpha(G) \leq 2$ has an immersion of $K_{\lceil \frac{n}{2} \rceil}$.*

In her paper, Vergara also showed that every such graph has an immersion of $K_{\lceil \frac{n}{3} \rceil}$. Conjecture 5 was also verified if G is H-free, where H is such that $|V(H)| \leq 4$ and $\alpha(H) \leq 2$ (QUIROZ (2021a)) or when H is a house graph (QUIROZ (2021b)).

The closest approximation to Conjecture 5 was presented by GAUTHIER *et al.* (2019). We go into some of their techniques in greater depth in Chapter 3.

Theorem 7 (Gauthier et al., 2017). *Let G be a graph on n vertices. If $\alpha(G) \leq 2$, then $K_{2\lfloor n/5 \rfloor} \subseteq_{IM} G$.*

An alternative approach to Conjecture 5 is to weaken the structure we desire, for example replacing the immersion of a complete graph by the immersion of a complete bipartite graph. BOTLER *et al.* (2024) proved that every graph with independence number 2 contains an immersion of $K_{\ell, \lceil \frac{n}{2} \rceil - \ell}$ for any $\ell \leq \lceil \frac{n}{2} \rceil - 1$. In Chapter 2 we study the proof structure they proposed, and some of its natural consequences.

Theorem 8 (Botler et al., 2023). *Let G be a graph on n vertices with independence number 2, and $\ell \leq \lceil \frac{n}{2} \rceil - 1$ be a positive integer. Then G contains an immersion of $K_{\ell, \lceil \frac{n}{2} \rceil - \ell}$.*

Finally, in Section 4.4 we prove an original result, that explores another restriction. Theorem 9 reaches the bound of $\frac{n}{2}$ proposed by Conjecture 5 but considering the case of graphs with bounded maximum degree as follows.

Theorem 9. *Let G be a graph with n vertices. If $\alpha(G) \leq 2$ and $\Delta(G) < 19n/29 - 1$, then G contains an immersion of $K_{\lceil n/2 \rceil}$.*

Our proof applies to a slightly stronger statement, as in fact $V(G)$ can be partitioned into two sets A, B such that A induces a clique in G and G contains an immersion of a clique whose branch vertices are precisely B.

In Chapter 4 we explore our approaches and attempts to settle Conjecture 5. Finally, in Chapter 5 we propose a linear program for finding large clique immersions in graphs, and present an implementation of such program in Sagemath. In Chapter 6 we present our conclusions, reviewing our contributions and proposing a possible path for further research.

Chapter 2

Immersions of complete bipartite graphs

Recall that BOTLER *et al.* (2024) proved that every graph with independence number 2 contains an immersion of $K_{\ell,\lceil\frac{n}{2}\rceil-\ell}$ for any $\ell \leq \lceil\frac{n}{2}\rceil - 1$. We recall Theorem 8:

Theorem 8 (Botler et al., 2023)**.** *Let G be a graph on n vertices with independence number 2, and $\ell \leq \lceil\frac{n}{2}\rceil - 1$ be a positive integer. Then G contains an immersion of $K_{\ell,\lceil\frac{n}{2}\rceil-\ell}$.*

During our research, we studied in depth the proof of Theorem 8 in an attempt to extend it. In this chapter, we discuss a structure proposed by BOTLER *et al.* (2024), and present some natural lemmas that follow from the definitions of such structure, and properties we discovered during this research. These lemmas help to understand the behavior of the studied graphs, and part of them are used in more intricate proofs in Chapter 4.

Let G be a minimal counterexample to Conjecture 5. Formally, suppose that G is a graph for which $\alpha(G) \leq 2$ and with no immersions of $K_{\lceil n/2\rceil}$, and assume that every graph G' with $\alpha(G') \leq 2$ for which $V(G') \subseteq V(G)$ and $E(G') \subset E(G)$ contains an immersion of $K_{\lceil v(G')/2\rceil}$. Observe that G is minimal with respect to $\alpha(G) \leq 2$, so the removal of any edge forms an independent set of size 3. That means that, for any edge, there is a vertex that is not adjacent to either of its vertices, or rather:

Lemma 1. *If G is a minimal graph with $\alpha(G) \leq 2$, then for every edge $uv \in E(G)$ we have $N[u] \cup N[v] \neq V(G)$.*

Proof. Let G' be the graph obtained from G by removing uv. Then, by the minimality of G, we have $\alpha(G') = 3$. So there is a vertex $w \in V(G')$ for which u, v, and w form an independent set. But then w is not adjacent to both u and w, and hence $N[u] \cup N[v] \neq V(G)$ as desired. □

This, together with the fact that $\alpha(G) \leq 2$, can be rewritten as Lemma 2. We say a set $X \subseteq V(G)$ *dominates* a set $Y \subseteq V(G)$ if $Y \subseteq \bigcup_{x \in X} N(x)$.

Lemma 2. *Let G be a minimal graph with $\alpha(G) \leq 2$. For every edge $uv \in E(G)$, the set $\{u, v\}$ does not dominate $V(G)$. Moreover, for every $uv \notin E(G)$, the set $\{u, v\}$ dominates G.*

Also, note that G is not a complete graph, i.e., there is at least one pair of vertices $x, y \in V(G)$ for which $xy \notin E(G)$. Let $x, y \in V(G)$ be such that $xy \notin E(G)$ and that maximize the size of $C = N(x) \cap N(y)$. Let $X \subseteq V(G)$ (resp. Y) be the subset of vertices that are non-adjacent to y (resp. x). Since $\alpha(G) \leq 2$, note that each vertex of G is adjacent to at least one between x and y, and hence $V(G) = C \cup X \cup Y$, and $X = N[x] \setminus N(y)$. Then, X and Y are partitioned further. Let $X_C \subseteq X$ (resp. $Y_C \subseteq Y$) be the set containing all vertices $v \in X$ (resp. $v \in Y$) for which $C \subseteq N(v)$, and let $\overline{X_C} = X \setminus X_C$ (resp. $\overline{Y_C} = Y \setminus Y_C$) (see Figure 2.1).

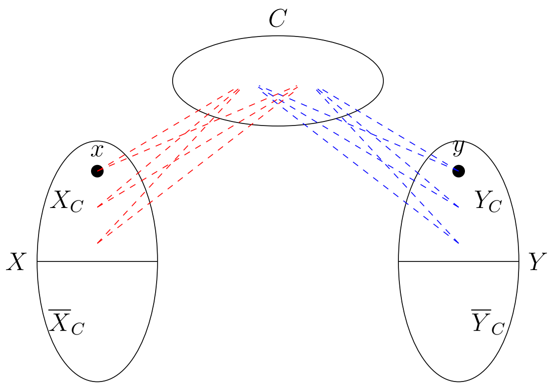

Figure 2.1: Partition of G according to non-adjacent vertices x and y and their common neighbourhood C.

BOTLER *et al.* (2024) presented a construction of an immersion $\mathcal{I}$ of a complete bipartite graph. Let us call A and B the partition of the branch vertices in $\mathcal{I}$. In order to find an immersion of $K_{n/2}$, one would only need to find additional edge-disjoint paths pairwise connecting vertices from A and pairwise connecting vertices from B. Unfortunately, despite the strong structure of the given graph, this is a hard task.

This construction already gives us a lot of information about the graph. We do not prove their theorem here but note, for instance, that $[C, X_C]$ is a complete bipartite graph, and the same happens to the pairs $\{C, Y_C\}$, $\{X_C, \overline{X_C}\}$ and $\{Y_C, \overline{Y_C}\}$. Also, $X_C, Y_C, \overline{X_C}$ and $\overline{Y_C}$ are all cliques. In fact, G is very similar to a *blow-up* of an induced C_5, which is further explained in Section 4.1. This becomes even clearer with the following proposition.

Proposition 3. $E(X_C, \overline{Y_C}) = E(X_C, Y_C) = E(\overline{X_C}, Y_C) = \emptyset$.

Proof. Suppose there are vertices $u \in X_C$, $v \in \overline{Y_C}$ such that $uv \in E(G)$. Since $C \cup X \subseteq N[u]$ and $Y \subseteq N[v]$, it follows that $N[u] \cup N[v] = V(G)$, which contradicts Claim 1. Therefore, there is no $uv \in E(G)$. The proofs for $E(\overline{X_C}, \overline{Y_C})$ and $E(\overline{X_C}, Y_C)$ are analogous. □

It is also not too difficult to find some basic relation between the sizes of the considered parts of G.

Proposition 4. *$|C| \geq |X_C|$, $|C| \geq |Y_C|$, $|C \cup X| \geq \lceil \frac{n}{2} \rceil > |Y|$ and $|C \cup Y| \geq \lceil \frac{n}{2} \rceil > |X|$.*

Proof. First, let us prove that $|C \cup X| \geq \lceil \frac{n}{2} \rceil > |Y|$. From the structure of G, we have that Y is a clique. If $|Y| \geq \lceil \frac{n}{2} \rceil$, then Y as a clique is a trivial immersion with the desired size, contradicting the fact that G is a counterexample to Vergara's conjecture. So $|Y| < \lceil \frac{n}{2} \rceil$, and $|V(G) \setminus Y| = |C \cup X| \geq \lceil \frac{n}{2} \rceil$. The proof for $|C \cup Y| \geq \lceil \frac{n}{2} \rceil > |X|$ is analogous.

