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Compositions, decompositions, and conformability for total coloring on power of cycle graphs[☆]

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ABSTRACT

Power of cycle graphs C_n^k have been extensively studied with respect to coloring problems, being both the vertex and the edge-coloring problems already solved in the class. The total coloring problem (of determining the minimum number of colors needed to color the vertices and the edges of a graph in a manner that no two adjacent elements are colored the same), however, is still open for power of cycle graphs. Actually, despite partial results for specific values of n and k, not even the well-known Total Coloring Conjecture is settled in the class. A remarkable conjecture by Campos and Mello of 2007 states that if C_n^k is neither a cycle nor a complete graph, then it has total chromatic number $\chi_T = \Delta+2$ if n is odd and $n < 3(k+1)$, and $\chi_T = \Delta+1$ otherwise. We provide strong evidences for this conjecture: we settle a dichotomy for all power of cycle graphs with respect to conformability (a well-known necessary condition for a graph to have $\chi_T = \Delta + 1$) and we develop a framework which may be used to prove that, for any fixed k, the number of C_n^k graphs with $\chi_T \neq \Delta + 1$ is finite. Moreover, we prove this finiteness for any even k and for $k \in \{3, 5, 7\}$. We also use our composition technique to provide a proof of Campos and Mello's conjecture for all C_n^k graphs with $k \in \{3, 4\}$.

1. A remarkable conjecture

The *Total Chromatic Number* $\chi_T(G)$ is the least number of colors needed to color the vertices and the edges of a graph G in a way that no incident or adjacent elements (vertices and edges) receive the same color. Clearly, $\chi_T(G) \geq \Delta(G)+1$ for every graph G (here we use $\Delta(G)$ to denote the *maximum degree* of G; for further preliminary definitions and notation, we refer the reader to Section 1.1). If a graph G can be totally colored using only $\Delta(G)+1$ colors, then it is called *Type 1*; if G cannot be colored with $\Delta(G)+1$ colors, but can be colored with $\Delta(G)+2$ colors, then G is called *Type 2*. The well-known *Total Coloring Conjecture* (TCC), proposed independently by Behzad [2] and Vizing [3], states that $\chi_T(G) \leq \Delta(G)+2$ for every simple graph G. That is, it is an open question whether simple graphs can be fully classified into Type 1 and Type 2 graphs. Even so, McDiarmid and Sánchez-Arroyo [4] proved that the *total coloring problem* of determining the total chromatic number is $\mathcal{NP}$-hard even when restricted to regular bipartite graphs.

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A power of cycle graph, denoted C_n^k, is a graph defined by $V(C_n^k) = \{v_0, v_1, \ldots, v_{n-1}\}$, wherein $v_0, v_1, \ldots, v_{n-1}$ is a spanning cycle (referred to as the *main spanning cycle* of C_n^k), and $E(C_n^k) = E^1 \cup \cdots \cup E^k$, wherein $E^i = \{v_j v_{j+i} \mid 0 \leq j \leq n-1\}$. In this work, when we refer to a vertex $v_i \in V(C_n^k)$ we mean $v_{i \bmod n}$. Power of cycle graphs are a well studied class of graphs with respect to several variations of coloring problems. The vertex-coloring problem can be solved in the class with a greedy algorithm [5] and the edge-coloring problem can be solved simply by looking at the parity of the order of the input graph [6].

The total coloring problem remains open for the class of power of cycle graphs, despite much effort through the last decades [1,5,7–10]. By definition, power of cycle graphs C_n^k generalize the cycle graphs (when $k = 1$) and the complete graphs (when $k \geq \lfloor n/2 \rfloor$). Since the total coloring problem is fully solved for cycles and complete graphs, research on total coloring power of cycle graphs C_n^k addresses the cases wherein $2 \leq k < \lfloor n/2 \rfloor$, i.e. the graph is neither a cycle nor complete.

A complete graph is Type 2 if n is even, and Type 1 otherwise. A cycle is Type 1 if $n \equiv 0 \pmod 3$, and Type 2 otherwise. Also, the total chromatic number of complete graphs is fully classified with respect to the *conformable* condition (a regular graph is *conformable* if it admits a $(\Delta + 1)$-vertex-coloring such that every color class has the same parity as the order of the graph): a complete graph is Type 1 if it is conformable, and Type 2 otherwise. However, the conformable condition does not characterize the Type 1 cycle graphs, since there are conformable cycle graphs which are Type 2 (for example C_7).

Campos [5] showed that if a power of cycle graph C_n^k which is neither a cycle nor a complete graph with odd n is conformable, then $n \geq 3(k+1)$. Ergo, for $2 \leq k < \lfloor n/2 \rfloor$, if n is odd and $n < 3(k+1)$, then C_n^k cannot be Type 1. Campos and Mello [7] then proposed a remarkable conjecture that those would be precisely the Type 2 power of cycle graphs which are neither cycles nor complete graphs, being Type 1 all the other power of cycle graphs:

Conjecture 1 (*[7]*). *A power of cycle graph $G = C_n^k$ which is neither a cycle nor a complete graph is: Type 2 if n is odd and $n < 3(k+1)$; Type 1 otherwise.*

In the present work, we provide a strong evidence for Conjecture 1 by proving that a power of cycle graph $G = C_n^k$ which is neither a cycle nor a complete graph is conformable if n is even, or if n is odd and $n \geq 3(k+1)$. Along with Campos's previous results, this implies the following dichotomy.

Theorem 1. *A power of cycle graph $G = C_n^k$ which is neither a cycle nor a complete graph is: non-conformable (therefore* not *Type 1) if n is odd and $n < 3(k+1)$; conformable otherwise.*

Not even the Total Coloring Conjecture is settled for power of cycle graphs. In a recent unpublished manuscript, a proof for $\chi_T(C_n^k) \leq \Delta(C_n^k) + 3$ has been announced for power of cycle graphs [10]. The following are results from the literature on total coloring on power of cycle graphs C_n^k for specific values of n and k (in all the statements below we assume $n > 2k + 1$, since otherwise we would be actually dealing with complete graphs):

- if n is odd and $n < 3(k+1)$, then C_n^k is not Type 1 [5];
- if $k = 2$, then C_n^k is: Type 2 if $n = 7$, and Type 1 otherwise [5];
- if n is even or $n \equiv 0 \pmod{k+1}$, then the TCC for such C_n^k graphs holds [7];
- if $n \equiv 0 \pmod{2k+1}$, then C_n^k is Type 1;[1]
- if $k = 3$, then C_n^k is Type 2 if $n \in \{9, 11\}$, and Type 1 otherwise [8][2];
- if $k = 4$, then C_n^k is Type 2 if $n \in \{11, 13\}$, and Type 1 otherwise [8].[2]

Table 1 summarizes these results, the first column $n = 2k + 1$ represents the Type 1 complete graphs since n is odd. Further results for specific values of n and k can also be found in two unpublished manuscripts [9,10].

Conjecture 1, if true, would imply that TCC holds for power of cycle graphs, but it is a stronger statement. Besides this implication, Conjecture 1 also states that the set of k-power of cycle graphs that are not Type 1 is finite. Since this finiteness property may still be valid even if TCC fails, the following can be stated as a weaker version of Conjecture 1, which does not imply TCC.

Conjecture 2. *Every power of cycle graph $G = C_n^k$ with $n \geq 3(k+1)$ and $k > 1$ is Type 1.*

We say that a decision problem Π is *trivial* if every *no*-instance of Π has size bounded by a constant. Notice that although Conjecture 2 does not imply that TCC holds, it implies that both TCC as well as the total coloring problem on k-power of cycle graphs would be trivial problems, i.e. to fully solve these problems in that class it would be sufficient to analyze a finite set of graphs.

In this paper, corroborating with the conjecture that the set of k-power of cycle graphs that are not Type 1 is finite, we show that the following hold.

[1] This result is specially interesting because it is a corollary of the pullback algorithm by Figueiredo, Meidanis, and Mello [11] and of the Very Greedy Neighborhood Coloring (*VGNC*) algorithm by Golumbic [12].

[2] The results by Almeida et al. [8] were presented at I Latin American Workshop on Cliques in Graphs in 2014 and they are published in the conference book of abstracts (with no detailed proofs).

Table 1
The state of the art of the two problems (TCC and total coloring) in the graphs C_n^k. The rows of the table represent the parameter k and the columns represent the parameter n as a function of k. The entry $\Delta+1$ represents graphs that are Type 1; $\Delta+2$ represents graphs that are Type 2; TCC represents that the TCC has been proved but the classification between Type 1 and Type 2 is unknown; NC represents the graphs that are not Type 1 because they are known *not* to be conformable. Label NC was omitted in the entries that already have another label, but every entry with odd n and $2k+1 < n < 3(k+1)$ is also an NC entry; and O represents the graphs where both problems are unknown. The row $k = 2$ is fully solved [5]. The rows $k = 3$ and $k = 4$ were announced in [8]. The column $2k+1$ consists of complete graphs, since $2k+1$ is odd, the graphs are Type 1 [13]. The column $2k+2$ is fully solved [13]. The TCC is proved for the columns $2k+3$ to $2k+5$ [13]. The TCC is proved for even n [7]. The TCC is proved for C_n^k, with $n = r(2k+1)+k$ and $r > 0$ [5]. Note that this result also proves the TCC for some specific values of odd n. The graphs C_n^k with $n = r(2k+1)$ and $r > 0$ are Type 1 [5].

k \ n	2k+1	2k+2	2k+3	2k+4	2k+5	2k+6	2k+7	...	odd n < 3(k+1)	r(2k+1)	r(2k+1) + k	...	even n	odd n ≥ 3(k+1)
2	Δ+1	Δ+1	Δ+2	Δ+1	Δ+1	Δ+1	Δ+1	...	NC	Δ+1	Δ+1	...	Δ+1	Δ+1
3	Δ+1	Δ+1	Δ+2	Δ+1	Δ+2	Δ+1	Δ+1	...	NC	Δ+1	Δ+1	...	Δ+1	Δ+1
4	Δ+1	Δ+1	Δ+2	Δ+1	Δ+2	Δ+1	Δ+1	...	NC	Δ+1	Δ+1	...	Δ+1	Δ+1
5	Δ+1	Δ+1	Δ+2	TCC	Δ+2	TCC	O	...	NC	Δ+1	TCC	...	TCC	O
6	Δ+1	Δ+1	Δ+2	TCC	Δ+2	TCC	O	...	NC	Δ+1	TCC	...	TCC	O
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
k	Δ+1	Δ+1	Δ+2	TCC	Δ+2	TCC	O	...	NC	Δ+1	TCC	...	TCC	O

Theorem 2. *Every graph C_n^k with even $k \geq 2$ and $n \geq 4k^2 + 2k$ is Type 1.*

Theorem 3. *Every graph C_n^k with $k \in \{3, 5, 7\}$ and $n \geq 4k^2 + 2k$ is Type 1.*

Now, let $C_n^k[N[v]]$ be the graph induced in $G = C_n^k$ by the closed neighborhood of a vertex $v \in V(G)$ where $n \geq 3k+1$, and let $f(k)$ be the number of distinct total colorings of $C_n^k[N[v]]$ using $\Delta + 1$ colors.

