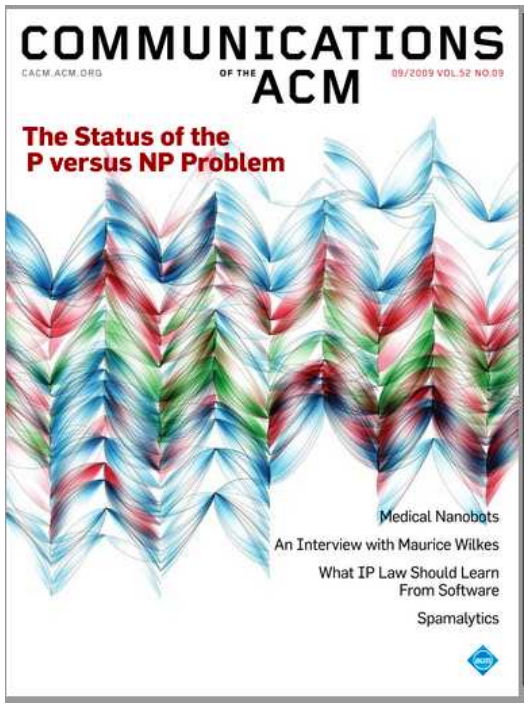
The millennium problem of computational intractability

Celina Miraglia Herrera de Figueiredo

Overview

Central problem in theoretical computer science: the P vs. NP problem

Are there questions whose answer can be quickly checked, but which require an impossibly long time to solve by any direct procedure?



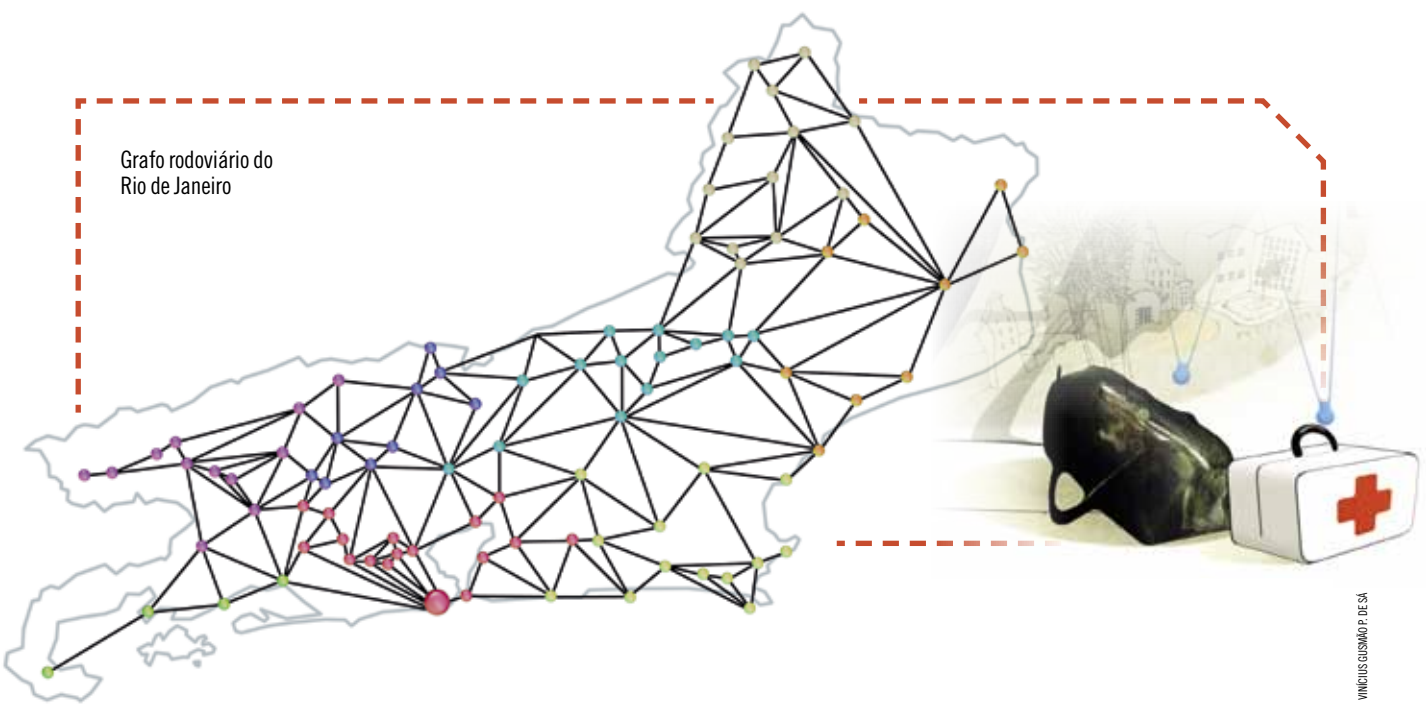
september 2009

The Millennium Prize Problems

In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) established seven *Prize Problems*. The Prizes were conceived to record some of the most difficult problems with which mathematicians were grappling at the turn of the second millennium; to elevate in the consciousness of the general public the fact that in mathematics, the frontier is still open and abounds in important unsolved problems; to emphasize the importance of working towards a solution of the deepest, most difficult problems; and to recognize achievement in mathematics of historical magnitude.