To prove that $|C| \geq |X_C|$, we claim that $\overline{X_C} \neq \emptyset$ and $\overline{Y_C} \neq \emptyset$. In fact, for any $a \in C$ if $\overline{X_C} = \emptyset$ then the edge ay could dominate the graph, which would contradict Lemma 2. Analogously we deduce that $\overline{Y_C} \neq \emptyset$. This is equivalent to Claim 10 in BOTLER *et al.* (2024).

Take a vertex $v \in \overline{X_C}$. Since v is not in X_C, there is a vertex $c \in C$ such that $vc \notin E(G)$. However, $X_C \subset N[v] \cap N[c]$, and because of the maximality of C that means $|C| \geq |X_C|$. Analogously, $|C| \geq |Y_C|$. □

Our goal is to find an immersion with half the vertices of G. First, let us assume by induction that there is an immersion of a complete graph in $G[C]$ with at least $\lceil \frac{|C|}{2} \rceil$ branch vertices, and let us call the set of branch vertices of this immersion C^*. Since C^* has at least half of the vertices in C, either $|X \cup C^*| \geq \lceil \frac{n}{2} \rceil$ or $|Y \cup C^*| \geq \lceil \frac{n}{2} \rceil$. Assume, without loss of generality, that $|X \cup C^*| \geq \lceil \frac{n}{2} \rceil$. If we find edge-disjoint paths connecting every pair of vertices in $X \cup C^*$, we are done. Of course, any pair in X is connected directly, and C^* is connected to every vertex in X_C. Finally, we can use the immersion that exists in C to connect all vertices in C^* to each other, and all that is left is to connect C^* to $\overline{X_C}$ using only edges in $E(\overline{X_C}, \overline{Y_C})$, $E(C, \overline{X_C})$, $E(C^*, Y)$ and $G[Y]$.

The three bipartite subgraphs are not necessarily complete nor necessarily empty, but their behavior is not as chaotic as it might seem at first. For any vertex $a \in C$, we call $X_a := N(a) \cap X$ and $Y_a := N(a) \cap Y$, also $\overline{X_a} := X \setminus X_a$ and $\overline{Y_a} := Y \setminus Y_a$.

Claim 1. *For every vertex $a \in C$, $G[\overline{X_a}, \overline{Y_a}]$ is a complete bipartite graph.*

Proof. Assume there is some pair $x \in \overline{X_a}$, $y \in \overline{Y_a}$ such that $xy \notin E(G)$. This would mean that $\{x, y, a\}$ is an independent set of size 3 in G, a contradiction. □

Claim 2. *For every edge $uv \in E(\overline{X_C}, \overline{Y_C})$, there is a vertex $a \in C$ such that $u \in \overline{X_a}$ and $v \in \overline{Y_a}$.*

Proof. This follows from Lemma 2. Since uv is an edge of G, there must be some vertex $a \in V(G)$ that is not adjacent to both u and v. However $a \notin X$, or it would be adjacent to x. Also $a \notin Y$, or it would be adjacent to y. So $a \in C$. □

With these claims, we can understand $E(\overline{X_C}, \overline{Y_C})$ as the union of at most $|C|$ complete bipartite graphs.

Claim 3. *For every vertex $a \in C$, $\overline{X_a} \neq \emptyset$ and $\overline{Y_a} \neq \emptyset$.*

Proof. Assume $\overline{X_a} = \emptyset$. Then $N[y] \cup N[a] \supseteq (Y \cup C) \cup X = V(G)$, which contradicts Lemma 2. So $\overline{X_a} \neq \emptyset$, and analogously $\overline{Y_a} \neq \emptyset$. □

Claim 4. *For every vertex $a \in C$, $|\overline{X_a}|, |\overline{Y_a}| \leq |C|$.*

Proof. From Claim 3, consider $b \in \overline{X_a}$. We know from Proposition 3 that $ab \notin E(G)$, and since $\overline{Y_a} \subset N[b] \cup N[y]$, from the maximality of C, we have that $|\overline{Y_a}| \leq |C|$ (resp. $|\overline{X_a}| \leq |C|$). □

Claim 5. $|C| \geq \frac{n}{5}$.

Proof. To show that $|C| \geq \frac{n}{5}$, we must simply choose a vertex $a \in C$ and show that $|X_a|, |Y_a|, |\overline{X_a}|, |\overline{Y_a}| \leq |C|$. From Claim 4 we know the latter two hold.

Let us take $b \in \overline{X_a}$, which we know must exist. Now if we consider $N[a] \cup N[b] \supseteq X_a$, from the definition of C, it follows that $|X_a|, |Y_a| \leq |C|$. □

While this sequence of claims shows that the sizes of $\overline{X_a}$ and $\overline{Y_a}$ are well bounded, because $\overline{X_C}$ is the union of all $|C|$ different $\overline{X_a}$, its size cannot be bounded in the same way.

Chapter 3

Immersions of size $2n/5$

The closest result we have to Conjecture 5 in the bibliography, Theorem 7, comes from GAUTHIER *et al.* (2019). In this chapter, we present an alternative, arguably more elementary proof of the same result, as well as new proofs of Conjecture 5 for certain classes of graphs.

3.1 An alternative proof of the existence of immersions of $K_{2n/5}$

Throughout the text, if G has a subgraph (resp. an induced subgraph) isomorphic to a graph H, we simply say that G has a *copy* (resp. *induced copy*) of H. Let us restate Theorem 7.

Theorem 7 (Gauthier et al., 2017)**.** *Let G be a graph on n vertices. If $\alpha(G) \leq 2$, then $K_{2\lfloor n/5 \rfloor} \subseteq_{IM} G$.*

Proof. Let G be a graph on n vertices for which $\alpha(G) \leq 2$. The proof follows by induction on $n + e(G)$. One can easily check that if $n \leq 9$ then $K_{2\lfloor n/5 \rfloor} \subseteq_{IM} G$. Since we seek an immersion with only $2\lfloor n/5 \rfloor$ vertices, we may also assume $n = 5t$ for some integer $t \geq 2$, and thus $n \geq 10$.

If $\alpha(G - e) \leq 2$ for some edge $e \in E(G)$, then, by the induction hypothesis, $K_{2n/5} \subseteq_{IM} G - e \subseteq G$, as desired. Therefore, we may assume that G is minimal with $\alpha(G) \leq 2$. We may assume G is not complete, otherwise it clearly contains an immersion of $K_{2n/5}$.

Also, since $\alpha(G) \leq 2$, the set $\overline{N(u)} = V(G) \setminus N(u)$ induces a clique for every $u \in V(G)$. Thus, if $|\overline{N(u)}| \geq 2n/5$ for some vertex $u \in V(G)$, then $\overline{N(u)}$ induces the desired immersion. Therefore, we may assume that $|\overline{N(u)}| < 2n/5$ for every $u \in V(G)$. This implies that $|N(u)| > 3n/5$ for every $u \in V(G)$.

Claim 6. *G has an induced copy of C_5.*

Proof. First, we claim that G contains an induced path of length 2. Indeed, since G is not a complete graph, there is at least a pair u and v of nonadjacent vertices. Let P be a shortest path joining u and v. Then P must be an induced path, and since u and v are nonadjacent, P contains an induced path of length 2 as desired.

Let $v_1v_2v_3 \subseteq G$ be an induced path of length 2. By Lemma 2, there is a vertex v_4 that is nonadjacent to both v_1 and v_2; and there is a vertex v_0 that is nonadjacent to both v_2 and v_3. Since v_1 is nonadjacent to both v_3 and v_4, then $v_3v_4 \in E(G)$ and thus $v_4 \neq v_0$. Analogously, $v_1v_0, v_4v_0 \in E(G)$, and hence $\{v_1, v_2, v_3, v_4, v_0\}$ induces a copy of C_5 in G as desired. □

Let C be an induced copy of C_5 in G, and assume that $V(C) = \{v_1, v_2, v_3, v_4, v_0\}$. By the induction hypothesis, we have $K_{2n/5-2} \subseteq_{IM} G \setminus V(C)$. Let K' be such an immersion, and let $I \subseteq V(G) \setminus V(C)$ be the branch vertices of K'. The goal is to prove that there are two vertices from C that can be used as additional branch vertices to enlarge K' into an immersion of $K_{2n/5}$.