Inspired by the classical pumping lemma for regular languages, we developed a framework that may be used as a tool to demonstrate that for any fixed $k > 1$, the set of k-power of cycle graphs that are not Type 1 is finite. Type 1-compatibility, a term appearing in the statement of Theorem 4, shall be properly defined and discussed in Section 2.

Theorem 4. *Let $\mathcal{L}_k$ be the class of Type 1 k-power of cycle graphs, then there exists a number $t \geq 1$ such that every graph $G = C_n^k$ in $\mathcal{L}_k$ with $n \geq t$ can be decomposed into power of cycle graphs*

$$C_{n_1}^k, C_{n_2}^k, \dots, C_{n_\ell}^k, \text{ where}$$

- $n = n_1 + n_2 + \cdots + n_\ell$;
- $n_i \leq f(k)$ *for any* $i \in [\ell]$;
- $C_{n_i}^k$ *and* $C_{n_{i+1}+\cdots+n_\ell}^k$ *are Type 1-compatible for each* $1 \leq i < \ell$.

In addition, $C_{c_1 \cdot n_1 + c_2 \cdot n_2 + \cdots + c_\ell \cdot n_\ell}^k$ is Type 1 for every $c_1, c_2, \dots, c_\ell \geq 1$.

Observe that Conjecture 1 is solved for $k \in \{3, 4\}$ by the result announced by Almeida et al. [8]. However, since the result has only been presented at a conference and published in the conference book of abstracts with no detailed proofs, we provide a full independent proof in this paper.

Table 2 summarizes the impact of our results in the state of the art of total coloring on power of cycle graphs.

The paper is organized as follows. In the remaining of this section we establish some further preliminaries for technical definitions and notation. In Section 2 we define operations and operators that are used in Section 3 to present a threshold for an even-power of cycle graph to be Type 1. In Section 3 we present the proofs of Theorems 2 and 3. In Section 4 we present a framework to decompose a power of cycle graph into a set of smaller power of cycle graphs, and a proof of Theorem 4. In Section 5 we present the proof of Theorem 1. In Section 6 we provide a proof for the result announced by Almeida et al. [8] concerning the third and the fourth powers of cycles. In Section 7 we present our conclusions and possible further directions and applications of the developed tools.

1.1. Further preliminaries

All graphs considered in this work are simple. The degree of a vertex u in a graph G is denoted $d_G(u)$. The set of neighbors of a vertex u is denoted $N_G(u)$, while $N_G[u]$ denotes the closed neighborhood of u (i.e. the set consisting of u and all its neighbors). The maximum degree of G is denoted $\Delta(G)$. Whenever free of ambiguity, G can be omitted from the notation.

A *total coloring* is a function $\mathscr{C} : E(G) \cup V(G) \rightarrow [t]$, where $[t] = \{0, \dots, t\}$, and adjacent elements must receive distinct colors. A color of a vertex v_i is denoted $\mathscr{C}(v_i)$, similarly a color of an edge v_iv_j is denoted $\mathscr{C}(v_iv_j)$. The *Total Chromatic Number* is the least number of colors needed in a total coloring of a graph G, which we denote by $\chi_T(G)$ following Campos and Mello [7] although the alternative notation $\chi''(G)$ is also found in the literature.

Table 2
The impact of our results in the state of the art of total coloring power of cycle graphs C_n^k. The highlighted cells correspond to our contributions. The new entry C corresponds to the graphs which we proved to be conformable in Theorem 1. The column $n \geq 4k^2 + 2k$ is due to Theorems 2 and 3. The rows $k = 3$ or $k = 4$ remain highlighted, since for them we show a full proof which is independent from the result announced by Almeida et al. [8]. The other $\Delta + 1$ entries which are highlighted (in the columns $4k + 2, 4k + 3, 4k + 4$) follow from our composition technique discussed in Sections 2 and 3.

k \ n	2k+1	2k+2	2k+3	...	even n	odd n < 3(k+1)	...	4k+1	4k+2	4k+3	4k+4	4k+5	...	odd n ≥ 3(k+1)	...	n ≥ 4k²+2k
3	Δ+1	Δ+1	Δ+2	...	Δ+1	NC	...	Δ+1	Δ+1	Δ+1	Δ+1	Δ+1	...	Δ+1	...	Δ+1
4	Δ+1	Δ+1	Δ+2	...	Δ+1	NC	...	Δ+1	Δ+1	Δ+1	Δ+1	Δ+1	...	Δ+1	...	Δ+1
5	Δ+1	Δ+1	Δ+2	...	C	NC	...	C	Δ+1	Δ+1	Δ+1	C	...	C	...	Δ+1
6	Δ+1	Δ+1	Δ+2	...	C	NC	...	C	Δ+1	Δ+1	Δ+1	C	...	C	...	Δ+1
7	Δ+1	Δ+1	Δ+2	...	C	NC	...	C	Δ+1	Δ+1	Δ+1	C	...	C	...	Δ+1
8	Δ+1	Δ+1	Δ+2	...	C	NC	...	C	Δ+1	Δ+1	Δ+1	C	...	C	...	Δ+1
9	Δ+1	Δ+1	Δ+2	...	C	NC	...	C	C	C	C	C	...	C	...	C
10	Δ+1	Δ+1	Δ+2	...	C	NC	...	C	Δ+1	Δ+1	Δ+1	C	...	C	...	Δ+1
11	Δ+1	Δ+1	Δ+2	...	C	NC	...	C	C	C	C	C	...	C	...	C
12	Δ+1	Δ+1	Δ+2	...	C	NC	...	C	Δ+1	Δ+1	Δ+1	C	...	C	...	Δ+1
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
even k	Δ+1	Δ+1	Δ+2	...	C	NC	...	C	Δ+1	Δ+1	Δ+1	C	...	C	...	Δ+1
odd k	Δ+1	Δ+1	Δ+2	...	C	NC	...	C	C	C	C	C	...	C	...	C

As already discussed, the *Total Coloring Conjecture* (TCC), proposed more than 50 years ago, states that $\chi_T(G) \leq \Delta(G)+2$ for every simple graph G. It is important to note that there is no known *implication relation* between the validity of the TCC in a class and the complexity of the total coloring problem. For example, for bipartite graphs, the TCC is solved, but the total coloring problem is $\mathcal{NP}$-hard [4]. On the other hand, for graphs with bounded tree-width, the total coloring problem is polynomial, but the TCC is not yet settled [14]. Very recently, a proof for the TCC has been announced by T. S. Murthy [15].

The TCC is the total coloring analogous of the Vizing Theorem that states that $\chi'(G) \leq \Delta(G)+1$ for every simple graph, where $\chi'(G)$ is the *chromatic index* of G, i.e., the least number of colors needed to color the edges of G. The TCC has been settled for restricted graph families, such as the complete r-partite graphs [16], cubic graphs [17], split graphs [18], dually chordal graphs [11], and graphs with large maximum degree [19]. However, the TCC remains open even when restricted to chordal graphs, to regular graphs, or to power of cycle graphs.

There exists a strong parallel between edge coloring and total coloring. Similarly as $\Delta(G)$ is a natural lower bound for $\chi'(G)$, it is easy to see that $\chi_T(G) \geq \Delta(G) + 1$. Remark that the TCC would imply the total coloring classification of graphs into Type 1 and Type 2, in the same way that Vizing's theorem implies the edge-coloring classification problem into Class 1 (graphs satisfying $\chi'(G) = \Delta(G)$) and Class 2 (graphs satisfying $\chi'(G) = \Delta + 1$).

The total coloring problem was first proved to be $\mathcal{NP}$-hard [20] by a reduction from edge coloring (also $\mathcal{NP}$-hard). This could suggest that total coloring would be a harder problem, nevertheless, we know classes where edge coloring is $\mathcal{NP}$-hard but total coloring is polynomial [21]. Effort has been done aiming at defining sufficient or necessary conditions for a graph to be Class 1 or Type 1, hoping that these conditions can be easily verified. For edge-coloring, we have the *subgraph-overfull* condition as sufficient for a graph to be Class 2 (therefore, the *non-subgraph-overfull* condition as necessary for being Class 1) [22]. For total coloring, the *conformable* condition is a necessary condition for a graph to be Type 1. As already introduced, restricted to regular graphs (as all the graphs dealt in this paper), a graph is *conformable* if it admits a *conformable vertex-coloring*, that is, a vertex-coloring using $\Delta + 1$ colors such that every color class has the same parity as the order of the graph [23]. The complexity of verifying the conformable condition is unknown, even for regular graphs.

2. Compatibility and bottom-up composition

In this section, we define two operations used as part of the process to construct a Type 1 total coloring of a graph C_n^k, from two other Type 1 total colorings of graphs $C_{n_1}^k$ and $C_{n_2}^k$, such that $n_1 + n_2 = n$.

A power of path graph, denoted by P_n^k, is a graph where $V(P_n^k) = \{v_0, v_1, \dots, v_{n-1}\}$, where $v_0, v_1, \dots, v_{n-1}$ is a spanning path, and $E(P_n^k) = E^1 \cup \dots \cup E^k$, where $E^i = \{v_j v_{(j+i)} \mid 0 \leq j \leq n-i-1\}$. Note that, in power of path graphs P_n^k, the indices of the vertices are not taken by using the modular operation.