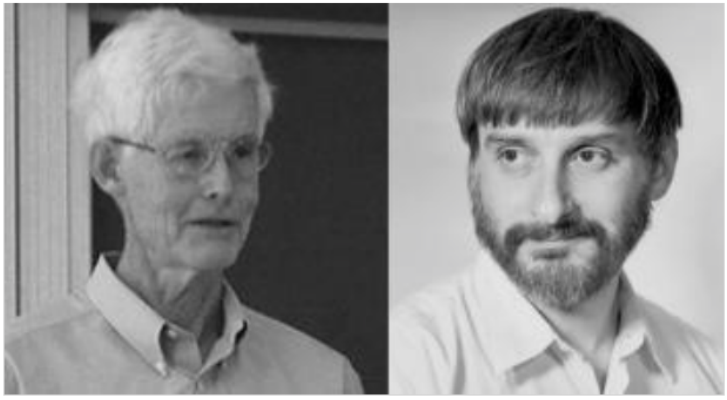
P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit (by car), how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily (given the methods I know) find a solution.

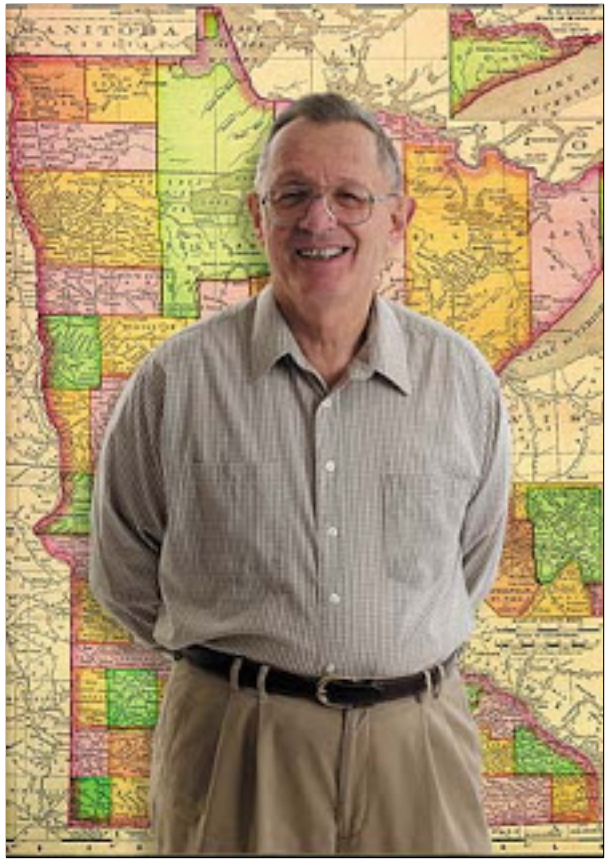


VINICIUS GUSMÃO P. DE SÁ

P vs NP Problem



Suppose that you are organizing housing accommodations for a group of four hundred university students. Space is limited and only one hundred of the students will receive places in the dormitory. To complicate matters, the Dean has provided you with a list of pairs of incompatible students, and requested that no pair from this list appear in your final choice. This is an example of what computer scientists call an NP-problem, since it is easy to check if a given choice of one hundred students proposed by a coworker is satisfactory (i.e., no pair taken from your coworker's list also appears on the list from the Dean's office), however the task of generating such a list from scratch seems to be so hard as to be completely impractical. Indeed, the total number of ways of choosing one hundred students from the four hundred applicants is greater than the number of atoms in the known universe! Thus no future civilization could ever hope to build a supercomputer capable of solving the problem by brute force; that is, by checking every possible combination of 100 students. However, this apparent difficulty may only reflect the lack of ingenuity of your programmer. In fact, one of the outstanding problems in computer science is determining whether questions exist whose answer can be quickly checked, but which require an impossibly long time to solve by any direct procedure. Problems like the one listed above certainly seem to be of this kind, but so far no one has managed to prove that any of them really are so hard as they appear, i.e., that there really is no feasible way to generate an answer with the help of a computer. Stephen Cook and Leonid Levin formulated the P (i.e., easy to find) versus NP (i.e., easy to check) problem independently in 1971.