Claim 7. *Every vertex in $V(G) \setminus V(C)$ is adjacent to three consecutive vertices in C.*

Proof. Let $u \in V(G) \setminus V(C)$. If u is adjacent to every vertex of C, then clearly u is adjacent to three consecutive vertices of C. Suppose, without loss of generality, that u is nonadjacent to v_1. Since $v_1v_3, v_1v_4 \notin E(G)$ and $\alpha(G) \leq 2$, u is adjacent to v_3 and v_4. Now, since $v_2v_0 \notin E(C)$ and $\alpha(G) \leq 2$, u is adjacent to v_2 or to v_0, as desired. □

Partition $V(G) \setminus (I \cup V(C))$ into five sets, $Z_1, \dots, Z_5$ such that if $u \in Z_i$ then $v_{i-1}, v_i, v_{i+1} \in N(u)$, where the subscripts are taken modulo 5. Observe that a vertex $u \notin V(C)$ may fit in more than one such Z_i. When this is the case, we choose one such Z_i arbitrarily. Then $|Z_1| + \cdots + |Z_5| = |V(G) \setminus (I \cup V(C))| = n - (2n/5 - 2 + 5) = 3(n-5)/5$, and hence there is an i for which $|Z_i| \leq \frac{3}{25}(n-5)$. We may assume, without loss of generality, that

$$|Z_2| \leq \frac{3}{25}(n-5). \tag{3.1}$$

Then we show that v_1 and v_3 will be the additional branch vertices. Now, for $i \in \{1, 3\}$, let

$$X_i = I \setminus N(v_i) \qquad \text{and} \qquad Y_i^+ = N(v_i) \setminus (I \cup V(C)).$$

Since $v_1v_3 \notin E(G)$, it follows that $Y_1^+ \cup Y_3^+ = V(G) \setminus (I \cup V(C))$, and $X_1 \cap X_3 = \emptyset$. The next claim is an important step in this proof.

Claim 8. *There are $Y_1 \subseteq Y_1^+ \setminus Z_2$ and $Y_3 \subseteq Y_3^+ \setminus Z_2$ such that*

A) $|Y_1| = |X_1|$ *and* $|Y_3| = |X_3|$*;*

B) $Y_1 \cap Y_3 = \emptyset$.

Proof. Let $i \in \{1,3\}$, and note that $N(v_i) = |Y_i^+| + |I \setminus X_i| + 2$. Observe that $|I \setminus X_i| = |I| - |X_i| = \frac{2n}{5} - 2 - |X_i|$, and hence

$$|Y_i^+| + \frac{2n}{5} - 2 - |X_i| + 2 = |Y_i^+| + |I \setminus X_i| + 2 = |N(v_i)| > \frac{3n}{5}.$$

Therefore, we have

$$|Y_i^+| > \frac{n}{5} + |X_i|. \tag{3.2}$$

Recall that $|Z_2| \leq \frac{3}{25}(n-5) < \frac{3n}{25} < \frac{n}{5}$. We choose $Y_1 \subseteq Y_1^+ \setminus Z_2$ with $|Y_1| = |X_1|$, giving priority to vertices not in $Y_1^+ \cap \; Y_3^+$. This choice implies that either $Y_1 \subseteq Y_1^+ \setminus Y_3^+$ or $Y_1^+ \setminus Y_3^+ \subseteq Y_1$. If $Y_1 \subseteq Y_1^+ \setminus Y_3^+$, then, by (3.1) and (3.2), we have $|Y_3^+ \setminus Z_2| > \frac{n}{5} + |X_3| - \frac{3}{25}(n-5) > |X_3|$, and we can choose Y_3 as desired. On the other hand, if $Y_1^+ \setminus Y_3^+ \subseteq Y_1$, then we have $V(G) \setminus (I \cup V(C)) = Y_1^+ \cup Y_3^+ = Y_1 \cup (Y_3^+ \setminus Y_1)$, and since $Y_1 \cap (Y_3^+ \setminus Y_1) = \emptyset$, we have

$$\begin{aligned}
|Y_3^+ \setminus (Y_1 \cup Z_2)| \geq |Y_3^+ \setminus Y_1| - |Z_2| &= |V(G) \setminus (I \cup V(C))| - |Y_1| - |Z_2| \\
&= |V(G) \setminus (I \cup V(C))| - |Z_2| - |X_1| \\
&\geq \frac{3}{5}(n-5) - \frac{3}{25}(n-5) - |X_1| \\
&> \frac{3}{5}(n-5) - \frac{1}{5}(n-5) - \frac{2}{5}(n-5) + |X_3| \\
&= |X_3|,
\end{aligned}$$

where we used that $|X_1| + |X_3| \leq \frac{2}{5}(n-5)$ because X_1 and X_3 are disjoint sets in I. Therefore, we can choose Y_3 as desired. □

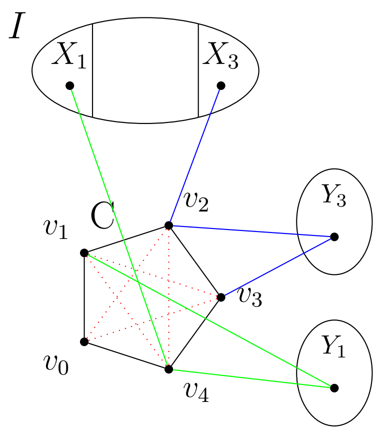

Figure 3.1: The current structure, including possible paths from v_1 to X_1, and from v_3 to X_3

Finally, note that every vertex in $V(G) \setminus (I \cup V(C) \cup Z_2)$ has two neighbors in $\{v_2, v_4, v_0\}$. Moreover, every vertex in $X_1 \cup X_3$ has two neighbors in $\{v_2, v_4, v_0\}$. Therefore, every pair of vertices x, y with $x \in X_1 \cup X_3$ and $y \in Y_1 \cup Y_3$ has at least one common neighbor $v_{xy} \in \{v_2, v_4, v_0\}$.

Now, let $X_1 = \{x_1, \dots, x_{\ell_1}\}$ and $Y_1 = \{y_1, \dots, y_{\ell_1}\}$, and for each $i \in \{1, \dots, \ell_1\}$, let $P_{v_1 x_i} = v_1 y_i v_{x_i y_i} x_i$ be a path that joins v_1 with x_i. Analogously, we can define the paths $P_{v_3 x'}$ that join v_3 to each vertex $x' \in X_3$ (see Figure 3.1). It is not difficult to check that, since $X_1 \cap X_3 = Y_1 \cap Y_3 = \emptyset$, these paths are edge-disjoint, and we can add v_1 and v_3 to K', obtaining an immersion of $K_{2n/5}$ whose set of branch vertices is $I \cup \{v_1, v_3\}$, as desired. $\square$

Chapter 4

Immersions in blown-up cycles

A *blow-up* of a graph G is commonly defined as any graph obtained from G by replacing every vertex $v \in V(G)$ by an independent set V_v of a certain size, and every edge $uv \in E(G)$ by a complete bipartite graph with parts V_u and V_v (see OLIVEIRA *et al.* (2014), HATAMI *et al.* (2014)). It is worth noting that for every graph H which is a blow-up of G, H is also homomorphic to G.

4.1 The blow-up of C_5

In this proof, we use a slightly modified structure called a *clique-blow-up* of a graph G. As the name implies, one of the main differences is that every vertex $v \in V(G)$ is replaced by a clique instead of an independent set. Also, in this structure, it is possible for the cliques to be of different sizes, that is, $|V_v| \neq |V_u|$ for some $u, v \in V(G)$. If H is a clique-blow-up of G, the sets V_u with $u \in V(G)$ are referred to as the *parts* of H.

The goal of Section 4.1 is to show that every graph which is a clique-blow-up of the C_5 cycle graph contains an immersion of a large clique. To do that, we first present Lemma 3, which deals with blow-ups of the P_4, and is the basis for the more advanced proofs.

Lemma 3 (P_4 Lemma). *Let H be a clique-blow-up of the P_4 with parts X_1, X_2, X_3, X_4, and such that $|X_3| \geq |X_1|$ and $|X_2| \geq |X_4|$. Then there is an immersion of $K_{|X_1|+|X_4|}$ with branch vertices $X_1 \cup X_4$.*

Proof. Let $X_i = x_{i,1}, \ldots, x_{i,n_i}$ be an ordering for each X_i. For every vertex $x_{1,k} \in X_1$, we choose vertex $x_{3,k} \in X_3$ to be its pair. It follows that, for every $x_{1,j}, x_{1,k} \in X_1$, if $j \neq k$, then $x_{3,j} \neq x_{3,k}$. Furthermore, since $|X_3| \geq |X_1|$, every $x_{1,k} \in X_1$ will have a pair. Similarly, we pair every $x_{4,k} \in X_4$ with $x_{2,k} \in X_2$, which is possible because $|X_2| \geq |X_4|$.

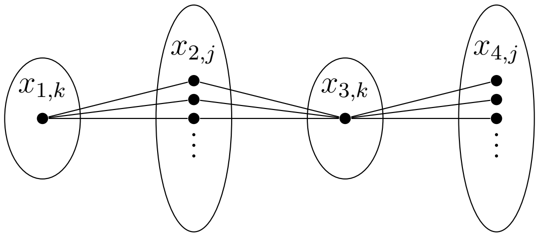

Figure 4.1: Edge-disjoint paths connecting X_1 to X_4.

