Theoretical Computer Science 801 (2020) 175–191

Contents lists available at ScienceDirect

Theoretical Computer Science

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The graph tessellation cover number: Chromatic bounds, efficient algorithms and hardness ☆

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ARTICLE INFO

Article history:
Received 6 April 2018
Received in revised form 22 August 2019
Accepted 5 September 2019
Available online 11 September 2019
Communicated by R. Klasing

Keywords:
Graph tessellation cover
Graph coloring
Clique graph
Staggered quantum walk

ABSTRACT

A tessellation of a graph is a partition of its vertices into vertex disjoint cliques. A tessellation cover of a graph is a set of tessellations that covers all of its edges, and the tessellation cover number is the size of the smallest tessellation cover. These concepts are motivated by their application to quantum walk models, in special, the evolution operator of the staggered model is obtained from a graph tessellation cover. We show that the minimum between the chromatic index of the graph and the chromatic number of its clique graph, which we call chromatic upper bound, is tight with respect to the tessellation cover number for star-octahedral and windmill graphs; whereas for $(3, p)$-extended wheel graphs, the tessellation cover number is 3 and the chromatic upper bound is $3p$. The t-TESSELLABILITY problem aims to decide whether there is a tessellation cover of the graph with t tessellations. Using graph classes whose tessellation cover numbers achieve the chromatic upper bound, we obtain that t-TESSELLABILITY is polynomial-time solvable for bipartite, {triangle, proper major}-free, threshold, and diamond-free K-perfect graphs; whereas is $\mathcal{NP}$-complete for triangle-free for $t \geq 3$, unichord-free for $t \geq 3$, planar for $t = 3$, biplanar for $t \geq 3$, chordal $(2, 1)$-graphs for $t \geq 4$, $(1, 2)$-graphs for $t \geq 4$, and diamond-free with diameter at most five for $t = 3$. We improve the complexity of 2-TESSELLABILITY problem to linear time.

1. Introduction

Random walks play an important role in Computer Science mainly in the area of algorithms and it is expected that quantum walks, which are the quantum counterpart of random walks, will play at least a similar role in Quantum Computation. In fact, the interest in quantum walks has grown considerably in the last decades, especially because they can be used to build quantum algorithms that outperform their classical counterparts [2].

Recently, the staggered quantum walk model [3] was proposed. This model is defined by an evolution operator, which is described by a product of local unitary matrices obtained from a *graph tessellation cover*. A *tessellation* is a partition of the

☆ An extended abstract was presented in LATIN 2018, The 13th Latin American Theoretical Informatics Symposium [1].

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https://doi.org/10.1016/j.tcs.2019.09.013

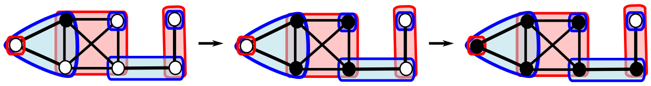

Fig. 1. The spreading of a walker subject to locality across a 2-tessellable graph. At each step, the walker may be observed at filled vertices that represent non-zero amplitudes, meaning that a measurement of the position can reveal the walker at one of those vertices. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

vertices of a graph into vertex disjoint cliques, and a tessellation cover is a set of tessellations so that the union covers the edge set of the graph. To cover the entire edge set is important because an edge that would not be in the tessellation cover would play no role in the quantum walk dynamics. In order to fully understand the possibilities of the staggered model, it is fundamental to introduce the t-TESSELLABILITY problem. This problem aims to decide whether a given graph can be covered by t tessellations.

The simplest evolution operators are the product of few local unitary matrices and, to obtain a non-trivial quantum walk, at least two matrices (corresponding to 2-tessellable graphs) are required [3]. There is a recipe to build a local unitary matrix based on a tessellation. Each clique in a tessellation is associated with a unit vector, and the set of those unit vectors spans a subspace of the model's Hilbert space. A subspace has an associated orthogonal projection Π, which is used to define the local unitary operator $(2\Pi - I)$ associated with the tessellation. Each clique of the partition establishes a neighborhood around which the walker can move under the action of the local unitary matrix. The evolution operator of the quantum walk is the product of the unitary operators associated with the tessellations of a tessellation cover. Fig. 1 depicts an example of how a quantum walker could spread across the vertices of a graph, given a particular tessellation cover, where the filled vertices represent that the probability of finding the walker is non-zero. Note that after each step the walker spreads across the cliques in the corresponding tessellation.

The study of tessellations in the context of Quantum Computing was proposed by Portugal et al. [3] with the goal of obtaining the dynamics of quantum walks. Portugal analyzed the 2-tessellable case in [4], showing that a graph is 2-tessellable if and only if its clique graph is bipartite, and examples for the t-tessellable case are available in [5]. The present paper is the first systematic attempt to study the graph tessellation cover as a branch of Graph Theory. Our aim is the study of graph classes whose tessellation cover number is close or equal to chromatic upper bounds, efficient algorithms, and hardness.

In Section 2, we establish a chromatic upper bound as the minimum between the chromatic index of the graph and the chromatic number of its clique graph, and we present infinite families of star-octahedral graphs and windmill graphs, showing that this bound is tight. We also present the infinite family of extended wheel graphs whose tessellation cover number is far from the chromatic upper bound. We describe the tessellation cover number for the classes of bipartite graphs and {triangle, proper major}-free graphs, and we prove that t-TESSELLABILITY for these classes is polynomial-time solvable, while is $\mathcal{NP}$-complete for triangle-free graphs, when $t \geq 3$. In Section 3, we present extremal graph classes, i.e., classes whose tessellation cover numbers reach the chromatic upper bound. Such classes are useful to establish hardness results in Section 4. We obtain proofs of $\mathcal{NP}$-completeness for t-TESSELLABILITY problem of planar graphs for $t = 3$, biplanar graphs for $t \geq 3$, chordal $(2, 1)$-graphs for $t \geq 4$, $(1, 2)$-graphs for $t \geq 4$, and diamond-free graphs with diameter at most five for $t = 3$. Moreover, we describe a linear-time algorithm for 2-TESSELLABILITY by improving the algorithm proposed by Peterson [6] for line graph of bipartite multigraph recognition. In Section 5, we summarize in Table 1 the extremal graph classes analyzed in Section 2.2 and in Section 3, whereas in Table 2 the complexity of the t-TESSELLABILITY problem for the graph classes analyzed in Section 4. Moreover, we leave open questions and discuss future work, such as whether every minimum tessellation cover contains tessellations with at least one maximum clique, and whether two minimum tessellation covers in a same graph have different quantum walk dynamics.

2. Preliminaries on the tessellation cover number

In this section, we present the main definitions of this paper, we introduce the chromatic upper bound, and we show that this bound is tight by presenting infinite families of graphs whose tessellation cover numbers achieve this chromatic upper bound. On the other hand, we present an infinite family of graphs whose tessellation cover number is far from the chromatic upper bound.

2.1. Definitions and upper bounds

A *clique* is a subset of vertices of a graph such that its induced subgraph is complete, and a *d-clique* is a clique of size d. The size of a maximum clique of a graph G is denoted by $\omega(G)$. The *clique graph* $K(G)$ is the intersection graph of the maximal cliques of G. A *partition of the vertices of a graph into cliques* is a collection of vertex disjoint cliques, where the union of these cliques is the vertex set. Clique graphs play a central role in tessellation covers. See [7] for an extensive survey on clique graphs and [8] for omitted graph theory terminologies.

Definition 1. A *tessellation* $\mathcal{T}$ is a partition of the vertices of a graph into cliques. An edge *belongs* to the tessellation $\mathcal{T}$ if and only if its endpoints belong to the same clique in $\mathcal{T}$. The set of edges belonging to $\mathcal{T}$ is denoted by $\mathcal{E}(\mathcal{T})$.

Definition 2. Given a graph G with edge set $E(G)$, a *tessellation cover* of size t of G is a set of t tessellations $\mathcal{T}_1, \ldots, \mathcal{T}_t$, whose union $\cup_{i=1}^{t} \mathcal{E}(\mathcal{T}_i) = E(G)$. A graph G is called t*-tessellable* if there is a tessellation cover of size at most t. The t-TESSELLABILITY PROBLEM aims to decide whether a graph G is t-tessellable. The *tessellation cover number* $T(G)$ is the size of a smallest tessellation cover of G.