Kenneth Appel

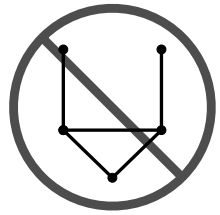
1932 – 2013

A Perfect Path from Computational Biology to Quantum Computing

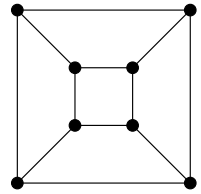
Celina Miraglia Herrera de Figueiredo

COPPE
UFRJ

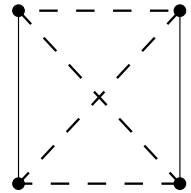
XI LAGOS 2023



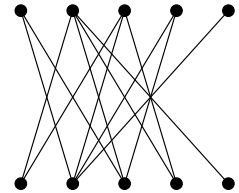
Celina, 1991



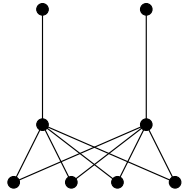
Luerbio, 1998



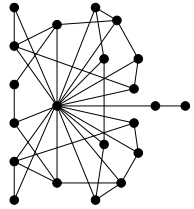
Simone, 2002



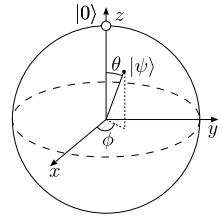
Vânia, 2004



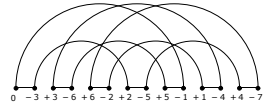
Cláudia, 2005



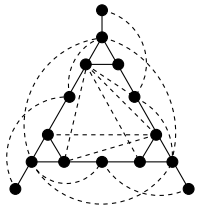
Vinicius, 2006



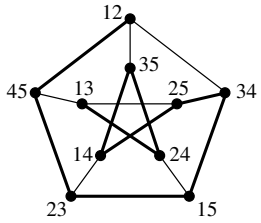
Luis Kowada, 2006



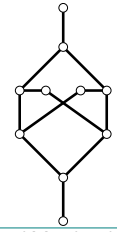
Rodrigo, 2007



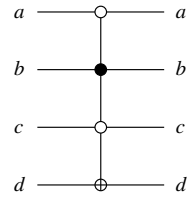
Rafael Bernardo, 2008



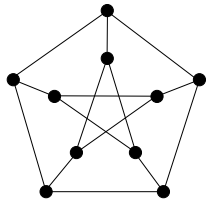
Letícia, 2009



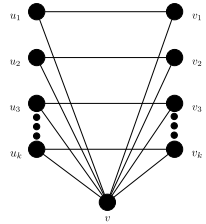
