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Total tessellation cover: Bounds, hardness, and applications

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ABSTRACT

The concept of graph tessellation cover was defined in the context of quantum walk models, and is a current research area in graph theory. In this work, we propose a generalization called total tessellation cover. A tessellation of a graph G is a partition of its vertex set $V(G)$ into vertex disjoint cliques. A tessellation cover of G is a set of tessellations that covers its edge set $E(G)$. A total tessellation cover of G consists of a tessellation cover together with a compatible vertex coloring, such that the color of each vertex is different from the tessellation labels of the edges incident to the vertex. The total tessellation cover number $T_t(G)$ is the size of a minimum total tessellation cover of G. We present lower bounds $T_t(G) \geq \omega(G)$ and $T_t(G) \geq s(G)+1$, where $\omega(G)$ is the size of a maximum clique, and $s(G)$ is the number of edges of a maximum induced star subgraph. A graph G is called a good total tessellable if $T_t(G) = \omega(G)$ or $T_t(G) = s(G)+1$. We study the complexity of the k-TOTAL TESSELLABILITY problem, which aims to decide whether a given graph G has $T_t(G) \leq k$. We prove that k-TOTAL TESSELLABILITY is in $\mathcal{P}$ for good total tessellable graphs. We establish the $\mathcal{NP}$-completeness of the k-TOTAL TESSELLABILITY when restricted to the following graph classes: bipartite graphs, line graphs of triangle-free graphs, universal graphs, (2, 1)-chordal graphs and planar graphs.

1. Graph tessellations and quantum walks

A *tessellation* of a graph $G = (V, E)$ is a partition of V into vertex disjoint cliques, where each clique of a tessellation is called a *tile*. An edge *belongs to a tile* if its endpoints are in the tile. A k-*tessellation cover* of G is a set of k tessellations that covers $E(G)$. A graph is k-tessellable if it admits a k-tessellation cover. The *tessellation cover number* $T(G)$ of a graph G is the size of a minimum tessellation cover. The k-TESSELLABILITY problem aims to decide whether a given graph G has $T(G) \leq k$. Fig. 1 depicts the Hajós graph H and a 3-tessellation cover with tessellations T_1, T_2, and T_3. The fact that the clique graph of G is a K_4 implies that $T(H) = 3$ [13].

Portugal et al. [15] introduced the concept of tessellations on graphs by showing a close relation between tessellations and quantum walk models and by establishing a modern application of tessellation covers to quantum computing. These results inspired us to addressed in [2,3] the computational complexity of the k-TESSELLABILITY problem, and to describe

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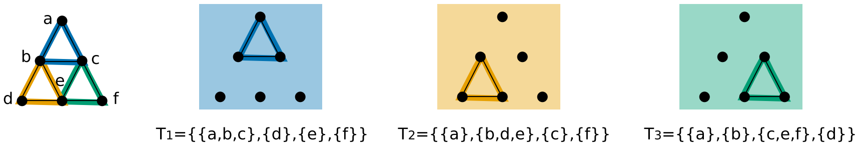

Fig. 1. Hajós graph H with a 3-tessellation cover. We depict in the first example the trivial tiles of size 1, but in the sequel we omit them and depict only nontrivial tiles of a tessellation.

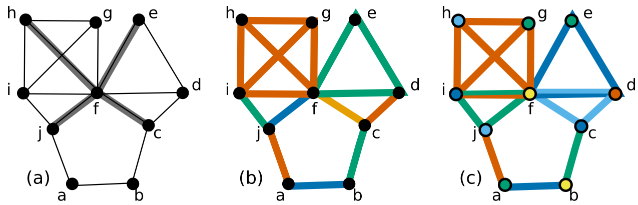

Fig. 2. (a) A graph G with $s(G) = 4$. (b) A 4-tessellation cover of G given by $T_1 = \{\{a, b\}, \{f, j\}\}$, $T_2 = \{\{c, f\}\}$, $T_3 = \{\{b, c\}, \{e, d, f\}, \{i, j\}\}$, $T_4 = \{\{a, j\}, \{c, d\}, \{f, g, h, i\}\}$. In a tessellation cover, trivial tiles of size 1 are omitted. (c) A 5-total tessellation cover of G given by $T_1 = \{\{a, b\}, \{e, d, f\}, \{c\}, \{i\}\}$, $T_2 = \{\{b\}, \{f\}\}$, $T_3 = \{\{b, c\}, \{f, j, i\}, \{a\}, \{e\}, \{g\}\}$, $T_4 = \{\{a, j\}, \{i, f, g, h\}, \{d\}\}$, $T_5 = \{\{c, d, f\}, \{j\}, \{h\}\}$. In a total tessellation cover, only nontrivial tiles are considered, and subsets of size 1 define a proper vertex coloring. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

polynomial-time algorithms and $\mathcal{NP}$-completeness proofs for several graph classes. The concept of minimum tessellation cover was independently proposed as equivalence dimension by Duchet [7], and the relation between these concepts is described in [3].

The interest in quantum walks, which are mathematical models of the motion of a particle on a graph, has grown considerably in the last decades. They are powerful tools in the development of quantum algorithms. When the dynamics of a quantum walk is described using graph tessellations, each tile of a tessellation establishes a neighborhood around which the walker can move under the action of the associated local evolution operator. Tessellation-based quantum walks were used in the development of new quantum algorithms that outperform their classical counterparts [14,17], and they can be implemented in quantum computers using less resources than other quantum walk models [4].

Graph-theoretical studies have related tessellation covers to the challenging edge and vertex coloring problems. Abreu et al. [2] proved that $\chi'(G)$ and $\chi(K(G))$ are upper bounds for $T(G)$. They also proved the hardness of k-TESSELLABILITY for planar graphs, (2, 1)-chordal graphs, (1, 2)-graphs, and showed that 2-TESSELLABILITY is solved in linear time. Since $T(G) = \chi'(G)$ if G is a triangle-free graph, we have that k-TESSELLABILITY is hard for triangle-free graphs [10]. Moreover, Posner et al. [16] showed that k-TESSELLALBILITY is $\mathcal{NP}$-complete for line graphs of triangle-free graphs. Abreu et al. [3] proved that $s(G)$ is a lower bound for $T(G)$, where $s(G)$ is the number of edges in a maximum induced star of G. They proved the hardness of k-TESSELLABILITY for universal graphs and the hardness of GOOD TESSELLABLE RECOGNITION, which aims to decide whether G has $T(G) = s(G)$. Fig. 2(a) depicts a graph G where a maximum induced star subgraph has four edges, i.e., $s(G) = 4$, and Fig. 2(b) depicts a 4-tessellation cover of G. cover of G. Note that we omit trivial tiles and depict only nontrivial tiles of a tessellation. Since $T(G) \geq s(G) = 4$, we have $T(G) = 4$.

The present paper proposes the concept of total tessellation cover, which consists of a tessellation cover together with a compatible vertex coloring defined over the same label set, i.e., the color of each vertex is different from the labels of the edges incident to the vertex. We remark that the color of each vertex must be different from the label of each nontrivial tile containing the vertex. Please refer to Fig. 2(c), where a 5-total tessellation cover of G is depicted. The formal definition is presented in Section 2. We discuss practical applications of total tessellation covers, such as a new quantum walk model in which the walker hops to both vertices and edges and a variation of the frequency assignment problem.

The paper is organized as follows. We end Section 1 with basic definitions and notation. Section 2 presents detailed definitions about the total tessellation cover and its practical and theoretical applications to quantum walks, frequency assignments, and coloring problems. Section 3 presents a detailed study about bounds on the total tessellation cover number parameter. Section 4 shows efficient algorithms to obtain the target parameter for good total tessellable graphs, and hardness proofs when restricted to the following well-known graph classes: universal graphs, line graphs, chordal graphs and planar graphs. Section 5 proposes open questions.

Basic definitions

Let $G = (V, E)$ be a simple graph. The set $N(v) = \{u \mid uv \in E(G)\}$ is the *neighborhood* of the vertex v in V and the *degree* of v is $d(v) = |N(v)|$. The maximum size of the neighborhood of a vertex of the graph G is denoted by $\Delta(G)$ (or simply Δ). A *subgraph* H of G has $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* H of G has $V(H) \subseteq V(G)$ and $E(H) = \{uv \mid uv \in E(G), u \in V(H) \text{ and } v \in V(H)\}$.

We say that a subset of vertices of V is an *independent set* (resp. *clique*) if there is no edge between any pair of its vertices (resp. there are edges between all pairs of its vertices). The size of a maximum independent set (resp. maximum clique) of G is denoted by $\alpha(G)$ (resp. $\omega(G)$). We say that a subset of edges of E is a *matching* if there is no pair of edges that share a common endpoint. The size of a maximum matching of a graph G is denoted by $\mu(G)$.