A *coloring* (resp. an *edge-coloring*) of a graph is a labeling of the vertices (resp. edges) with colors such that no two adjacent vertices (resp. adjacent edges) have the same color. A k*-colorable* (resp. k*-edge-colorable*) graph is the one which admits a *coloring* (resp. an *edge-coloring*) with at most k colors. The *chromatic number* $\chi(G)$ (resp. *chromatic index* $\chi'(G)$) of a graph G is the smallest number of colors needed to color the vertices (resp. edges) of G.

Note that an edge-coloring of a graph G induces a tessellation cover of G. Each color class induces a partition of the vertex set into disjoint cliques of size two (vertices incident to edges of that color) and cliques of size one (vertices not incident to edges of that color), which forms a tessellation. Moreover, a coloring of $K(G)$ induces a tessellation cover of G. As presented in [5], two vertices of the same color in $K(G)$ correspond to disjoint maximal cliques of G and every edge of G is in at least one maximal clique. So, each color in $K(G)$ defines a tessellation in G by possibly adding cliques of size one (vertices that do not belong to the maximal cliques of G related to vertices of $K(G)$ with that color), such that the union of these tessellations is the edge set of G. Hence, we have the *chromatic upper bound*, denoted by $\text{cub}(G)$, as the minimum between $\chi'(G)$ and $\chi(K(G))$.

Theorem 1. *If G is a graph, then $T(G) \le \text{cub}(G) = \min\{\chi'(G), \chi(K(G))\}$.*

Portugal [4] characterized the 2-tessellable graphs as those whose clique graphs are bipartite graphs. Note that if $K(G)$ is bipartite, then $\chi(K(G)) = 2$, while $\chi'(G)$ may be arbitrarily large due to the fact that this parameter is related to the maximum degree $\Delta(G)$. In order to characterize t-tessellable graphs, for $t \ge 3$, we find graph classes such that $T(G) = 3$, with $\chi'(G)$ and $\chi(K(G))$ arbitrarily large, and graph classes whose tessellation cover number reaches the chromatic upper bound of Theorem 1, i.e., $T(G) = \chi'(G)$ but $\chi(K(G))$ arbitrarily large; and $T(G) = \chi(K(G))$ but $\chi'(G)$ arbitrarily large, some of those examples were described in [5], and further developed in Section 2.2.

An interesting case occurs for a triangle-free graph. Note that any of its tessellations can only be formed by cliques of size two or one. Hence, we have that if G is a triangle-free graph, then $T(G) = \chi'(G) = \chi(K(G)) = \chi(L(G))$, where $L(G)$ is the line graph of G. Therefore, t-TESSELLABILITY is polynomial-time solvable for bipartite graphs and for {triangle, proper major}-free graphs, and there are also polynomial-time algorithms to obtain a minimum tessellation cover for these graph classes [9,10]. On the other hand, it is known that Δ-EDGE COLORABILITY of triangle-free graphs for $\Delta \ge 3$ is $\mathcal{NP}$-complete [11]. Therefore, t-TESSELLABILITY of triangle-free graphs for $t \ge 3$ is also $\mathcal{NP}$-complete. Similarly, we know that Δ-EDGE COLORABILITY of regular unichord-free graphs with girth at least 15 for $\Delta \ge 3$ is $\mathcal{NP}$-complete [12]. As this graph class is triangle-free, we conclude that the same hardness proof holds for t-TESSELLABILITY for $t \ge 3$.

2.2. Three infinite families

We present three infinite families of graphs G that illustrate some interesting situations: (i) $T(G) = \chi'(G)$ with $\chi(K(G))$ arbitrarily large; (ii) $T(G) = \chi(K(G))$ with $\chi'(G)$ arbitrarily large; and (iii) $T(G) = 3$ with both upper bounds arbitrarily large. Note that the first two situations are illustrated by families of graphs whose tessellation cover numbers achieve the chromatic upper bound.

A *coalescence* [13] of disjoint graphs G_1 and G_2 is obtained by identifying a vertex of G_1 with another vertex of G_2. The first family of *star-octahedral graphs* G_p is the coalescence of the graphs S_{2p} and O_p — where S_{2p} is the star graph with $2p$ leaves and O_p is the p dimensional octahedral graph defined by the $(2p-2)$-regular graph with $2p$ vertices — by identifying a leaf of S_{2p} into any vertex of O_p. Fig. 2 depicts the star-octahedral graph G_4.

Next, we establish that the tessellation cover number of star-octahedral graph G_p is equal to its chromatic index.

Theorem 2. *Let G_p be a star-octahedral graph. Then:*

1. $T(G_p) = \Delta(G_p) = \chi'(G_p) = \chi(K(G_p)) = 2p$, *for* $p \in \{2, 3\}$, *and;*
2. $T(G_p) = \Delta(G_p) = \chi'(G_p) = 2p$ *and* $\chi(K(G_p)) = 2^{p-1} + 1$, *for* $p \ge 4$.

Proof. We know that the clique graph of the octahedral graph O_p is the octahedral graph $O_{2^{p-1}}$ [14]. Moreover, as the star-octahedral graph has only one vertex with maximum degree, we know that $\chi'(G_p) = 2p$. The tessellation cover number of S_{2p} is equal to $2p$, hence $T(G_p) \ge 2p$. The proof is divided into two cases:

1. Consider $p = 2$. We know that $K(O_2) = O_2$ and $K(S_4) = K_4$, which are induced subgraphs of $K(G_2)$. Since G_2 has a vertex $v \in V(S_4)$ identified with a vertex of O_2, it follows that $K(G_2)$ has a vertex $u \in V(K(S_4))$ that is a neighbor of two vertices of $K(O_2)$. Hence, the largest maximal clique of $K(G_2)$ has size 4, and $\chi(K(G_2)) = 2p = 4$. Moreover, $\Delta(G_2) = \Delta(S_4) = \chi'(G_2) = 2p = 4$. Then, from Theorem 1, it follows that $T(G_2) = \chi'(G_2) = \chi(K(G_2)) = 2p$. The proof is analogous for $p = 3$.

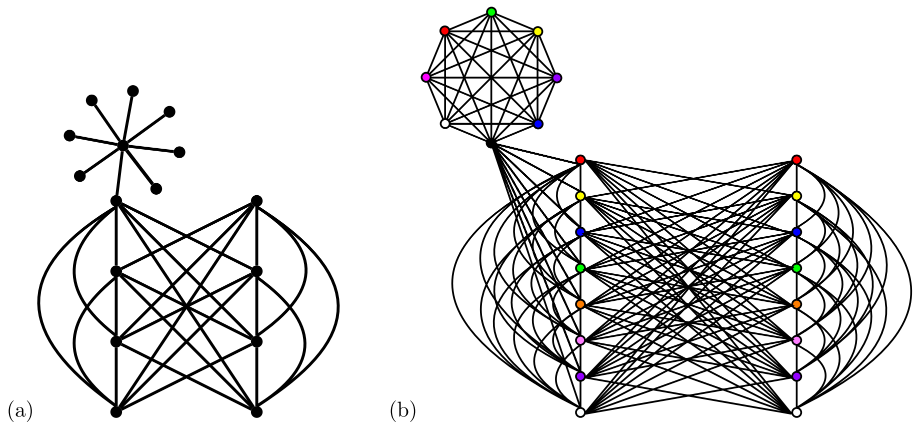

Fig. 2. (a) The star-octahedral graph G_4, i.e., the coalescence between the octahedral graph O_4 and the star graph S_8. (b) The clique graph $K(G_4)$. Notice that $T(G_4) = \chi'(G_4) = 8$, while $\chi(K(G_4)) = 9$.

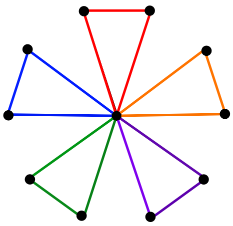

Fig. 3. The windmill graph $\mathrm{Wd}_{3,5}$, composed by 5 copies of the complete graph K_3. Notice that this graph has $T(W_{3,5}) = \chi(K(\mathrm{Wd}_{3,5})) = 5$, since its clique graph $K(\mathrm{Wd}_{3,5})$ is the complete graph K_5.