Raphael Machado, 2010



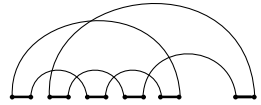
André, 2013



Diana, 2013

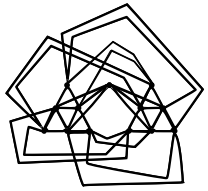


Hélio, 2014

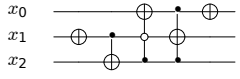


Luís Felipe, 2017

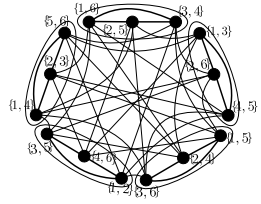
Ana Luísa, 2017



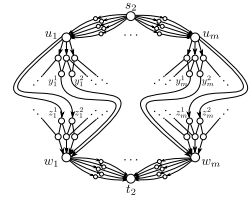
Alexandre, 2020



Edinelço, 2021

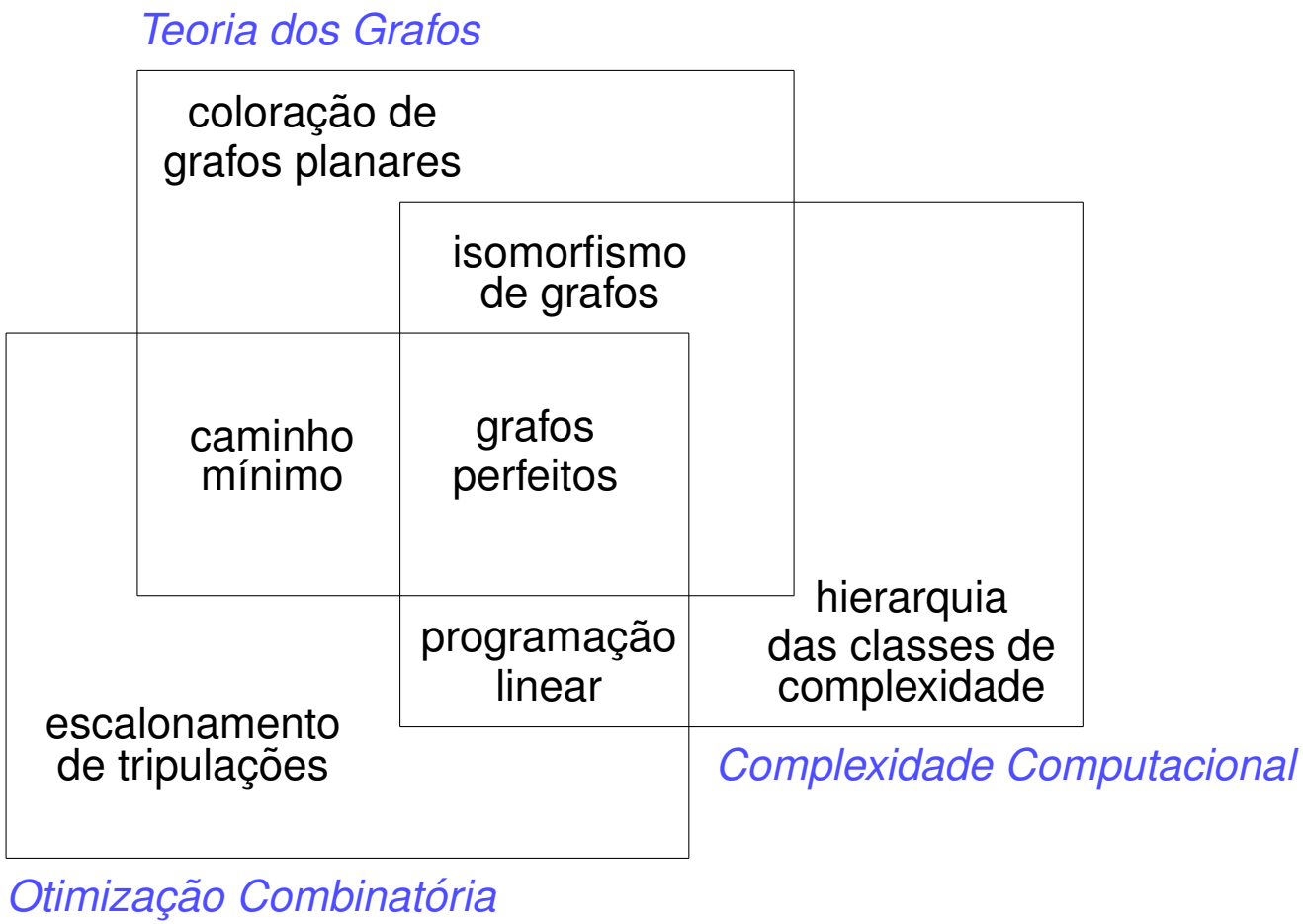


Caroline, 2021

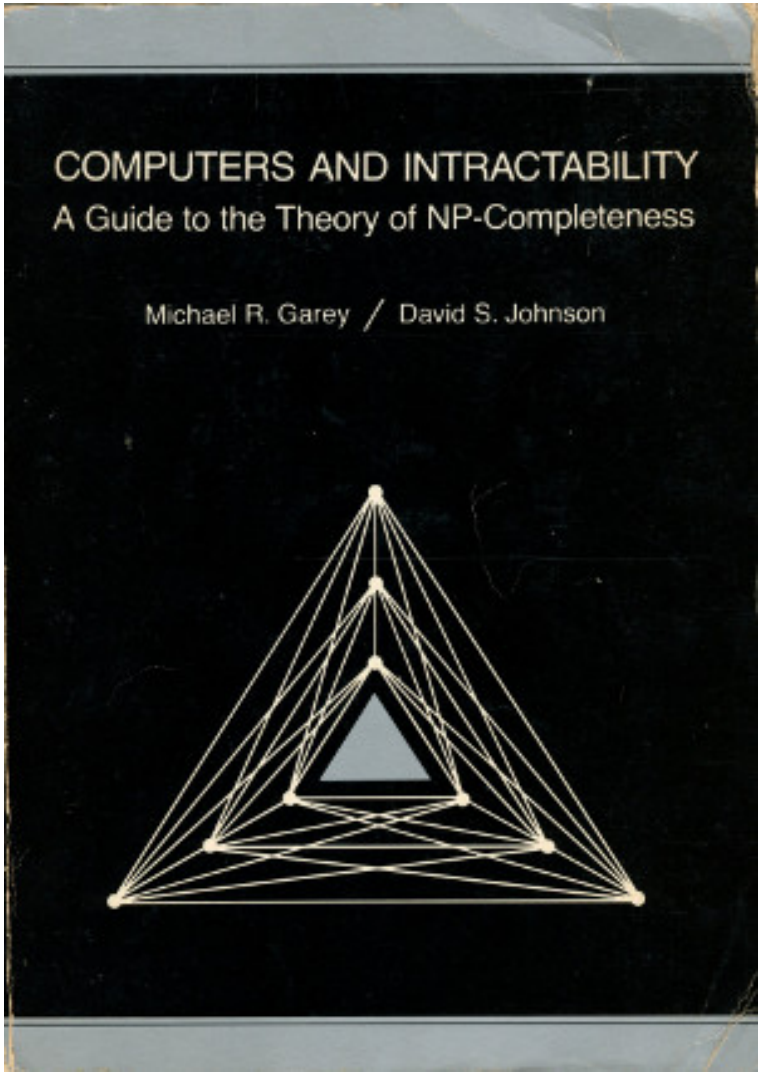


Alexsander, 2022

Origem e desenvolvimento da área de pesquisa



The Guide – Computers and Intractability



"Despite that 23 years have passed since its publication, I consider Garey and Johnson the single most important book on my office bookshelf. Every computer scientist should have this book on their shelves as well. NP-completeness is the single most important concept to come out of theoretical computer science and no book covers it as well as Garey and Johnson."