The *union* $G = G_1 \cup G_2$ of two graphs G_1 and G_2 has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *join* $G = G_1 \vee G_2$ of two graphs G_1 and G_2 has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$. The *line graph* $L(G)$ of a graph G has a vertex for each edge of G and there is an edge between two vertices of $L(G)$ if the respective edges of G share a common endpoint. The *clique graph* $K(G)$ of a graph G has a vertex for each maximal clique of G and there is an edge between two vertices of $K(G)$ if the respective maximal cliques of G share a common vertex. A *universal graph* is a graph G that has a vertex v with $N(v) = V(G) \setminus v$. A *planar graph* is a graph G that can be drawn on the plane in such a way that the edges of G intersect only at their endpoints. A graph G is a *chordal graph* if it has no induced cycle of size four or more. A graph G is (k, ℓ) if its vertex set can be partitioned into k independent sets and ℓ cliques.

A *k-coloring* (resp. *k-edge coloring*) is a partition of the vertices of the graph into k independent sets (resp. k matchings). A *k-total coloring* of a graph G is a coloring of vertices and edges with k colors such that no two adjacent vertices have the same color, nor two adjacent edges, and nor an edge and its endpoints. A graph G is *k-colorable* (resp. *k-edge colorable* and *k-total colorable*) if it admits a k-coloring (resp. k-edge coloring and k-total coloring). We denote by $\chi(G)$, $\chi'(G)$, and $\chi_t(G)$ the minimum k such that G is k-colorable, k-edge colorable, k-total colorable, respectively.

2. Total tessellations and applications

We now precisely define the total tessellation cover of a graph, which takes into account not only the tessellation labels, but also the vertex labels. This provides a quantum walk scenario in which the walker hops to both vertices and edges.

Definition 1. Let $G = (V, E)$ be a graph and Σ a non-empty label set. A *total tessellation cover* comprises a proper vertex coloring and a tessellation cover of G both with labels in Σ such that, for any vertex $v \in V$, there is no edge $e \in E$ incident to v so that e belongs to a tile in a tessellation with label equal to the color of v.

Note that an alternative way to define a tessellation is by describing the edges that belong to the tessellation. A k-tessellation cover of $G = (V, E)$ is a function h that assigns to each edge of E a nonempty subset in the power set of Σ, $\mathcal{P}(\Sigma)$, where $\Sigma = \{1, \ldots, k\}$, such that the set of edges having the same label corresponds to a tessellation, i.e., induces a partition of V into cliques. A *k-total tessellation cover* of a graph G simultaneously assigns labels in Σ to V as a proper vertex coloring f and labels in $\mathcal{P}(\Sigma) \setminus \emptyset$ to E as a tessellation cover with function h, such that each $uv \in E$ satisfies $f(u) \notin h(uv)$ and $f(v) \notin h(uv)$.

Definition 2. The *total tessellation cover number* $T_t(G)$ of a graph G is the minimum size of the set of labels Σ for which G has a total tessellation cover.

The k-TOTAL TESSELLABILITY problem aims to decide whether a given graph G has $T_t(G) \leq k$. Fig. 2(c) and the lower graph of Fig. 3 depict examples of total tessellation covers of size 5 and 4, respectively. Note that both vertices and edges have label assignments, and an edge that is covered by more than one tessellation receives more than one label. A vertex label must be different from the labels of the vertices of its neighborhood and from the labels of its incident edges. An edge label must be different from the labels of its endpoint vertices and from the labels of the other tiles that contain these endpoint vertices.

2.1. Application to quantum walks

The total tessellation cover can be used to define a new form of quantum walk, which allows the walker to hop to vertices and edges alike. The new description simplifies the underlying graph structure of the quantum walk compared to the standard description and explores extra available resources in terms of where to hop. This has an interesting consequence on the dispersion of the wave function. It is known from the postulates of quantum mechanics that the walker can be in more that one place simultaneously. Besides, the time-evolution operations explore this feature, but they must also fulfill the postulates. It is a nontrivial task to describe recipes that generate allowed evolution operations. The notion of total tessellation helps in this task and provides a new form of dispersion, which is depicted in Fig. 3, where the walker's position is shown by a mark. The upper row shows the walker's position's dispersion as time goes to the right in discrete time-steps when we use the standard quantum walk model. The lower row shows the new dispersion when the walker is driven by the total tessellation-based quantum walk.

Here we show how to simulate the new dynamics in terms of the standard quantum walk model to highlight the economy of resources that the concept of total tessellation cover enables. In fact, we show that the dynamics based on a total tessellation cover of G is described in terms of the standard dynamics based on the original tessellation cover of the total graph $\mathrm{Tot}(G)$.

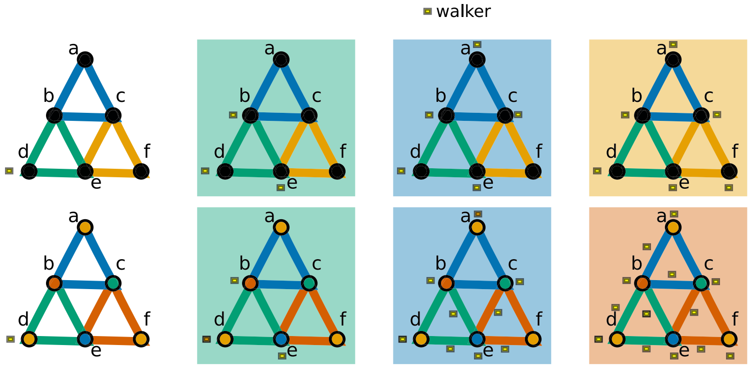

Fig. 3. Upper row: A 3-tessellation cover of the Hajós graph given by $T_1 = \{\{a, b, c\}\}$, $T_2 = \{\{c, e, f\}\}$, $T_3 = \{\{b, d, e\}\}$, with colors blue, yellow, and green, respectively, and the associated dispersion of the walker's position when the successive time-evolution operations based on the tessellations T_3, T_1 and T_2 take place. Lower row: A 4-total tessellation cover of the same graph given by $T_1 = \{\{a, b, c\}, \{e\}\}$, $T_2 = \{\{a\}, \{d\}, \{f\}\}$, $T_3 = \{\{b, d, e\}, \{c\}\}$, $T_4 = \{\{c, e, f\}, \{b\}\}$ and the corresponding dispersion based on the tessellations T_3, T_1 and T_4. The unoccupied edges ac and cf are reached by a further application of tessellation T_3. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

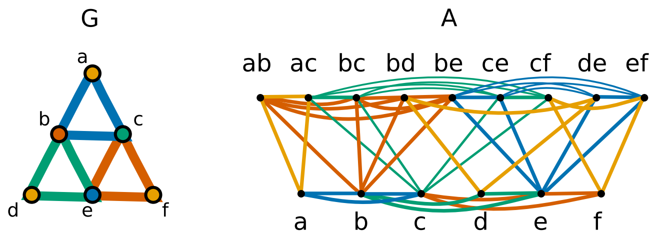

Fig. 4. A 4-total tessellation cover of G given by $T_1 = \{\{a, b, c\}, \{e\}\}$, $T_2 = \{\{a\}, \{d\}, \{f\}\}$, $T_3 = \{\{b, e, d\}, \{c\}\}$, $T_4 = \{\{c, e, f\}, \{b\}\}$ and the associated 4-tessellation cover of A.

The *total graph* $\mathrm{Tot}(G)$ of G has $V(\mathrm{Tot}(G)) = V(G) \cup E(G)$ and $E(\mathrm{Tot}(G)) = E(G) \cup \{(u, uw) |\ u \in V(G), uw \in E(G)\} \cup \{(uv, vw) |\ uv \in E(G) \text{ and } vw \in E(G)\}$.

Let $A = \mathrm{Tot}(G)$, $A[E(G)] = Y$ and $A[V(G)] = X$. Subgraph Y is isomorphic to the line graph $L(G)$ of G, and X is isomorphic to the original G. We define the clique $K_v = \{v\} \cup \{vw \mid vw \in E(G)\}$ of A.

Consider a total tessellation cover of a graph G. Define an associated tessellation cover of A as follows. Assign the labels of the edges of G to the respective edges of X and assign the color of each vertex v of G to the edges of $A[K_v]$. We simulate the total quantum walk on G with tessellations on A by considering the vertices of G as the corresponding vertices of X in A, and the edges of G as the corresponding vertices of Y in A. Fig. 4 depicts a total tessellation cover of a graph G and the associated tessellation cover of $A = \mathrm{Tot}(G)$.