2. Notice that each vertex of O_p belongs to 2^{p-1} maximal cliques. Since G_p has a vertex $v \in V(S_{2p})$ identified with a vertex of O_p, it follows that v belongs to $2^{p-1}+1$ maximal cliques. Hence, $\chi(K(G_p)) \geq \omega(K(G_p)) \geq 2^{p-1}+1$. One can obtain a $(2^{p-1}+1)$-coloring of $K(G_p)$ as displayed in Fig. 2(b). Hence, $\chi(K(G_p)) = 2^{p-1}+1$.

As $\Delta(G_p) = \Delta(S_{2p}) = 2p$, then $\chi'(G_p) = 2p$. From Theorem 1, it follows that $T(G_p) \leq \min\{\chi'(G_p), \chi(K(G_p))\} = \chi'(G_p) = 2p$, since $2p \leq 2^{p-1}+1$ for $p \geq 4$. As $T(S_{2p}) = \chi'(S_{2p}) = 2p \leq T(G_p)$, we conclude that $T(G_p) = \chi'(G_p) = 2p$. □

The second family of *windmill graphs* $\mathrm{Wd}_{p,q}$ is obtained by identifying q copies of the complete graph K_p at a universal vertex. Note that $T(\mathrm{Wd}_{p,q}) = \chi(K(\mathrm{Wd}_{p,q})) = q$ for $p \geq 2$ and $\chi'(\mathrm{Wd}_{p,q}) = (p-1)q$ [5]. Fig. 3 depicts an example of the windmill graph $\mathrm{Wd}_{3,5}$, that is composed by 5 copies of the complete graph K_3. Clearly, its clique graph $K(\mathrm{Wd}_{3,5})$ is a complete graph K_5, then $\chi(K(\mathrm{Wd}_{3,5})) = 5$, which is equal to the tessellation cover number of $\mathrm{Wd}_{3,5}$. On the other hand $\chi'(\mathrm{Wd}_{3,5}) = 10$.

The third family of (k, p)*-extended wheel graphs* $E_{k,p}$, for $k \geq 3$ and $p \geq 2$, is defined by adding to the wheel graph W_{kp} (defined by a cycle C_{kp}, $V(C_{kp}) = \{0, 1, 2, \ldots, kp-1\}$, after adding a universal vertex with label kp.) the following edges: $\{ki, kj\}$, $\{ki+1, kj+1\}$, ..., $\{ki+k-1, kj+k-1\}$, for $0 \leq i < j < p$. When focusing on the case $k = 3$, we show that $T(E_{3,p}) = 3$, $\chi'(G) = 3p$, and $\chi(K(E_{3,p})) = 3p+3$. The class $E_{3,p}$ comprises 3-tessellable graphs with arbitrarily large chromatic index, whose clique graphs have arbitrarily large chromatic numbers. It shows that the tessellation cover number does not necessarily depend neither on $\chi'(G)$ nor on $\chi\big(K(G)\big)$.

Lemma 1. *The maximal cliques of $E_{3,p}$ are 3-cliques or $(p+1)$-cliques. The number of maximal cliques is $3p+3$. The maximal cliques are the 3-cliques of the spanning wheel W_{3p}, plus three new $(p+1)$-cliques. All maximal cliques share the vertex with label $3p$, which is the universal vertex.*

Proof. Each of the $3p$ vertex sets $\{0, 1, 3p\}, \{1, 2, 3p\}, \ldots, \{3p-2, 3p-1, 3p\}, \{3p-1, 0, 3p\}$ is a maximal clique K_3 because it induces a triangle of the spanning wheel graph and the vertex set $\{i : 0 \leq i < 3p\}$ contains no maximal clique of size 3 in $E_{3,p}$.

Now consider the three sets of vertices $\{0, 3, 6, ..., 3p-3, 3p\}$, $\{1, 4, 7, ..., 3p-2, 3p\}$, and $\{2, 5, 8, ..., 3p-1, 3p\}$, each of them with cardinality $p+1$. We claim that each one is a maximal clique K_{p+1}. Consider the set $\{0, 3, 6, ..., 3p-3, 3p\}$ (analogous for the other ones). All vertices in this set are adjacent because every pair of vertices is either $\{3i, 3j\}$ for some $0 \leq i, j < p$ or $\{3i, 3p\}$ for some $0 \leq i < p$. In the first case, these edges were added to W_{3p} to define $E_{3,p}$, and in the second case the edges belong to the spanning wheel graph. If a new vertex is added, it must have the form $3i+1$ or $3i+2$

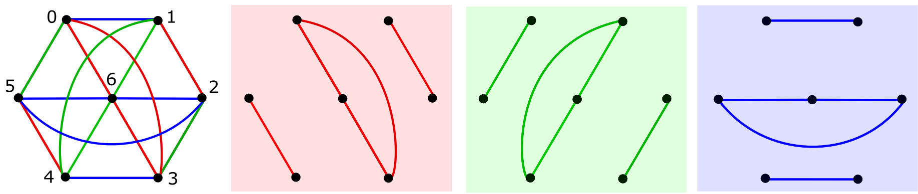

Fig. 4. An example of the (3, 2)-extended wheel graph. Notice that the tessellations applied in this graph are $\mathcal{T}_1 = \{\{0,3,6\},\{1,2\},\{4,5\}\}$, $\mathcal{T}_2 = \{\{1,4,6\},\{2,3\},\{5,0\}\}$, and $\mathcal{T}_3 = \{\{2,5,6\},\{3,4\},\{0,1\}\}$, as described in the proof of Theorem 3.

for some $0 \le i < p$ and it will not be adjacent to all vertices of set $\{0, 3, 6, ..., 3p-3, 3p\}$. Hence, there are three maximal cliques of size $p+1$ in $E_{3,p}$. Then, the total number of maximal cliques is $3p+3$ and all of them share the vertex with label $3p$. □

Theorem 3. *Let $E_{3,p}$ be a $(3, p)$-extended wheel graph. Then, $T(E_{3,p}) = 3$ for $p \ge 2$.*

Proof. Let us show that $E_{3,p}$ is 3-tessellable by describing explicitly three tessellations that cover the edges of $E_{3,p}$. The tessellations are the following ones:

$$\mathcal{T}_1 = \{\{0,3,6...,3p-3,3p\},\{1,2\},\{4,5\},...,\{3p-5,3p-4\},\{3p-2,3p-1\}\};$$
$$\mathcal{T}_2 = \{\{1,4,7,...,3p-2,3p\},\{2,3\},\{5,6\},...,\{3p-4,3p-3\},\{3p-1,0\}\};$$
$$\mathcal{T}_3 = \{\{2,5,8,...,3p-1,3p\},\{3,4\},\{6,7\},...,\{3p-3,3p-2\},\{0,1\}\}.$$

Let us show that $\mathcal{T}_1$ is a well defined tessellation (analogous for the other ones) by checking each item of the following list: (1) Each vertex set in $\mathcal{T}_1$ must induce a clique, (2) the vertex sets in $\mathcal{T}_1$ must be pairwise disjoint, and (3) the union of the vertex sets in $\mathcal{T}_1$ must be the vertex set of $E_{3,p}$. Item (1) holds by Lemma 1, since the set {0,3,6,...,3p-3,3p} is a clique, and the remaining sets define edges of the spanning wheel. Item (2) holds since the set $\{0, 3, 6, ..., 3p-3, 3p\}$ is comprised of vertices that are multiple of 3 while the remaining sets are disjoint and contain no multiple of 3. Item (3) holds since the union of the sets in $\mathcal{T}_1$ is the vertex set. Since no edge belongs to more than one tessellation and each tessellation covers $p(p+3)/2$ edges, the union $\mathcal{E}(\mathcal{T}_1) \cup \mathcal{E}(\mathcal{T}_2) \cup \mathcal{E}(\mathcal{T}_3)$ covers $3p(p+3)/2$ edges, which is the number of edges of $E_{3,p}$. It is not possible to cover the edges of $E_{3,p}$ with less than three tessellations because if $T(E_{3,p}) = 2$ then $\chi(K(E_{3,p})) = 2$ [3]. However, the chromatic number of the clique graph of $E_{3,p}$ is $3p+3$. Then, $T\left(E_{3,p}\right) = 3$ for $p \ge 2$. □

Fig. 4 depicts the (3, 2)-extended wheel graph $E_{3,2}$.

