MaxCut is hard when restricted to geometric intersection model graph classes

Celina Miraglia Herrera de Figueiredo

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Based on

Maximum cut on interval graphs of interval count four is NP-complete

with Alexsander A. de Melo, Fabiano S. Oliveira, Ana Silva

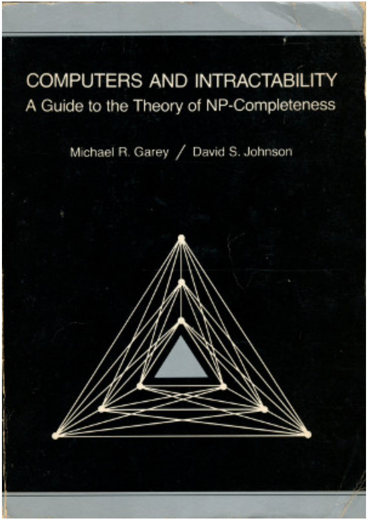
arxiv.org/abs/2012.09804

MaxCut on Permutation Graphs is NP-complete

with Alexsander A. de Melo, Fabiano S. Oliveira, Ana Silva

arxiv.org/abs/2202.13955

The Guide – Computers and Intractability



"Despite that 23 years have passed since its publication, I consider Garey and Johnson the single most important book on my office bookshelf. Every computer scientist should have this book on their shelves as well. NP-completeness is the single most important concept to come out of theoretical computer science and no book covers it as well as Garey and Johnson."

Lance Fortnow, "Great Books: Computers and Intractability: A Guide to the Theory of NP-Completeness"

Ongoing Guide – Graph Restrictions and Their Effect

Graph Class	Member	IndSet	Clique	CliPar	ChrNum	ChrInd	HamCir	DomSet	MaxCut	StTree	GraIso
Trees/Forests	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [GJ]	P [T]	P [GJ]
Almost Trees (k)	P	P [24]	P [T]	P?	P?	P?	P?	P [45]	P?	P?	P?
Partial k-Trees	P [2]	P [1]	P [T]	P?	P [1]	O?	P [3]	P [3]	P?	P?	O?
Bandwidth-k	P [68]	P [64]	P [T]	P?	P [64]	P?	P?	P [64]	P [64]	P?	P [58]
Degree-k	P [T]	N [GJ]	P [T]	N [GJ]	N [GJ]	N [49]	N [GJ]	N [GJ]	N [GJ]	N [GJ]	P [58]
Planar	P [GJ]	N [GJ]	P [T]	N [10]	N [GJ]	O	N [GJ]	N [GJ]	P [GJ]	N [35]	P [GJ]
Series Parallel	P [79]	P [75]	P [T]	P?	P [74]	P [74]	P [74]	P [54]	P [GJ]	P [82]	P [GJ]
Outerplanar	P	P [6]	P [T]	P [6]	P [67]	P [67]	P [T]	P [6]	P [GJ]	P [81]	P [GJ]
Halin	P	P [6]	P [T]	P [6]	P [74]	P [74]	P [T]	P [6]	P [GJ]	P?	P [GJ]
k-Outerplanar	P	P [6]	P [T]	P [6]	P [6]	O?	P [6]	P [6]	P [GJ]	P?	P [GJ]
Grid	P	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [51]	N [55]	P [T]	N [35]	P [GJ]
$K_{3,3}$-Free	P [4]	N [GJ]	P [T]	N [10]	N [GJ]	O?	N [GJ]	N [GJ]	P [5]	N [GJ]	O?
Thickness-k	N [60]	N [GJ]	P [T]	N [10]	N [GJ]	N [49]	N [GJ]	N [GJ]	N [7]	N [GJ]	O?
Genus-k	P [34]	N [GJ]	P [T]	N [10]	N [GJ]	O?	N [GJ]	N [GJ]	O?	N [GJ]	P [61]
Perfect	O!	P [42]	P [42]	P [42]	P [42]	O?	N [1]	N [14]	O?	N [GJ]	I [GJ]
Chordal	P [76]	P [40]	P [40]	P [40]	P [40]	O?	N [22]	N [14]	O?	N [83]	I [GJ]
Split	P [40]	P [40]	P [40]	P [40]	P [40]	O?	N [22]	N [19]	O?	N [83]	I [15]
Strongly Chordal	P [31]	P [40]	P [40]	P [40]	P [40]	O?	O?	P [32]	O?	P [83]	O?
Comparability	P [40]	P [40]	P [40]	P [40]	P [40]	O?	N [1]	N [28]	O?	N [GJ]	I [GJ]
Bipartite	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [1]	N [28]	P [T]	N [GJ]	I [GJ]
Permutation	P [40]	P [40]	P [40]	P [40]	P [40]	O?	O	P [33]	O?	P [23]	P [21]
Cographs	P [T]	P [40]	P [40]	P [40]	P [40]	O?	P [25]	P [33]	O?	P [23]	P [25]
Undirected Path	P [39]	P [40]	P [40]	P [40]	P [40]	O?	O?	N [16]	O?	O?	I [GJ]
Directed Path	P [38]	P [40]	P [40]	P [40]	P [40]	O?	O?	P [16]	O?	P [83]	O?
Interval	P [17]	P [44]	P [44]	P [44]	P [44]	O?	P [53]	P [16]	O?	P [83]	P [57]
Circular Arc	P [78]	P [44]	P [50]	P [44]	N [36]	O?	O?	P [13]	O?	P [83]	O?
Circle	P [71]	P [GJ]	P [50]	O?	N [36]	O?	P [12]	O?	O?	P [70]	O?
Proper Circ. Arc	P [77]	P [44]	P [50]	P [44]	P [66]	O?	P [12]	P [13]	O?	P [83]	O?
Edge (or Line)	P [47]	P [GJ]	P [T]	N [GJ]	N [49]	O?	N [11]	N [GJ]	O?	N [70]	I [15]
Claw-Free	P [T]	P [63]	O?	N [GJ]	N [49]	O?	N [11]	N [GJ]	O?	N [70]	I [15]