We then have the desired immersion. All vertices in the clique X_1 are adjacent to each other, and the same happens in X_4. To connect any vertices $x_{1,k} \in X_1$ and $x_{4,j} \in X_4$, we use the unique path $x_{1,k}x_{2,j}x_{3,k}x_{4,j}$ (see Figure 4.1). □

With a similar strategy, we can also prove a lemma for clique-blow-ups of the P_3.

Lemma 4 (P_3 Lemma)**.** *Let H be a clique-blow-up of the P_3 with parts X_1, X_2, X_3, and such that $|X_2| \geq |X_1|$ and $|X_2| \geq |X_3|$. Then there is an immersion of $K_{|X_1|+|X_3|}$ with branch vertices $X_1 \cup X_3$.*

Proof. Without loss of generality, consider $|X_1| \geq |X_3|$. Let $|X_1| = k$ and $X_1 = \{v_0, \ldots, v_{k-1}\}$. Let also $X_2' \subseteq X_2$ be such that $|X_2'| = |X_1| = k$, and $X_2' = \{u_0, \ldots, u_{k-1}\}$. We can construct k edge-disjoint perfect matchings between X_1 and X_2', labelled $\{M_0, \ldots, M_{k-1}\}$, simply by taking $v_j u_{(j+i) \bmod k} \in M_i$, with $0 \leq j \leq k-1$. Finally, we associate each M_i with a vertex $w_i \in X_3$, which is possible because $|X_1| = k \geq |X_3|$ (see Figure 4.2).

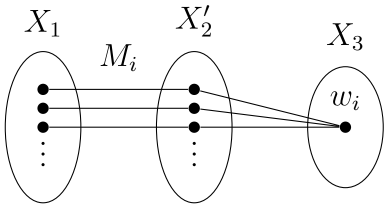

Figure 4.2: Edge-disjoint paths connecting X_1 to X_3.

We then have the desired immersion. All vertices in the clique X_1 are adjacent to each other, and the same happens in X_3. To connect each $w_i \in X_3$ to all vertices in X_1, we use the set of edge-disjoint paths induced by $X_1 \cup M_i \cup X_2' \cup w_i$. □

We now verify Conjecture 5 for the class of graphs that are clique-blow-ups of the C_5.

Theorem 10. *Let G be a clique-blow-up of the C_5 with n vertices. Then G has an immersion of $K_{\lceil \frac{n}{2} \rceil}$.*

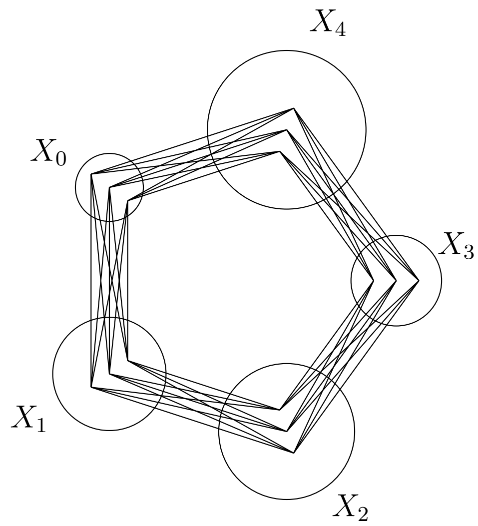

Figure 4.3: A clique-blow-up of the C_5.

Proof. Let G be a clique-blow-up of the C_5 with parts X_0, X_1, X_2, X_3, X_4 (See Figure 4.3). All indexes and index operations in this proof are to be taken modulo 5.

Claim 9. *For any $i \in \{0, 1, 2, 3, 4\}$, either $|X_i| + |X_{i+1}| < n/2$ or $X_i \cup X_{i+1}$ is a clique with at least $n/2$ vertices.*

Since a clique with size at least $n/2$ is already an immersion as desired, we can assume that $|X_i| + |X_{i+1}| < n/2$. It follows then that, for any $i \in \{0, 1, 2, 3, 4\}$, $|X_i| + |X_{i+1}| + |X_{i+2}| \geq n/2$. In what follows, we show that there is always an immersion of a clique where the branch vertices consist of three consecutive X_i.

If there is an empty part, say $X_0 = \emptyset$, then there are n vertices to be distributed between the two halves $\{X_1, X_2\}$ and $\{X_3, X_4\}$, which would mean one of them has at least half the vertices, in contradiction with $|X_i| + |X_{i+1}| < n/2$. Therefore there are no empty parts.

Let $X_0 \neq \emptyset$ be the smallest X_i. In particular, $|X_2| \geq |X_0|$ and $|X_3| \geq |X_0|$. We now divide the problem into four cases, depending on the relation between the pairs $\{X_2, X_4\}$ and $\{X_3, X_1\}$.

- If $|X_{|}2 \geq |X_4|$ and $|X_3| \geq |X_1|$, we can think of $G - X_0$ as a clique-blow-up of the P_4 with parts X_1, X_2, X_3, X_4 and use Lemma 3 to find a clique immersion with branches $X_1 \cup X_4$. As X_0 is complete to X_1 and X_4, we can unite it with the immersion to find a clique immersion using X_1, X_4, and X_0, as desired.

- If $|X_2| < |X_4|$ and $|X_3| < |X_1|$, we can think of $G - X_1$ as a clique-blow-up of the P_4 with parts X_2, X_3, X_4, X_0. Because $|X_4| \geq |X_2|$ and $|X_3| \geq |X_0|$ we can use Lemma 3 to find a clique immersion with branches $X_2 \cup X_0$. As X_1

is complete to X_2 and X_0, we can unite it with the immersion to find a clique immersion using X_1, X_2, and X_0, as desired.

- If $|X_2| \geq |X_4|$ and $|X_3| < |X_1|$, we can think of $G - X_4$ as a clique-blow-up of the P_4 with parts X_0, X_1, X_2, X_3. Because $|X_1| \geq |X_3|$ and $|X_2| \geq |X_0|$ we can use Lemma 3 to find a clique immersion with branches $X_3 \cup X_0$. As X_4 is complete to X_3 and X_0, we can unite it with the immersion to find a clique immersion using X_3, X_4, and X_0, as desired.

- If $|X_2| < |X_4|$ and $|X_3| \geq |X_1|$, we can think of $G - X_1$ as a clique-blow-up of the P_4 with parts X_2, X_3, X_4, X_0. Because $|X_3| \geq |X_5|$ and $|X_4| \geq |X_2|$ we can use Lemma 3 to find a clique immersion with branches $X_2 \cup X_5$. As X_1 is complete to X_2 and X_5, we can unite it with the immersion to find a clique immersion using X_1, X_2, and X_5, as desired.

In all cases, we have found the desired immersion. □

4.2 Generalization for the blow-up of odd cycles

Cycles longer than C_5 (and their respective clique-blow-ups) do not fall under Conjecture 5 with the restriction $\alpha(G) \leq 2$, but we can still use Lemma 3 to show the existence of certain immersions in them.

Theorem 11. *Let G be a clique-blow-up with n vertices of an odd cycle C_{2k+1}. Then G has $K_{\lceil \frac{n}{k} \rceil}$ as an immersion.*

Proof. Let G be a clique-blow-up of any odd cycle, with parts A_0 through A_{2k}. We say a subgraph $H \subseteq G$ is *good* if it is a clique-blow-up of the P_4 with parts $A'_1, \ldots, A'_4$ and such that $|A'_2| \geq |A'_4|$ and $|A'_3| \geq |A'_1|$. Assume that every subgraph defined by the union of four consecutive parts in G is bad. Then, for the set $\{A_0, A_1, A_2, A_3\}$, we either have $|A_1| < |A_3|$ or $|A_2| < |A_0|$. Without loss of generality, assume the former. We then look at the parts $\{A_1, A_2, A_3, A_4\}$. We already know that $|A_1| < |A_3|$, and since this set must also be bad, it follow that $|A_2| < |A_4|$.

If we repeat this process enough times, we end up with the chain of inequalities $|A_1| < |A_3| < \cdots < |A_{2k-1}| < |A_0| < \cdots < |A_{2k}| < |A_1|$, an impossibility. In Figure 4.4 we see an instance of this, where every vertex represents an entire part in a clique-blow-up of the C_9, and the arrows go from a smaller part to a greater one.

Therefore, there must be at least one good clique-blow-up of the P_4 in G. Without loss of generality, let us say the good set is $\{A_{2k-2}, A_{2k-1}, A_{2k}, A_0\}$. Using Lemma 3, we can make an immersion $\mathcal{I}_{P4}$ of a complete bipartite graph between A_{2k-2} and A_0. Then we replace each path in this immersion with an edge connecting the two

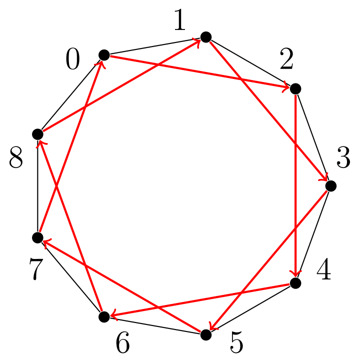

Figure 4.4: Size relations on the clique-blow-up of the C_9.