It is straightforward to extend those results and to prove that $T(E_{k,p}) \le k$, $\chi'(G) = kp$, and $\chi\left(K(E_{k,p})\right) = k(p+1)$. Therefore, we are able to provide examples of classes of k-tessellable graphs with arbitrarily large chromatic index, whose clique graphs have arbitrarily large chromatic number for any $k \ge 3$.

3. Extremal graph classes

In this section, we show extremal graph classes, by presenting constructions that force the tessellation cover number of some graphs to be equal to the chromatic upper bound. An *extremal graph* is a graph whose tessellation cover number is equal to the chromatic upper bound of Theorem 1. We are particularly interested in constructing graphs with tessellation cover number corresponding or close to the chromatic upper bound. Note that the family of star-octahedral, windmill, triangle-free, bipartite, {triangle, proper major}-free, and unichord-free graphs with girth at least 15, analyzed in Section 2, are examples of extremal graph classes. For the sake of convenience, we may omit one-vertex cliques inside tessellations in our proofs.

Construction 1. Let H be obtained from a graph G by adding a star with $\chi'(G)$ leaves and identifying one of these leaves with a minimum degree vertex of G. See Fig. 5.

The tessellation cover number of H, obtained from Construction 1 on a non-regular graph G, is equal to $\chi'(G)$, i.e., $T(H) = \chi'(H) = \chi'(G)$. For regular graphs, if $\chi'(G) = \Delta(G)+1$, then $T(H) = \chi'(H) = \chi'(G)$. Otherwise, $T(H) = \chi'(H) = \chi'(G)+1$. Construction 1 also implies that every non-regular graph G is a subgraph of a graph H with $T(H) = \chi'(H) = \chi'(G)$.

Additionally, Construction 2 in diamond-free graphs G forces the tessellation cover number of the obtained graph H to be equal to the chromatic number of the clique graph $\chi(K(G))$. First, we define a property of the cliques on a tessellation called exposed maximal clique. Such a property helps us with particular cases of diamond-free graphs.

Fig. 5. Example of Construction 1. $T(G) = 3$, and $T(H) = \chi'(H) = \chi'(G) = 4$.

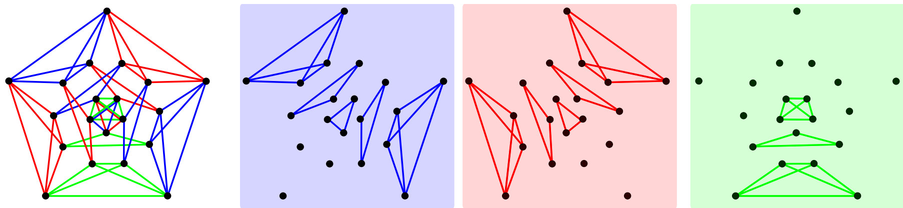

Fig. 6. Example of a 3-tessellable graph G whose clique graph is the Mycielskian of a C_5, with $\chi(K(G)) = 4$ but $T(G) = 3$. Each tessellation is depicted separately.

Definition 3. A maximal clique K of a graph G is said to be *exposed* by a tessellation cover $\mathcal{C}$ if $E(K) \nsubseteq \mathcal{E}(\mathcal{T})$ for all $\mathcal{T} \in \mathcal{C}$, that is, the edges of K are not covered by any single tessellation of $\mathcal{C}$.

Lemma 2. *A graph G admits a minimum tessellation cover with no exposed maximal cliques if and only if $T(G) = \chi(K(G))$.*

Proof. Given a minimum tessellation cover $\mathcal{C} = \{\mathcal{T}_1, \ldots, \mathcal{T}_t\}$ of G, if there are no exposed maximal cliques in G, then $\mathcal{C}$ induces a coloring of $K(G)$. In fact, suppose that C_v is a maximal clique of G associated with vertex $v \in V(K(G))$. If C_v is covered by tessellation $\mathcal{T}_i$ then v receives color c_i (if C_v is covered by more than one tessellation, we have more than one choice for coloring v). Using the definitions of tessellation and tessellation cover, we conclude that this method produces a coloring of K(G) with colors $c_1, \ldots, c_t$, which implies that $\chi(K(G))$ is at most t. And Theorem 1 implies the equality $\chi(K(G)) = t = T(G)$.

Conversely, the proof of Theorem 1 describes a minimum tessellation cover of size t with no exposed maximal clique when $\chi(K(G)) = T(G) = t$. □

In the remaining part of this section we consider diamond-free graphs, which have the following properties [6]: (1) their clique-graphs are diamond-free, and (2) any two maximal cliques intersect in at most one vertex.

Theorem 4. *If G is a diamond-free graph with $\chi(K(G)) = \omega(K(G))$, then $T(G) = \chi(K(G))$.*

Proof. Let $d = \chi\big(K(G)\big) = \omega(K(G))$. Hence, there is a complete graph K_d, where $V(K_d) = \{v_1, \ldots, v_d\}$ in $K(G)$. Let $C_{v_1}, \ldots, C_{v_d}$ be the maximal cliques in G, such that each C_{v_i} is associated with vertex v_i in $K(G)$.

Since G is diamond-free, the cliques $C_{v_1}, \ldots, C_{v_d}$ compose an induced subgraph H and these cliques share exactly one vertex in G, that is universal in H, because any two maximal cliques of a diamond-free graph intersect in at most one vertex and each edge belongs to exactly one maximal clique. Since $\chi(K(G)) = d$, this coloring induces a tessellation cover with d tessellations in H, that is optimal for H, then $T(G) \geq \chi(K(G))$. By Theorem 1 $T(G) \leq \chi(K(G))$, then $T(G) = \chi(K(G))$. □

A graph is K-*perfect* if its clique graph is perfect [15]. Since a diamond-free K-perfect graph G satisfies the premises of Theorem 4, we have $T(G) = \chi(K(G))$. Note that the size of the clique graph of a diamond-free graph is polynomially bounded by the size of the original graph [6]. Moreover, there is a polynomial-time algorithm to obtain an optimal coloring of $K(G)$ with $\omega(K(G))$ colors [16] and, by Theorem 1, a coloring of $K(G)$ with t colors yields that G is t-tessellable. Thus, both the tessellation cover number and a minimum tessellation cover of diamond-free K-perfect graphs are obtained in polynomial time.

Interestingly, there are diamond-free graphs whose clique graphs have chromatic number greater than the tessellation cover number. Fig. 6 illustrates an example of a 3-tessellable diamond-free graph whose clique graph has chromatic number 4 (the clique graph $K(G)$ is the Grötzsch graph, i.e. Mycielskian of a 5-cycle graph). Note that any minimum tessellation cover of this graph necessarily has an exposed maximal clique. Moreover, this graph shows that the upper bound of Theorem 5 is tight.

Lemma 3. *Let G be a 3-tessellable diamond-free graph. If C_1 and C_2 are two maximal cliques of G with a common vertex, then C_1 and C_2 cannot be both exposed by a minimum tessellation cover.*

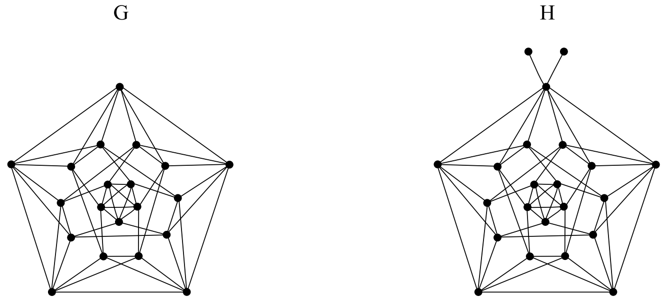

Fig. 7. Example of Construction 2. $T(G) = 3$ and $\chi(K(G)) = 4$, but $T(H) = 4$ and $\chi(K(H)) = 4$.