Consider the walker located on a vertex a of G. If we apply the evolution operation associated with the color of a, the walker hops to the edges incident to a (the edges ab and ac). If we apply the evolution operation associated with the label of an edge incident to a, the walker hops to the vertices in the tile of the tessellation of the same label that contains a (the vertices b and c). The same happens by considering the walker located on a vertex a in X. In a tessellation, each tile establishes a neighborhood around which the walker can move under the action of the associated local evolution operator. If we apply the evolution operation associated with the labels of the edges of $A[K_a]$, the walker hops to the vertices ab and ac of Y, and if we apply the evolution operation associated with the label of an edge of X incident to a, the walker hops to the vertices b and c of X. Consider the walker located on an edge ab of G. If we apply the operator associated with the color of a (or b), the walker hops to a (or b) and to the edges incident to it. The same happens by considering the walker located on a vertex ab in Y. If we apply the operator associated with the labels of the edges of $A[K_a]$ (or $A[K_b]$), the walker hops to vertices of K_a (or K_b). Otherwise, the walker stays put in both G and A.

2.2. Application to frequency assignment problems

Frequency assignment problems [1] constitute another practical application of total tessellations unrelated to quantum theory. Assume we have a group of people that requires private and shared types of communications. Each person has subgroups of contacts with whom the person is required to communicate. Moreover, each person in a subgroup may communicate with either one person in the subgroup using a private frequency or all people in the subgroup using a shared frequency.

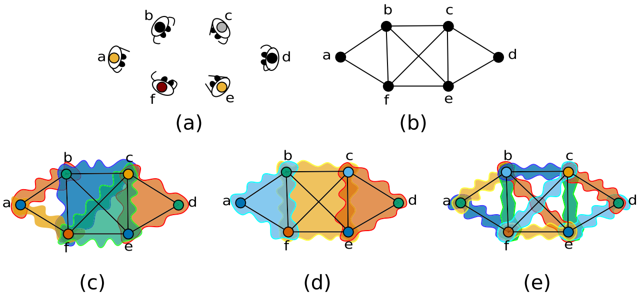

Fig. 5. Three total tessellation covers and their associated frequency assignments. (a) a group of six people named a to f; (b) the associated communication graph G that fulfills their demands; (c) a 4-total tessellation cover given by $T_1 = \{\{a\}, \{e\}\{b, c, f\}\}$, $T_2 = \{\{a, f\}, \{c\}\}$, $T_3 = \{\{c, e, f\}, \{d\}, \{b\}\}$, and $T_4 = \{\{a, b\}, \{f\}, \{c, d, e\}\}$; (d) a 5-total tessellation cover given by $T_1 = \{\{a\}, \{e\}\}$, $T_2 = \{\{b, c, e, f\}\}$, $T_3 = \{\{b\}, \{d\}\}$, $T_4 = \{\{c, e, d\}, \{f\}\}$, $T_5 = \{\{a, b, f\}, \{c\}\}$ and; (e) a 5-total tessellation cover given by $T_1 = \{\{e\}, \{a, f\}, \{b, c\}\}$. $T_2 = \{\{c\}, \{a, b\}, \{e, f\}\}$, $T_3 = \{\{a\}, \{d\}, \{b, f\}, \{c, e\}\}$, $T_4 = \{\{f\}, \{b, e\}, \{c, d\}\}$, and $T_5 = \{\{b\}, \{c, f\}, \{e, d\}\}$.

Table 1
Computational complexity of the problems associated to the parameters $\chi'(G)$, $\chi_t(G)$, $T(G)$, and $T_t(G)$. The only unknown class example is the open problem, in order to fully achieve the $\mathcal{P}$ *vs* $\mathcal{NP}$-complete dichotomy. The $[2|V(G)|, G^c]$ graphs are discussed in Section 4 (Construction 1).

	$\chi'(G)$	$T(G)$
$[2\|V(G)\|, G^c]$	$\mathcal{P}$	$\mathcal{NP}$c
Line graph of Bipartite	$\mathcal{NP}$c	$\mathcal{P}$

	$T(G)$	$\chi_t(G)$
Bipartite	$\mathcal{P}$	$\mathcal{NP}$c
$[2\|V(G)\|, G^c]$	$\mathcal{NP}$c	$\mathcal{P}$

	$\chi'(G)$	$\chi_t(G)$
$G \cup K_{\Delta(G)+1}$ Δ even	$\mathcal{P}$	$\mathcal{NP}$c
$G \cup K_{\Delta(G)+1}$ Δ odd	$\mathcal{NP}$c	$\mathcal{P}$

	$T(G)$	$T_t(G)$
Bipartite	$\mathcal{P}$	$\mathcal{NP}$c
$G \cup K_{3\Delta(G)}$	$\mathcal{NP}$c	$\mathcal{P}$

	$\chi'(G)$	$T_t(G)$
$[2\|V(G)\|, G^c]$	$\mathcal{P}$	$\mathcal{NP}$c
Line graph of Bipartite $\omega(G) \geq 6$	$\mathcal{NP}$c	$\mathcal{P}$

	$\chi_t(G)$	$T_t(G)$
$G \cup K_{\Delta(G)+1}$ Δ odd	$\mathcal{P}$	$\mathcal{NP}$c
Unknown (Open)	$\mathcal{NP}$c	$\mathcal{P}$

This model has some restrictions. First, to avoid interference we do not allow two people that need to communicate (i.e., there is an edge between them in the graph) to use the same private frequency. Second, we do not allow a private frequency to be equal to a shared frequency, or the person cannot identify whether the received message is personal or shared. Third, we do not allow a person to have the same shared frequency in two different subgroups, or the person cannot select for which subgroup to send the specific shared information.

Fig. 5(a) depicts a group of six people named a to f. In Fig. 5(b) we have the associated communication graph G that fulfills their demands. In Fig. 5(c) we describe a 4-total tessellation cover of G, which in this case represents that the group of people $\{b, c, f\}$ shares the same blue frequency, and they can communicate among themselves. Moreover, anyone in this group can send a private information to person f using the red frequency. Note that we may assign these frequencies in different ways. In Fig. 5(d) it is needed a 5-total tessellation cover for G by choosing each maximal clique as a tile. In Fig. 5(e) we describe a 5-total tessellation cover for G which corresponds to a 5-total coloring of G. The frequency assignments of Fig. 5(d) and Fig. 5(e) are 5-total tessellation covers, hence, they both require five different frequencies, whereas the 4-total tessellation cover of Fig. 5(c) requires less different frequencies. Therefore, the total tessellation cover problem provides a minimum spectrum of required frequencies for this situation.

2.3. Application to the complexity of graph coloring problems

A theoretical motivation is the study of the behavior of the computational complexity of different graph theory problems. In Section 4, we establish efficient solutions and hardness results of the total tessellation cover problem for different graph classes. Such results allow us to compare the computational complexity of k-TOTAL TESSELLABILITY with the other known related problems: k-COLORABILITY, k-EDGE COLORABILITY, and k-TOTAL COLORABILITY. For instance, these four problems are in $\mathcal{P}$ when restricted to complete graphs, star graphs and trees, whereas for triangle-free graphs, the four problems are $\mathcal{NP}$-complete. Table 1 summarizes the obtained results about the target parameter $T_t(G)$, by giving examples of graph classes for which the computational complexity of these four problems disagree.

3. Bounds on $T_t(G)$

An important issue when we address a new parameter in graph theory is to establish bounds on graph invariants or other parameters. For instance, we have the well-known Brooks' Theorem, which states that $\chi(G) \leq \Delta(G)$ when G is not an odd cycle nor a complete graph; the Vizing's Theorem, which states that $\chi'(G) \leq \Delta + 1$; and the Behzad's Conjecture, which states that $\chi_T(G) \leq \Delta(G) + 2$.

Observe that a total coloring defines a particular total tessellation cover, where each color of the total coloring corresponds to a label of the total tessellation cover so that no two same-colored vertices are adjacent, nor two same-colored edges are adjacent, and nor same-colored edge and vertex are incident to each other. Thus,

$$T_t(G) \leq \chi_t(G). \tag{1}$$

For triangle-free graphs, a tile on a tessellation contains at most one edge. Therefore, for this graph class, any total tessellation cover with $T_t(G)$ tessellations induces a total coloring with $T_t(G) = \chi_t(G)$ colors. Hence, considering that $(\Delta + 1)$-TOTAL COLORABILITY is $\mathcal{NP}$-complete for bipartite graphs [12], $(\Delta+1)$-TOTAL TESSELLABILITY is also hard even when restricted to bipartite graphs.

The labels of the vertices (resp. the edges) of a total tessellation cover also induce a coloring (resp. a tessellation cover) of the graph. Furthermore, we obtain a total tessellation cover of a graph by using the same $\chi(G)$ labels of a coloring to its vertices and another $T(G)$ different labels of a tessellation cover to its edges. Thus,

$$\max\{\chi(G), T(G)\} \leq T_t(G) \leq \chi(G) + T(G). \tag{2}$$

The key idea of Lemma 3.1 followed by its extension is to reduce the upper bound of Eq. (2) by using vertex colors as labels for the tessellations. We claim that for each three colors in $\{1, \ldots, \chi(G)\}$ we can reduce the tessellation labels in $\{\chi(G) + 1, \ldots, \chi(G) + T(G)\}$, and therefore the upper bound of Eq. (2), by one unit.