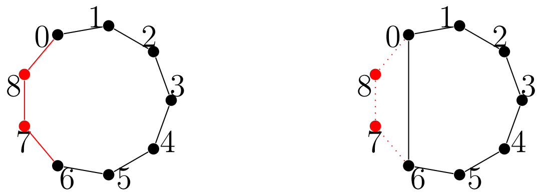

Figure 4.5: Lifting of two adjacent parts.

endpoints, effectively removing A_{2k-1} and A_{2k}. With this, we find a different clique-blow-up of the C_{2k-1}. In Figure 4.5, we see this happening on the same instance of the clique-blow-up of the C_9 seen in Figure 4.4.

By induction, we can assume there is a immersion $\mathcal{I}$ of $K_{\lceil \frac{n'}{k-1} \rceil}$ in any clique-blow-up of the C_{2k-1}, where $n' = |V(H)|$.

If $|A_{2k} \cup A_{2k+1}| \geq \frac{n}{k}$, $A_{2k} \cup A_{2k+1}$ would be in itself a trivial immersion of $K_{\lceil \frac{n}{k} \rceil}$. Therefore, we may safely assume that $|A_{2k} \cup A_{2k+1}| < \frac{n}{k}$. It follows that $n' = |V(H)| = |V(G)| - |A_{2k}| - |A_{2k+1}| > \frac{k-1}{k}n$.

So there is in H an immersion $\mathcal{I}$ with at least $\frac{k-1}{k}n \times \frac{1}{(k-1)} = \frac{n}{k}$ vertices. To find a similar immersion $\mathcal{I}'$ in G with the desired size, we simply replace any edges used between A_{2k-1} and A_1 in $\mathcal{I}$ by the corresponding path in $\mathcal{I}_{P4}$. $\square$

Theorem 10 is a direct corollary of Theorem 11.

4.3 Andrásfai graphs

Let G be a triangle-free graph with n vertices. ANDRÁSFAL (1964) showed that if $\delta(G) > 2n/5$, then G is bipartite. This result was generalized in many directions, one of which is the following. HÄGGKVIST (1982) proved that if $\delta(G) > 3n/8$, then G is 3-colorable, and JIN (1995) weakened this minimum degree condition proving that if $\delta(G) > 10n/29$, then G is 3-colorable. CHEN *et al.* (1997) strengthened Jin's result showing that G is 3-colorable by exposing the structure of the graph. Specifically, they proved that if $\delta(G) > 10n/29$, then G is homomorphic to Γ_d for some $d \in \mathbb{N}$, where Γ_d is the graph (V_d, E_d) for which $V_d = [3d-1]$ and $E_d = \{xy : y = x + i \text{ with } i \in [d, 2d-1]\}$, where the sum is taken modulo $3d-1$. Note also that these graphs Γ_d for $d \in \mathbb{N}$ are the so called Andrásfai graphs.

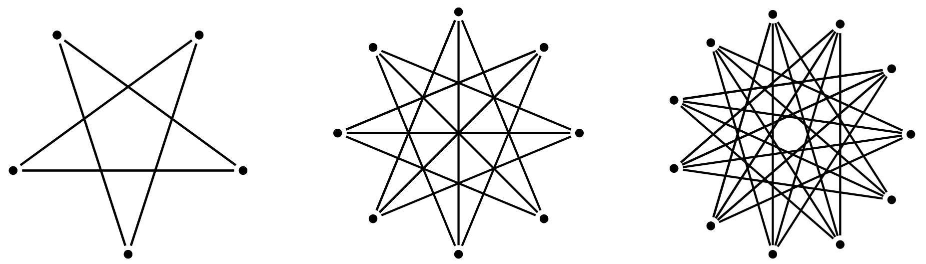

Figure 4.6: The Andrásfai graphs Γ_2, Γ_3, and Γ_4.

There is another equivalent construction of Γ_d, which gives rise to a slightly different drawing for the graphs (See Figure 4.7). To determine the edges, we can take each integer $x \in V_d$ and connect it to every other vertex of the format $y = x \pm d$, where the distances $d \equiv 1 \mod 3$ and $d \leq \lfloor \frac{3d-1}{2} \rfloor$.

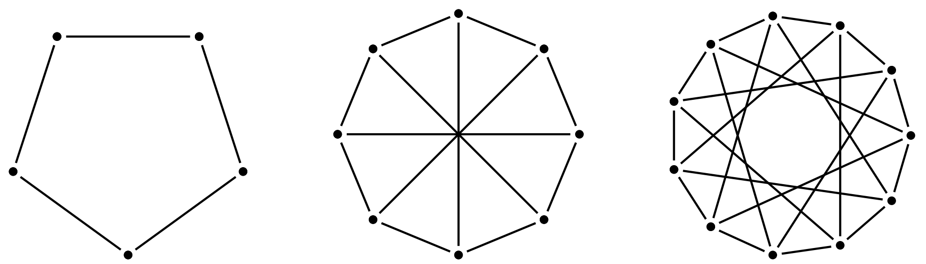

Figure 4.7: Alternative embedding of the Andrásfai graphs Γ_2, Γ_3, and Γ_4.

Theorem 12 (Chen–Jin–Koh, 1997)**.** *If G is a triangle-free graph with n vertices for which $\delta(G) > 10n/29$, then G is homomorphic to Γ_d for some d.*

In the next proofs we use the following property of the graph Γ_d. We denote by $\overline{H}$ the complement of a graph H.

Lemma 5. *Let $d \in \mathbb{N}$, and let D_1 be a maximal independent set of Γ_d. Then Γ_d admits a 3-coloring $\{D_1, D_2, D_3\}$ such that $\overline{\Gamma_d}[D_2 \cup D_3]$ has no induced C_4.*

Proof. Consider d and Γ_d defined as above and embedded as in Figure 4.6. We first observe that the maximal independent sets of Γ_d consist precisely of sequences of d consecutive (modulo $3d-1$) vertices of Γ_d. Indeed, by the definition of E_d, u and v are adjacent if and only if u and v have (circular) distance at least d. Now, let D_1 be a maximum independent set of Γ_d. Assume, without loss of generality, that $D_1 = \{1, \ldots, d\}$. Let $D_2 = \{d+1, \ldots, 2d\}$ and $D_3 = \{2d+1, \ldots, 3d-1\}$. As observed above, D_2 and D_3 are independent sets of Γ_d. By the definition of

E_d, if $u, u' \in D_2$, then either $N_{D_3}(u) \subseteq N_{D_3}(u')$ or $N_{D_3}(u') \subseteq N_{D_3}(u)$. Now, we claim that $\overline{\Gamma_d}[D_2 \cup D_3]$ has no induced C_4. This is equivalent to claiming that $\Gamma_d[D_2 \cup D_3]$ has no induced matching with two edges. Suppose, for a contradiction that $\Gamma_d[D_2 \cup D_3]$ has an induced matching M with two edges. Since D_2 and D_3 are independent sets, the edges of M must join vertices from D_2 to vertices of D_3. Say, $M = \{e, e'\}$ with $e = uv$ and $e' = u'v'$ with $u, u' \in D_2$ and $v, v' \in D_3$. Assume, without loss of generality, that $N_{D_3}(u') \subseteq N_{D_3}(u)$. Then we have $v' \in N_{D_3}(u)$, and hence $\Gamma_d[\{u, u', v, v'\}]$ is not a matching. □

Lemma 6. *Let G be a maximal triangle-free graph homomorphic to Γ_d for some $d \in \mathbb{N}$, and let I_1 be a maximum independent set of G. Then G admits a 3-coloring $\{I_1, I_2, I_3\}$ such that $\overline{G}[I_2 \cup I_3]$ has no induced C_4.*

Proof. Let $h\colon V(G) \to V(\Gamma_d)$ be a homomorphism from G to Γ_d. For each $i \in [3d-1]$ let $V_i = h^{-1}(i) = \{u \in V(G) : h(u) = i\}$ be the set of vertices of G mapped to i. Note that every independent set of G is mapped to an independent set of Γ_d. Moreover, every maximal independent set of G is mapped to a maximal independent set of Γ_d. Now, let I_1 be a maximum independent set of G. Then $D_1 = h(I_1) = \{h(u) : u \in I_1\}$ is a maximal independent set of Γ_d. Let $\{D_1, D_2, D_3\}$ be the coloring of Γ_d given by Lemma 5 for D_1, and, for $i = 2, 3$, let $I_i = h^{-1}(D_i) = \{u \in V(G) : h(u) \in D_i\}$. Naturally, I_i is an independent set for $i = 1, 2, 3$, and, since $V(\Gamma_d) = D_1 \cup D_2 \cup D_3$, we have $V(G) = I_1 \cup I_2 \cup I_3$. Because Γ_d is triangle-free, by the maximality of G, we have that $uv \in E(G)$ if and only if $h(u)h(v) \in E_d$. Now if $G[\{u, u', v, v'\}]$ is a matching with two edges then we may assume, without loss of generality, that $u, u' \in I_2$ and $v, v' \in I_3$. Moreover, u and u' (resp. v and v') are in different sets V_i. But this implies that $\Gamma_d[h(\{u, u', v, v'\})]$ is a matching with two edges, a contradiction. □

4.4 Dense graphs with bounded maximum degree

In this section we prove Theorem 9. The strategy is to use that, if G is a triangle-free graph with n vertices, independence number 2, and maximum degree less than $19n/29 - 1$, then $\overline{G}$ admits a 3-coloring as in Lemma 6, and then to show that every graph whose complement admits such a 3-coloring contains an immersion of $K_{\lceil n/2 \rceil}$. For that, given a positive integer k, a *k-clique coloring* of a graph G is a partition $\{D_1, \ldots, D_k\}$ of $V(G)$ such that D_i is a clique of G for every $i \in [k]$. For $X, Y \subseteq V(G)$, we use $N_X(Y)$ to denote the set of neighbors of Y in X, and, if $X \cap Y = \emptyset$, then we denote by $G[X, Y]$ the bipartite subgraph of G with vertex set $X \cup Y$ and all edges of G between X and Y.