Proof. For the sake of the contradiction, assume that $v \in V(C_1 \cap C_2)$, and that C_1 and C_2 are both exposed maximal cliques. Since G is diamond-free, the vertex v is the only vertex in the intersection between cliques C_1 and C_2. Let $\mathcal{C}$ be a minimum tessellation cover with size at most 3. Let us focus on cliques of size at least two in tessellations that cover edges of C_1. In a tessellation, no clique of size at least two covers all edges from v to its neighbors because C_1 is an exposed maximal clique. Then, v belongs to at least two cliques of size at least two in different tessellations, where each one covers a proper subset of the edges of C_1. The same is true for C_2, that is, v belongs to at least two cliques of size at least two in different tessellations, where each one covers a proper subset of the edges of C_2. This means that at least four cliques of size at least two intersect on v and all of them must belong to different tessellations. This contradiction shows that C_1 and C_2 cannot be both exposed by a minimum tessellation cover, if G is a diamond-free graph. □

Theorem 5. *If G is a diamond-free graph with $T(G) = 3$, then* $3 \le \chi(K(G)) \le 4$.

Proof. By Theorem 1, we have that $3 \le \chi(K(G))$. Given a minimum tessellation cover $\mathcal{C} = \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$ of G, Lemma 3 implies that the set of vertices in $K(G)$ that are associated with the exposed maximal cliques in G is a stable set in $K(G)$. $K(G)$ can be colored with four colors in the following way: the vertices in $K(G)$ that correspond to exposed cliques have color c_4; the vertices in $K(G)$ that correspond to a maximal clique fully contained in $\mathcal{T}_i$ have color c_i. This coloring shows that $\chi\big(K(G)\big) \le 4$. □

Now we present a construction which forces the tessellation cover number of a graph H, obtained from Construction 2 on a diamond-free graph G, to be $T(H) = \chi(K(H)) = \chi(K(G))$. If G has $T(G) < \chi(K(G))$, then there is no vertex of G that belongs to $\chi(K(G))$ maximal cliques. The graph H obtained from G by Construction 2 satisfies $\chi(K(H)) = \chi(K(G))$ and contains a vertex that belongs to $\chi(K(G))$ maximal cliques, which implies $T(H) = \chi(K(G))$.

Construction 2. Let H be obtained from a graph G by iteratively adding pendant vertices to a vertex of G until it belongs to $\chi(K(G))$ maximal cliques. See Fig. 7.

Construction 2 implies that every diamond-free graph G is a subgraph of a graph H with $T(H) = \chi(K(H)) = \chi(K(G))$. Note that this construction is not restricted to diamond-free graphs and it can also be applied several times to vertices that only belong to one maximal clique. The hardness proofs of Theorems 8 and 9 rely on this result.

We finish this section showing that threshold graphs are extremal graphs by proving that the tessellation cover numbers of these graphs achieve the chromatic upper bound. The class of threshold graphs is hereditary and self-complementary [17]. We can describe a threshold graph as $G = (C \cup S, E)$, where C represents a maximum clique of G, S represents an independent set of G with nested neighborhood, and E represents the edge set of G. Threshold graphs can be constructed from an empty graph by repeatedly adding either an isolated vertex or a universal vertex. Considering a connected threshold graph $G = (C \cup S, E)$, in the clique graph $K(G)$, C is represented by vertex v_C, and each maximal clique containing vertices $v_i \in S, i \in \{1, \dots, |S|\}$ is represented by a vertex v_{S_i}. Since there exists a universal vertex $u \in V(G)$, there are no disjoint maximal cliques in G, and the clique graph $K(G)$ is a complete graph with $|S| + 1$ vertices. Moreover, note that its chromatic index $\chi'(G)$ can be arbitrarily large.

Theorem 6. *If $G = (C \cup S, E)$ is a connected threshold graph, then $T(G) = \chi(K(G))$.*

Proof. Since G does not have disjoint maximal cliques and its clique graph is the complete graph with size $|S| + 1$, then $\chi(K(G)) = |S| + 1$. Hence, by Theorem 1, $T(G) \le |S| + 1$. By the fact of C is a maximal clique in G, there is no vertex $v \in S$ such that v is neighbor of all vertices in C, otherwise there would exist a maximal clique C', greater than C, containing the clique C and the vertex v. The vertices of S in a threshold graph G must have a nested neighborhood [18]. Hence, there is at least one vertex $w \in V(C)$ such that w is not neighbor of any vertex in S. Since G has a universal vertex u, then G has

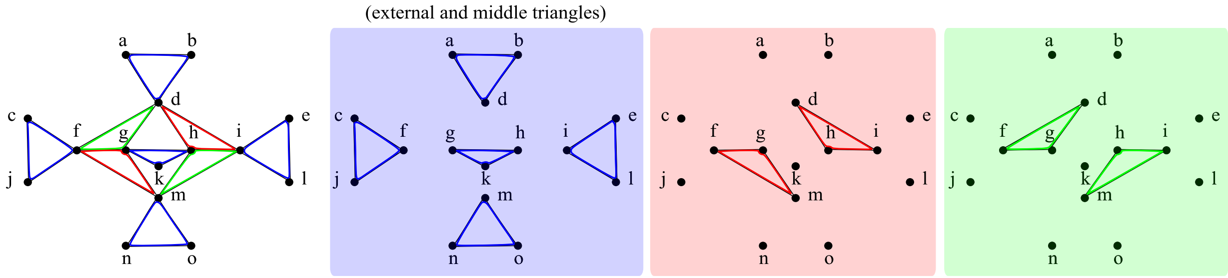

Fig. 8. The 3-tessellable *graph-gadget* of Lemma 5. Each tessellation is depicted separately. The external vertices are a, b, c, e, j, l, n, o, and the internal vertices are the remaining ones.

an induced star subgraph with $|S|+1$ leaves centered in u, all vertices of S and $w \in V(C)$ as its leaves. Therefore, such an induced star requires at least $|S|+1$ tessellations, i.e., $T(G) \geq |S|+1$.

We conclude that $T(G) = |S|+1 = X(K(G))$. □

Since to construct and to color the complete clique graph $K(G)$ can be done in polynomial-time for a threshold graph G, we conclude that t-TESSELLABILITY is polynomial-time solvable.

4. Computational complexity

Now, we focus on the computational complexity of t-TESSELLABILITY by firstly proving that the problem is in $\mathcal{NP}$. In Section 4.1, we use extremal graph classes obtained in the previous section to show $\mathcal{NP}$-completeness of planar graphs with maximum degree $\Delta(G) \leq 6$ for $t = 3$, biplanar graphs for $t \geq 3$, chordal $(2,1)$-graphs for $t \geq 4$, $(1,2)$-graphs for $t \geq 4$, and diamond-free graphs with diameter at most five for $t \geq 3$. In Section 4.2, we efficiently solve 2-TESSELLABILITY in linear time.

Lemma 4. *t-TESSELLABILITY is in $\mathcal{NP}$.*

Proof. Let G be an instance for t-TESSELLABILITY. If $t \geq \Delta(G)+1$, then by Theorem 1 and the well-known Vizing's theorem on Δ-EDGE COLORABILITY, the answer is always *YES*. When $t \leq \Delta(G)$, consider a certificate for t-TESSELLABILITY, which consists of at most t tessellations that cover the edge set $E(G)$. Note that each of these tessellations has at most $|E(G)|$ edges. One can easily verify in polynomial time if the at most $|E(G)|$ edges in each of the at most $t \leq \Delta(G)$ tessellations form disjoint cliques in G and if the at most $|E(G)|\Delta(G)$ edges in these tessellations cover $E(G)$. □

4.1. $\mathcal{NP}$-completeness

We remarked in Section 2.1 that the 3-TESSELLABILITY problem is $\mathcal{NP}$-complete for the triangle-free graphs. This result comes from the result presented by Koreas [11], who proved that Δ-EDGE COLORABILITY problem of triangle-free graphs with maximum degree three is $\mathcal{NP}$-complete. Since Δ-EDGE COLORABILITY of unichord-free graphs with girth at least 15 (which are triangle-free) for $\Delta \geq 3$ is $\mathcal{NP}$-complete [12], t-TESSELLABILITY for $t \geq 3$ is also $\mathcal{NP}$-complete for this graph class.