The updated NP-Completeness Column: An Ongoing Guide table 35 years later

Graph Class	Member	IndSet	Clique	CliPar	ChrNum	ChrInd	HamCir	DomSet	MaxCut	StTree	GraphIso
Trees/Forests	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [GJ]	P [T]	P [GJ]
Almost Trees (k)	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
Partial k-trees	P [OG]	P [5]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
Bandwidth-k	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [OG]	P [76]	P [OG]
Degree-k	P [T]	N [GJ]	P [T]	N [29]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [GJ]	N [GJ]	P [OG]
Planar	P [GJ]	N [GJ]	P [T]	N [78]	N [GJ]	O	N [GJ]	N [GJ]	P [GJ]	N [OG]	P [GJ]
Series Parallel	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [OG]	P [GJ]	P [OG]	P [GJ]
Outerplanar	P [OG]	P [OG]	P [T]	P [OG]	P [OG]	P [OG]	P [T]	P [OG]	P [GJ]	P [OG]	P [GJ]
Halin	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [T]	P [OG]	P [GJ]	P [118]	P [GJ]
k-Outerplanar	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [OG]	P [OG]	P [GJ]	P [76]	P [GJ]
Grid	P [OG]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [32]	P [T]	N [OG]	P [GJ]
$K_{3,3}$-Free*	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	P [OG]	N [GJ]	P [40]
Thickness-k	N [OG]	N [GJ]	P [T]	N [78]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [119]	N [GJ]	I [RJ]
Genus-k	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	O?	N [GJ]	P [OG]
Perfect	P [34]	P [OG]	P [OG]	P [OG]	P [OG]	N [28]	N [OG]	N [OG]	N [20]	N [GJ]	I [84]
Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	N [20]	N [OG]	I [84]
Split	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	N [20]	N [OG]	I [108]
Strongly Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	P [OG]	N [109]	P [OG]	I [111]
Comparability	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	N [28]	N [OG]	N [94]	N [102]	N [GJ]	I [22]
Bipartite	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [94]	P [T]	N [GJ]	I [22]
Permutation	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [44]	P [OG]	N [120]	P [OG]	P [OG]
Cographs	P [T]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	P [20]	P [OG]	P [OG]
Undirected Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [13]	N [OG]	N [20]	N [RJ]	I [22]
Directed Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [99]	P [OG]	N [1]	P [OG]	P [7]
Interval	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	N [1]	P [OG]	P [OG]
Circular Arc	P [OG]	P [OG]	P [OG]	P [OG]	N [OG]	O?	P [106]	P [OG]	N [1]	P [11]	P [80]
Circle	P [OG]	P [GJ]	P [OG]	N [73]	N [OG]	O?	**N** [39]	N [71]	N [26]	P [OG]	P [68]
Proper Circ. Arc	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	O?	P [11]	P [82]
Edge (or Line)	P [OG]	P [GJ]	P [T]	N [95]	N [OG]	N [28]	N [OG]	N [GJ]	P [59]	N [19]	I [OG]
Claw-Free	P [T]	P [OG]	N [103]	N [85]	N [OG]	N [28]	N [OG]	N [GJ]	N [20]	N [19]	I [OG]