Please refer to Fig. 1. We aim to decompose a power of cycle graph C_n^k into two power of path graphs $P_{n_1}^k$ and $P_{n_2}^k$, with $n_1 + n_2 = n$, and two sets of edges $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$, with $w_x, w_y \in V(C_n^k)$, in such way that we can use total colorings of the graphs $C_{n_1}^k$ and $C_{n_2}^k$ to totally color $P_{n_1}^k$ and $P_{n_2}^k$, respectively, using the colors of the edges of $E(C_{n_1}^k) \setminus E(P_{n_1}^k)$ and $E(C_{n_2}^k) \setminus E(P_{n_2}^k)$ to color the edges of $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$ to obtain a valid total color of C_n^k. The operation of transferring total colorings is called *pullback* and it was defined by Figueiredo, Meidanis, and Mello [11]. The pullback from a graph G to another graph G' is a function $f : V(G) \to V(G')$, such that: (i) f is a homomorphism, i.e., if $pq \in E(G)$, then $f(p)f(q) \in E(G')$; (ii) f is injective when restricted to $N[p]$, for $p \in V(G)$. Pullback functions are a powerful tool that allows us to transfer colorings from one graph to another.

We define more formally the operations and the operators of this procedure in Definitions 5 and 6.

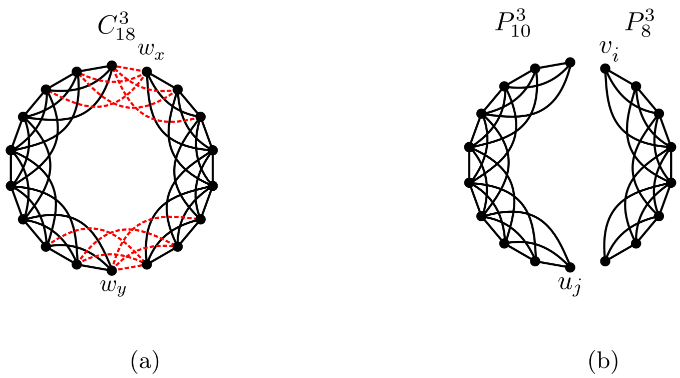

Fig. 1. The illustration of a decomposition process, note that the vertex $w_x \in C^k_n$ represents the vertex $v_i \in C^k_{n_1}$ and the vertex $w_y \in C^k_n$ represents the vertex $u_j \in C^k_{n_2}$. (a) A graph C^k_n, with $n = 18$ and $k = 3$. The red dashed edges represent the sets $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$. (b) The vertex v_i represents the first vertex of the graph P^3_8, analogously the vertex u_j is the first vertex of P^3_{10}. The edges of the sets $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$ were omitted to highlight the graphs P^3_8 and P^3_{10}, but the sets can be seen in 1(a).

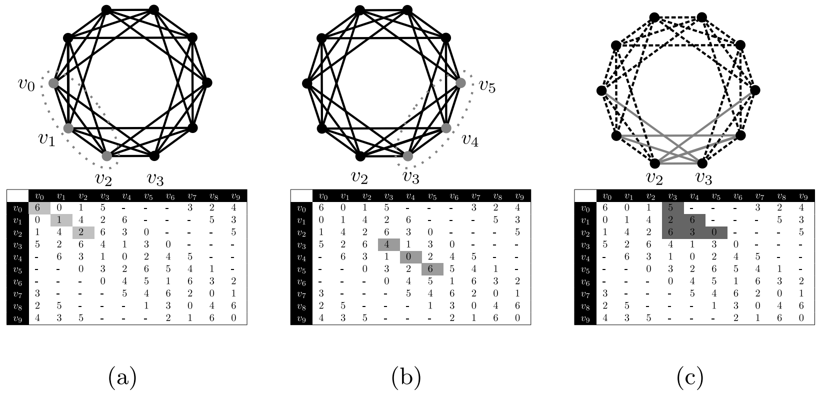

Fig. 2. (a) The vertices of the set $N^-(v_3)$ are the vertices v_0, v_1, v_2. (b) The vertices of the set $N^+(v_2)$ are the vertices v_3, v_4, v_5. (c) The edges of the set $N^-(v_3)N^+(v_2)$ are represented by the shaded gray edges.

A semi-cut of vertices of a graph C^k_n, is a set of k consecutive vertices in the main spanning cycle, considering the cyclic order. We denote two special semi-cuts of vertices: $N^-(w_x) = \{w_q \mid x - k \le q < x\}$ and $N^+(w_{x-1}) = \{w_q \mid x \le q < x + k\}$. A semi-cut of edges of a graph C^k_n is a set $N^-(w_x)N^+(w_{x-1})$ which represents the edges that have one extreme in $N^-(w_x)$ and the other extreme in $N^+(w_{x-1})$, more formally $N^-(w_x)N^+(w_{x-1}) = \{w_q w_l \mid x - k < q \le x, x < l \le x + k, \text{and } l - q \le k\}$. Fig. 2 shows a graph C^3_{10} and the sets $N^-(v_3)$, $N^+(v_2)$, and $N^-(v_3)N^+(v_2)$. Each set is a shade highlighted in one table. The tables show total colorings, in such way that the color of an edge $v_i v_j$ is represented by the cell i, j of the matrix. The color of the vertex v_i is represented by the cell i, i.

Definition 5. We say that a *k-power of path decomposition* of a power of cycle graph C^k_n is a decomposition into two power of path graphs $P^k_{n_1}$ and $P^k_{n_2}$, and two sets of edges $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$, with the distance between $w_x, w_y \in C^k_n$ in the main spanning cycle greater than $2k$. Vertex w_x represents the vertex $v_i \in P^k_{n_1}$ which is the first vertex of the induced path of $P^k_{n_1}$, and vertex w_y represents the vertex $u_j \in P^k_{n_2}$ which is the first vertex of the induced path of $P^k_{n_2}$. Note that every power of cycle graph C^k_n with $n \ge 4k + 2$ has a k-power of path decomposition. Fig. 1 illustrates a 3-power of path decomposition of C^3_{18}.

Definition 6. We say that two graphs $C^k_{n_1}$ and $C^k_{n_2}$ are Type 1-*compatible* if each graph has a Type 1 total coloring $\mathscr{C}_1$, respectively $\mathscr{C}_2$, and a pivot vertex v_i, respectively u_j, such that the following holds: (a) $N^-(v_i)$ is compatible with $N^+(u_{j-1})$, meaning $\mathscr{C}_1(v_{i-r}) \ne \mathscr{C}_2(u_{j+s})$, for every $r \in \{1, \dots, k\}$ and every $s \in \{0, \dots, k-r\}$; (b) $N^+(v_{i-1})$ is compatible with $N^-(u_j)$, meaning $\mathscr{C}_1(v_{i+s}) \ne \mathscr{C}_2(u_{j-r})$, for every $r \in \{1, \dots, k\}$ and every $s \in \{0, \dots, k-r\}$; (c) $N^-(v_i)N^+(v_{i-1})$ is compatible with $N^-(u_j)N^+(u_{j-1})$, meaning $\mathscr{C}_1(v_{i-r}v_{i+s}) = \mathscr{C}_2(u_{j-r}u_{j+s})$, for every $r \in \{1, \dots, k\}$ and every $s \in \{0, \dots, k-r\}$. By definition, every Type 1 graph is Type 1-compatible with itself.

3. A threshold for power of cycle graphs to be Type 1

In order to present the proofs of Theorems 2 and 3, which give a threshold for power of cycle graphs to be Type 1, for even-power of cycle graphs and for small order odd-power of cycle graphs, respectively, we need to prove Theorems 7 and 8, together with Lemma 9.

Theorem 7. *If two graphs $C^k_{n_1}$ and $C^k_{n_2}$ are Type 1-compatible, then the graph C^k_n, with $n = n_1 + n_2$, is Type 1-compatible with $C^k_{n_1}$ and $C^k_{n_2}$.*

Proof. Since $C^k_{n_1}$ and $C^k_{n_2}$ are Type 1-compatible, there are two Type 1 total colorings $\mathscr{C}_1$ and $\mathscr{C}_2$ and vertices $v_i \in V(C^k_{n_1})$ and $u_j \in V(C^k_{n_2})$ that fulfill the restrictions of Definition 6.

The graph $P^k_{n_1}$ is formed from the graph $C^k_{n_1}$ by removing the edges of $N^-(v_i)N^+(v_{i-1})$. And the graph $P^k_{n_2}$ from the graph $C^k_{n_2}$ by removing the edges of $N^-(u_j)N^+(u_{j-1})$.

Since the graph C^k_n has a k-power of path decomposition, we can use a pullback from the total coloring of the graphs $C^k_{n_1}$ and $C^k_{n_2}$ to the power of path graphs $P^k_{n_1}$ and $P^k_{n_2}$. Then we only need to color the edges of the sets $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$, again by the same pullback technique to the colors of $N^-(v_i)N^+(v_{i-1})$ and $N^-(u_j)N^+(u_{j-1})$. □

Theorem 8 describes a recoloring procedure from a famous *Latin square L* to a valid total coloring represented by a matrix M of order $2k+2$, in such a way that L and M represent Type 1-compatible total colorings of the graphs C^k_{2k+1} and C^k_{2k+2}, respectively. The procedure can be divided in three steps: First, from the conformable condition, we know that we have to change the colors of some vertices in such a way that every color colors an even number of vertices. For this, we have to *swap* some colors using the fact that the matrix M has cells that do not represent valid edges in the graph C^k_{2k+2}. The second step is to *complete* the cells in M that represent the one vertex and its adjacent edges that are in C^k_{2k+2} but not in C^k_{2k+1}. The third step is to *make* the two total colorings *compatible* with each other. Combining Theorem 8 with Theorem 7 we show that every power of cycle graph C^k_n, with even k, is Type 1 except for at most $2k^2 - k$ graphs.