In this section, we present the $\mathcal{NP}$-completeness of the t-TESSELLABILITY problem of planar graphs with maximum degree $\Delta(G) \leq 6$ for $t = 3$ in Theorem 7, biplanar graphs for $t \geq 3$ in Theorem 8, chordal $(2,1)$-graphs for $t \geq 4$ in Theorem 9, $(1,2)$-graphs for $t \geq 4$ in Theorem 10, and diamond-free graphs with diameter at most five for $t \geq 3$ in Theorem 11.

A graph is *planar* if it can be embedded in the plane such that no two edges cross each other. We show a polynomial transformation from the $\mathcal{NP}$-complete 3-COLORABILITY of planar graphs with maximum degree four [9] to 3-TESSELLABILITY of planar graphs with maximum degree six.

Lemma 5. *Any tessellation cover of size 3 of the* graph-gadget *depicted in Fig. 8 contains a tessellation that covers the middle and the external triangles.*

Proof. Consider any tessellation cover of size 3 for the *graph-gadget* of Fig. 8. For the sake of the contradiction, assume that the triangle $\{a,b,d\}$ is exposed, needing to be covered by 3 tessellations, one tessellation for each one of its edges. However, the remaining neighborhood of vertex d does not induce a clique, needing at least other 2 tessellations to be covered, a contradiction with the fact that the graph-gadget is 3-tessellable.

Now, without loss of generality, assume that the triangle $\{a,b,d\}$ is covered by tessellation 1. If we cover the triangle $\{d,g,h\}$ with tessellation 2, we will need more two tessellations to cover the edges $\{d,f\}$ and $\{d,i\}$, a contradiction. Therefore, we need to cover the triangle $\{d,f,g\}$ with tessellation 3 and the triangle $\{d,i,h\}$ with tessellation 2.

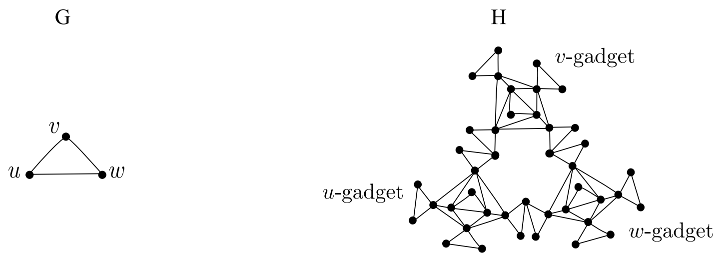

Fig. 9. Example of Construction 3.

Now, the middle triangle $\{g, h, k\}$ needs to be covered by tessellation 1, otherwise, we need the edge $\{g, h\}$ in tessellation 1 and as the set of vertices $\{g, k, m\}$ does not induce a clique, there will be two edges to be covered in the neighborhood of g with only one remaining tessellation, a contradiction. Next, we need to cover the triangles $\{f, g, m\}$ and $\{h, i, m\}$ with tessellations 2 and 3, respectively.

Finally, to obtain a 3-tessellation of this graph, the other external triangles $\{c, f, j\}$, $\{e, i, l\}$, and $\{m, n, o\}$ must be covered by tessellation 1. □

Construction 3. Let graph H be obtained from a graph G by local replacements of each vertex u of G for a graph-gadget of Fig. 8 denoted by u-gadget. Each edge uv of G represents the intersection of the u-gadget with the v-gadget by identifying two external vertices of external triangles of those graph-gadgets. See Fig. 9.

Theorem 7. 3-TESSELLABILITY *of planar graphs with* $\Delta(G) \leq 6$ *is* $\mathcal{NP}$*-complete.*

Proof. Let G be an instance graph of 3-COLORABILITY of planar graphs with $\Delta(G) \leq 4$ and H be obtained by Construction 3 on G. Notice that applying Construction 3 on a planar graph with $\Delta(G) \leq 4$ results on a planar graph with $\Delta(H) \leq 6$.

Suppose that G is 3-colorable. Then, H is 3-tessellable because the middle and the external triangles of a v-gadget can be covered by the tessellation related to the color of v and the remaining triangles of the v-gadget can be covered by the other two tessellations.

Suppose that H is 3-tessellable. Then, G is 3-colorable because the color of v in G can be related to the tessellation that covers the middle triangle of the v-gadget. This assignment is a 3-coloring because by Lemma 5 all external triangles of the v-gadget belong to the same tessellation of the middle triangle. The external triangles of the v-gadget are connected to the external triangles of the graph-gadgets of the neighborhood of v. Then, the tessellations of the latter external triangles must differ from the external triangles of the v-gadget. This implies that the neighborhood of vertex v receives different colors from the color of v. □

The next construction allows us to show a hardness proof of t-TESSELLABILITY, with any fixed $t \geq 3$ of biplanar graphs. A graph $G = (V, E)$ is biplanar if we can partition the edge set E into at most two sets E_1 and E_2 such that $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are planar graphs. Biplanar graphs are known as graphs of thickness ≤ 2. The polynomiality of Δ-EDGE COLORABILITY for planar graphs with $\Delta(G) \geq 8$ suggests that t-TESSELLABILITY for planar graphs might be polynomial-time solvable for large enough t. On the other hand, by Theorem 8, we know that t-TESSELLABILITY of biplanar graphs remains $\mathcal{NP}$-complete, even for a large value of t.

Construction 4. Let t be an integer and H be a graph obtained from a graph G as follows. Initially H is equal to G. Add a star S_t with t leaves. Add three paths P^1, P^2, and P^3 with $2|V(G)| + 1$ vertices each one. Identify the first vertex in each one of these paths with three different leaves of S_t. Let $V(P^1) = \{p_{1,1}, p_{1,2}, \ldots, p_{1,2|V(G)|+1}\}$, $V(P^2) = \{p_{2,1}, p_{2,2}, \ldots, p_{2,2|V(G)|+1}\}$, and $V(P^3) = \{p_{3,1}, p_{3,2}, \ldots, p_{3,2|V(G)|+1}\}$. For each edge of type $(p_{i,2j+1}, p_{i,2j+2})$ (for $1 \leq i \leq 3$ and $0 \leq j \leq |V(G)| - 1$), add $t - 1$ vertices adjacent to both endpoints of the edge and, for each of these $t - 1$ vertices, add $t - 1$ pendant vertices adjacent to it. Add a stable set $U = \{u_1, u_2, \ldots, u_{|V(G)|}\}$ and relate each one of these vertices with $V(G) = \{v_1, v_2, \ldots, v_{|V(G)|}\}$. For each vertex $u_k \in U$ add the edges: $(u_k, p_{1,2k})$, $(u_k, p_{1,2k+1})$, $(u_k, p_{2,2k})$, $(u_k, p_{2,2k+1})$, $(u_k, p_{3,2k})$, and $(u_k, p_{3,2k+1})$. For each vertex $u_k \in U$ add the edge (u_k, v_k). For each vertex u_k, add the vertices $w_{k,m}$ (for $1 \leq m \leq t - 3$) adjacent to both u_k and v_k, and add $t - 1$ pendant vertices for each of these $w_{k,m}$ (for $1 \leq m \leq t - 3$) vertices. See Fig. 10.

Theorem 8. t-TESSELLABILITY *of biplanar graphs for* $t \geq 3$ *is* $\mathcal{NP}$*-complete.*

Proof. Let G be an instance graph for 3-COLORABILITY of planar graphs with $\Delta(G) \leq 4$, B be the graph obtained from Construction 3 on G, and H be the graph obtained from Construction 4 on B. We claim that H is t-tessellable (for $t \geq 3$) if and only if B is 3-tessellable. Therefore, the $\mathcal{NP}$-completeness follows immediately from Theorem 7.

