

MaxCut is hard when restricted to geometric intersection model graph classes

Celina Miraglia Herrera de Figueiredo



CLAIO 2022



Based on

Maximum cut on interval graphs of interval count four is NP-complete

with Alexander A. de Melo, Fabiano S. Oliveira, Ana Silva

arxiv.org/abs/2012.09804



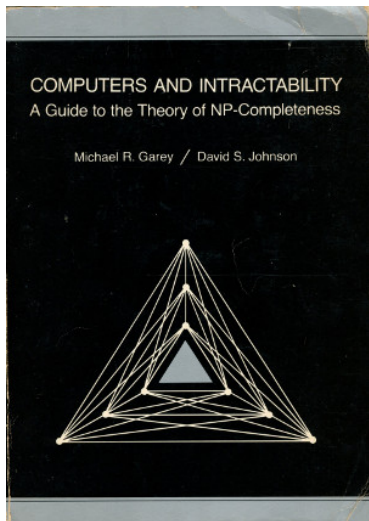
MaxCut on Permutation Graphs is NP-complete

with Alexander A. de Melo, Fabiano S. Oliveira, Ana Silva

arxiv.org/abs/2202.13955



The Guide – Computers and Intractability



“Despite that 23 years have passed since its publication, I consider Garey and Johnson the single most important book on my office bookshelf. Every computer scientist should have this book on their shelves as well. NP-completeness is the single most important concept to come out of theoretical computer science and no book covers it as well as Garey and Johnson.”

Lance Fortnow, “Great Books: Computers and Intractability: A Guide to the Theory of NP-Completeness”

Ongoing Guide – Graph Restrictions and Their Effect

GRAPH CLASS	MEMBER	INDSET	CLIQUE	CLIPAR	CHRNUM	CHRIND	HAMCIR	DOMSET	MAXCUT	STREE	GRAISO
Trees/Forests	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [GJ]	P [T]	P [GJ]
Almost Trees (k)	P	P [24]	P [T]	P?	P?	P?	P?	P [45]	P?	P?	P?
Partial k -Trees	P [2]	P [1]	P [T]	P?	P [1]	O?	P [3]	P [3]	P?	P?	O?
Bandwidth- k	P [68]	P [64]	P [T]	P?	P [64]	P?	P?	P [64]	P [64]	P?	P [58]
Degree- k	P [T]	N [GJ]	P [T]	N [GJ]	N [GJ]	N [49]	N [GJ]	N [GJ]	N [GJ]	N [GJ]	P [58]
Planar	P [GJ]	N [GJ]	P [T]	N [10]	N [GJ]	O	N [GJ]	N [GJ]	P [GJ]	N [35]	P [GJ]
Series Parallel	P [79]	P [75]	P [T]	P?	P [74]	P [74]	P [74]	P [54]	P [GJ]	P [82]	P [GJ]
Outerplanar	P	P [6]	P [T]	P [6]	P [67]	P [67]	P [T]	P [6]	P [GJ]	P [81]	P [GJ]
Halin	P	P [6]	P [T]	P [6]	P [74]	P [74]	P [T]	P [6]	P [GJ]	P?	P [GJ]
k -Outerplanar	P	P [6]	P [T]	P [6]	P [6]	O?	P [6]	P [6]	P [GJ]	P?	P [GJ]
Grid	P	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [51]	N [55]	P [T]	N [35]	P [GJ]
$K_{3,3}$ -Free	P [4]	N [GJ]	P [T]	N [10]	N [GJ]	O?	N [GJ]	N [GJ]	P [5]	N [GJ]	O?
Thickness- k	N [60]	N [GJ]	P [T]	N [10]	N [GJ]	N [49]	N [GJ]	N [GJ]	N [7]	N [GJ]	O?
Genus- k	P [34]	N [GJ]	P [T]	N [10]	N [GJ]	O?	N [GJ]	N [GJ]	O?	N [GJ]	P [61]
Perfect	O!	P [42]	P [42]	P [42]	P [42]	O?	N [1]	N [14]	O?	N [GJ]	I [GJ]
Chordal	P [76]	P [40]	P [40]	P [40]	P [40]	O?	N [22]	N [14]	O?	N [83]	I [GJ]
Split	P [40]	P [40]	P [40]	P [40]	P [40]	O?	N [22]	N [19]	O?	N [83]	I [15]
Strongly Chordal	P [31]	P [40]	P [40]	P [40]	P [40]	O?	O?	P [32]	O?	P [83]	O?
Comparability	P [40]	P [40]	P [40]	P [40]	P [40]	O?	N [1]	N [28]	O?	N [GJ]	I [GJ]
Bipartite	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [1]	N [28]	P [T]	N [GJ]	I [GJ]
Permutation	P [40]	P [40]	P [40]	P [40]	P [40]	O?	O	P [33]	O?	P [23]	P [21]
Cographs	P [T]	P [40]	P [40]	P [40]	P [40]	O?	P [25]	P [33]	O?	P [23]	P [25]
Undirected Path	P [39]	P [40]	P [40]	P [40]	P [40]	O?	O?	N [16]	O?	O?	I [GJ]
Directed Path	P [38]	P [40]	P [40]	P [40]	P [40]	O?	O?	P [16]	O?	P [83]	O?
Interval	P [17]	P [44]	P [44]	P [44]	P [44]	O?	P [53]	P [16]	O?	P [83]	P [57]
Circular Arc	P [78]	P [44]	P [50]	P [44]	N [36]	O?	O?	P [13]	O?	P [83]	O?
Circle	P [71]	P [GJ]	P [50]	O?	N [36]	O?	P [12]	O?	O?	P [70]	O?
Proper Circ. Arc	P [77]	P [44]	P [50]	P [44]	P [66]	O?	P [12]	P [13]	O?	P [83]	O?
Edge (or Line)	P [47]	P [GJ]	P [T]	N [GJ]	N [49]	O?	N [11]	N [GJ]	O?	N [70]	I [15]
Claw-Free	P [T]	P [63]	O?	N [GJ]	N [49]	O?	N [11]	N [GJ]	O?	N [70]	I [15]