Theorem 8. *The graphs C^k_{2k+1} and C^k_{2k+2}, with even k, are Type 1-compatible.*

Proof. Let M be a matrix to represent the Type 1-total coloring of the graph C^k_{2k+2}. The order of M is $2k+2$ and it shall have the same properties of the Latin square L, which is used to total coloring complete graphs [13]. The diagonal cells of M shall represent the colors of the vertices and the off-diagonal cells shall represent the colors of the edges. Start by setting each cell $L[i,j] = M[i,j] = (i+j) \bmod 2k+1$, with $0 \le i,j \le 2k$. Fig. 3(a) represents the matrix L, being the corresponding graph $C^k_{2k+1} = K_{2k+1}$ with $k = 4$ illustrated in Fig. 4, and Fig. 3(b) represents the matrix M. Note that L defines a Type 1-total coloring of the graph C^k_{2k+1}. Note also that the cells $M[x, k+1+x]$, with $x \in \{0,\dots,k\}$ represent edges that do not exist in the graph C^k_{2k+2}, due to the distance between the vertices, so our procedure uses these colors to construct the coloring of C^k_{2k+2}.

We make a recoloring procedure on the matrix M, this procedure can be made in three steps. Figs. 3(c), 3(d) 3(e) illustrate the procedure (ref. Steps 1, 2, and 3, respectively), being the graph C^k_{2k+2} displayed in Figs. 5 and 6, with the total colorings of Figs. 3(d) and 3(e), respectively. In all the steps, we consider only changes in one of the triangular parts of matrix M, including the main diagonal, assuming that all changes performed also reflect in the other part.

Step 1. *Swapping*

1. *For $s \in \{1,\dots,k/2\}$, we swap, for all $r \in \{0,\dots,k/2\}$, the color of the cell $M[k+s-r, k+s+r]$ with the color of the cell $M[k+s-r-1, k+s+r]$. Note that, when $r = k/2$, the cells $M[k+s-k/2-1, k+s+k/2] = M[k/2+s-1, 3k/2+s]$ represent the edges that do not exist in the graph C^k_{2k+2}. So, the only conflicts of colors in M created are in cells $M[k/2+s-1, 3k/2+s]$ and $M[k/2+s-1, (3k/2+s+1) \bmod (2k+1)]$, which have both the same color.*
2. *For $s = k/2+1$ we swap, for all $r \in \{0,\dots,k/2-1\}$, the color of the cell $M[k+s-r, k+s+r]$ with the color of the cell $M[k+s-r-1, k+s+r]$. Note that the cell $M[k+s-(k/2-1)-1, k+s+(k/2-1)] = M[k+1, 2k]$ receives the original color of the cell $M[k+2, 2k] = k+1 = M[0, k+1]$, which is a cell that represents an edge that does not exist in C^k_{2k+2}. So, by also assigning to $M[0, k+1]$ the original color of $M[k+1, 2k] = k$, we create no further conflict of colors in M.*
3. *For the case where $s \in \{k/2+2,\dots,k\}$ (subject to $k > 2$), we swap, for all $r \in \{0,\dots,k/2-1\}$, the color of the cell $M[k+s-r, k+s+r]$ with the color of the cell $M[k+s-r-1, k+s+r]$. Note that, in this case, we have to perform the arithmetic operations modulo $2k+1$. Note also that the cell $M[k+s-(k/2-1)-1, k+s+(k/2-1)] = M[k/2+s, (k+k/2+s-1) \bmod (2k+1)]$ receives the original color of the cell $M[k/2+s, (k+k/2+s) \bmod (2k+1)] = 2s-1$, which represents an edge that does not exist in the graph C^k_{2k+2}. So, the only conflicts of colors in M created are in cells $M[k/2+s, (k+k/2+s-1) \bmod (2k+1)]$ and $M[k/2+s, (k+k/2+s) \bmod (2k+1)]$.*

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	0
2	2	3	4	5	6	7	8	0	1
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

(a)

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	
1	1	2	3	4	5	6	7	8	0	
2	2	3	4	5	6	7	8	0	1	
3	3	4	5	6	7	8	0	1	2	
4	4	5	6	7	8	0	1	2	3	
5	5	6	7	8	0	1	2	3	4	
6	6	7	8	0	1	2	3	4	5	
7	7	8	0	1	2	3	4	5	6	
8	8	0	1	2	3	4	5	6	7	
9										

(b)

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	4	7	6	8	
1	1	2	3	4	5	6	7	8	0	
2	2	3	4	5	6	7	8	1	1	
3	3	4	5	6	7	8	1	0	3	
4	4	5	6	7	8	1	0	3	2	
5	4	6	7	8	1	0	3	2	5	
6	7	7	8	1	0	3	2	5	4	
7	6	8	1	0	3	2	5	4	7	
8	8	0	1	3	2	5	4	7	6	
9										

(c)

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4		7	6	8	5
1	1	2	3	4	5	6		8	0	7
2	2	3	4	5	6	7	8		1	0
3	3	4	5	6	7	8	1	0		2
4	4	5	6	7	8	1	0	3	2	
5		6	7	8	1	0	3	2	5	4
6	7		8	1	0	3	2	5	4	6
7	6	8		0	3	2	5	4	7	1
8	8	0	1		2	5	4	7	6	3
9	5	7	0	2		4	6	1	3	8

(d)

	0	1	2	3	4	5	6	7	8	9
0	1	0	2	3	4		7	6	8	5
1	0	2	3	4	5	6		8	1	7
2	2	3	4	5	6	7	8		0	1
3	3	4	5	6	7	8	0	1		2
4	4	5	6	7	8	0	1	3	2	
5		6	7	8	0	1	3	2	5	4
6	7		8	0	1	3	2	5	4	6
7	6	8		1	3	2	5	4	7	0
8	8	1	0		2	5	4	7	6	3
9	5	7	1	2		4	6	0	3	8

(e)

Fig. 3. The red cells represent the edges that do not exist in the graph C^k_{2k+2}; the green cells represent the cells of the set $N^-(u_4)N^+(u_3)$; the light blue cells represent the set $N^-(u_4)$; the dark blue cells represent the set $N^+(u_3)$. (3(a)) The Latin square L represents a Type 1-total coloring of the graph C^k_{2k+1} (see Fig. 4). (3(b)) The matrix M at the start of the procedure. The highlighted non-red cells represent the pairs of colors to be swapped by the procedure, being each pair identified by a distinct shade. (3(c)) The matrix M after *swapping* colors (Step 1). (3(d)) The matrix M after *completing* (Step 2) the cells that are in M but not in L (see Fig. 5). (3(e)) The matrix M after the step *Making compatible* (Step 3), representing a Type 1-total coloring of the graph C^k_{2k+2} (see Fig. 6), which is compatible with the coloring of the graph C^k_{2k+1} with respect to the pivots $v_4 \in V(C^k_{2k+1})$ and $u_4 \in V(C^k_{2k+2})$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

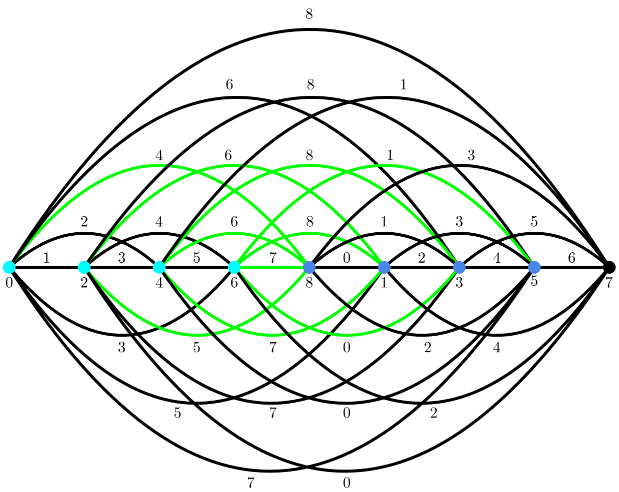

Fig. 4. The graph $C^k_{2k+1} = K_{2k+1}$ total colored as in Fig. 3(a).

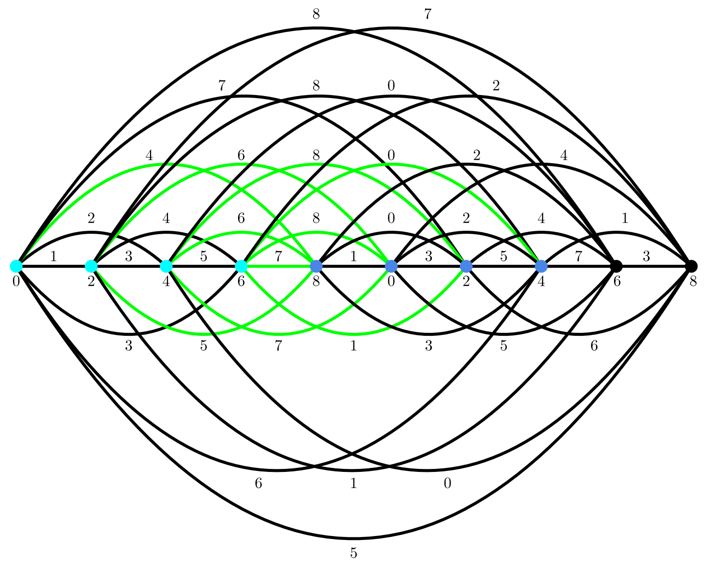

Fig. 5. The graph C^k_{2k+2} total colored as in Fig. 3(d).

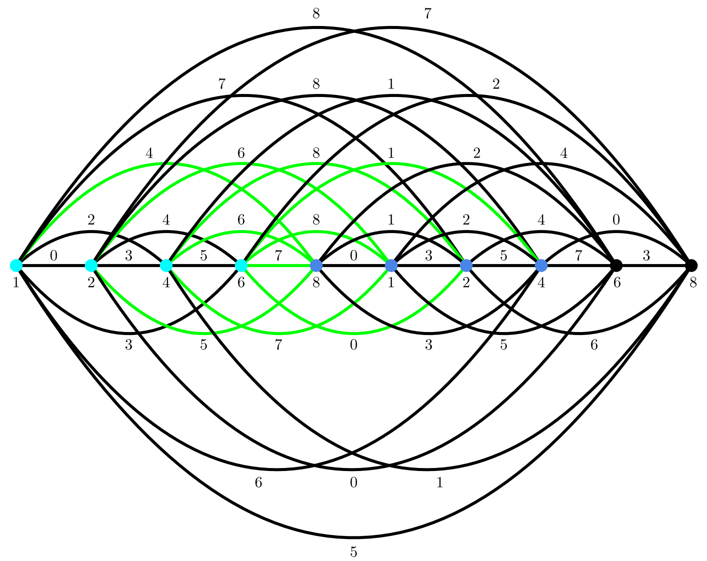

Fig. 6. The graph C^k_{2k+2} total colored as in Fig. 3(e).