For instance, if $\chi(G) = 3$, the replacement is made by relabeling as tessellation 1 all edges of tessellation $\chi(G) + T(G)$ with endpoint vertices with colors different from 1, as tessellation 2 all edges of tessellation $\chi(G) + T(G)$ with endpoint vertices with colors different from 2, and as tessellation 3 all edges of tessellation $\chi(G)+T(G)$ with endpoint vertices with colors different from 3. Note that each edge of tessellation $\chi(G)+T(G)$ appears in at least one of these three tessellations, since an edge has two endpoints.

Now we establish a relation between the chromatic number $\chi(G)$ of a graph and its total tessellation cover number $T_t(G)$. Such relation is important for several hardness proofs in Section 4.

Lemma 3.1. *If $\chi(G) \geq 3T(G)$, then $T_t(G) = \chi(G)$.*

Proof. Let f be a proper vertex coloring and $\mathcal{C} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{T(G)}\}$ be a $T(G)$-tessellation cover for G. We define $\mathcal{C}'$ a tessellation cover for G with $3T(G)$ labels such that $\mathcal{C}'$ is compatible with f as follows. Each tessellation $\mathcal{T}'_i$, $1 \leq i \leq 3T(G)$, of $\mathcal{C}'$ is associated with a color i. Since $\chi(G) \geq 3T(G)$, there are enough colors. The edges of tessellations $\mathcal{T}'_{3j-2}$, $\mathcal{T}'_{3j-1}$, and $\mathcal{T}'_{3j}$ are given by the edges of the tessellation $\mathcal{T}_j$, $1 \leq j \leq T(G)$, such that $\mathcal{T}'_{3j-2}$ (resp. $\mathcal{T}'_{3j-1}$, $\mathcal{T}'_{3j}$) consists of the edges of $\mathcal{T}_j$ that do not have an endpoint with color $3j - 2$ (resp. $3j - 1$, $3j$). Note that the tiles of tessellation $\mathcal{T}'_{3j-2}$ of $\mathcal{C}'$ (resp. $\mathcal{T}'_{3j-1}$ and $\mathcal{T}'_{3j}$) are obtained by removing edges incident to vertices with color $3j-2$ (resp. $3j-1$ and $3j$) from the tiles of tessellation $\mathcal{T}_j$ of $\mathcal{C}$. Since an edge has two endpoints, any edge in a tessellation $\mathcal{T}_j$ of the tessellation cover $\mathcal{C}$ appears in at least one tessellation $\mathcal{T}'_{3j-2}$, $\mathcal{T}'_{3j-1}$ or $\mathcal{T}'_{3j}$ of $\mathcal{C}'$ i.e., $\mathcal{C}'$ is a tessellation cover of G. □

Lemma 3.1 can be extended to any graph with $\chi(G) < 3T(G)$ to reduce the upper bound $T_t(G) \leq \chi(G)+T(G)$ of Eq. (2). Indeed, for each three colors in $\{1, \ldots, \chi(G)\}$, we may reduce by one unit the upper bound $\chi(G)+T(G)$ of $T_t(G)$. Therefore, such upper bound is given by $T_t(G) \leq \chi(G) + T(G) - \lfloor \frac{\chi(G)}{3} \rfloor = T(G) + \lceil \frac{2\chi(G)}{3} \rceil$. Hence, using an argument similar to the one used in the proof of Lemma 3.1, we can rewrite the upper bound of Eq. (2) as follows

$$T_t(G) \leq \max\{\chi(G), T(G) + \lceil 2\chi(G)/3 \rceil\}. \tag{3}$$

We now describe some special situations regarding Eqs. (2), (3), and Lemma 3.1. If $\chi(G) \geq 3T(G)$, then $T_t(G) = \chi(G)$. Otherwise, $\chi(G) \leq 3T(G)$ and $T(G) \leq T_t(G) \leq 3T(G)$. Fig. 6 depicts an example of a graph G with $\chi(G) < 3T(G)$ and $T_t(G) > \chi(G)$. Particularly, when $\chi(G) = 3$, we have $T(G) \leq T_t(G) \leq T(G) + 2$.

Note that $T_t(G) = \chi(G) + T(G)$ requires $\chi(G) < 2$ (i.e., G must be a bipartite graph). On the other hand, if G is a bipartite graph, then $T_t(G) = \chi_T(G)$ because G is triangle-free, and $T_t(G)$ is either $T_t(G) = \chi(G) + T(G) = \Delta(G) + 2$ or $T_t(G) = \chi(G) + T(G) - 1 = \Delta(G) + 1$.

We know that $T(G) \leq \Delta(G)$ and, by Brooks' Theorem, we know that $\chi(G) \leq \Delta(G)$ (except odd cycles or complete graphs). Using these two pieces of information in the upper bound of Eq. (3), we obtain $T_t(G) \leq \frac{5\Delta(G)+3}{3}$. Particularly, when $\Delta(G) = 3$, we have $T(G) \leq T_t(G) \leq 6$.

In Section 4 we present the definition of Good tessellable graph, for which the following Lemma 3.2 is important.

Lemma 3.2. $T_t(G) \geq \max\limits_{v \in V(G)} \{\chi(G^c[N(v)])\} + 1 \geq \max\limits_{v \in V(G)} \{\omega(G^c[N(v)])\} + 1 = s(G) + 1.$

Proof. Consider a total tessellation cover of a graph G, a vertex v of G, and $G^c[N(v)]$, which is the complement graph of the graph induced by the neighborhood of v. In any tessellation, the endpoints of the edges that are incident to v and belong to the tessellation induce a clique, hence the vertices of this clique are an independent set in $G^c[N(v)]$. Therefore, the

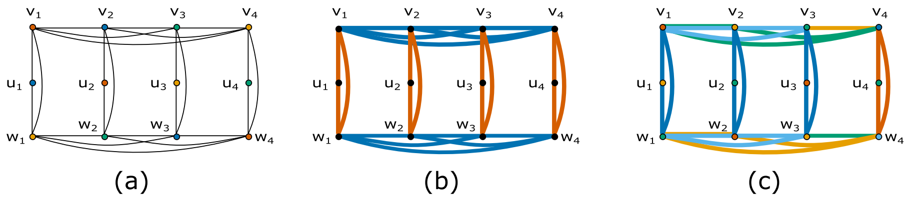

Fig. 6. A graph G with $\chi(G) < 3T(G)$ and $T_t(G) > \chi(G)$. (a) a 4-coloring giving by color classes $c_1 = \{v_1, u_2, w_4\}$, $c_2 = \{u_1, v_2, w_3\}$, $c_3 = \{w_2, v_3, u_4\}$, and $c_4 = \{w_1, u_3, v_4\}$; (b) a 2-tessellation cover with tessellations $T_1 = \{\{v_1, v_2, v_3, v_4\}, \{w_1, w_2, w_3, w_4\}\}$ and $T_2 = \{\{v_1, u_1, w_1\}, \{v_2, u_2, w_2\}, \{v_3, u_3, w_3\}, \{v_4, u_4, w_4\}\}$; and (c) a 5-total tessellation cover with tessellations $T_1 = \{\{v_1, u_1, w_1\}, \{v_2, u_2, w_2\}, \{v_3, u_3, w_3\}, \{v_4\}\}$, $T_2 = \{\{v_3, v_4\}, \{w_1, w_2, w_4\}, \{u_1\}, \{v_2\}, \{w_3\}\}$, $T_3 = \{\{v_1, v_2, v_4\}, \{w_3, w_4\}, \{u_2\}, \{v_3\}, \{u_4\}, \{w_1\}\}$, $T_4 = \{\{v_1\}, \{w_2\}, \{u_3\}, \{v_4, u_4, w_4\}\}$, and $T_5 = \{\{v_1, v_2, v_3\}, \{w_1, w_2, w_3\}, \{w_4\}\}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

tessellations with edges incident to a vertex v induce a vertex coloring of $G^c[N(v)]$, and the number of these tessellations is at least $\chi(G^c[N(v)])$. Moreover, these tessellations have labels that are different from the color of vertex v. Therefore, $T_t(G) \geq \chi(G^c[N(v)]) + 1$. Note that $s(G[N[v]]) = \alpha(G[N(v)]) = \omega(G^c[N(v)])$ and $s(G) = \max_{v \in V(G)} s(G[N[v]])$. $\square$

Graphs with $T_t(G) = T(G) = k$ have no induced subgraph $K_{1,k}$ because $T_t(G) \geq s(G) + 1 \geq k + 1$. Moreover, there is no tile of size k in any tessellation of a total tessellation cover. If $T_t(G) = T(G) = 3$, then G is $K_{1,3}$-free and there is no clique of size three in any tessellation. Therefore, the total tessellation cover of G induces a total coloring of G, and the only graphs for which $T_t(G) = T(G) = 3$ are the odd cycles with n vertices such that $n \equiv 0 \mod 3$. For bipartite graphs, $T(G) = \Delta(G)$ and $T_t(G) > T(G)$. For triangle-free graphs, $T_t(G) = T(G)$ if $\chi'(G) = \chi_t(G) = \Delta + 1$. It follows that deciding whether $T_t(G) = T(G) = \Delta(G) + 1$ is $\mathcal{NP}$-complete from the proof that $(\Delta + 1)$-TOTAL COLORABILITY is $\mathcal{NP}$-complete for triangle-free snarks [18], which are graphs with $\chi'(G) = \Delta + 1$.

