The sandwich problem for almost monotone properties

Celina Miraglia Herrera de Figueiredo

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UFRJ

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Graph sandwich problem

$G_1 = (V, E_1)$ is a spanning subgraph of $G_2 = (V, E_2)$ if $E_1 \subseteq E_2$

$G = (V, E)$ is a sandwich graph for pair G_1, G_2 if $E_1 \subseteq E \subseteq E_2$

E_1 forced edges, $E_2 \setminus E_1$ optional edges, E_3 edges not in E_2

Graph sandwich problem for property Π
instance: Vertex set V, forced edge set E_1, forbidden edge set E_3
question: Is there a graph $G = (V, E)$ such that $E_1 \subseteq E$ and $E \cap E_3 = \emptyset$ that satisfies property Π ?

M. Golumbic, H. Kaplan, R. Shamir, J Algorithms 1995

Generalized recognition problem

Sandwich problem generalizes graph recognition problem with respect to a property Π

A recognition problem has one graph as input

A sandwich problem has two graphs as input

In a sandwich problem, we look for a third graph such that

- its edge set lies between the edge sets of two given graphs
- it satisfies a property Π

Sandwich problem is a coloring problem

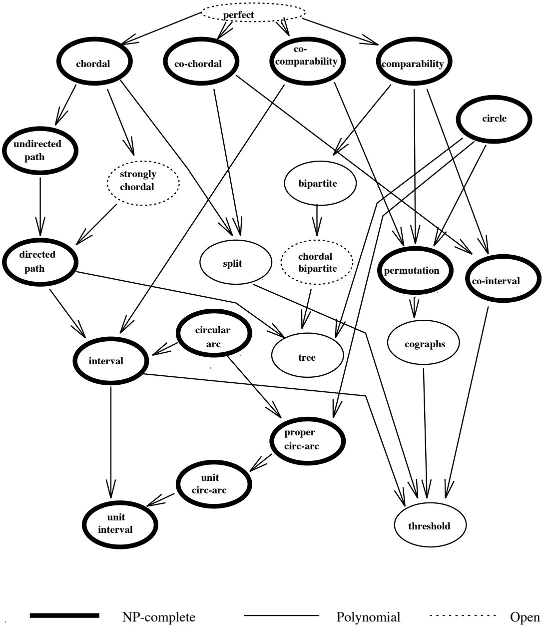
Triangulating a colored graph (TCG)
instance: Graph $G = (V, E)$, proper vertex coloring $c : V \to Z$.
question: Does there exist a supergraph $G_T = (V, E_T)$ of G that is chordal and also properly colored by c?

TCG is NP-complete even when each color class has exactly two vertices

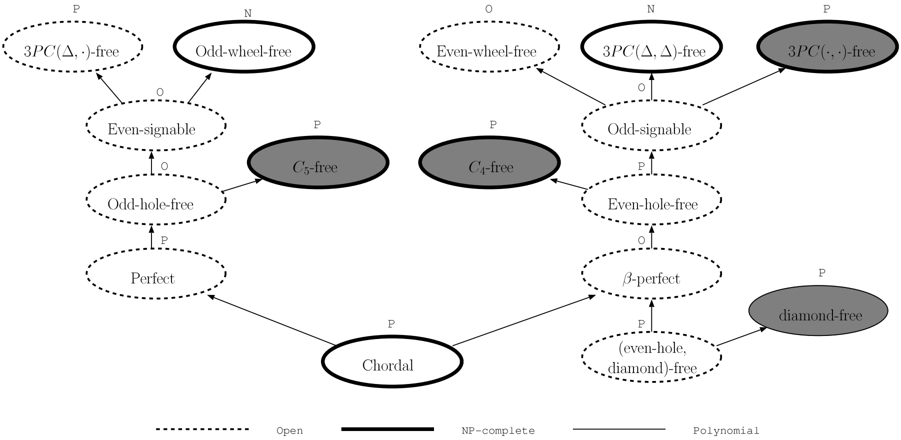
NP-completeness of GRAPH SANDWICH PROBLEM FOR CHORDAL GRAPHS follows from NP-completeness of TCG:
coloring c defines forbidden edge set E_3
(pairs of vertices with same color),
E defines forced edge set E_1

TCG is polynomially equivalent to the Perfect Phylogeny problem

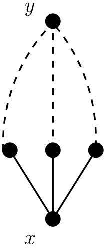
Graph classes

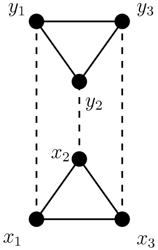


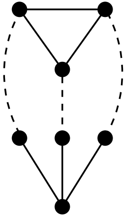
Graph classes defined by forbidden induced subgraphs