Step 2. *Completing*

To color the column $2k+1$ of M (and, by symmetry, the row $2k+1$), we use the following procedure: for $i \in \{0, \dots, k-1\}$, we set $M[i, 2k+1] = L[i, k+1+i]$, representing the colors of the edges which do not exist in the graph, $M[k+1+i, 2k+1] = L[i, k+i]$ if $i < k/2$, and $M[k+1+i, 2k+1] = L[i+1, k+1+i]$ if $i \geq k/2$, with $M[k+i+1, 2k+1]$ representing the colors of the edges which do not exist in the graph. And we set the cell $M[2k+1, 2k+1] = 2k$, the only color possible for the new vertex. This step is illustrated in Fig. 3(d).

In order to show that the matrix M is a valid total coloring for C^k_{2k+2} we have to show that:

- *The diagonal cells represent a proper vertex coloring: Since we use as base the colors of L, in which all the diagonal cells are distinct, we only change the colors of the cells $M[k+i+1, k+i+1]$ to the color $L[k+i, k+i+1] = (2k+2i+1) \bmod 2k+1 = 2i$, with $i \in \{0, \dots, k-1\}$ and add the color $2k$ to the cell $M[2k+1, 2k+1]$. As the color $2i$ is used as a diagonal color in the cell $M[i, i]$ and in the cell $M[k+i+1, k+i+1]$, this can be used as a proper vertex coloring of the graph C^k_{2k+2}, since the only repetitions of colors happen in vertices which have distance $k+1$, and therefore are not adjacent.*

- *Every row and column of M, excluding the cells which represent an edge that does not exist in C^k_{2k+2}, is a permutation of $2k+1$ elements: Since we use L as base and by construction of the recoloring procedure we only stop the recoloring when the conflict was in one of the cells that represents the edges that do not exist in the graph C^k_{2k+2}, then M is a valid total coloring of C^k_{2k+2}.*

Step 3. *Making compatible*

For each $i \in \{0, \dots, k/2-2\}$ (subject to $k > 2$), let α_i be the color of $M[i, i]$ and let β_i be the color of $M[i, i+1]$. Swap α_i with β_i throughout M. Note that this further swapping does not change any property of the matrix M.

Now M and L represent the Type 1 total colorings of C^k_{2k+2} and C^k_{2k+1}, respectively, in such a way that the first vertex in the main spanning cycle is represented by the first row (or column) in the respective matrix, the second vertex is represented by the second row in the matrix and so on.

In order to show that C^k_{2k+2} and C^k_{2k+1} are compatible, we have to fulfill the conditions of Definition 6. So let us call C^k_{2k+1} by $C^k_{n_1}$ and C^k_{2k+2} by $C^k_{n_2}$, the vertex $v_i \in C^k_{n_1}$, with $i = k$, and $u_j \in C^k_{n_2}$ also with $j = k$.

To prove (a) of Definition 6: Note that $N^-(v_i)$ has the vertices v_p with $p \in \{0, \dots, k-1\}$ and $\mathscr{C}_1(v_p) = 2p$. Now note that $N^+(u_{j-1})$ has the vertices u_q with $q \in \{k, \dots, 2k-1\}$ and: $\mathscr{C}_2(u_q) = 2k$ if $q = k$; $\mathscr{C}_2(u_q) = 2(q-k-1)+1$ if $k < q < 3k/2$; $\mathscr{C}_2(u_q) = 2(q-k-1)$ if $q \geq 3k/2$. So, the distance between the vertices which have the same color is $k+1$. Ergo, $N^-(v_i)$ and $N^+(u_{j-1})$ are clearly compatible.

To prove (b) of Definition 6: Note that $N^+(v_{i-1})$ has the vertices v_p with $p \in \{k, \dots, 2k-1\}$ and $\mathscr{C}_1(v_p) = 2(p-k-1)+1$ if $p > k$, and $\mathscr{C}_1(v_p) = 2k$ if $p = k$. Note that $N^-(u_j)$ has the vertices u_q with $q \in \{0, \dots, k-1\}$ and $\mathscr{C}_2(u_q) = 2q+1$ if $q \leq k/2-2$, and $\mathscr{C}_2(u_q) = 2q$ otherwise. Again, the distance between the vertices which have the same color is $k+1$, implying the compatibility between $N^+(v_{i-1})$ and $N^-(u_j)$.

To prove (c) of Definition 6: Observe that we use the same Latin square L to generate the total colorings and the only parts of the recoloring procedure that changes the color of a cell which represents an edge in $N^-(u_j)N^+(u_{j-1})$ are the *Swapping* step and the *Making compatible* step, the latter applied exactly to the colors in $N^-(u_j)N^+(u_{j-1})$ that are affected by the Swapping step. Therefore, the sets $N^-(v_i)N^+(v_{i-1})$ and $N^-(u_j)N^+(u_{j-1})$ are compatible. Fig. 3(e) shows the matrix M representing a valid total coloring that is compatible with the coloring represented by the Latin square L. $\square$

Lemma 9 explicitly solves the equation that guarantees the claimed threshold.

Lemma 9. *Let a and z be non-negative integers. If $z \geq a^2 - a$, then there exist $x, y \in \mathbb{Z}_{\geq 0}$ such that $z = xa + y(a+1)$.*

Proof. The proof is by induction on z. If $z = a^2 - a$ and $a = 0$, we have also $z = 0$, then we can take $y = 0$ and any value for x. If $z = a^2 - a$ and $a > 0$, then $z = xa + 0(a+1)$, with $x = a-1$. Now, suppose that for any $z_1 \geq a^2 - a$, we have $z_1 = x_1 a + y_1(a+1)$. For $z_2 = z_1 + 1$: if $x_1 > 0$, then $z_2 = (x_1 - 1)a + (y_1 + 1)(a+1)$. If $x_1 = 0$, then: $z_1 = 0a + y_1(a+1)$ and as $z_1 > a^2 - a$, $y_1 \geq a-1$. So we can write $z_1 = (a+1)(a-1) + (y_1 - (a-1))(a+1) = a^2 - 1 + (y_1 - (a-1))(a+1)$ and then $z_2 = x_2 a + y_2(a+1)$, with $x_2 = a$ and $y_2 = y_1 - (a-1)$. $\square$

Note that the threshold given in Lemma 9 is tight, as this is a classic linear diophantine equation, it is known that if $z = a^2 - a - 1$, then $z = xa + y(a+1)$ has no valid integer solutions.

At this point, we are ready to present the proof of Theorem 2.

Proof of Theorem 2. Following Lemma 9, $n = x(2k+1) + y(2k+2)$, and by Theorems 7 and 8, we only need to compose x times the graph C^k_{2k+1} and y times C^k_{2k+2}. $\square$

Proposition 1. *For a fixed even k, with $2k+1 \leq n \leq 4k^2 + 2k$, at most $2k^2 - k$ graphs C^k_n are not Type 1.*

Proof. The proof is by induction in s, which denotes the number of graphs that will compose the target graph. For $s = 2$: We have the graphs C^k_{4k+2}, C^k_{4k+3}, and C^k_{4k+4}, which have Type 1 colorings by doing compositions of the graphs C^k_{2k+1} and C^k_{2k+1}, C^k_{2k+1} and C^k_{2k+2}, and C^k_{2k+2} and C^k_{2k+2}, respectively. Note that $2k-2$ — the graphs C^k_n with $2k+3 \leq n \leq 4k+1$ — were not totally colored. For $3 \leq s < 2k$: We have the graphs $C^k_{(2s)k+s+l}$, with $l \in \{0, \dots, s\}$, obtained by collating the graphs C^k_{2k+1} and C^k_{2k+2} in the graphs obtained in the step $s-1$; Note that $2k-s$ graphs were not totally colored. And for any C^k_n, with $n \geq 4k^2 + 2k$, all the graphs are Type 1.

Note that for each $2 \leq s \leq 2k$, only $2k-s$ graphs were not totally colored. Hence, in total, less than $2k^2 - k$ graphs were not colored. $\square$

The following result shows that for small values of odd k, the technique presented in Theorem 8 is also valid. Thus, Theorem 3 holds.

Theorem 10. *The graphs C^k_{2k+1} and C^k_{2k+2} are Type 1-compatible for $k = 3$, $k = 5$ and $k = 7$.*

Proof. We show Type 1-compatible total colorings for the graphs in Figs. 7–10, wherein the matrices represent the Type 1-compatible total coloring of the graphs C^k_{2k+1} and C^k_{2k+2}, for $k \in \{3, 5, 7\}$. $\square$

	0	1	2	3	4	5	6
0	0	4	1	5	2	6	3
1	4	1	5	2	6	3	0
2	1	5	2	6	3	0	4
3	5	2	6	3	0	4	1
4	2	6	3	0	4	1	5
5	6	3	0	4	1	5	2
6	3	0	4	1	5	2	6

(a)

	0	1	2	3	4	5	6	7
0	0	3	1	5		2	6	4
1	3	1	4	2	6		5	0
2	1	4	2	6	3	0		5
3	5	2	6	3	1	4	0	
4		6	3	1	0	5	4	2
5	2		0	4	5	1	3	6
6	6	5		0	4	3	2	1
7	4	0	5		2	6	1	3

(b)

Fig. 7. (a) The total coloring of C_7^3. (b) The total coloring of C_8^3. Observe that the two colorings are compatible with respect to the pivots $v_3 \in V(C_7^3)$ and $u_3 \in V(C_8^3)$.