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The updated NP-Completeness Column: An Ongoing Guide table 35 years later

Graph Class	Member	IndSet	Clique	CliPar	ChrNum	ChrInd	HamCir	DomSet	MaxCut	StTree	GraphIso
Trees/Forests	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [GJ]	P [T]	P [GJ]
Almost Trees (k)	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
Partial k-trees	P [OG]	P [5]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
Bandwidth-k	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [5]	P [OG]	P [76]	P [OG]
Degree-k	P [T]	N [GJ]	P [T]	N [29]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [GJ]	N [GJ]	P [OG]
Planar	P [GJ]	N [GJ]	P [T]	N [78]	N [GJ]	O	N [GJ]	N [GJ]	P [GJ]	N [OG]	P [GJ]
Series Parallel	P [OG]	P [OG]	P [T]	P [105]	P [5]	P [17]	P [5]	P [OG]	P [GJ]	P [OG]	P [GJ]
Outerplanar	P [OG]	P [OG]	P [T]	P [OG]	P [OG]	P [OG]	P [T]	P [OG]	P [GJ]	P [OG]	P [GJ]
Halin	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [T]	P [OG]	P [GJ]	P [118]	P [GJ]
k-Outerplanar	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [OG]	P [OG]	P [GJ]	P [76]	P [GJ]
Grid	P [OG]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [32]	P [T]	N [OG]	P [GJ]
$K_{3,3}$-Free*	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	P [OG]	N [GJ]	P [40]
Thickness-k	N [OG]	N [GJ]	P [T]	N [78]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [119]	N [GJ]	I [RJ]
Genus-k	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	O?	N [GJ]	P [OG]
Perfect	P [34]	P [OG]	P [OG]	P [OG]	P [OG]	N [28]	N [OG]	N [OG]	N [20]	N [GJ]	I [84]
Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	N [20]	N [OG]	I [84]
Split	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	N [20]	N [OG]	I [108]
Strongly Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	P [OG]	N [109]	P [OG]	I [111]
Comparability	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	N [28]	N [OG]	N [94]	N [102]	N [GJ]	I [22]
Bipartite	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [94]	P [T]	N [GJ]	I [22]
Permutation	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [44]	P [OG]	N [120]	P [OG]	P [OG]
Cographs	P [T]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	P [20]	P [OG]	P [OG]
Undirected Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [13]	N [OG]	N [20]	N [RJ]	I [22]
Directed Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [99]	P [OG]	N [1]	P [OG]	P [7]
Interval	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	N [1]	P [OG]	P [OG]
Circular Arc	P [OG]	P [OG]	P [OG]	P [OG]	N [OG]	O?	P [106]	P [OG]	N [1]	P [11]	P [80]
Circle	P [OG]	P [GJ]	P [OG]	N [73]	N [OG]	O?	N [39]	N [71]	N [26]	P [OG]	P [68]
Proper Circ. Arc	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	O?	P [11]	P [82]
Edge (or Line)	P [OG]	P [GJ]	P [T]	N [95]	N [OG]	N [28]	N [OG]	N [GJ]	P [59]	N [19]	I [OG]
Claw-Free	P [T]	P [OG]	N [103]	N [85]	N [OG]	N [28]	N [OG]	N [GJ]	N [20]	N [19]	I [OG]

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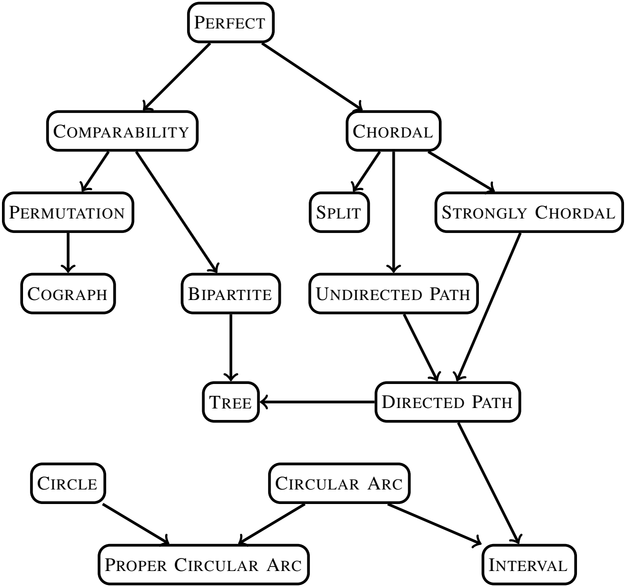
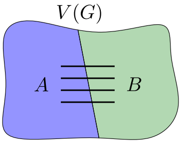
Containment relations for classes



MaxCut problem

Given a graph G and $k \in \mathbb{Z}_0^+$, MaxCut asks whether

$$\mathrm{mc}(G) = \max_{[A,B]} |E_G(A,B)| \geq k.$$



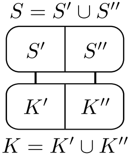
Classical NP-complete problem

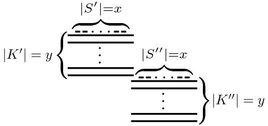
(Garey, Johnson, Stockmeyer, 1976).