Lance Fortnow, "Great Books: Computers and Intractability: A Guide to the Theory of NP-Completeness"

Ongoing Guide – Graph Restrictions and Their Effect

Graph Class	Member		IndSet		Clique		CliPar		ChrNum	
Trees/Forests	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[T]
Almost Trees (k)	P		P	[24]	P	[T]	P?		P?	
Partial k-Trees	P	[2]	P	[1]	P	[T]	P?		P	[1]
Bandwidth-k	P	[68]	P	[64]	P	[T]	P?		P	[64]
Degree-k	P	[T]	N	[GJ]	P	[T]	N	[GJ]	N	[GJ]
Planar	P	[GJ]	N	[GJ]	P	[T]	N	[10]	N	[GJ]
Series Parallel	P	[79]	P	[75]	P	[T]	P?		P	[74]
Outerplanar	P		P	[6]	P	[T]	P	[6]	P	[67]
Halin	P		P	[6]	P	[T]	P	[6]	P	[74]
k-Outerplanar	P		P	[6]	P	[T]	P	[6]	P	[6]
Grid	P		P	[GJ]	P	[T]	P	[GJ]	P	[T]
$K_{3,3}$-Free	P	[4]	N	[GJ]	P	[T]	N	[10]	N	[GJ]
Thickness-k	N	[60]	N	[GJ]	P	[T]	N	[10]	N	[GJ]
Genus-k	P	[34]	N	[GJ]	P	[T]	N	[10]	N	[GJ]
Perfect	O!		P	[42]	P	[42]	P	[42]	P	[42]
Chordal	P	[76]	P	[40]	P	[40]	P	[40]	P	[40]
Split	P	[40]	P	[40]	P	[40]	P	[40]	P	[40]
Strongly Chordal	P	[31]	P	[40]	P	[40]	P	[40]	P	[40]
Comparability	P	[40]	P	[40]	P	[40]	P	[40]	P	[40]
Bipartite	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[T]
Permutation	P	[40]	P	[40]	P	[40]	P	[40]	P	[40]
Cographs	P	[T]	P	[40]	P	[40]	P	[40]	P	[40]
Undirected Path	P	[39]	P	[40]	P	[40]	P	[40]	P	[40]
Directed Path	P	[38]	P	[40]	P	[40]	P	[40]	P	[40]
Interval	P	[17]	P	[44]	P	[44]	P	[44]	P	[44]
Circular Arc	P	[78]	P	[44]	P	[50]	P	[44]	N	[36]
Circle	P	[71]	P	[GJ]	P	[50]	O?		N	[36]
Proper Circ. Arc	P	[77]	P	[44]	P	[50]	P	[44]	P	[66]
Edge (or Line)	P	[47]	P	[GJ]	P	[T]	N	[GJ]	N	[49]
Claw-Free	P	[T]	P	[63]	O?		N	[GJ]	N	[49]

ChrInd		HamCir		DomSet		MaxCut		StTree		GraIso	
P	[GJ]	P	[T]	P	[GJ]	P	[GJ]	P	[T]	P	[GJ]
P?		P?		P	[45]	P?		P?		P?	
O?		P	[3]	P	[3]	P?		P?		O?	
P?		P?		P	[64]	P	[64]	P?		P	[58]
N	[49]	N	[GJ]	N	[GJ]	N	[GJ]	N	[GJ]	P	[58]
O		N	[GJ]	N	[GJ]	P	[GJ]	N	[35]	P	[GJ]
P	[74]	P	[74]	P	[54]	P	[GJ]	P	[82]	P	[GJ]
P	[67]	P	[T]	P	[6]	P	[GJ]	P	[81]	P	[GJ]
P	[74]	P	[T]	P	[6]	P	[GJ]	P?		P	[GJ]
O?		P	[6]	P	[6]	P	[GJ]	P?		P	[GJ]
P	[GJ]	N	[51]	N	[55]	P	[T]	N	[35]	P	[GJ]
O?		N	[GJ]	N	[GJ]	P	[5]	N	[GJ]	O?	
N	[49]	N	[GJ]	N	[GJ]	N	[7]	N	[GJ]	O?	
O?		N	[GJ]	N	[GJ]	O?		N	[GJ]	P	[61]
O?		N	[1]	N	[14]	O?		N	[GJ]	I	[GJ]
O?		N	[22]	N	[14]	O?		N	[83]	I	[GJ]
O?		N	[22]	N	[19]	O?		N	[83]	I	[15]
O?		O?		P	[32]	O?		P	[83]	O?	
O?		N	[1]	N	[28]	O?		N	[GJ]	I	[GJ]
P	[GJ]	N	[1]	N	[28]	P	[T]	N	[GJ]	I	[GJ]
O?		O		P	[33]	O?		P	[23]	P	[21]
O?		P	[25]	P	[33]	O?		P	[23]	P	[25]
O?		O?		N	[16]	O?		O?		I	[GJ]
O?		O?		P	[16]	O?		P	[83]	O?	
O?		P	[53]	P	[16]	O?		P	[83]	P	[57]
O?		O?		P	[13]	O?		P	[83]	O?	
O?		P	[12]	O?		O?		P	[70]	O?	
O?		P	[12]	P	[13]	O?		P	[83]	O?	
O?		N	[11]	N	[GJ]	O?		N	[70]	I	[15]
O?		N	[11]	N	[GJ]	O?		N	[70]	I	[15]