	0	1	2	3	4	5	6	7	8	9	10
0	0	6	1	7	2	8	3	9	4	10	5
1	6	1	7	2	8	3	9	4	10	5	0
2	1	7	2	8	3	9	4	10	5	0	6
3	7	2	8	3	9	4	10	5	0	6	1
4	2	8	3	9	4	10	5	0	6	1	7
5	8	3	9	4	10	5	0	6	1	7	2
6	3	9	4	10	5	0	6	1	7	2	8
7	9	4	10	5	0	6	1	7	2	8	3
8	4	10	5	0	6	1	7	2	8	3	9
9	10	5	0	6	1	7	2	8	3	9	4
10	5	0	6	1	7	2	8	3	9	4	10

(a)

	0	1	2	3	4	5	6	7	8	9	10	11
0	0	2	1	6	3	8		9	5	10	7	4
1	2	1	0	7	8	3	9		4	5	10	6
2	1	0	2	8	7	9	4	10		6	5	3
3	6	7	8	3	2	4	10	5	0		9	1
4	3	8	7	2	4	10	5	0	6	1		9
5	8	3	9	4	10	5	1	2	7	0	6	
6		9	4	10	5	1	0	6	3	7	2	8
7	9		10	5	0	2	6	1	8	4	3	7
8	5	4		0	6	7	3	8	2	9	1	10
9	10	5	6		1	0	7	4	9	3	8	2
10	7	10	5	9		6	2	3	1	8	4	0
11	4	6	3	1	9		8	7	10	2	0	5

(b)

Fig. 8. (a) The total coloring of C_{11}^5. (b) The total coloring of C_{12}^5. Observe that the two colorings are compatible with respect to the pivots $v_5 \in V(C_{11}^5)$ and $u_5 \in V(C_{12}^5)$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
2	2	3	4	5	6	7	9	9	10	11	12	13	14	0	1
3	3	4	5	6	7	8	10	10	11	12	13	14	0	1	2
4	4	5	6	7	8	9	10	11	12	13	14	0	1	2	3
5	5	6	7	8	9	10	11	12	13	14	0	1	2	3	4
6	6	5	7	8	10	11	12	13	14	0	1	2	3	4	5
7	7	8	9	10	11	12	13	14	0	1	2	3	4	5	6
8	8	9	10	11	12	13	14	0	1	2	3	4	5	6	7
9	9	10	11	12	13	14	0	1	2	3	4	5	6	7	8
10	10	11	12	13	14	0	1	2	3	4	5	6	7	8	9
11	11	12	13	14	0	1	2	3	4	5	6	7	8	9	10
12	12	13	14	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	0	1	2	3	4	5	6	7	8	9	10	11	12
14	14	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Fig. 9. The total coloring of C_{15}^7.

4. Top-down decomposition and the pumping lemma

In Section 3, we verify that every graph C_n^k with even $k \geq 2$ and $n \geq 4k^2 + 2k$ is Type 1. Next, we present a general framework that allows us to identify for a given integer $k \geq 0$ whether there is a threshold t such that every graph C_n^k with $n \geq t$ is Type 1.

Our framework is based on reducing such a problem in finding an integer t such that for any $n \geq t$ there are integers $n_1, n_2, \ldots, n_\ell$ such that

- $n_i \leq f(k)$ for any $i \in [\ell]$ (f is a function that depends only on k),
- $C_{n_i}^k$ an $C_{n_{i+1}+\cdots+n_\ell}^k$ are Type 1-compatible for each $1 \leq i < \ell$.
- and $c_1 \cdot n_1 + c_2 \cdot n_2 + \cdots + c_\ell \cdot n_\ell = n$ for some positive integers $c_1, c_2, \ldots, c_\ell$.

This result shows that we only need to consider a finite number of power of cycle graphs for a given fixed k.

Let $C_n^k[N[v]]$ be the graph induced in $G = C_n^k$ by the closed neighborhood of a vertex $v \in V(G)$ where $n \geq 3k + 1$, and let $f(k)$ be the number of distinct total colorings of $C_n^k[N[v]]$ using $\Delta + 1$ colors. Theorem 11 shows that every power of cycle with large order — in the sense that $n > f(k)$ — can be decomposed into smaller compatible graphs.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7		8	10	9	14	13	11	12
1	1	2	0	4	3	6	5	8	9		12	14	11	7	10	13
2	2	0	4	1	5	3	7	9	10	11		12	13	8	14	6
3	3	4	1	6	2	7	8	10	11	12	13		9	14	0	5
4	4	3	5	2	8	9	10	11	12	13	14	0		6	1	7
5	5	6	3	7	9	10	11	12	13	14	0	1	2		8	4
6	6	5	7	8	10	11	12	13	14	0	1	2	3	4		9
7	7	8	9	10	11	12	13	14	1	3	2	4	0	5	6	
8		9	10	11	12	13	14	1	0	4	3	5	6	2	7	8
9	8		11	12	13	14	0	3	4	2	6	7	5	1	9	10
10	10	12		13	14	0	1	2	3	6	4	8	7	9	5	11
11	9	14	12		0	1	2	4	5	7	8	6	10	11	13	3
12	14	11	13	9		2	3	0	6	5	7	10	8	12	4	1
13	13	7	8	14	6		4	5	2	1	9	11	12	10	3	0
14	11	10	14	0	1	8		6	7	9	5	13	4	3	12	2
15	12	13	6	5	7	4	9		8	10	11	3	1	0	2	14

Fig. 10. The total coloring of C^7_{16}. Observe that this coloring is compatible with the coloring of Fig. 9 with respect to the pivots $v_7 \in V(C^7_{15})$ and $u_7 \in V(C^7_{16})$.

Theorem 11. *If $n > f(k)$, then C^k_n can be decomposed into two power of cycle graphs $C^k_{n_1}$ and $C^k_{n_2}$, such that $n = n_1 + n_2$ and $C^k_{n_1}, C^k_{n_2}$ are compatible.*

Proof. Suppose that $G = C^k_n$ is r-total colorable and $n > f(k)$.

Since $n > f(k)$ then, by the pigeonhole principle, there are at least one pair of vertices $w_x, w_y \in V(C^k_n)$ such that $C^k_n[N[w_x]]$ and $C^k_n[N[w_y]]$ have the same total coloring (preserving the cyclic order of the elements). Thus, we can decompose C^k_n into two power of cycle graphs $C^k_{n_1}$ and $C^k_{n_2}$, with $n_1 + n_2 = n$, using the reverse operation applied in Theorem 7, with w_x representing the vertex v_i and w_y representing the vertex u_j. □

N. Trotignon and K. Vušković [24] defined that a decomposition is said to be *extremal* if at least one of the blocks is *basic*, i.e., it cannot be decomposed into smaller blocks. If we apply the same reasoning of Theorem 11 recursively, we can generate an extremal *total coloring tree decomposition*, such that: the root of the tree is the original power of cycle graph C^k_n, two sibling nodes are compatible graphs, the *collage* of these two nodes generate the father node, and every leaf node of the tree is a basic block - $C^k_{n_i}$ graph with $n_i \leq f(k)$. Note that the size of each node is smaller than the size of the original graph C^k_n and that the number of nodes of the decomposition tree is a linear function of n. In addition, the size of each leaf node is smaller than $f(k)$.

This observation implies some consequences from an algorithmic point of view.

First, note that given a graph $G = C^k_n$ to determine whether G is Type 1 it is sufficient to guess the leaves of an extremal total coloring tree decomposition of G, as well as the appropriate coloring of each leaf (for the collage). Since each leaf has size at most $f(k)$, the number of different sizes of leaves, as well as the number of the possible total colorings of each leaf, are bounded by a function of k.

Denote by *leaf pattern* as a total colored $C^k_{n_i}$ using colors from $\{1, \dots, \Delta + 1\}$, where $n_i \leq f(k)$. Recall that the number of distinct leaf patterns is bounded by a function of k.

Now, let S be a guessed set of leaf patterns to be used in an extremal total coloring tree decomposition of G. Let $\mathcal{P}_S$ be the pattern graph where each vertex represents a pattern in S and each edge represents that the corresponding patterns are compatible. Clearly, if S is the set of leaf patterns to be used in an extremal total coloring tree decomposition of G then $\mathcal{P}_S$ is a connected graph. Assuming that $\mathcal{P}_S$ is connected, to check whether from S it is possible to obtain a Type 1 total coloring of G, we can interpret each pattern as a coin, reducing the problem to a decision version of the popular Change-Making Problem, i.e, we are given a set of $|S|$ non-negative integers representing a coin system, and we are asked whether there is a set of coins that sum up to a given value x, where each type of coin can be used an unlimited number of times. Since each guessed pattern must be used at least once, we can set x as $n - (n_1 + n_2 + \cdots + n_\ell)$. The Change-Making Problem is a popular homework exercise in dynamic programming, where the textbook solution runs in $O(\ell \cdot x)$ time.

Therefore, the problem of determining whether a given $G = C^k_n$ is Type 1 total colorable is (Turing) reducible to solve $g(k)$ instances of Change-Making Problem, where $g(k)$ is a function that depends only on k.

It is easy to see that the kth power of cycle graphs is a class of graphs with treewidth $O(k)$, and that the property of being Type 1 can be expressed in $MSOL_2$ by a formula of polynomial-size with respect to Δ. Since $\Delta \leq 2k$ on k-power of cycles, by Courcelle's theorem, this implies that using monadic-second order model checking, one can determine whether a power of cycle graph is Type 1 in FPT-time when k (the power of the cycle) is the parameter. However, by analyzing every possible subset S of leaf patterns, checking the connectivity of $\mathcal{P}_S$, and then performing an algorithm for Change-Making Problem, we design a simpler FPT algorithm for Total Coloring on power of cycle graphs parameterized by k.

More interestingly, from Theorem 11 we also obtain a pumping lemma for Type 1 power of cycle graphs, similar to the classical pumping lemma for regular languages.

Lemma 12 (*Pumping Lemma for Type 1 Power of Cycle Graphs*). *Let $\mathcal{L}_k$ be the class of Type 1 k-power of cycle graphs, then there exists a number $t \geq 1$ such that every graph $G = C_n^k$ in $\mathcal{L}_k$ with $n \geq t$ can be decomposed into two Type 1 compatible power of cycle graphs*

$$C_{n_1}^k \text{ and } C_{n_2}^k,$$

with $n = n_1 + n_2$, $n_1 \leq t$, such that $C_{c_1 \cdot n_1 + n_2}^k$ is Type 1 for every integer $c_1 \geq 1$.