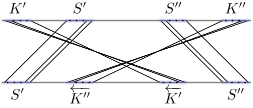
The key gadget to the NP-completeness

An (x, y)-grained gadget is a split graph $H\langle K, S\rangle$, such that

- $S = S' \cup S''$, $|S'| = |S''| = x$;
- $K = K' \cup K''$, $|K'| = |K''| = y$;
- $N_H(K') = K \cup S'$;
- $N_H(K'') = K \cup S''$.

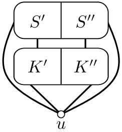




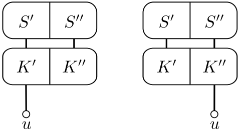


Possible intersections with a grained gadget

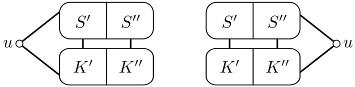
A graph G *respects the structure* of H if, $\forall\, u \in V(G) \setminus V(H)$, $N_G(v) \cap V(H) = \emptyset$ or u satisfies



Covering intersection



Weak intersection



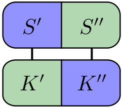
Strong intersection

The key property of grained gadgets

Let G be a graph and $[A, B]$ be a maximum cut of G.

If G respects the structure of an (x, y)-grained gadget H, then, for suitable x and y,

- either H is A-partitioned by $[A, B]$;
- or H is B-partitioned by $[A, B]$.

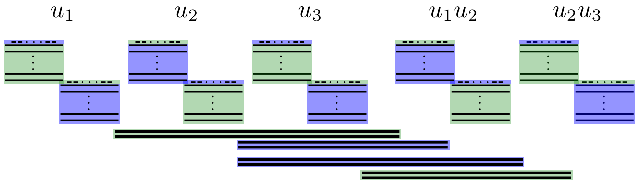


Adhikary, Bose, Mukherjee, and Roy's reduction

Polynomial-time reduction from MaxCut on cubic graphs.

Let G be a cubic graph, $\pi_V = (u_1, \ldots, u_n)$ and $\pi_E = (e_1, \ldots, e_m)$.

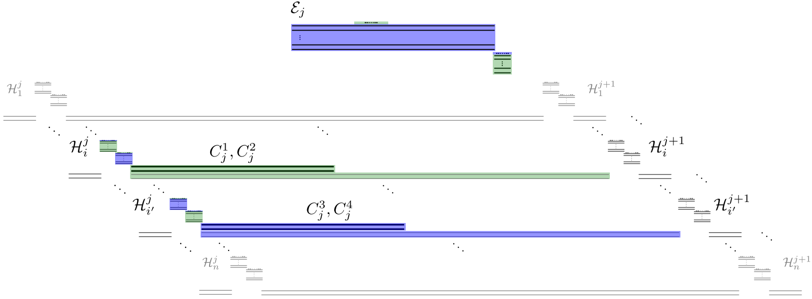
For suitable x, y, $\text{mc}(\mathbb{G}_{\mathcal{M}}) \geq \phi(n, k)$ iff $\text{mc}(G) \geq k$.



(Complexity of maximum cut on interval graphs. Adhikary, Bose, Mukherjee, Roy, 2021)

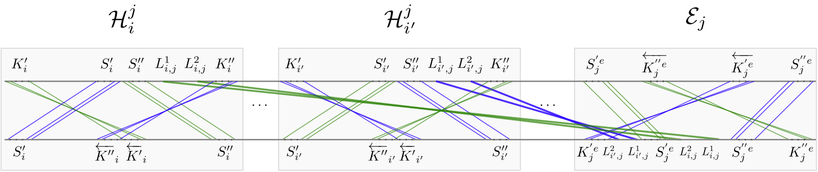
Our reduction: Interval count 4

$[A, B]$ is a max-cut of $\mathbb{G}_{\mathcal{M}}$



Our reduction: Permutation

$[A, B]$ is a max-cut of $\mathbb{G}_{\mathcal{M}}$



Main open questions

- Is MaxCut polynomial-time solvable on unit interval graphs?
- Is MaxCut polynomial-time solvable on interval permutation graphs?