Theorem 13. *Let G be a graph with $\alpha(G) = 2$ that admits a 3-clique coloring $\{D_1, D_2, D_3\}$ such that (i) D_1 is a maximum clique of G; and (ii) $G[D_2 \cup D_3]$ has no* *induced C_4. Then G contains an immersion of a clique whose set of branch vertices is precisely $D_2 \cup D_3$.*

Proof. Observe that $\{D_1, D_2, D_3\}$ is a 3-coloring of $\overline{G}$, and by hypothesis D_1 is a maximum independent set of $\overline{G}$. Let $i \in \{2, 3\}$ and let $C \subseteq D_i$. If $|N_{D_1}(C)| < |C|$, then the set $(D_1 \setminus N_{D_1}(C)) \cup C$ is independent and is larger than D_1, a contradiction to the maximality of D_1. Hence, $|N_{D_1}(C)| \geq |C|$ for every subset C of D_i. So, by Hall's Theorem, there is a matching M_i in $\overline{G}[D_i, D_1]$ that covers D_i.

Given a vertex $u \in D_2 \cup D_3$, note that there is precisely one edge in $M_2 \cup M_3$ that contains u, and let $r_u \in D_1$ be the vertex such that $ur_u \in M_2 \cup M_3$. Note that $r_u \notin N(u)$, and hence, because $\alpha(G) = 2$, r_u is adjacent to every vertex in $V(G) \setminus N[u]$. In what follows, let $E' = E\big(G[D_2, D_3]\big)$. Note that if $u \in D_2$ and $v \in D_3$ are such that $r_u = r_v = w$, then $uv \in E'$, otherwise u, v, w would be an independent set of size 3. Note that if $uv \notin E'$ we have $r_u \in N(v)$ and $r_v \in N(u)$, so, because D_1 is a clique, $r_ur_v \in E(G)$.

Now, for every $uv \notin E'$, let P_{uv} be the path ur_vr_uv. We claim that the paths P_e with $e \notin E'$ are pairwise edge-disjoint. Indeed, let $u, u' \in D_2$ and $v, v' \in D_3$ be such that $uv, u'v' \notin E'$ and $uv \neq u'v'$. Note that u and u' (resp. v and v') are not necessarily distinct, but $u \neq u'$ or $v \neq v'$. If $v \neq v'$, then $r_v \neq r_{v'}$ because M_3 is a matching. This implies that $ur_v \neq u'r_{v'}$ (even if $u = u'$). Analogously, we deduce that $vr_u \neq v'r_{u'}$. In what follows, we prove that $r_ur_v \neq r_{u'}r_{v'}$. Suppose, for a contradiction, that $r_ur_v = r_{u'}r_{v'}$. If $r_u = r_{u'}$ and $r_v = r_{v'}$, then $u = u'$ and $v = v'$, a contradiction. Thus, we must have $r_u = r_{v'}$ and $r_v = r_{u'}$. As noted above, this implies that $uv', vu' \in E'$ (see Figure 4.8). But then $\{u, v, u', v'\}$ induce a C_4 in $G[D_2 \cup D_3]$, a contradiction.

Because D_2 and D_3 are cliques, and the paths P_e with $e \notin E'$ are edge-disjoint, $G[D_2 \cup D_3] \cup \{P_e : e \notin E'\}$ is the desired immersion. □

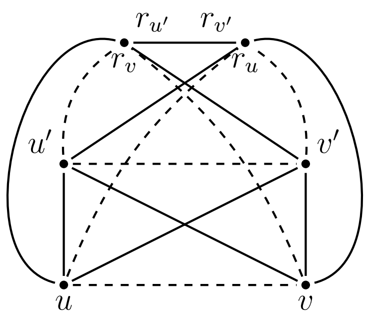

Figure 4.8: Situation in the proof of Theorem 13.

Now we can prove our main theorem.

Proof of Theorem 9. Let G be a graph with n vertices for which $\Delta(G) < 19n/29 - 1$ and $\alpha(G) \leq 2$. Observe that $\delta(\overline{G}) = (n-1) - \Delta(G) > 10n/29$, and hence, by

Theorem 12, $\overline{G}$ is homomorphic to Γ_d for some d. Now, let I_1 be a maximum independent set of $\overline{G}$. If $|I_1| \geq n/2$, then $G[I_1]$ is the desired immersion. Thus, assume $|I_1| < n/2$. By Lemma 6, there exists a 3-coloring $\{I_1, I_2, I_3\}$ of $\overline{G}$ such that $G[I_2 \cup I_3]$ has no induced C_4. Observe that $\{I_1, I_2, I_3\}$ is a 3-clique coloring of G such that I_1 is a maximal clique and $G[I_2 \cup I_3]$ has no induced C_4. Therefore, by Theorem 13, G contains an immersion of a clique whose set of branch vertices is precisely $I_2 \cup I_3$. Since $|I_2 \cup I_3| = n - |I_1| > n/2$, it follows that G contains an immersion of $K_{\lceil n/2 \rceil}$. □

Chapter 5

A linear programming model for finding immersions

During our research on the problem, we wanted to find large clique immersions in specific graphs. Although there exist in the bibliography algorithms to find minors in graphs, such as in HICKS (2004) and ADLER *et al.* (2011), we did not find any literature for immersion finding algorithms. Hence, we developed an integer linear programming model to solve this problem. In this chapter we present the model, as well as an implementation in SageMath.

We propose in Model 5.1 a standard form for the problem of finding large clique immersions in graphs. There is a variable b_v for each vertex $v \in V(G)$ and $b_v = 1$ if and only if v is a branch vertex of the desired immersion. The main idea is to color the edges of the given graph G so that each color corresponds to a distinct path that joins two branch vertices. For that, each (unordered) pair $(u, v) \in \binom{[n]}{2}$ of vertices of G corresponds to a color $c \in [\binom{n}{2}]$. If both u and v are taken as branch vertices in the immersion, then there must be a path of color c in the immersion connecting u to v. Then each edge $e \in E(G)$ corresponds to a set of variables $x_{e,c}$ with $c \in [\binom{n}{2}]$ for which we have $x_{e,c} = 1$ if and only if the edge e is colored with color c. Finally, for each vertex $v \in [n]$ and color $c \in [\binom{n}{2}]$, we have a variable $y_{v,c}$ whose value is 1 if and only if v is an interior vertex of the path of color c.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{v \in V(G)} b_v && && (5.1a)\\
\text{subject to} \quad & \sum_{c=1}^{m} x_{e,c} \le 1, && e \in E(G), && (5.1b)\\
& \sum_{e \in E(G): v \in e} x_{e,c} \le b_v, && c \in [m], v \in \{c_a, c_b\}, && (5.1c)\\
& \sum_{e \in E(G): v \in e} x_{e,c} \ge b_{c_a} + b_{c_b} - 1, && c \in [m], v \in \{c_a, c_b\}, && (5.1d)\\
& \sum_{e \in E(G): v \in e} x_{e,c} = 2y_{u,c}, && c \in [m], v \notin \{c_a, c_b\}, && (5.1e)\\
& b_v, x_{e,c}, y_{v,c} \in \{0,1\} && v \in V(G), e \in E(G), c \in [m] && (5.1f)
\end{aligned}
$$

Model 5.1: An integer linear programming model for finding large immersions

The objective function (Equation 5.1a) is to maximize the number of branch vertices of an immersion of a clique. Constraint 5.1b encodes the restriction that every edge is colored with at most one color.

Given any color c, we are able to find the pair of vertices $\{u, v\}$ which it is assigned to, which we denote by c_a and c_b. Any color c chosen as a path in the immersion appears in exactly one edge adjacent to each branch vertex c_a and c_b. This is modeled by Constraints 5.1c and 5.1d. Every interior vertex of a path of color c in the immersion has exactly two incident edges with color c, and this is encoded in Constraint 5.1e.

5.1 Implementation

In what follows we show how to encode these restrictions. All algorithms in this chapter should be understood as parts of a larger program created to find immersions in graphs. The first step is to create our objective function as in 5.1a and our variables b_v, $x_{e,c}$ and $y_{v,c}$, which we do in Algorithm 1.