Proof. Follows directly from Theorem 11, and the fact that $C_{n_1}^k$ is Type 1 compatible with itself. □

Again, by applying the same reasoning of Lemma 12 recursively, we have that Theorem 4 holds.

Notice that Theorem 4 can be seem as a strong evidence for the conjecture that the set of Type 2 k-power of cycle graphs is finite, as well as for the Total Coloring Conjecture, since from every $\Delta + 1$ total coloring of a C_n^k we can define a infinite family of Type 1 k-power of cycle graphs. More precisely, for a fixed $k > 1$, one can show that the set of k-power of cycle graphs that are not Type 1 is finite, by finding sets of power of cycles as described in Theorem 4 such that for every integer n greater than a threshold t, C_n^k can be generated from one of these sets (i.e., $n = c_1 \cdot n_1 + c_2 \cdot n_2 + \cdots + c_\ell \cdot n_\ell$). Note that Theorem 2 and Theorem 3 illustrate applications of this approach using only two k-power of cycle graphs.

5. A dichotomy for the conformability of power of cycle graphs

In this section we present the proof for the dichotomy announced in Theorem 1, which constitutes a strong evidence for Conjecture 1.

Proof of Theorem 1. It suffices to show that, if n is even, or if n is odd and $n \geq 3(k+1)$, then a power of cycle graph C_n^k which is neither a cycle nor a complete graph is conformable, since otherwise it has already been shown that G is not conformable by Campos [5], as discussed in Section 1.

Since we are excluding complete graphs and cycles by hypothesis (i.e. we are assuming $n > 2k+1$ and $k \geq 2$), we split the proof into the following cases:

Case 1. n is even and $n \geq 2(2k+1)$;
Case 2. n is odd and $n \geq 3(2k+1)$;
Case 3. n is even and $2k+1 < n < 2(2k+1)$;
Case 4. n is odd and $3(k+1) \leq n < 3(2k+1)$.

Proof for Case 1 and Case 2. First observe that if n is divisible than $2k+1$, then we can write $n = r(2k+1)$ for some positive integer r satisfying $n \equiv r \pmod 2$. In this case, it is easy to construct a conformable coloring for $G = C_n^k$: just assign to each vertex v_i of C_n^k the color $i \bmod (2k+1)$. Since $\Delta(C_n^k) = 2k$, this is a $(\Delta(C_n^k)+1)$-vertex-coloring of G in which each color is assigned to exactly r vertices; therefore, this is a conformable coloring as claimed.

Now let $n \geq 3$ and k be positive integers with either n even and $n \geq 4k+2$, or n odd and $n \geq 6k+3$. If n is not multiple of $2k+1$, then take $q = \lfloor n/(2k+1) \rfloor$ and

$$n_1 = \begin{cases} q(2k+1), & \text{if } q \equiv n \pmod 2; \\ (q-1)(2k+1), & \text{if } q \not\equiv n \pmod 2. \end{cases}$$

Clearly, n_1 can be written as $n_1 = r(2k+1)$, being r a positive integer satisfying $n \equiv r \pmod 2$ (which also implies $n \equiv n_1 \pmod 2$). By defining $\mathscr{C}(u_i) = i \bmod (2k+1)$ for $0 \leq i < n_1$ we get a conformable coloring for $H = C_{n_1}^k$ on vertices $u_0, \ldots, u_{n_1-1}$.

Since $n - n_1$ is even, we shall demonstrate how to add $n - n_1$ vertices to H, two of them at a time, coloring each pair of vertices added in order to obtain a power of cycle graph $G = C_n^k$ conformably colored. For each $i \in \{0, \ldots, (n-n_1)/2 - 1\}$, the i^{th} pair of vertices added shall be colored i. Because $n - n_1 < 4k+2$, at most $2k$ pairs of vertices shall be added. Since we start with a conformable coloring of H wherein the distance between every pair of vertices colored the same is $2k+1$, we can add, for each $i \in \{0, \ldots, (n-n_1)/2 - 1\}$, a vertex between u_{i+k} and u_{i+k+1} and a vertex between u_{n_1-2} and u_{n_1-1}. Note that this way we get no color conflicts, since we can still guarantee that the distance between every pair of vertices colored the same is at least $k+1$, so they will not be adjacent.

Proof for Case 3. If n is even and $2k+1 < n < 4k+2$, then the only graphs which remain to be colored are the graphs $C_{2k+2}^k, C_{2k+4}^k, \ldots, C_{2k+2k}^k$. But constructing a conformable $(2k+1)$-vertex-coloring for any $G = C_n^k$ among these graphs is easy: for each among $n/2$ colors selected, assign this color to two vertices of G in antipodal position in the cycle; assign the remaining colors to no vertex. Now we have a $(2k+1)$-vertex-coloring for G in which each color class has size 0 or 2, conforming to the parity of n.

Proof for Case 4. Now assume that n is odd and $3(k+1) \leq n < 3(2k+1)$. We shall construct a proper $(2k+1)$-vertex-coloring of C_n^k in which, among the $2k+1$ colors available, $x > 0$ of them shall color exactly 3 vertices each, and $y > 0$ of them shall color exactly 1 vertex each, implying the conformability of the coloring constructed. First we verify that such positive integers x and y exist. Since $x+y = 2k+1$ and $3x+y = n$, we have $x = (n-(2k+1))/2$ and $y = (3(2k+1)-n)/2$, which can be verified to be positive integers since n is odd and $3(k+1) \le n < 3(2k+1)$. Moreover,

$$\begin{aligned} x + \left\lfloor \frac{y}{3} \right\rfloor &\ge x + \frac{y-2}{3} \\ &= \frac{1}{6}\Big(3n - 3(2k+1) + 3(2k+1) - n - 4\Big) \\ &= \frac{n-2}{3} \ge \frac{3(k+1)-2}{3} = (k+1) - \frac{2}{3}, \end{aligned}$$

which, since $x + \lfloor y/3 \rfloor$ is an integer, implies $x + \lfloor y/3 \rfloor \ge k+1$ (remark that this can hold with equality; for instance, take $n = 9$ and $k = 2$, yielding $x = 2$ and $y = 3$). Ergo, our coloring can be defined by:

$$\begin{aligned} &\mathscr{C}(v_i) = i && \text{for } 0 \le i < x \text{ and for } x \le i < x + \left\lfloor \frac{y}{3} \right\rfloor; \\ &\mathscr{C}(v_i) = i - \left(x + \left\lfloor \frac{y}{3} \right\rfloor\right) && \text{for } x + \left\lfloor \frac{y}{3} \right\rfloor \le i < 2x + \left\lfloor \frac{y}{3} \right\rfloor; \\ &\mathscr{C}(v_i) = i - x && \text{for } 2x + \left\lfloor \frac{y}{3} \right\rfloor \le i < 2x + 2\left\lfloor \frac{y}{3} \right\rfloor; \\ &\mathscr{C}(v_i) = i - 2\left(x + \left\lfloor \frac{y}{3} \right\rfloor\right) && \text{for } 2x + 2\left\lfloor \frac{y}{3} \right\rfloor \le i < 3x + 2\left\lfloor \frac{y}{3} \right\rfloor; \\ &\mathscr{C}(v_i) = i - 2x && \text{for } 3x + 2\left\lfloor \frac{y}{3} \right\rfloor \le i < 3x + y. \end{aligned}$$

Observe that, in this coloring, each color in $X = \{0, \ldots, x-1\}$ is used exactly 3 times, and each color in the sets

$$\begin{aligned} Y_1 &= \left\{x, \ldots, x + \left\lfloor \frac{y}{3} \right\rfloor - 1\right\}, \\ Y_2 &= \left\{x + \left\lfloor \frac{y}{3} \right\rfloor, \ldots, x + 2\left\lfloor \frac{y}{3} \right\rfloor - 1\right\}, \quad \text{and} \\ Y_3 &= \left\{x + 2\left\lfloor \frac{y}{3} \right\rfloor, \ldots, x + y - 1\right\} \end{aligned}$$

is used exactly once. Observe further that, following the order $v_0, \ldots, v_{n-1}$ of the vertices along the main spanning cycle, we verify the order X, Y_1, X, Y_2, X, Y_3 for the sets in which the colors assigned to the vertices lie. Hence, every two vertices colored the same are at distance at least $x + \lfloor y/3 \rfloor \ge k+1$ and thus they are not adjacent. □

6. Total coloring the third and the fourth powers of cycles

As discussed in Section 1, Conjecture 1 has already been settled for graphs C_n^k with $k = 3$ or $k = 4$, implying that the total coloring problem is fully solved for these graphs. This result has been announced by Almeida et al. [8], presented at a conference and published in the book of abstracts with no detailed proofs. In this section we provide an independent proof for the result, using our composition techniques discussed in Sections 2 and 3.

Theorem 13. *A power of cycle graph C_n^k with $k \in \{3, 4\}$ which is not a complete graph is: Type 2 if n is odd and $n < 3(k+1)$; Type 1 otherwise.*

Proof. Since the graphs C_9^3, C_{11}^3, C_{11}^4, and C_{13}^4 are well-known [5] to be Type 2, it remains only to prove that if n is even, or if n is odd and $n \ge 3(k+1)$, then C_n^k is Type 1 (assuming $n > 2k+1$ since we are excluding complete graphs by hypothesis).

By taking the set of seeds $\{C_7^3, C_8^3, C_{10}^3, C_{12}^3\}$, whose Type 1 total colorings are presented in Table 3, we can compose every C_n^3 with $n \ge 14$. Similarly, by taking the set of seeds $\{C_9^4, C_{10}^4, C_{12}^4, C_{14}^4, C_{16}^4\}$, whose total colorings are presented in Table 4, we can compose every C_n^4 with $n \ge 18$.