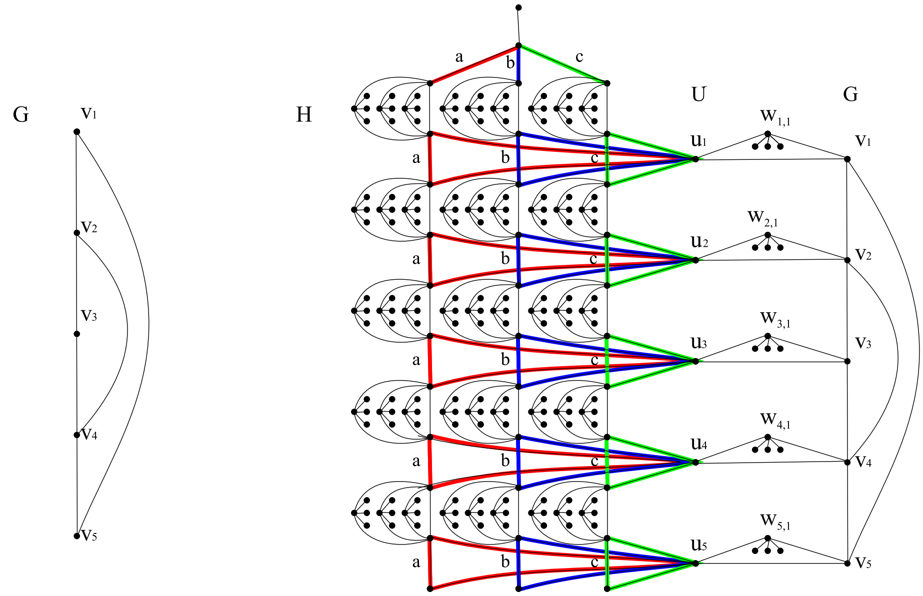

Fig. 10. Example of Construction 4, for $t = 4$. The graph H is biplanar since we can partition its edges into two planar graphs as follows. The edges of G, S_t, P^1, P^2, P^3, the triangles connected to these paths, and the pendant vertices incident to these vertices define a planar graph and the remaining edges (incident to vertices u_i and $w_{k,m}$) define other planar graph. Colors a, b, c highlight three tessellations.

Consider the case when H is t-tessellable. Let a, b, and c be three tessellations used to cover the edges of S_t which have one of their endpoints identified with P^1, P^2, and P^3, respectively. The $t-1$ triangles incident to the edges $(p_{1,2j-1}, p_{1,2j})$, for $1 \le j \le |V(G)|$, are not exposed since there are $t-1$ pendant vertices incident to a vertex in each of the triangles, which force them to be covered by $t-1$ tessellations. Therefore, for $j = 1$, the tessellation a cannot cover the corresponding $t-1$ triangles which implies that edge $(p_{1,2}, p_{1,3})$ must be covered by tessellation a, which in turn implies that all edges $(p_{1,2j}, p_{1,2j+1})$, for $1 \le j \le |V(G)|$, must be covered by tessellation a. The same holds with tessellation b and edges $(p_{2,2j}, p_{2,2j+1})$, and tessellation c and edges $(p_{3,2j}, p_{3,2j+1})$ (for $1 \le j \le |V(G)|$). Moreover, as the vertices $p_{1,2j-1}$ $(1 \le j \le |V(G)|)$ are in $t-1$ tessellations because the triangles incident to the edges $(p_{1,2j-1}, p_{1,2j})$, the triangles with vertices $u_k, p_{1,2k}, p_{1,2k+1}$ need to be not exposed and use the tessellation a. The same holds for tessellation b and the triangles with the vertices $u_k, p_{2,2k}, p_{2,2k+1}$, and for tessellation c and the triangles with the vertices $u_k, p_{3,2k}, p_{3,2k+1}$.

Now, as the $t-3$ vertices $w_{k,m}$ are not exposed (because they have $t-1$ pendant vertices incident to them), the triangles they are part with vertices of U and vertices of B need to be covered by a single tessellation. There are $t-3$ such $w_{k,m}$ vertices incident to each vertex of B and they are part of $t-3$ tessellations different from a, b, and c. Therefore, all vertices of B are part of these $t-3$ tessellations and it remains only three tessellations (a, b, and c) to cover the edges of the original graph B, i.e., if H is t-tessellable, then B is 3-tessellable.

Conversely, if B is 3-tessellable, we can cover the edges of the triangles in vertices $w_{k,m}$ with $t-3$ tessellations not used in B. Now, we can cover the edges of triangles $u_k, p_{1,2k}, p_{1,2k+1}$ with one of the three remaining tessellations a, b or c. Without loss of generality, let it be the tessellation a, the triangles $u_k, p_{2,2k}, p_{2,2k+1}$ be covered by tessellation b and the triangles $u_k, p_{3,2k}, p_{3,2k+1}$ be covered by tessellation c. Now we can cover the $t-1$ triangles which uses the edges of type $(p_{2,2j-1}, p_{2,2j})$ $(1 \le j \le |V(G)|)$ with the $t-1$ tessellations different from the one used to cover the edges $(p_{2,2j}, p_{2,2j+1})$. The remaining edges of pendant vertices are trivially covered by the non-used tessellations. Therefore, if B is 3-tessellable, then H is t-tessellable. □

A graph is (k, ℓ) if its vertex set can be partitioned into at most k stable sets and at most ℓ cliques. Next, we show a polynomial transformation from the $\mathcal{NP}$-complete 3-COLORABILITY [9] to 4-TESSELLABILITY of chordal $(2, 1)$-graphs, and then we generalize this proof for any fixed $t \ge 4$. This proof is based on a result of Bodlaender et al. [19] for 3-$L(0, 1)$-COLORABILITY of split graphs.

Construction 5. Let H be a graph obtained from a non-bipartite graph G as follows. Initially $V(H) = V(G) \cup E(G)$ and $E(H) = \emptyset$. Add edges to H so that the $E(G)$ vertices induce a clique. For each $e = vw \in E(G)$, add to H edges ve and we. For each vertex $v \in V(H) \cap V(G)$, add three pendant vertices adjacent to v. Add a vertex u adjacent to all $E(G)$ vertices. Add three pendant vertices adjacent to u. Denote all pendant vertices by V_2. See Fig. 11.

Theorem 9. *The t-TESSELLABILITY of chordal $(2, 1)$-graphs is $\mathcal{NP}$-complete, for any fixed $t \ge 4$.*

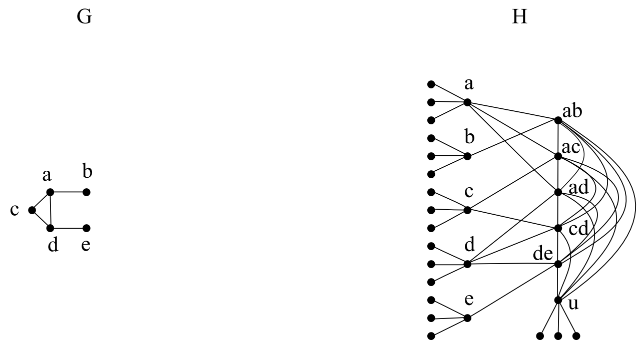

Fig. 11. Example of Construction 5.

Proof. Firstly, we show that the 4-TESSELLABILITY is $\mathcal{NP}$-complete for chordal $(2,1)$-graphs, and then, we extend to the cases when $t \geq 4$, for any fixed t.

Let G be a non-bipartite instance graph of 3-COLORABILITY. We show that G is 3-colorable if and only if H, obtained by Construction 5 on G, is 4-tessellable.

Let f be a 3-coloring of G. Consider the following tessellation cover: for any vertex $v \in V(G)$, cover the maximal clique it belongs with vertices of $E(G)$ in H with tessellation $f(v)$, and use tessellation 4 to cover the maximal clique of vertices of $E(G)$ with vertex u. Now, cover the three pendant vertices of each vertex of $V(G)$ and u with their 3 non incident tessellations. Note that all edges of H were covered and two maximal cliques between vertices of $V(G)$ and $E(G)$ in H can only share a vertex in $E(G)$. However, if the maximal cliques of these vertices share a vertex in $E(G)$, it means these two vertices are adjacent in G and, therefore, their maximal cliques are covered by different tessellations.

Conversely, consider a tessellation cover of H with 4 tessellations. We need the maximal clique given by u and the vertices of $E(G)$ not be exposed. Additionally, the tessellation used to cover it cannot cover any other maximal clique between vertices of $V(G)$ and $E(G)$. Therefore, there are only three remaining tessellations to cover them.