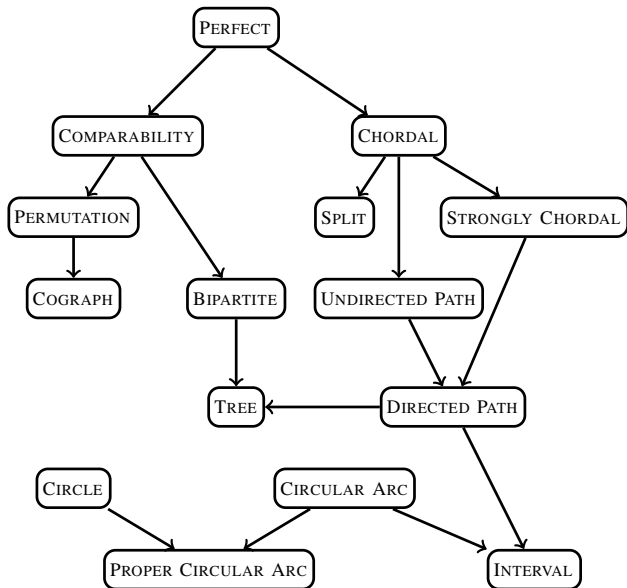
The updated NP-Completeness Column: An Ongoing Guide table 35 years later