The only graphs which remain to be Type 1 total colored are C_{13}^3, C_{15}^4, and C_{17}^4. For C_{13}^3 and C_{15}^4, a Type 1 total coloring was presented by Campos [5]. Hence, we conclude the proof presenting a Type 1 total coloring for C_{17}^4 in Table 5. □

7. Conclusion

We have proved that if n is bigger than a given threshold then every graph C_n^k with even k is Type 1. Our result is actually stronger, as we give a linear time optimal algorithm to total color such graphs. The result asymptotically settles for even k the conjecture proposed by Campos and Mello [7], reducing to a finite number the graphs that are still unknown to satisfy the conjecture.

Table 3
Initial seeds for the total coloring of C_n^3 with $n \geq 14$.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6
v_0	0	4	1	5	2	6	3
v_1	4	1	5	2	6	3	0
v_2	1	5	2	6	3	0	4
v_3	5	2	6	3	0	4	1
v_4	2	6	3	0	4	1	5
v_5	6	3	0	4	1	5	2
v_6	3	0	4	1	5	2	6

(a)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_0	0	3	1	5	-	2	6	4
v_1	3	1	4	2	6	-	5	0
v_2	1	4	2	6	3	0	-	5
v_3	5	2	6	3	1	4	0	-
v_4	-	6	3	1	0	5	4	2
v_5	2	-	0	4	5	1	3	6
v_6	6	5	-	0	4	3	2	1
v_7	4	0	5	-	2	6	1	3

(b)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_0	6	0	1	5	-	-	-	3	2	4
v_1	0	1	4	2	6	-	-	-	5	3
v_2	1	4	2	6	3	0	-	-	-	5
v_3	5	2	6	4	1	3	0	-	-	-
v_4	-	6	3	1	0	2	4	5	-	-
v_5	-	-	0	3	2	6	5	4	1	-
v_6	-	-	-	0	4	5	1	6	3	2
v_7	3	-	-	-	5	4	6	2	0	1
v_8	2	5	-	-	-	1	3	0	4	6
v_9	4	3	5	-	-	-	2	1	6	0

(c)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
v_0	0	3	4	5	-	-	-	-	-	1	2	6
v_1	3	1	5	2	6	-	-	-	-	-	0	4
v_2	4	5	2	6	3	0	-	-	-	-	-	1
v_3	5	2	6	3	0	1	4	-	-	-	-	-
v_4	-	6	3	0	4	2	1	5	-	-	-	-
v_5	-	-	0	1	2	5	3	4	6	-	-	-
v_6	-	-	-	4	1	3	0	2	5	6	-	-
v_7	-	-	-	-	5	4	2	1	3	0	6	-
v_8	-	-	-	-	-	6	5	3	2	4	1	0
v_9	1	-	-	-	-	-	6	0	4	3	5	2
v_{10}	2	0	-	-	-	-	-	6	1	5	4	3
v_{11}	6	4	1	-	-	-	-	-	0	2	3	5

(d)

Table 4
Initial seeds for the total coloring of C_n^4 with $n \geq 18$.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_0	0	5	1	6	2	7	3	8	4
v_1	5	1	6	2	7	3	8	4	0
v_2	1	6	2	7	3	8	4	0	5
v_3	6	2	7	3	8	4	0	5	1
v_4	2	7	3	8	4	0	5	1	6
v_5	7	3	8	4	0	5	1	6	2
v_6	3	8	4	0	5	1	6	2	7
v_7	8	4	0	5	1	6	2	7	3
v_8	4	0	5	1	6	2	7	3	8

(a)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_0	0	4	1	6	2	-	5	8	7	3
v_1	4	1	0	2	7	3	-	6	5	8
v_2	1	0	2	7	3	8	4	-	6	5
v_3	6	2	7	3	8	4	0	5	-	1
v_4	2	7	3	8	4	5	6	0	1	-
v_5	-	3	8	4	5	0	7	1	2	6
v_6	5	-	4	0	6	7	1	3	8	2
v_7	8	6	-	5	0	1	3	2	4	7
v_8	7	5	6	-	1	2	8	4	3	0
v_9	3	8	5	1	-	6	2	7	0	4

(b)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
v_0	8	0	1	6	2	-	-	-	7	4	3	5
v_1	0	1	5	2	7	3	-	-	-	6	8	4
v_2	1	5	2	7	3	8	4	-	-	-	0	6
v_3	6	2	7	3	8	4	0	5	-	-	-	1
v_4	2	7	3	8	5	1	6	0	4	-	-	-
v_5	-	3	8	4	1	0	2	6	5	7	-	-
v_6	-	-	4	0	6	2	8	3	1	5	7	-
v_7	-	-	-	5	0	6	3	1	8	2	4	7
v_8	7	-	-	-	4	5	1	8	2	0	6	3
v_9	4	6	-	-	-	7	5	2	0	3	1	8
v_{10}	3	8	0	-	-	-	7	4	6	1	5	2
v_{11}	5	4	6	1	-	-	-	7	3	8	2	0

(c)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
v_0	0	4	1	6	2	-	-	-	-	-	7	5	8	3
v_1	4	1	5	2	7	3	-	-	-	-	-	6	0	8
v_2	1	5	2	7	3	8	4	-	-	-	-	-	6	0
v_3	6	2	7	3	8	4	0	5	-	-	-	-	-	1
v_4	2	7	3	8	4	0	1	6	5	-	-	-	-	-
v_5	-	3	8	4	0	5	2	1	6	7	-	-	-	-
v_6	-	-	4	0	1	2	6	3	7	5	8	-	-	-
v_7	-	-	-	5	6	1	3	0	2	8	4	7	-	-
v_8	-	-	-	-	5	6	7	2	1	3	0	8	4	-
v_9	-	-	-	-	-	7	5	8	3	2	6	0	1	4
v_{10}	7	-	-	-	-	-	8	4	0	6	3	1	2	5
v_{11}	5	6	-	-	-	-	-	7	8	0	1	4	3	2
v_{12}	8	0	6	-	-	-	-	-	4	1	2	3	5	7
v_{13}	3	8	0	1	-	-	-	-	-	4	5	2	7	6

(d)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
v_0	0	4	1	6	2	-	-	-	-	-	-	-	5	7	3	8
v_1	4	1	0	2	7	3	-	-	-	-	-	-	-	6	8	5
v_2	1	0	2	7	3	8	4	-	-	-	-	-	-	-	5	6
v_3	6	2	7	3	8	4	0	5	-	-	-	-	-	-	-	1
v_4	2	7	3	8	4	0	1	6	5	-	-	-	-	-	-	-
v_5	-	3	8	4	0	5	2	1	6	7	-	-	-	-	-	-
v_6	-	-	4	0	1	2	6	3	7	5	8	-	-	-	-	-
v_7	-	-	-	5	6	1	3	7	2	0	4	8	-	-	-	-
v_8	-	-	-	-	5	6	7	2	0	4	3	1	8	-	-	-
v_9	-	-	-	-	-	7	5	0	4	1	6	2	3	8	-	-
v_{10}	-	-	-	-	-	-	8	4	3	6	2	5	7	0	1	-
v_{11}	-	-	-	-	-	-	-	8	1	2	5	3	6	4	7	0
v_{12}	5	-	-	-	-	-	-	-	8	3	7	6	4	1	0	2
v_{13}	7	6	-	-	-	-	-	-	-	8	0	4	1	5	2	3
v_{14}	3	8	5	-	-	-	-	-	-	-	1	7	0	2	6	4
v_{15}	8	5	6	1	-	-	-	-	-	-	-	0	2	3	4	7

(e)

In addition to our main results for even k, for odd k we also have computational results for specific values of k, namely $k = 5$ and $k = 7$. These results point to a path to follow towards a more general technique which can also be applied for odd k.

All techniques presented in this work can be expanded: we can extend the set of graphs colored by the technique presented in Section 3 by simply increasing the size of the seed set used to generate the total colorings. As Definition 6 does not present any restriction on the number of colors, the same idea can be used to prove the TCC as well, by giving $(\Delta(G)+2)$-compatible total colorings. The framework presented in Section 4 can be extended to totally color other classes of graphs, mostly the classes of graphs that have similar structural decomposition, for example circulant graphs. Another graph class in which our results may have an impact is the class of power of paths graphs, since a power of path is a spanning subgraph of a power of cycle.

The total coloring tree decomposition proposed in this paper for power of cycles is closed related to the notion of tree decomposition, as well as treewidth, defined by Robertson and Seymour [25]. Therefore, another interesting direction would be to extend our technique in order to improve the state of art of the total coloring problem for classes of graphs with bounded treewidth.

Table 5
A Type 1 total coloring for C_{17}^4.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
v_0	0	4	1	6	2	-	-	-	-	-	-	-	-	5	8	3	7
v_1	4	1	0	2	7	3	-	-	-	-	-	-	-	-	5	8	6
v_2	1	0	2	7	3	8	4	-	-	-	-	-	-	-	-	6	5
v_3	6	2	7	3	8	4	0	5	-	-	-	-	-	-	-	-	1
v_4	2	7	3	8	4	1	5	0	6	-	-	-	-	-	-	-	-
v_5	-	3	8	4	1	0	2	6	5	7	-	-	-	-	-	-	-
v_6	-	-	4	0	5	2	1	3	7	6	8	-	-	-	-	-	-
v_7	-	-	-	5	0	6	3	2	1	4	7	8	-	-	-	-	-
v_8	-	-	-	-	6	5	7	1	3	0	2	4	8	-	-	-	-
v_9	-	-	-	-	-	7	6	4	0	5	1	2	3	8	-	-	-
v_{10}	-	-	-	-	-	-	8	7	2	1	0	3	5	6	4	-	-
v_{11}	-	-	-	-	-	-	-	8	4	2	3	1	6	0	7	5	-
v_{12}	-	-	-	-	-	-	-	-	8	3	5	6	2	7	0	1	4
v_{13}	5	-	-	-	-	-	-	-	-	8	6	0	7	3	1	4	2
v_{14}	8	5	-	-	-	-	-	-	-	-	4	7	0	1	6	2	3
v_{15}	3	8	6	-	-	-	-	-	-	-	-	5	1	4	2	7	0
v_{16}	7	6	5	1	-	-	-	-	-	-	-	-	4	2	3	0	8

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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