The updated NP-Completeness Column: An Ongoing Guide table 35 years later

Graph Class	Member		IndSet		Clique		CliPar		ChrNum		ChrInd		HamCir		DomSet		MaxCut		StTree		GraphIso	
Trees/Forests	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[GJ]	P	[T]	P	[GJ]
Almost Trees (k)	P	[OG]	P	[OG]	P	[T]	P	[105]	P	[5]	P	[17]	P	[5]	P	[5]	P	[20]	P	[76]	P	[17]
Partial k-trees	P	[OG]	P	[5]	P	[T]	P	[105]	P	[5]	P	[17]	P	[5]	P	[5]	P	[20]	P	[76]	P	[17]
Bandwidth-k	P	[OG]	P	[OG]	P	[T]	P	[105]	P	[5]	P	[17]	P	[5]	P	[5]	P	[OG]	P	[76]	P	[OG]
Degree-k	P	[T]	N	[GJ]	P	[T]	N	[29]	N	[GJ]	N	[OG]	N	[GJ]	N	[GJ]	N	[GJ]	N	[GJ]	P	[OG]
Planar	P	[GJ]	N	[GJ]	P	[T]	N	[78]	N	[GJ]	O		N	[GJ]	N	[GJ]	P	[GJ]	N	[OG]	P	[GJ]
Series Parallel	P	[OG]	P	[OG]	P	[T]	P	[105]	P	[5]	P	[17]	P	[5]	P	[OG]	P	[GJ]	P	[OG]	P	[GJ]
Outerplanar	P	[OG]	P	[OG]	P	[T]	P	[OG]	P	[OG]	P	[OG]	P	[T]	P	[OG]	P	[GJ]	P	[OG]	P	[GJ]
Halin	P	[OG]	P	[OG]	P	[T]	P	[OG]	P	[5]	P	[17]	P	[T]	P	[OG]	P	[GJ]	P	[118]	P	[GJ]
k-Outerplanar	P	[OG]	P	[OG]	P	[T]	P	[OG]	P	[5]	P	[17]	P	[OG]	P	[OG]	P	[GJ]	P	[76]	P	[GJ]
Grid	P	[OG]	P	[GJ]	P	[T]	P	[GJ]	P	[T]	P	[GJ]	N	[OG]	N	[32]	P	[T]	N	[OG]	P	[GJ]
$K_{3,3}$-Free*	P	[OG]	N	[GJ]	P	[T]	N	[78]	N	[GJ]	O?		N	[GJ]	N	[GJ]	P	[OG]	N	[GJ]	P	[40]
Thickness-k	N	[OG]	N	[GJ]	P	[T]	N	[78]	N	[GJ]	N	[OG]	N	[GJ]	N	[GJ]	N	[119]	N	[GJ]	I	[RJ]
Genus-k	P	[OG]	N	[GJ]	P	[T]	N	[78]	N	[GJ]	O?		N	[GJ]	N	[GJ]	O?		N	[GJ]	P	[OG]
Perfect	P	[34]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	N	[28]	N	[OG]	N	[OG]	N	[20]	N	[GJ]	I	[84]
Chordal	P	[OG]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	O?		N	[93]	N	[OG]	N	[20]	N	[OG]	I	[84]
Split	P	[OG]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	O?		N	[93]	N	[OG]	N	[20]	N	[OG]	I	[108]
Strongly Chordal	P	[OG]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	O?		N	[93]	P	[OG]	N	[109]	P	[OG]	I	[111]
Comparability	P	[OG]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	N	[28]	N	[OG]	N	[94]	N	[102]	N	[GJ]	I	[22]
Bipartite	P	[T]	P	[GJ]	P	[T]	P	[GJ]	P	[T]	P	[GJ]	N	[OG]	N	[94]	P	[T]	N	[GJ]	I	[22]
Permutation	P	[OG]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	O?		P	[44]	P	[OG]	N	[120]	P	[OG]	P	[OG]
Cographs	P	[T]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	O?		P	[OG]	P	[OG]	P	[20]	P	[OG]	P	[OG]
Undirected Path	P	[OG]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	O?		N	[13]	N	[OG]	N	[20]	N	[RJ]	I	[22]
Directed Path	P	[OG]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	O?		N	[99]	P	[OG]	N	[1]	P	[OG]	P	[7]
Interval	P	[OG]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	O?		P	[OG]	P	[OG]	N	[1]	P	[OG]	P	[OG]
Circular Arc	P	[OG]	P	[OG]	P	[OG]	P	[OG]	N	[OG]	O?		P	[106]	P	[OG]	N	[1]	P	[11]	P	[80]
Circle	P	[OG]	P	[GJ]	P	[OG]	N	[73]	N	[OG]	O?		N	[39]	N	[71]	N	[26]	P	[OG]	P	[68]
Proper Circ. Arc	P	[OG]	P	[OG]	P	[OG]	P	[OG]	P	[OG]	O?		P	[OG]	P	[OG]	O?		P	[11]	P	[82]
Edge (or Line)	P	[OG]	P	[GJ]	P	[T]	N	[95]	N	[OG]	N	[28]	N	[OG]	N	[GJ]	P	[59]	N	[19]	I	[OG]
Claw-Free	P	[T]	P	[OG]	N	[103]	N	[85]	N	[OG]	N	[28]	N	[OG]	N	[GJ]	N	[20]	N	[19]	I	[OG]