4. Good and hard total tessellable graphs

Since the concept of good tessellable graphs introduced in [3] has provided keen insights into the hardness of finding minimum-sized tessellation covers, we define the concept of good total tessellable graphs in order to further explore hardness results related to total tessellation covers. In the quantum computation context, we are interested in graph classes which use as few color labels as possible so that the number of time-evolution operations for total tessellation-based quantum walks is as low as possible. Therefore, we need $T_t(G)$ to be close to the lower bounds obtained in Section 3.

Definition 3. A graph G is *good total tessellable* of *Type I* if $T_t(G) = \omega(G)$, and of *Type II* if $T_t(G) = s(G)+1$ and $T_t(G) \neq \omega(G)$.

Now we show that k-TOTAL TESSELLABILITY is in $\mathcal{P}$ if we know beforehand that the graph is good total tessellable Type I or Type II. The *Lovász number* $\vartheta(G)$ is a real number such that $\omega(G^c) \leq \vartheta(G) \leq \chi(G^c)$ [9]. We denote $\psi(G)$ the integer nearest to $\vartheta(G)$. The value of $\psi(G)$ can be determined in polynomial time [9].

For Type I graphs, $T_t(G) = \omega(G)$. Since Eq. (2) implies that $\omega(G) \leq \chi(G) \leq T_t(G)$, we have $\omega(G) = \chi(G) = T_t(G) = \psi(G^c)$.

For Type II graphs, $T_t(G) = s(G) + 1$. For any vertex $v \in V(G)$, $\omega(G^c[N(v)]) \leq \psi(G[N(v)]) \leq \chi(G^c[N(v)])$, and by Lemma 3.2, $T_t(G) \geq \psi(G[N(v)]) + 1$. Since $T_t(G) = s(G) + 1$, by Lemma 3.2 there is a vertex $u \in V(G)$ such that $T_t(G) = \omega(G^c[N(u)])+1$. In this case, $\omega(G^c[N(u)])+1 = \chi(G^c[N(u)])+1$, and we determine $\omega(G^c[N(u)])$ using $\psi(G^c[N(u)])$. Therefore, $T_t(G) = \max_{v \in V(G)} \{\psi(G[N(v)])\}+1$.

The same method used to determine $T_t(G)$ for Type II graphs can be applied for good tessellable graphs in order to determine $T(G)$, where $T(G) = \max_{v \in V(G)} \{\psi(G[N(v)])\}$.

As presented in Section 3, $(\Delta+1)$-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete for bipartite graphs, which have $s(G)+1 = \Delta+1$ and $\omega(G) = 2$. Now, we show that k-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete for the following cases: line graph of triangle-free graphs with $k = \omega(G) \geq 9$ and $s(G) + 1 = 3$; universal graphs with k very far apart from both $s(G) + 1$ and $\omega(G)$; $(2, 1)$-chordal graphs with $k = s(G) + 1 = \omega(G) + 3$; and planar graphs with $k = 4 = \omega(G) = s(G) + 1$.

4.1. Line graph of triangle-free graphs

Machado et al. [11] proved that k-EDGE COLORABILITY is $\mathcal{NP}$-complete for 3-colorable k-regular triangle-free graphs if $k \geq 3$.

The key idea of the proof of Theorem 4.1 is to verify that $T_t(L(G)) = \chi'(G)$, for k-regular triangle-free graphs with $k \geq 9$.

Theorem 4.1. *k-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete for line graphs $L(G)$ of 3-colorable k-regular triangle-free graphs G for any $k \geq 9$.*

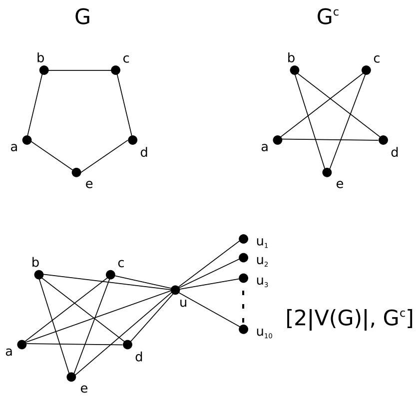

Fig. 7. $[10, C_5^c]$ is the graph obtained from Construction 1.

Proof. The edges incident to any vertex v of graph G correspond to a clique of $L(G)$, whose size is the degree of v. If two vertices of G are non-adjacent, then the corresponding cliques in $L(G)$ share no vertices. Hence, we cover the edges of the cliques of $L(G)$ incident to the vertices of each of the three color classes of the 3-coloring of G with a tessellation related to the color class because these cliques share no vertices. Therefore, since $T(L(G)) = 3$ and $\chi(L(G)) \geq 9 \geq 3T(L(G))$, by Lemma 3.1, $T_t(L(G)) = \chi(L(G)) = \chi'(G)$. Note that in this case $k = \omega(L(G))$ and $s(L(G)) + 1 = 3$. □

4.2. Universal graphs

Abreu et al. [3] established the $\mathcal{NP}$-completeness of k-TESSELLABILITY for universal graphs by relating this problem to q-INDEPENDENT SET. We present a similar argument to establish the $\mathcal{NP}$-completeness of k-TOTAL TESSELLABILITY for universal graphs. Let G be an instance of q-COLORABILITY. In Construction 1, we construct from a graph G, the graph $[2|V(G)|, G^c]$ as follows.

Construction 1. *Let $[2|V(G)|, G^c]$ be the graph obtained from a graph G in the following way:*

- *Obtain G^c, the complementary graph of G;*
- *Obtain a graph H by adding a vertex u in G^c such that $H = G^c \vee \{u\}$;*
- *Add $2|V(G)|$ pendant vertices to H adjacent to u.*

Fig. 7 depicts an example of Construction 1, when $G = C_5$.

Theorem 4.2. k-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for universal graphs.*

Proof. Let $[2|V(G)|, G^c]$ be a graph of Construction 1. The total tessellation cover number of the constructed graph $[2|V(G)|, G^c]$ is given by $2|V(G)| + \chi(G) + 1$. It is obtained using: labels $1, \ldots, \chi(G)$ to cover the edges incident to u that belong to the subgraph induced by $V(G^c \cup \{u\})$; labels $\chi(G) + 1, \ldots, \chi(G) + 2|V(G)|$ to cover the edges incident to the pendant vertices; and labels $\chi(G) + 1, \ldots, \chi(G) + |V(G)|$ are enough to cover the edges of G^c. Assign to u the color $2|V(G)|+\chi(G)+1$, to the pendant vertices color 1, and to the remaining vertices colors $\chi(G)+|V(G)|+1, \ldots, \chi(G)+2|V(G)|$. The minimality follows from Lemma 3.2. Therefore, $T_t([2|V(G)|, G^c]) = 2|V(G)| + \chi(G) + 1$, and G is q-colorable if and only if $[2|V(G)|, G^c]$ is k-total tessellable, with $k = 2|V(G)| + \chi(G) + 1$. □

Note that $s(C_5 \vee \{u\}) = 2$, $T_t(C_5 \vee \{u\}) = 4$, and any minimum total tessellation cover of $C_5 \vee \{u\}$ has at least three labels assigned to the edges incident to u and a fourth label assigned to u. Thus, $T_t([2|V(G)|, G^c \cup C_5]) = T_t([2|V(G)|, G^c]) + 3$; $s([2|V(G)|, G^c \cup C_5]) = s([2|V(G)|, G^c]) + 2$; and $\omega([2|V(G)|, G^c \cup C_5]) = \omega([2|V(G)|, G^c])$. Therefore, each addition of a C_5 increases the gap between the total tessellation cover number and both the sizes of a maximum induced star and a maximum clique.

As long as the number of the C_5's is polynomially bounded by the size of G, k-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete even if k is far apart from $s(G)$ and $\omega(G)$.

4.3. (2, 1)-Chordal graphs

Recall that a graph G is (2, 1) if its vertex set can be partitioned into two independent sets and one clique. Hence, a graph G is a (2, 1)-chordal graph if it is a (2, 1)-graph and at the same time G is chordal. Since 3-EDGE COLORABILITY is $\mathcal{NP}$-complete for 3-regular graphs [11], 3-VERTEX COLORABILITY is also $\mathcal{NP}$-complete for 4-regular line graphs. Let G be a 4-regular line graph. We construct a (2, 1)-chordal graph H from G as follows.