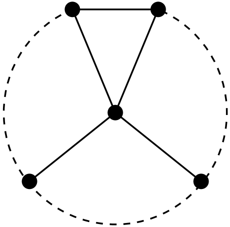


Path configurations and a wheel









Complementary properties

G satisfies complementary property Π^c iff G^c satisfies Π

Π^c sandwich problem has the same complexity of Π sandwich problem

(G_1, G_2) is a YES instance for the Π^c sandwich problem

iff

(G_2^c, G_1^c) is a YES instance for the Π sandwich problem

Ancestral, hereditary, monotone

Π is ancestral if
for all $G = (V, E)$ that satisfy Π and $E \subseteq E'$,
$G' = (V, E')$ also satisfies Π

Π is hereditary if
for all $G = (V, E)$ that satisfy Π and $E' \subseteq E$,
$G' = (V, E')$ also satisfies Π

Π is monotone if
Π is ancestral or hereditary

If Π is ancestral, then Π^c is hereditary, and vice versa

For monotone properties,
the sandwich problem reduces to the recognition problem for G_1 or G_2

Almost monotone properties

A more general notion of monotonicity. Reduce solving the sandwich problem to solving a polynomial number of recognition problems.

Π is k-edge monotone when, for all sandwich instances (G_1, G_2), if there exists a sandwich graph G that satisfies Π, then there exists a sandwich graph G' that satisfies Π and $|E(G') \setminus E(G_1)| \leqslant k$ or $|E(G_2) \setminus E(G')| \leqslant k$

Let C be a set of graphs.

G is C-free if no induced subgraph of G is isomorphic to a graph in C

C is almost edge monotone if there exists a k such that the property of not being C-free is k-edge monotone

The set of odd holes is almost edge monotone

Let (G_1, G_2) be a sandwich instance such that there is a sandwich graph for (G_1, G_2) that contains an odd hole

Let G be the sandwich graph for (G_1, G_2) with $|E(G_2) \setminus E(G)|$ minimum subject to G containing an odd hole, and let C be an odd hole in G

All edges in $E(G_2) \setminus E(G)$ have both endpoints in C

The addition of $e \in E(G_2) \setminus E(G)$ splits C into two smaller induced cycles, one is odd but not an odd hole, it is a triangle and defines vertex $v(e)$

If there are two edges e_1, e_2 such that vertices $v(e_1), v(e_2)$ are not adjacent, then $|C| = 5$, and $|E(G_2) \setminus E(G)| \leqslant 5$
Else, vertices $v(e)$ define a clique in C, and $|E(G_2) \setminus E(G)| \leqslant 2$

Berge, perfect

G is Berge if
G contains no odd hole and no odd antihole as induced subgraph

G is perfect if
for each H induced subgraph of G,
the clique number of H equals the chromatic number of H

The strong perfect graph theorem says that G is Berge iff G is perfect

Berge graphs can be recognized in polynomial time
but
the recognition of graphs containing an odd hole is an open challenge

Berge trigraphs

A trigraph is defined as a sandwich pair (G_1, G_2)

A trigraph (G_1, G_2) satisfies property Π if
there is no sandwich graph G for (G_1, G_2) which does not satisfy Π

Trigraphs are complementary to sandwich graphs

Berge graphs can be recognized in polynomial time
but
the recognition of Berge trigraphs was previously open

Theorem 1 yields that recognizing Berge trigraphs is polynomial;
equivalently, the imperfect graph sandwich problem is polynomial.

Recognizing Berge trigraphs is polynomial

(G_1, G_2) is a Berge trigraph iff (G_1, G_2) is a No instance for the imperfect graph sandwich problem.
The property of containing an odd hole is almost monotone, and the property of containing an odd antihole is almost monotone as well.

Let (G_1, G_2) be a trigraph. Suppose that (G_1, G_2) is not Berge. Then there is a sandwich graph for (G_1, G_2) which contains an odd hole or an odd antihole, and so there is a sandwich graph G which differs little from G_1 or G_2, and which is not Berge. We can check by using the Berge recognition algorithm.

If we find a sandwich graph that is not Berge, then (G_1, G_2) is not a Berge trigraph. If all of the graphs we checked are Berge, then no sandwich graph for (G_1, G_2) contains an odd hole or an odd antihole, and (G_1, G_2) is a trigraph.

Four additional monotone properties

Theorem 2
The sandwich problems for the following properties can be solved in polynomial time:

- containing a pyramid as an induced subgraph
- containing a theta as an induced subgraph
- containing a theta or a prism as an induced subgraph
- containing an even hole

Is the not C-free easier than the C-free sandwich problem?

In the not C-free sandwich problem, we are asking if there exists a sandwich graph in which there exists an induced subgraph isomorphic to a graph of set C

In the C-free sandwich problem, we are testing if there exists a sandwich graph G such that for every induced subgraph H of G, H is not in set C

If the recognition problem for C-free graphs is hard,
then the not C-free sandwich problem is hard

Is there a set C such that recognition of C-free graphs is polynomial,
but the not C-free sandwich problem is hard?

Is there a set C such that the C-free sandwich problem is polynomial,
but the not C-free sandwich problem is hard?

Sophie, Celina, Maria