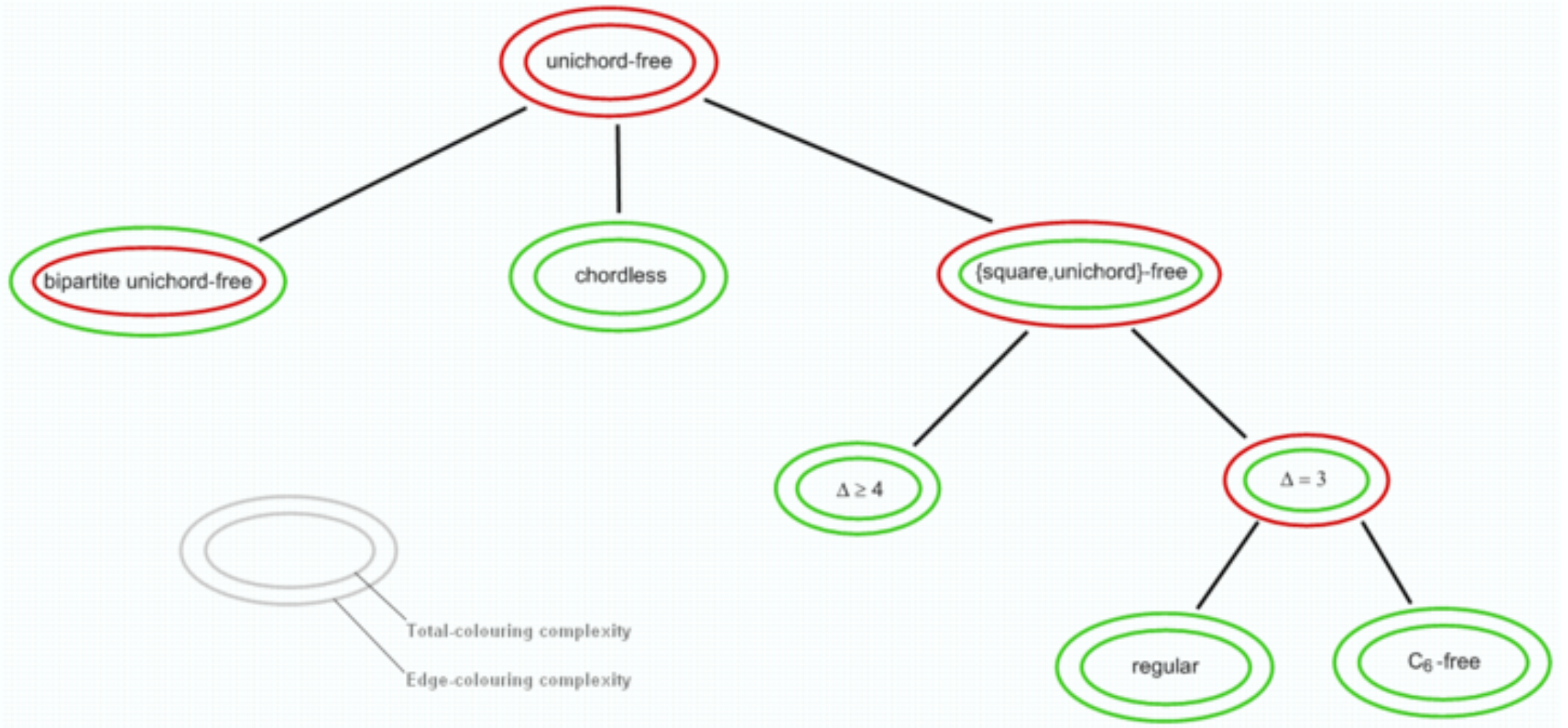
www.cos.ufrj.br/~celina/ftp/j/RJ-current.pdf

Complexity-separating graph classes for vertex, edge and total coloring

Celina de Figueiredo

COPPE
UFRJ

Edge and total coloring complexity-separating classes



When restricted to {square,unichord}-free graphs,
edge coloring is NP-complete whereas total coloring is polynomial