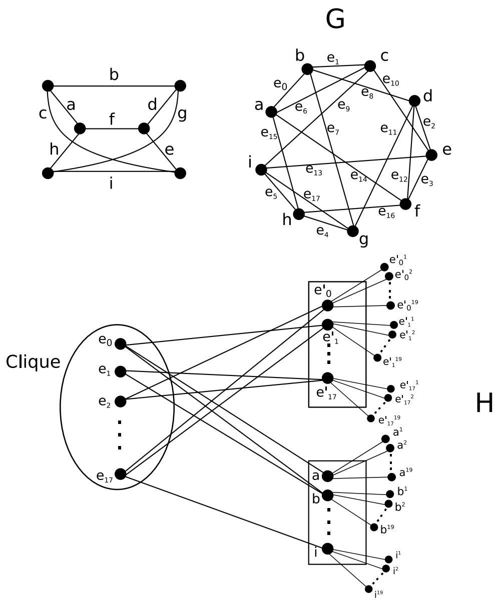

Fig. 8. Top: G is the 4-regular line graph of the leftmost one. Bottom: the (2, 1)-chordal graph H is obtained from G by Construction 2.

Construction 2. *Let graph H be obtained from a 4-regular line graph G so that:*

- *$V(H)$ contains a clique $\{e_0, \ldots, e_{|E(G)|-1}\}$ where each e_i, $0 \le i \le |E(G)| - 1$, is associated with a distinct edge of G.*
- *$V(H)$ contains an independent set $\{e'_0, \ldots, e'_{|E(G)|-1}\}$ such that each e'_i is adjacent to all e_j with $j \neq i$ and $j \neq i + 1 \mod |E(G)|$.*
- *$V(H)$ contains an independent set $\{v_0, \ldots, v_{|V(G)|-1}\}$, where each v_i, $0 \le i \le |V(G)| - 1$, is associated with a distinct vertex of G.*
- *Each $e_j \in \{e_0, \ldots, e_{|E(G)|-1}\}$ is adjacent to vertices $v_r, v_s \in \{v_0, \ldots, v_{|V(G)|-1}\}$ such that $e_j = v_r v_s$.*
- *$V(H)$ contains an independent set P comprising $(|V(G)| + |E(G)|)(|E(G)| + 1)$ pendant vertices such that each vertex of $\{v_0, \ldots, v_{|V(G)|-1}\} \cup \{e'_0, \ldots, e'_{|E(G)|-1}\}$ is adjacent to $|E(G)| + 1$ pendant vertices.*

Fig. 8 depicts an example of Construction 2.

Theorem 4.3. *k-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete for chordal graphs.*

Proof. We claim that $T_t(H) = |E(G)| + 3$ if and only if $\chi(G) = 3$. Consider a 3-coloring c of G. Obtain a k-total tessellation cover of H with $k = |E(G)| + 3$ as follows. Assign colors in $\{1, \ldots, |E(G)|\}$ to the vertices of the clique $\{e_0, \ldots, e_{|E(G)|-1}\}$. Assign to vertex e'_i, for $1 \le i \le |E(G)|$, the same color of the vertex e_i. For $0 \le i \le |E(G)| - 1$, the tile with vertices $\{e'_i\} \cup \{e_j \mid j \neq i \text{ and } j \neq i + 1 \mod |E(G)|\}$ is in the tessellation with label $i + 2 \mod |E(G)|$. Note that if two vertices v_i and v_k of G are not adjacent, then the cliques $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$ and $\{v_k\} \cup \{e_j \mid v_k \text{ is endpoint of } e_j \text{in } G\}$ are disjoint. Thus, the tile with vertices $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$ is in the tessellation with label $c(v_i) + |E(G)|$. Finally, greedily assign colors and labels to the remaining vertices and edges of H. Consider a total tessellation cover of H with $k = |E(G)| + 3$ labels. Note that we require $|E(G)|$ tessellations to cover the edges between the vertices $\{e_0, \ldots, e_{|E(G)|-1}\} \cup \{e'_0, \ldots, e'_{|E(G)|-1}\}$ in any total tessellation cover of H. Moreover, a tile in each of those $|E(G)|$ tessellations contains $|E(G)| - 2$ vertices of the clique $\{e_0, \ldots, e_{|E(G)|-1}\}$. Since each tile $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$, for $0 \le i \le |V(G)| - 1$, contains four vertices of the clique $\{e_0, \ldots, e_{|E(G)|-1}\}$, there are only three tessellation labels used by the tiles $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$, for $0 \le i \le |V(G)| - 1$. Moreover, if two vertices v_i and v_k are adjacent in G, then the tiles $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$ and $\{v_k\} \cup \{e_j \mid v_k \text{ is endpoint of } e_j \text{ in } G\}$ share a vertex $e_j = v_i v_k$ in H and they are tiles belonging to different tessellations. Hence, we obtain a 3-coloring c of G as follows. Assign the label of the tile $\{v_i\} \cup \{e_j \mid v_i \text{ is endpoint of } e_j \text{ in } G\}$ to the color of v_i in c.

Therefore, G has a 3-coloring if and only if H has a total tessellation cover with $|E(G)|+3$ labels. Note that $k = s(H)+1 = \omega(H)+3 = |E(G)|+3$. $\square$

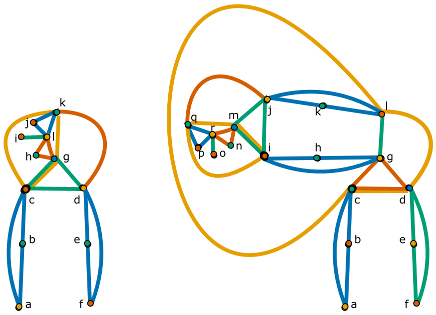

Fig. 9. Equal Gadget (left graph): A 4-total tessellation cover given by $T_1 = \{\{a, b, c\}, \{d, e, f\}, \{k, l, j\}, \{g\}\}$, $T_2 = \{\{c, g, k\}, \{l\}, \{a\}, \{d\}\}$, $T_3 = \{\{c, d, g\}, \{j, l\}, \{b\}, \{e\}, \{h\}, \{k\}\}$, $T_4 = \{\{l, g, h\}, \{d, k\}, \{f\}, \{c\}, \{i\}, \{j\}\}$. In any 4-total tessellation cover, the edges of the two external triangles belong to tiles in a same tessellation (blue, T_1). NotEqual Gadget (right graph): A 4-total tessellation cover given by $T_1 = \{\{a, b, c\}, \{g, h, i\}, \{l, k, j\}, \{p, q, r\}, \{d\}, \{m\}\}$, $T_2 = \{\{c, d, l\}, \{i, m, q\}, \{a\}, \{e\}, \{g\}, \{r\}, \{j\}\}$, $T_3 = \{\{d, e, f\}, \{i, j, m\}, \{l, g\}, \{r, o\}, \{h\}, \{k\}, \{n\}, \{q\}\}$, $T_4 = \{\{c, d, g\}, \{m, n, r\}, \{j, q\}, \{b\}, \{f\}, \{l\}, \{i\}, \{o\}, \{p\}\}$. The edges of the two external triangles always belong to tiles in different tessellations (blue and green, T_1 and T_3, resp). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

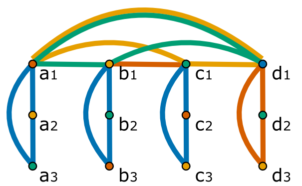

Fig. 10. Graph G of *Lemma 4.1* and a 4-total tessellation cover given by $T_1 = \{\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}, \{c_1, c_2, c_3\}, \{d_1\}\}$, $T_2 = \{\{a_1, c_1, d_1\}, \{a_2\}, \{b_1\}, \{c_3\}, \{d_3\}\}$, $T_3 = \{\{a_1, b_1, d_1\}, \{a_3\}, \{b_2\}, \{c_1\}, \{d_2\}\}$, $T_4 = \{\{d_1, d_2, d_3\}, \{b_1, c_1\}, \{a_1\}, \{b_3\}, \{c_2\}\}$. In any 4-total tessellation cover of G, the edges of three triangles have the same color (blue, T_1) and the edges of the fourth triangle have a different color (orange, T_4). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4.4. Planar graphs

We show in this subsection that the 4-TOTAL TESSELLABILITY problem for planar graphs is $\mathcal{NP}$-complete in stark contrast to the fact that the computational complexity of the TOTAL COLORING problem for planar graphs is still open. We also show that deciding whether a graph has both $T_t(H) = s(H) + 1$ and $T_t(H) = \omega(H)$ is $\mathcal{NP}$-complete even for planar graphs.

To prove the main theorem (Theorem 4.4), we need four graph gadgets, which are described sequentially. We start with the two gadgets of Fig. 9 (called Equal Gadget and NotEqual Gadget), which are constructed using the graph G of Lemma 4.1.