GRAPH CLASS	MEMBER	INDSET	CLIQUE	CLIPAR	CHRNUM	CHRIND	HAMCIR	DOMSET	MAXCUT	STTREE	GRAPHISO
TREES/FORESTS	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [GJ]	P [T]	P [GJ]
ALMOST TREES (K)	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
PARTIAL K-TREES	P [OG]	P [5]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
BANDWIDTH-K	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [OG]	P [76]	P [OG]
DEGREE-K	P [T]	N [GJ]	P [T]	N [29]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [GJ]	N [GJ]	P [OG]
PLANAR	P [GJ]	N [GJ]	P [T]	N [78]	N [GJ]	O	N [GJ]	N [GJ]	P [GJ]	N [OG]	P [GJ]
SERIES PARALLEL	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [OG]	P [GJ]	P [OG]	P [GJ]
OUTERPLANAR	P [OG]	P [OG]	P [T]	P [OG]	P [OG]	P [OG]	P [T]	P [OG]	P [GJ]	P [OG]	P [GJ]
HALIN	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [T]	P [OG]	P [GJ]	P [118]	P [GJ]
K-OUTERPLANAR	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [OG]	P [OG]	P [GJ]	P [76]	P [GJ]
GRID	P [OG]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [32]	P [T]	N [OG]	P [GJ]
K _{3,3} -FREE*	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	P [OG]	N [GJ]	P [40]
THICKNESS-K	N [OG]	N [GJ]	P [T]	N [78]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [119]	N [GJ]	I [RJ]
GENUS-K	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	O?	N [GJ]	P [OG]
PERFECT	P [34]	P [OG]	P [OG]	P [OG]	P [OG]	N [28]	N [OG]	N [OG]	N [20]	N [GJ]	I [84]
CHORDAL	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	N [20]	N [OG]	I [84]
SPLIT	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	N [20]	N [OG]	I [108]
STRONGLY CHORDAL	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	P [OG]	N [109]	P [OG]	I [111]
COMPARABILITY	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	N [28]	N [OG]	N [94]	N [102]	N [GJ]	I [22]
BIPARTITE	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [94]	P [T]	N [GJ]	I [22]
PERMUTATION	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [44]	P [OG]	N [120]	P [OG]	P [OG]
COGRAPHS	P [T]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	P [20]	P [OG]	P [OG]
UNDIRECTED Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [13]	N [OG]	N [20]	N [RJ]	I [22]
DIRECTED PATH	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [99]	P [OG]	N [11]	P [OG]	P [7]
INTERVAL	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	N [11]	P [OG]	P [OG]
CIRCULAR ARC	P [OG]	P [OG]	P [OG]	P [OG]	N [OG]	O?	P [106]	P [OG]	N [11]	P [11]	P [80]
CIRCLE	P [OG]	P [GJ]	P [OG]	N [73]	N [OG]	O?	N [39]	N [71]	N [26]	P [OG]	P [68]
PROPER CIRC. ARC	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	O?	P [11]	P [82]
EDGE (OR LINE)	P [OG]	P [GJ]	P [T]	N [95]	N [OG]	N [28]	N [OG]	N [GJ]	P [59]	N [19]	I [OG]
CLAW-FREE	P [T]	P [OG]	N [103]	N [85]	N [OG]	N [28]	N [OG]	N [GJ]	N [20]	N [19]	I [OG]