Dániel Marx plenary talk at ICGT 2014

Every graph is easy or hard: dichotomy theorems for graph problems

Dániel Marx[1]

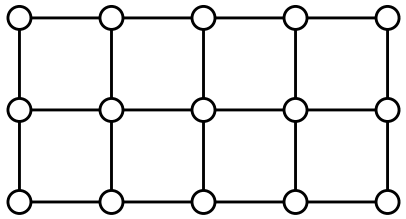
[1]Institute for Computer Science and Control,
Hungarian Academy of Sciences (MTA SZTAKI)
Budapest, Hungary

ICGT 2014
Grenoble, France
July 3, 2014

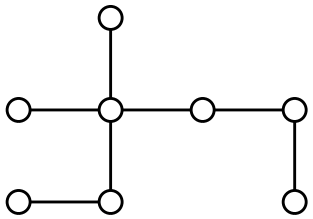
Grid embedding

Graph theory: The recognition of partial grids is often stated as an open problem.

Graph drawing: Deciding whether a graph admits a VLSI layout with unit-length edges is NP-complete.



the grid $G_{3,5}$



embedding for $\{1, 2, 4\}$-tree

A. Brandstädt, V.B. Le, et al. – Information system on graph class inclusions.

http://wwwteo.informatik.uni-rostock.de/isgci/, 2002

S. N. Bhatt, S. S. Cosmadakis – *Inform. Process. Lett.* 1987

P vs. N dichotomy for degree-constrained partial grids

D	D-graphs	D-trees
{1}	P	P
{2}	P	—
{3}	P	—
{4}	P	—
{1,2}	P	P
{1,3}	N	O
{1,4}	P	P
{2,3}	N	—

D	D-graphs	D-trees
{2,4}	N	—
{3,4}	P	—
{1,2,3}	N [G89]	N [G89]
{1,2,4}	N [BC87]	N [BC87]
{1,3,4}	N	N
{2,3,4}	N	—
{1,2,3,4}	N [BC87]	N [BC87]

Is $\{1,3\}$-partial-grid recognition a complexity-separating problem?

S. N. Bhatt, S. S. Cosmadakis – *Inform. Process. Lett.* 1987
A. Gregori – *Inform. Process. Lett.* 1989

"Complexity dichotomy on degree-constrained VLSI layouts with unit-length edges"
submitted to *LATIN 2010* (with Vinícius Sá, Guilherme Fonseca, Raphael Machado)

Most significant publications

- FIGUEIREDO, C. M. H. · KLEIN, S. · KOHAYAKAWA, Y. · REED, B.
  Finding skew partitions efficiently
  Journal of Algorithms (2000)

- FIGUEIREDO, C. M. H. · MAFFRAY, F.
  Optimizing bull-free perfect graphs
  SIAM Journal on Discrete Mathematics (2004)

- FARIA, L. · FIGUEIREDO, C. M. H. · SYKORA, O. · VRTO, I.
  An improved upper bound on the crossing number of the hypercube
  Journal of Graph Theory (2008)

- ALCON, L. · FARIA, L. · FIGUEIREDO, C. M. H. · GUTIERREZ, M.
  The complexity of clique graph recognition
  Theoretical Computer Science (2009)

- FIGUEIREDO, C. M. H.
  The P vs. NP-complete dichotomy of some challenging problems in graph theory
  Discrete Applied Mathematics (2012)

Most significant publications

- CUNHA, L. F. I. · KOWADA, L. A. B. · HAUSEN, R. A. · FIGUEIREDO, C. M. H.
  A faster 1.375-approximation algorithm for sorting by transpositions
  Journal of Computational Biology (2015)

- MACÊDO, H. B. · MACHADO, R. C. S. · FIGUEIREDO, C. M. H.
  Hierarchical complexity of 2-clique-colouring weakly chordal graphs and perfect graphs having cliques of size at least 3
  Theoretical Computer Science (2016)

- CHUDNOVSKY, M. · FIGUEIREDO, C. M. H. · SPIRKL, S.
  The sandwich problem for decompositions and almost monotone properties
  Algorithmica (2018)

- MELO, A. A. · FIGUEIREDO, C. M. H. · SOUZA, U. S.
  A multivariate analysis of the strict terminal connection problem
  Journal of Computer and System Sciences (2020)

- ABREU, A. · CUNHA, L. · FIGUEIREDO, C. · KOWADA, L. · MARQUEZINO, F. · POSNER, D. · PORTUGAL, R.
  The graph tessellation cover number: Chromatic bounds, efficient algorithms and hardness
  Theoretical Computer Science (2020)

DOI:10.1145/3460351

Advances in algorithms, machine learning, and hardware can help tackle many NP-hard problems once thought impossible.

BY LANCE FORTNOW

COMMUNICATIONS OF THE ACM | JANUARY 2022

Fifty Years of P vs. NP and the Possibility of the Impossible