Algorithm 1 Integer Linear Program in Sagemath for finding size of largest clique immersion - Setting Variables and Objective

```
def max_clique_immersion(G):

    # Setting maximization problem
    prob = MixedIntegerLinearProgram(maximization = True)

    # b[v] = 1 if and only if vertex v is a branch vertex in the immersion
    b = prob.new_variable(binary = True)

    # The objective is to maximize branch vertices in the immersion
    prob.set_objective(sum(b[v] for v in G.vertices()))

    # x[e,c] = 1 if and only if the edge e is colored with color c
    x = prob.new_variable(binary = True)

    # y[v,c] = 1 if and only if v is a interior vertex of a path of color c
    y = prob.new_variable(binary = True)
```

Given any color c, we want our algorithm to be able to obtain c_a and c_b. This is implemented on Algorithm 2 as `color_edge_dictionary`. Similarly, the set of all colors which connects u to other vertices is well defined, and is implemented through the use of the dictionary `vertex_color_dictionary`. Although the creation of these dictionaries is a straightforward procedure, we present them here for completeness and as a way to illustrate the program which is being written.

Algorithm 2 Integer Linear Program in Sagemath for finding size of largest clique immersion - Setting path colors

```
    # vertex_color_dictionary[u] is the set of colors which connects u to other vertices
    vertex_color_dictionary = {}
    for u in G.vertices():
        vertex_color_dictionary[u] = []

    # color_edge_dictionary[c] is the pair of vertices connected by color c
    color_edge_dictionary = {}

    # At first, associate each pair of vertices with one color
    color = 0
    for u,v in Combinations(G.vertices(),2):
        vertex_color_dictionary[u].append(color)
        vertex_color_dictionary[v].append(color)
        color_edge_dictionary[color] = (u,v)
        color += 1
```

Since any vertex in G has the potential to be chosen as a branch vertex, any (unordered) pair $(u, v) \in \binom{[n]}{2}$ of vertices of G might require a colored path connecting them. Therefore we set $m = \binom{[n]}{2}$ as the number of colors.

Finally, we are ready to set the constraints of the problem, which we do in Algorithm 3. Constraint 5.1b, which limits each edge to at most one color, can be implemented such as in lines 4 through 7.

Next we need to set the constraints limiting, for each vertex $v \in V(G)$, the amount of incident edges of any color c. Constraints 5.1c and 5.1d can be implemented such as in lines 9 through 19. Constraint 5.1e can be implemented as in lines 20 through 25.

Algorithm 3 Integer Linear Program in Sagemath for finding size of largest clique immersion - Setting Constraints

```
1 n = G.order()
2 m = n*(n-1)//2
3
4 # Each edge receives at most one color
5 for e in G.edges():
6     equation = sum(x[e,c] for c in range(m))
7     prob.add_constraint(equation <= 1)
8
9 for c in range(m):
10     a,b = color_edge_dictionary[c]
11     for u in (a,b):
12         summation = sum(x[e,c] for e in G.edges_incident(u))
13         # if c is a color which connects u to another vertex,
14         # then u has at most one edge of the color c
15         prob.add_constraint(summation <= b[a])
16         prob.add_constraint(summation <= b[b])
17         # if both a and b are chosen as branch vertices,
18         # this equation has to equal 1
19         prob.add_constraint(summation >= b[a] + b[b] - 1)
20     for u in G.vertices:
21         equation = sum(x[e,c] for e in G.edges_incident(u))
22         if u != a and u != b:
23             # if u is an interior vertex for the path of color c,
24             # then there are two edges of color c adjacent to u
25             prob.add_constraint(equation == 2*y[u,c])
```

In total, this model contains $O(n^4)$ variables and $\Theta(n^3)$ constraints, with most of them originating from Constraint 5.1e, as it creates one restriction per vertex not in $\{c_a, c_b\}$ per color.

The final part of the code solves the problem using the preferred solver, and prepares the information to return. Algorithm 4 returns the size of the maximum clique and one possible set of branch vertices. Lines 4 through 8 of Algorithm 4,

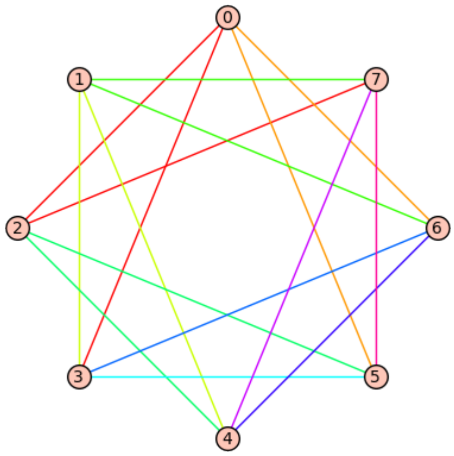

Figure 5.1: Immersion of K_4 on $\overline{\Gamma_3}$, with branch vertices 3, 4, 5 and 6.

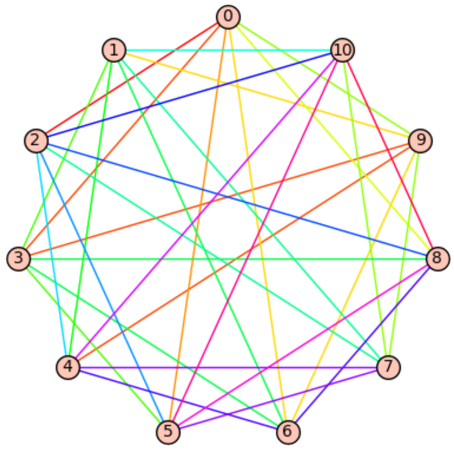

Figure 5.2: Immersion of K_7 on $\overline{\Gamma_4}$, with branch vertices 0, 1, 2, 4, 5, 8 and 10

although not directly used in this version, can be altered to produce images showing the encountered immersions. We display in Figures 5.1 and 5.2 some examples of immersions found by this algorithm. Our main subjects for testing the algorithm were the complements of Andrásfai graphs, denoted by $\overline{\Gamma_n}$.

Algorithm 4 Integer Linear Program in Sagemath for finding size of largest clique immersion - Solving and Coloring

```
    # Solving the problem
    max_clique_immersion = int(prob.solve())

    #Coloring the edges of the graph
    x_dictionary = prob.get_values(x)
    for e,c in x_dictionary:
        if x_dictionary[(e,c)] == 1:
            G.set_edge_label(e[0],e[1],c)

    #Listing the branch vertices to return
    branch_vertices = []
    b_dictionary = prob.get_values(b)
    for u in b_dictionary:
        if b_dictionary[u] == 1:
            branch_vertices.append(u)

    return max_clique_immersion, branch_vertices
```

In our tests, the code proved itself efficient, GLPK was used as the solver, and we were able to find immersions in graphs with up to 20 vertices in a reasonable time frame. The fraction of branch vertices in $\overline{\Gamma_n}$ seems to approach 2/3 as n gets larger. Interestingly, for every tested n, the largest clique immersions in $\overline{\Gamma_n}$ used every single edge available.

Chapter 6

Conclusion

Our original goal in this research was to tackle Conjecture 5.

Conjecture 5 (Vergara, 2017)**.** *Every graph G with n vertices and $\alpha(G) \leq 2$ has an immersion of $K_{\lceil \frac{n}{2} \rceil}$.*

Our main result, Theorem 9, represents a significant step in this direction, which hopefully will help to bridge the gap towards a full proof in the future.

Theorem 9. *Let G be a graph with n vertices. If $\alpha(G) \leq 2$ and $\Delta(G) < 19n/29 - 1$, then G contains an immersion of $K_{\lceil n/2 \rceil}$.*

In particular, the condition on the maximum degree seems counterintuitive, as a larger maximum degree would mean more edges and more possible paths to use for the immersions. We believe, then, that Conjecture 5 holds, and the maximum degree condition may be sidestepped if the problem is approached with the correct strategy or with new additional ideas. Theorem 13 could be particularly useful in that regard, as it does not rely directly on the maximum degree but indirectly through Lemma 6, which could be generalized or replaced.

Theorem 13. *Let G be a graph with $\alpha(G) = 2$ that admits a 3-clique coloring $\{D_1, D_2, D_3\}$ such that (i) D_1 is a maximum clique of G; and (ii) $G[D_2 \cup D_3]$ has no induced C_4. Then G contains an immersion of a clique whose set of branch vertices is precisely $D_2 \cup D_3$.*

A possible direction for improving Theorem 9 lies in exploring Vega graphs. While Vega graphs are less well behaved than Andrásfai graphs, BRANDT (2005) showed that they also appear as a homomorphism class for triangle-free graphs, and could potentially weaken the maximum degree condition in Theorem 9 to $\Delta(G) < n/3$ or further.

Finally, the linear program presented in Chapter 5 is another new contribution to the literature on the topic, and can be applied to find clique immersions on any specific graphs of moderate size (the largest explored graph, $\overline{\Gamma}_7$, has 20 vertices and 120 edges) in a reasonable amount of time. This allows for the exploration of highly structured graphs, such as Andrásfai graphs, Vega graphs and other classes of graphs. This might provide insights and ease visualization into the immersions in specific instances, hopefully helping the development of future proofs.