Lemma 4.1. *Any 4-total tessellation cover of the graph G of Fig. 10 has the following property: The edges of three triangles belong to tiles in a same tessellation and the edges of the remaining triangle belong to a tile in a different tessellation.*

Proof. The proof follows after analyzing all possibilities of total tessellation covers with four labels. □

In any 4-total tessellation cover of the Equal Gadget, the edges of the two external triangles always belong to tiles in a same tessellation (blue in the left graph of Fig. 9) because the two internal triangles (red and blue) share a common vertex (yellow) and Lemma 4.1 demands that exactly three triangles are in a same tessellation. In any 4-total tessellation cover of the NotEqual Gadget, the edges of the two external triangles always belong to tiles in different tessellations (blue and green in the right graph of Fig. 9) because the two external triangles of the internal Equal Gadget are tiles in a same tessellation (blue in Fig. 9).

Now we introduce the $2d$-Duplicator Gadget, which is a circular pattern of $2d$ Equal Gadgets with the addition of $2d$ extra Equal Gadgets, as depicted in Fig. 11 for the case $d = 2$. In the 4-total tessellation cover described in Fig. 11, we consider the colors blue, yellow, green, and orange as color classes 1, 2, 3, and 4, resp. The edges of the external triangles of the extra Equal Gadgets have color class 1 (blue), and the degree-2 vertices of the external triangles have the color classes 2 and 3 (yellow and green). We need only two color classes for the degree-2 vertices because $2d$ is even. By rotating the color classes modulo 4, we obtain total tessellation covers that have different color classes for the degree-2 vertices and edges of the external triangles. For instance, if the external triangles are tiles in tessellation with color class 2, then the degree-2 vertices have color classes 3 and 4, and so on. We use the rotating method in the proof of the main theorem.

Now we introduce the Shifter Gadget, which admits only two independent 4-total tessellation covers depending on the colors of two specific vertices, a and d. Fig. 12(a) depicts a 4-total tessellation cover of the Shifter Gadget when a and

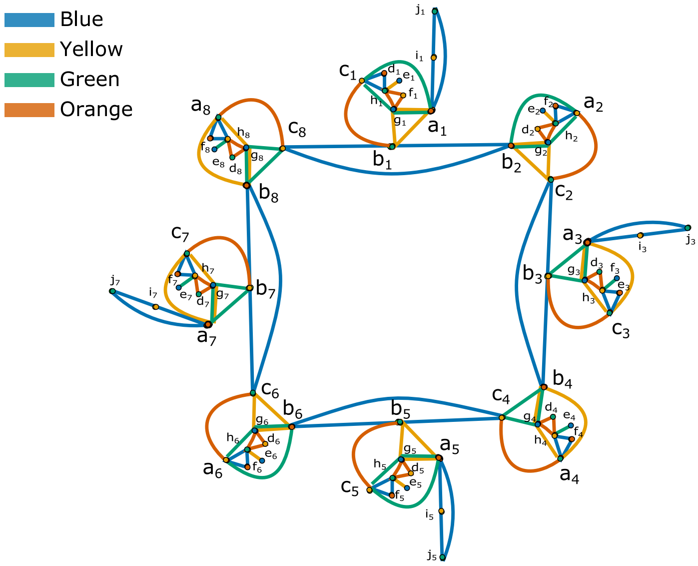

Fig. 11. A 4-total tessellation cover given by $T_1 = \{\{a_1, i_1, j_1\}, \{c_1, d_1, h_1\}\}, \{e_1\}, \{g_1\}, \{a_2, f_2, h_2\}, \{e_2\}, \{g_2\}, \{a_3, i_3, j_3\}, \{c_e, e_3, h_3\}, \{f_3\}, \{g_3\}, \{a_4, f_4, h_4\}, \{e_4\}, \{g_4\}, \{a_5, i_5, j_5\}, \{c_5, f_5, h_5\}, \{e_5\}, \{g_5\}, \{a_6, f_6, h_6\}, \{e_6\}, \{g_6\}, \{a_7, i_7, j_7\}, \{c_7, f_7, h_7\}, \{f_7\}, \{g_7\}, \{a_8, f_8, h_8\}, \{f_8\}, \{g_8\}, \{b_1, b_2, c_8\}, \{b_3, b_4, c_2\}, \{b_5, b_6, c_4\}, \{b_7, b_8, c_6\}$, $T_2 = \{\{a_1, b_1, g_1\}, \{e_1, h_1\}, \{c_1\}, \{f_1\}, \{i_1\}, \{b_2, c_2, g_2\}, \{e_2, h_2\}, \{a_2\}, \{d_2\}, \{a_3, c_3, g_3\}, \{h_3\}, \{i_3\}, \{a_4, b_4, g_4\}, \{a_4, g_4\}, \{c_4\}, \{h_4\}, \{a_5, b_5, g_5\}, \{e_5, h_5\}, \{c_5\}, \{d_5\}, \{i_5\}, \{b_6, c_6, g_6\}, \{e_6, h_6\}, \{a_6\}, \{d_6\}, \{a_7, c_7, g_7\}, \{b_7\}, \{h_7\}, \{i_7\}, \{a_8, b_8, g_8\}, \{c_8\}, \{h_8\}\}$, $T_3 = \{\{a_1, c_1, g_1\}, \{b_1\}, \{h_1\}, \{j_1\}, \{c_2\}, \{a_2, b_2, g_2\}, \{h_2\}, \{a_3, b_3, g_3\}, \{f_3, h_3\}, \{c_3, d_3, j_3\}, \{b_4, c_4, g_4\}, \{e_4, h_4\}, \{a_4\}, \{d_4\}, \{a_5, c_5, g_5\}, \{b_5\}, \{h_5\}, \{j_5\}, \{a_6, b_6, g_6\}, \{c_6\}, \{h_6\}, \{a_7, b_7, g_7\}, \{e_7, h_7\}, \{c_7\}, \{d_7\}, \{j_7\}, \{c_8, d_8, g_8\}, \{e_8, h_8\}, \{a_8\}, \{d_8\}\}$, $T_4 = \{\{f_1, h_1, g_1\}, \{b_1, c_1\}, \{a_1\}, \{d_1\}\{d_2, g_2, h_2\}, \{a_2, c_2\}, \{f_2\}, \{d_3, g_3, h_3\}, \{b_3, c_3\}, \{a_3\}, \{e_3\}, \{d_4, g_4, h_4\}, \{a_4, c_4\}, \{b_4\}, \{f_4\}, \{d_5, g_5, h_5\}, \{b_5, c_5\}, \{a_5\}, \{f_5\}, \{d_6, h_6, g_6\}, \{a_6, c_6\}, \{b_6\}, \{f_6\}, \{d_7, g_7, h_7\}, \{b_7, c_7\}, \{a_7\}, \{f_7\}, \{d_8, g_8, h_8\}, \{a_8, c_8\}, \{b_8\}, \{f_8\}\}$ for an example of a $2d$-Duplicator Gadget with $d = 2$. The tiles of its external triangles belong to a same tessellation (color class blue) and the two degree-2 vertices of each of these tiles have their color classes respectively yellow and green. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

d have the same color. Fig. 12(b) depicts the other case when the colors are different. Note that in both cases we may consistently permute the labels of the tessellations of the most external Equal Gadgets so that the most external vertices of the Shifter Gadget may have any color different from its tile's label.

Lemma 4.2. *Any 4-total tessellation cover of the Shifter Gadget has the following property: Triangles T_1 and T_4 are tiles in a same tessellation and triangles T_2 and T_3 are tiles in another tessellation.*

Proof. Consider any 4-total tessellation cover of the Shifter Gadget. The tiles incident to vertex b (resp. c and e) need three different labels as depicted in Fig. 12(a) and (b), and by Lemma 4.1 the Equal Gadget incident to vertices a and d has external triangles in a same tessellation.

Fig. 12(a) considers the case when vertices a and d have the same color. Tiles T_1 and $\{b, d, e\}$ have the same label because vertices a and d have the same color (yellow) and the tiles of the external triangles of the incident Equal Gadget have the same label (green). By symmetry, tiles T_2 and $\{a, b, c\}$ have the same label (orange). Note that the color of vertex b is different from the color of vertices a and d and from the labels of tiles T_1 and T_2. Therefore, vertex c and tile T_1 have the same label. By symmetry, e and T_2 have the same label. Tile $\{c, e, f\}$ and vertex f receive labels different from the labels of tiles T_1 and T_2. This implies that tiles T_1 and T_4 receive the same label. By symmetry, T_2 and T_3 have the same label, and the label of T_1 and T_4 is different from the label of T_2 and T_3. This completes the proof when vertices a and d have the same color because we have explicitly shown a 4-total tessellation cover in Fig. 12(a).

With a similar argument, the same result holds when vertices a and d have different colors and the proof is complete because we have explicitly shown a 4-total tessellation cover for the second case in Fig. 12(b). □

The proof of the main theorem uses a graph H constructed as follows.