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ALMOST TREES (K)	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
PARTIAL K-TREES	P [OG]	P [5]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
BANDWIDTH-K	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [OG]	P [76]	P [OG]
DEGREE-K	P [T]	N [GJ]	P [T]	N [29]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [GJ]	N [GJ]	P [OG]
PLANAR	P [GJ]	N [GJ]	P [T]	N [78]	N [GJ]	O	N [GJ]	N [GJ]	P [GJ]	N [OG]	P [GJ]
SERIES PARALLEL	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [OG]	P [GJ]	P [OG]	P [GJ]
OUTERPLANAR	P [OG]	P [OG]	P [T]	P [OG]	P [OG]	P [OG]	P [T]	P [OG]	P [GJ]	P [OG]	P [GJ]
HALIN	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [T]	P [OG]	P [GJ]	P [118]	P [GJ]
K-OUTERPLANAR	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [OG]	P [OG]	P [GJ]	P [76]	P [GJ]
GRID	P [OG]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [32]	P [T]	N [OG]	P [GJ]
K _{3,3} -FREE*	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	P [OG]	N [GJ]	P [40]
THICKNESS-K	N [OG]	N [GJ]	P [T]	N [78]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [119]	N [GJ]	I [RJ]
GENUS-K	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	O?	N [GJ]	P [OG]
PERFECT	P [34]	P [OG]	P [OG]	P [OG]	P [OG]	N [28]	N [OG]	N [OG]	N [20]	N [GJ]	I [84]
CHORDAL	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	N [20]	N [OG]	I [84]
SPLIT	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	N [20]	N [OG]	I [108]
STRONGLY CHORDAL	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	P [OG]	N [109]	P [OG]	I [111]
COMPARABILITY	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	N [28]	N [OG]	N [94]	N [102]	N [GJ]	I [22]
BIPARTITE	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [94]	P [T]	N [GJ]	I [22]
PERMUTATION	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [44]	P [OG]	N [120]	P [OG]	P [OG]
COGRAPHS	P [T]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	P [20]	P [OG]	P [OG]
UNDIRECTED Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [13]	N [OG]	N [20]	N [RJ]	I [22]
DIRECTED PATH	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [99]	P [OG]	N [11]	P [OG]	P [7]
INTERVAL	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	N [11]	P [OG]	P [OG]
CIRCULAR ARC	P [OG]	P [OG]	P [OG]	P [OG]	N [OG]	O?	P [106]	P [OG]	N [11]	P [11]	P [80]
CIRCLE	P [OG]	P [GJ]	P [OG]	N [73]	N [OG]	O?	N [39]	N [71]	N [26]	P [OG]	P [68]
PROPER CIRC. ARC	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	O?	P [11]	P [82]
EDGE (OR LINE)	P [OG]	P [GJ]	P [T]	N [95]	N [OG]	N [28]	N [OG]	N [GJ]	P [59]	N [19]	I [OG]
CLAW-FREE	P [T]	P [OG]	N [103]	N [85]	N [OG]	N [28]	N [OG]	N [GJ]	N [20]	N [19]	I [OG]

Containment relations for classes



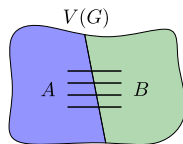
MaxCut problem

Given a graph G and $k \in \mathbb{Z}_0^+$, **MAXCUT** asks whether

$$\text{mc}(G) = \max_{[A,B]} |E_G(A, B)| \geq k.$$

Classical NP-complete problem

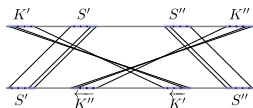
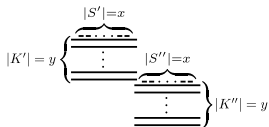
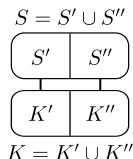
(Garey, Johnson, Stockmeyer, 1976).



The key gadget to the NP-completeness

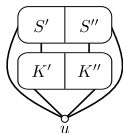
An (x, y) -grained gadget is a split graph $H\langle K, S\rangle$, such that

- ▶ $S = S' \cup S''$, $|S'| = |S''| = x$;
- ▶ $K = K' \cup K''$, $|K'| = |K''| = y$;
- ▶ $N_H(K') = K \cup S'$;
- ▶ $N_H(K'') = K \cup S''$.

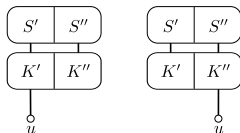


Possible intersections with a grained gadget

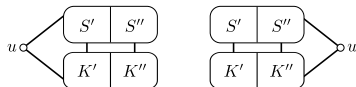
A graph G respects the structure of H if, $\forall u \in V(G) \setminus V(H)$,
 $N_G(v) \cap V(H) = \emptyset$ or u satisfies



Covering intersection



Weak intersection



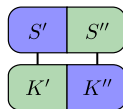
Strong intersection

The key property of grained gadgets

Let G be a graph and $[A, B]$ be a maximum cut of G .

If G respects the structure of an (x, y) -grained gadget H , then, for suitable x and y ,

- ▶ either H is A -partitioned by $[A, B]$;
- ▶ or H is B -partitioned by $[A, B]$.