References

BONDY, J., MURTY, U. *Graph Theory*. 1st ed. New York City, Springer Publishing Company, Incorporated, 2008. ISBN: 1846289696.

BOTLER, F., COLLARES, M., MARTINS, T., et al. *Combinatória*. 1st ed. Rio de Janeiro, IMPA, 2021. ISBN: 978-65-89124-53-5.

MANTEL, W. "Vraagstuk XXVIII", *Wiskundige Opgaven met de Oplossingen*, v. 10, n. 2, pp. 60–1, 1907.

TURÁN, P. "On an extremal problem in graph theory", *Matematikai és Fizikai Lapok*, v. 48, pp. 436–451, 1941.

MCKAY, B. D. "A note on the history of the four-colour conjecture". 2012. Available in: <`https://arxiv.org/abs/1201.2852`>.

WILSON, R., NASH, C. "Four colours suffice: how the map problem was solved", *The Mathematical Intelligencer*, v. 25, pp. 80–83, 12 2003. doi: 10.1007/BF02984867.

F.G. "Tinting Maps". June 10th, 1854.

APPEL, K., HAKEN, W. "The solution of the four-color-map problem", *Scientific American*, v. 237, n. 4, pp. 108–121, 1977.

KURATOWSKI, C. "Sur le problème des courbes gauches en Topologie", *Fundamenta Mathematicae*, v. 15, n. 1, pp. 271–283, 1930. Available in: <`http://eudml.org/doc/212352`>.

WAGNER, K. "Über eine Eigenschaft der ebenen Komplexe", *Mathematische Annalen*, v. 114, pp. 570–590, 1937. Available in: <`http://eudml.org/doc/159935`>.

HADWIGER, H. "Über eine Klassifikation der Streckenkomplexe", *Vierteljschr. Naturforsch. Ges. Zürich*, v. 88, n. 2, pp. 133–142, 1943.

ROBERTSON, N., SEYMOUR, P., THOMAS, R. "Hadwiger's conjecture for K6-free graphs", *Combinatorica*, v. 13, n. 3, pp. 279–361, set. 1993. ISSN: 0209-9683. doi: 10.1007/BF01202354.

TOFT, B. "A survey of Hadwiger's conjecture", *Congressus Numerantium*, pp. 249–283, 1996.

SEYMOUR, P. D. "Hadwiger's Conjecture". In: *Open Problems in Mathematics*, 2016. Available in: <https://api.semanticscholar.org/CorpusID:264379143>.

BALOGH, J., KOSTOCHKA, A. "Large minors in graphs with given independence number", *Discrete Mathematics*, v. 311, n. 20, pp. 2203–2215, 2011. ISSN: 0012-365X. doi: https://doi.org/10.1016/j.disc.2011.07.003. Available in: <https://www.sciencedirect.com/science/article/pii/S0012365X11003128>.

ERDŐS, P., FAJTLOWICZ, S. "On the conjecture of Hajós", *Combinatorica*, v. 1, n. 2, pp. 141–143, jun 1981. doi: 10.1007/bf02579269. Available in: <https://doi.org/10.1007/bf02579269>.

CATLIN, P. A. "Hajós' graph-coloring conjecture: Variations and counterexamples", *J. Comb. Theory B*, v. 26, pp. 268–274, 1979. Available in: <https://api.semanticscholar.org/CorpusID:31608676>.

LESCURE, F., MEYNIEL, H. "On a problem upon configurations contained in graphs with given chromatic number". In: Andersen, L. D., Jakobsen, I. T., Thomassen, C., et al. (Eds.), *Graph Theory in Memory of G.A. Dirac*, v. 41, *Annals of Discrete Mathematics*, Elsevier, pp. 325–331, Amsterdam, 1988. doi: https://doi.org/10.1016/S0167-5060(08)70470-9. Available in: <https://www.sciencedirect.com/science/article/pii/S0167506008704709>.

ABU-KHZAM, F. N., LANGSTON, M. A. "Graph coloring and the immersion order". In: Warnow, T., Zhu, B. (Eds.), *Computing and Combinatorics*, pp. 394–403, Berlin, Heidelberg, 2003. Springer Berlin Heidelberg. ISBN: 978-3-540-45071-9.

GAUTHIER, G., LE, T.-N., WOLLAN, P. "Forcing clique immersions through chromatic number", *European Journal of Combinatorics*, v. 81, pp. 98–118, 2019. ISSN: 0195-6698. doi: https://doi.org/10.1016/j.ejc.2019.04.003. Available in: <https://www.sciencedirect.com/science/article/pii/S0195669819300435>.

BUSTAMANTE, S., QUIROZ, D. A., STEIN, M., et al. "Clique immersions and independence number", *European Journal of Combinatorics*, v. 106, pp. 103550, dez. 2022. ISSN: 0195-6698. doi: 10.1016/j.ejc.2022.103550. Available in: <http://dx.doi.org/10.1016/j.ejc.2022.103550>.

BUSTAMANTE, S., QUIROZ, D., STEIN, M., et al. "Large immersions in graphs with independence number 3 and 4", *Electronic Notes in Theoretical Computer Science*, v. 346, pp. 221–228, 2019. ISSN: 1571-0661. doi: https://doi.org/10.1016/j.entcs.2019.08.020. Available in: <https://www.sciencedirect.com/science/article/pii/S1571066119300702>. Proceedings of the X Latin and American Algorithms, Graphs and Optimization Symposium.

VERGARA, S. "Complete graph immersions in dense graphs", *Discrete Mathematics*, v. 340, n. 5, pp. 1019–1027, 2017.

QUIROZ, D. A. "Clique immersions in graphs of independence number two with certain forbidden subgraphs", *Discrete Mathematics*, v. 344, n. 6, pp. 112365, jun. 2021a. ISSN: 0012-365X. doi: 10.1016/j.disc.2021.112365. Available in: <http://dx.doi.org/10.1016/j.disc.2021.112365>.

QUIROZ, D. A. "Complete immersions in graphs with independence number two and small forbidden subgraphs", *Procedia Computer Science*, v. 195, pp. 343–349, 2021b. ISSN: 1877-0509. doi: https://doi.org/10.1016/j.procs.2021.11.042. Available in: <https://www.sciencedirect.com/science/article/pii/S1877050921021815>. Proceedings of the XI Latin and American Algorithms, Graphs and Optimization Symposium.

BOTLER, F., JIMÉNEZ, A., LINTZMAYER, C., et al. "Biclique immersions in graphs with independence number 2", *European Journal of Combinatorics*, v. 122, pp. 104042, 2024. ISSN: 0195-6698. doi: https://doi.org/10.1016/j.ejc.2024.104042. Available in: <https://www.sciencedirect.com/science/article/pii/S0195669824001276>.

OLIVEIRA, C., DE LIMA, L., NIKIFOROV, V. "Spectra of the blow-up graphs". 2014. Available in: <https://arxiv.org/abs/1412.0319>.

HATAMI, H., HIRST, J., NORINE, S. "The inducibility of blow-up graphs", *Journal of Combinatorial Theory, Series B*, v. 109, pp. 196–212, 2014. ISSN: 0095-8956. doi: https://doi.org/10.1016/j.jctb.2014.06.005. Available in: <https://www.sciencedirect.com/science/article/pii/S0095895614000744>.

ANDRÁSFAL, B. "Graphentheoretische Extremalprobleme", *Acta Mathematica Hungarica*, v. 15, n. 3-4, pp. 413–438, 1964.

HÄGGKVIST, R. "Odd cycles of specified length in non-bipartite graphs". In: Bollobás, B. (Ed.), *Graph Theory*, v. 62, *North-Holland Mathematics Studies*, North-Holland, pp. 89–99, Amsterdam, 1982. doi: https://doi.org/10.1016/S0304-0208(08)73552-7. Available in: <https://www.sciencedirect.com/science/article/pii/S0304020808735527>.

JIN, G. "Triangle-free four-chromatic graphs", *Discrete Mathematics*, v. 145, n. 1-3, pp. 151–170, 1995.

CHEN, C.-C., JIN, G. P., KOH, K. M. "Triangle-free graphs with large degree", *Combinatorics, Probability and Computing*, v. 6, n. 4, pp. 381–396, 1997.

HICKS, I. V. "Branch decompositions and minor containment", *Networks*, v. 43, n. 1, pp. 1–9, 2004. doi: https://doi.org/10.1002/net.10099. Available in: <https://onlinelibrary.wiley.com/doi/abs/10.1002/net.10099>.

ADLER, I., DORN, F., FOMIN, F. V., et al. "Faster parameterized algorithms for minor containment", *Theoretical Computer Science*, v. 412, n. 50, pp. 7018–7028, 2011. ISSN: 0304-3975. doi: https://doi.org/10.1016/j.tcs.2011.09.015. Available in: <https://www.sciencedirect.com/science/article/pii/S0304397511007912>.

BRANDT, S. "Dense triangle-free graphs are four-colorable : A solution to the Erdős-Simonovits problem". 2005. Available in: <https://api.semanticscholar.org/CorpusID:15565912>.