Construction 3. *Let H be the graph obtained from an instance of* 3-COLORABILITY *of a planar graph G with degree at most four in the following way:*

- *Add a universal vertex u to G, i.e., obtain the graph $G \vee \{u\}$;*
- *Replace each vertex z of degree $d(z)$ of $G \vee \{u\}$ by a $2d$-Duplicator Gadget with $d = d(z)$ and, simultaneously, replace each edge vw of $G \vee \{u\}$ by a NotEqual Gadget merging its external triangles with the external triangles of the Duplicator Gadgets that replace v and w. Ignore d external triangles of each $2d$-Duplicator Gadget in the merging process.*

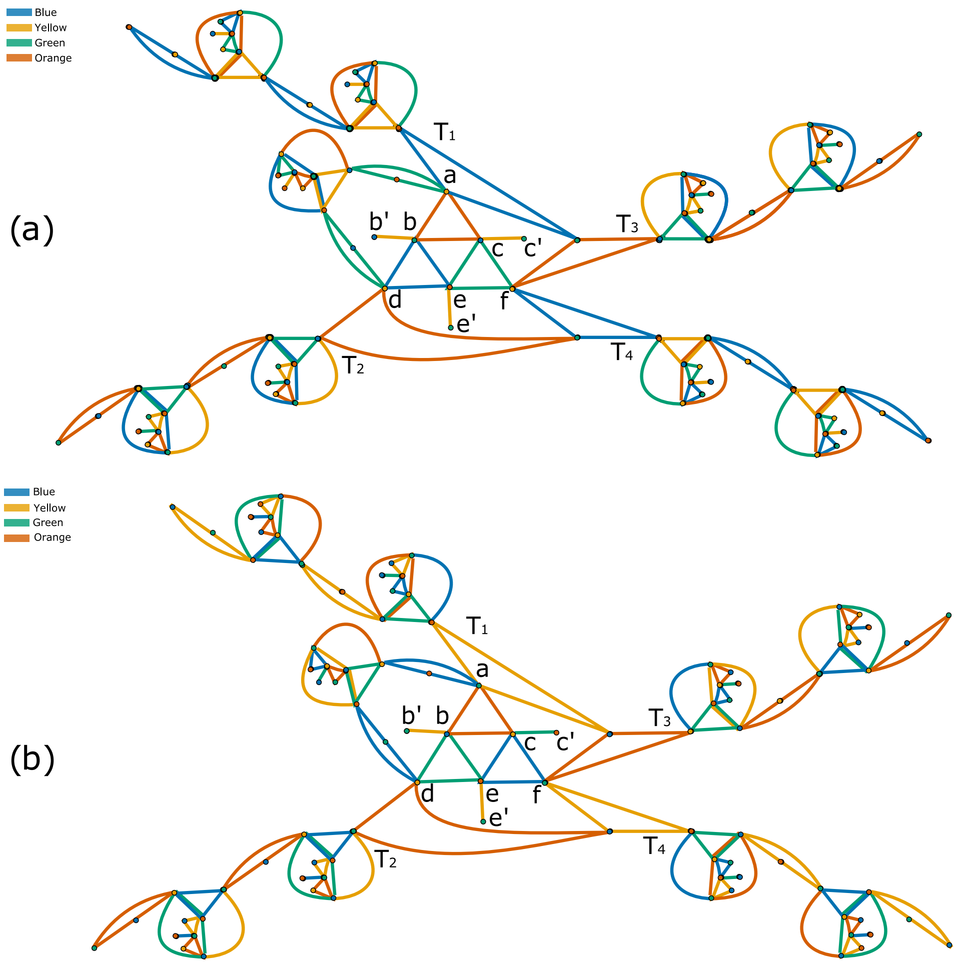

Fig. 12. Two independent 4-total tessellation covers of the Shifter Gadget: (a) when vertices a and d have the same color and (b) when they have different colors. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

- *Replace crossing triangles by Shifter Gadgets.*

In the first step of Construction 3, the only crossing edges of $G \vee \{u\}$ have u as one of their endpoints. In the second step, these crossing edges become crossing triangles. In the third step, when a pair of crossing triangles is removed, there are four free vertices that are used to dock a Shifter Gadget with compatible labels. The crossing triangles are replaced by Shifter Gadgets because the resulting graph H is planar.

Theorem 4.4. 4-TOTAL TESSELLABILITY *is $\mathcal{NP}$-complete for planar graphs.*

Proof. Let G be an instance of 3-COLORABILITY of planar graphs with degree at most four [8] and let H be a graph obtained in Construction 3. We have that $G \vee \{u\}$ has a 4-coloring if and only if G has a 3-coloring. We claim that the resulting planar graph H of Construction 3 has a 4-total tessellation cover if and only if G has a 3-coloring.

Consider a 4-total tessellation cover of H. If two vertices v and w are adjacent in $G \vee \{u\}$, then the external triangles of the $2d$-Duplicator Gadgets that replace v and w belong to different tessellations because they are connected by a NotEqual Gadget. Therefore, we obtain a 4-coloring of $G \vee \{u\}$ by assigning the color of a vertex as the label of the tile of the external triangles of the $2d$-Duplicator Gadget that replaces the vertex. Then, G has a 3-coloring.

Now suppose that G has a 3-coloring. We obtain a 4-total tessellation cover of H induced by this 3-coloring following the steps of Construction 3. Let f be a 4-coloring of $G \vee \{u\}$ induced by the 3-coloring of G. For each degree-d vertex v, assign label $f(v)$ to the external triangles of the $2d$-Duplicator Gadget of H and label the remaining vertices and tiles of the $2d$-Duplicator Gadget using the rotating method as described when we introduced this gadget.

Consider the external triangles of two $2d$-Duplicator Gadgets that will be merged with a NotEqual Gadget. The labels a and b of these two external triangles, A and B, are different because they correspond to adjacent vertices in the original graph. The labels of the degree-2 vertices of A (B) can be interchanged. Let c (d) be the label of the most external degree-2 vertex of A (B). If the cardinality of set $\{a, b, c, d\}$ is 4, we need to interchange the labels of the degree-2 vertices of A or B. When we select the labels so that the cardinality of set $\{a, b, c, d\}$ is at most 3, the 4-tessellation cover of the remaining part of the NotEqual Gadget is straightforward.

When we replace the crossing triangles by Shifter Gadgets in the third step of Construction 3, note that the labels of the crossing triangles are different because the color of the universal vertex, which is the label of one of these triangles, must be different from the colors of all other vertices of G. As described in Lemma 4.2 and Fig. 12, the extremal Equal Gadgets of a Shifter Gadget allow us to replicate the exact same colors of the four free vertices that appear when we delete a pair of crossing triangles, allowing a consistent docking procedure in terms of vertex and tile labels.

We have described a compatible 4-total tessellation cover of all gadgets, which is a 4-total tessellation cover of H induced by a 3-coloring of G. □

As a corollary of Theorem 4.4, to decide whether a graph has both $T_t(H) = s(H)+1$ and $T_t(H) = \omega(H)$ is $\mathcal{NP}$-complete for planar graphs because any graph H obtained in Construction 3 has $s(H) + 1 = \omega(H) = 4$.

A reader may wonder whether it is really necessary to use a non-planar graph $G \vee \{u\}$ in the proof of Theorem 4.4. This seems unavoidable because both 4-COLORABILITY for planar graphs and 3-TOTAL TESSELLABILITY are in $\mathcal{P}$. On the other hand, 3-COLORABILITY for planar graphs is $\mathcal{NP}$-complete.

5. Open problems

We leave as an open problem to search for graphs with at least 3 vertices satisfying $T_t(G) = 3T(G)$ and $T_t(G) > \chi(G)$. Moreover, it would be interesting to define graph classes with $T_t(G) = T(G) = k$ for $k \geq 4$, since for $k = 3$ the only such graphs are the odd cycles C_n with $n \equiv 0 \mod 3$.

We have shown that 4-TOTAL TESSELLABILITY is $\mathcal{NP}$-complete for planar graphs satisfying $s(G) + 1 = \omega(G) = 4$. This is important since the hardness of k-EDGE COLORABILITY and k-TOTAL COLORABILITY for planar graphs are still open. On the other hand, we know that planar graphs with large maximum degree have edge and total colorings as small as possible [5,6]. We leave the following open problems: to find a threshold for $T_t(G)$ for which all planar graphs are Type II; to find a graph class for which k-TOTAL COLORABILITY is $\mathcal{NP}$-complete and k-TOTAL TESSELLABILITY is in $\mathcal{P}$. We have not identified a class answering the latter open problem, because so far all known $\mathcal{NP}$-completeness proofs of k-TOTAL COLORABILITY are restricted to graph classes with $\chi_t(G) = T_t(G)$, in order to fully achieve the $\mathcal{P}$ *vs* $\mathcal{NP}$-complete dichotomy in Table 1.

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