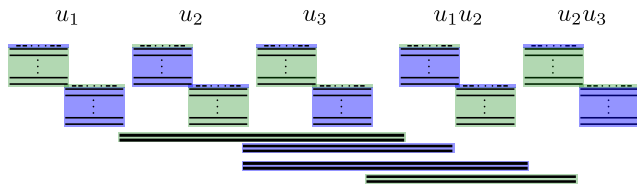


Adhikary, Bose, Mukherjee, and Roy's reduction

Polynomial-time reduction from **MAXCUT** on **cubic graphs**.

Let G be a cubic graph, $\pi_V = (u_1, \dots, u_n)$ and $\pi_E = (e_1, \dots, e_m)$.

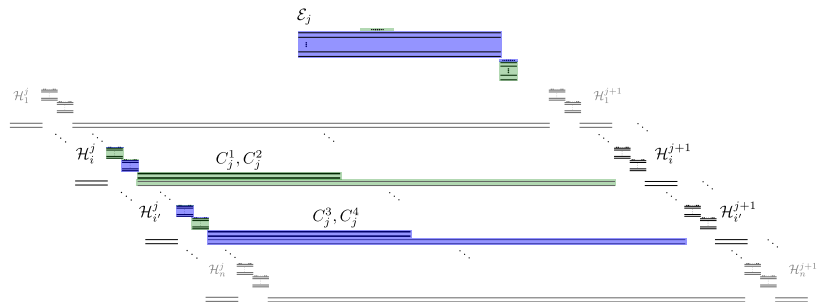
For suitable x, y , $\text{mc}(\mathbb{G}_{\mathcal{M}}) \geq \phi(n, k)$ iff $\text{mc}(G) \geq k$.



(Complexity of maximum cut on interval graphs. Adhikary, Bose, Mukherjee, Roy, 2021)

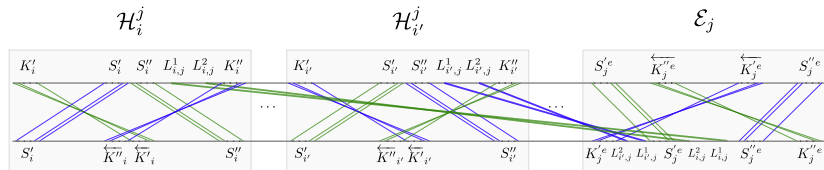
Our reduction: Interval count 4

$[A, B]$ is a max-cut of \mathbb{G}_M



Our reduction: Permutation

$[A, B]$ is a max-cut of \mathbb{G}_M



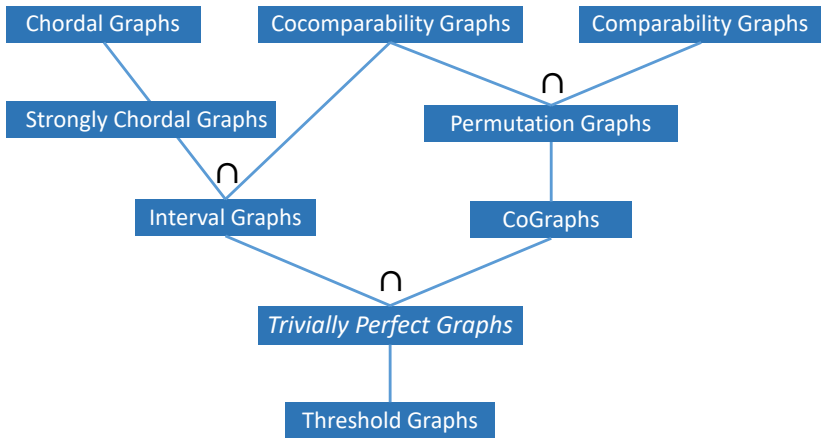


Figura 1. A hierarchy of graph classes

Main open questions

- ▶ Is `MAXCUT` polynomial-time solvable on `unit interval graphs`?
- ▶ Is `MAXCUT` polynomial-time solvable on `interval permutation graphs`?