Each vertex of $V(G)$ in H has 4 maximum cliques incident to them sharing only one vertex. Thus, all maximal cliques incident to them must not be exposed. Note that if two vertices v and w of G are adjacent, then their related maximal cliques between vertices of $V(G)$ and $E(G)$ share a vertex vw. Therefore, the colors of $f(v)$ and $f(w)$, which are related to the tessellations that cover these maximal cliques, are different.

This proof holds for t-TESSELLABILITY, with $t \geq 5$, of chordal $(2,1)$-graphs. The idea is to use the same proof considering $(t-1)$-COLORABILITY of G instead of 3-COLORABILITY, and adding the necessary number of pendant vertices to H to force all its maximal cliques not to be exposed.

Note that the vertices of H can be partitioned into one clique and two stable sets: The vertices in H related with $E(G)$ and vertex u define a clique, the vertices in H related with $V(G)$ define a stable set, and the pendant vertices define another stable set. Moreover, clearly H is chordal as the induced graph by the vertices related with $E(G)$ and $V(G)$ is a split graph (a subclass of chordal graph), and the addition of pedant vertices does not create any cycles in the graph, i.e., H is chordal. □

Construction 6. Let H' be a graph obtained from the graphs G and H of Construction 5 by transforming the stable set S of H corresponding to $V(G)$ into a clique, removing one pendant vertex of each vertex of S, and adding a vertex u' adjacent to all vertices of S with three new pendant vertices adjacent to it. See Fig. 12.

Theorem 10. *The t-TESSELLABILITY of $(1,2)$-graphs is $\mathcal{NP}$-complete, for any fixed $t \geq 4$.*

Proof. Firstly, we show that the 4-TESSELLABILITY is $\mathcal{NP}$-complete for $(1,2)$-graphs, and then, we extend to the cases when $t \geq 4$, for any fixed t.

Consider the graph H', obtained from Construction 6 on graph H of Theorem 9 for 4-TESSELLABILITY. Clearly, H' is a $(1,2)$-graph. We will show that H is 4-tessellable if and only if H' is 4-tessellable.

In the 4-tessellation cover given by the proof of Theorem 9, an edge of a pendant vertex of each of $V(G)$'s vertices is covered by tessellation 4 (the same tessellation of the maximal clique of u and the vertices of $E(G)$). Define 3 tessellations of H' using the first three tessellations of H. Now, we cover the edges in the maximal clique of $V(G)$'s vertices and u' with tessellation 4 and the three remaining edges of the pendant vertices incident to u' with tessellations 1, 2, and 3.

Consider a tessellation cover of H' with 4 tessellations. First, the maximal clique of vertices of $V(G)$ and u' must be covered by the same tessellation of the maximal clique of vertices of $E(G)$ and u. For the sake of the contradiction, assume these two maximal cliques are covered by different tessellations. Therefore, now there are only two available tessellations to cover maximal cliques between vertices of $V(G)$ and $E(G)$ in H'. However, these maximal cliques are related to a coloring of vertices of G and if we could obtain a tessellation cover of them using only two colors, then G would be a bipartite graph (which we exclude from the 3-COLORABILITY instance graphs), a contradiction.

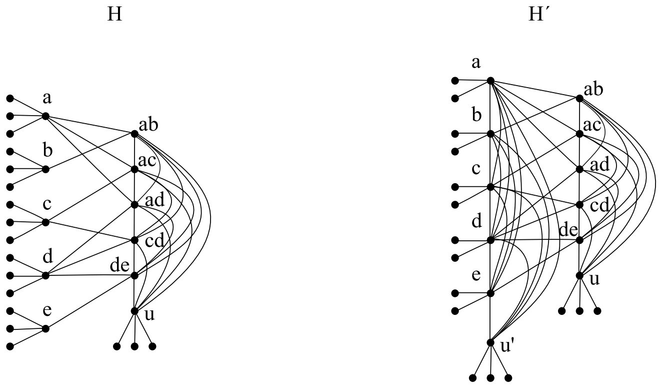

Fig. 12. Example of Construction 6.

Now we obtain a tessellation cover of H with the same number of tessellations as follows. We remove the edges of the maximal clique of $V(G)$ and u' (which are all covered by the tessellation 4). Then, we remove the vertex u' and its pendant vertices. Moreover, we add a pendant vertex to each vertex of $V(G)$ with the tessellation 4 covering their edges.

This proof holds for t-TESSELLABILITY with $t \geq 5$ of $(1,2)$-graphs considering the t-TESSELLABILITY of chordal $(2,1)$-graphs with $t \geq 5$ presented in Theorem 9. Moreover, the $\mathcal{NP}$-completeness of $(t-1)$-COLORABILITY for non-$(t-2)$-colorable graphs, for $t \geq 5$, holds by the following facts: (1) an edge coloring of a graph Γ is equivalent to a vertex coloring of its line graph $L(\Gamma)$; (2) the k-EDGE-COLORABILITY problem is $\mathcal{NP}$-complete for any fixed $k = \Delta(\Gamma) \geq 3$ [20], and; (3) the line graph of a graph Γ is non-$(\Delta(\Gamma)-1)$-colorable because a vertex of degree $\Delta(\Gamma)$ of Γ implies a clique of size $\Delta(\Gamma)$ in $L(\Gamma)$. □

Next, we show a polynomial transformation from the $\mathcal{NP}$-complete problem NAE 3-SAT [9] to 3-TESSELLABILITY of diamond-free graphs with diameter at most five. The NAE 3-SAT problem consists in finding, in a given set U of literals and a set C of clauses all of size three, if we can assign true/false values to each literal in U satisfying all clauses in C, with the restriction that in each clause at least one literal must have true value and at least one literal must have false value. Hereinafter, we do not consider clauses with two repeated variable, i.e., if there is a clause (v_1, v_1, v_2), we create a new instance I' with a new variable x and exchange the previous clause for two clauses (x, v_1, v_2) and $(\overline{x}, v_1, v_2)$, such that I is satisfiable if and only if I' is satisfiable.

This proof is given in two phases: given an instance I of NAE 3-SAT we construct a graph B for which we show that there is a 3-coloring of B if and only if I is satisfiable; subsequently, we show that there is a construction of a diamond-free graph G with diameter at most five for which G is 3-tessellable if and only if $K(G)$ is 3-colorable and $K(G)$ is isomorphic to B.

Construction 7. Let B be a graph obtained from an instance of NAE 3-SAT as follows. For each variable v of I, include a P_2 with vertices v and $\overline{v}$ in B. Moreover, add a vertex u adjacent to all P_2's vertices. And, for each clause $\{a \vee b \vee c\}$ of I, add a triangle with vertices T_a, T_b, T_c in B and three edges aT_a, bT_b, and cT_c. See Fig. 13.

Lemma 6. *Let B be obtained from Construction 7 on a* NAE 3-SAT *instance I. Then B is 3-colorable if and only if I is satisfiable.*

Proof. If B is 3-colorable, then there are no three vertices connected to a clause's triangle with the same color. Moreover, without loss of generality, the color 1 given to the vertex u in a 3-coloring cannot be used in any vertex of a P_2. Besides, each one of the literal vertices v and $\overline{v}$ of a P_2 receives either the color 2 or 3. Assume without loss of generality that a literal is true if its color is 2, and false otherwise. Therefore, the above assignment of values to literals gives a satisfiable solution to the instance.

Conversely, if I is satisfiable, then one may assign color 2 to each literal vertex which is true and color 3 to its negation. Moreover, vertex u receives color 1. Since there are no three literal vertices with the same color adjacent to the clause triangles, one may assign colors to the vertices of the triangles in a 3-coloring where one vertex of the triangle adjacent to a vertex with color 2 receives color 3, and one vertex adjacent to a vertex with color 3 receives color 2. The remaining vertex receives color 1. □

Next, we construct a graph G whose clique graph $K(G)$ is isomorphic to graph B obtained from Construction 7.

Construction 8. Let G be obtained from the graph B (of Construction 7), which is isomorphic to the clique graph $K(G)$ of G, as follows. For each clause's triangle in B, add a star with three leaves in G, where each of those leaves represents a literal of this clause. Next, all P_2's triangles in B are represented in G by a clique C of size the number of P_2's. Each vertex of this

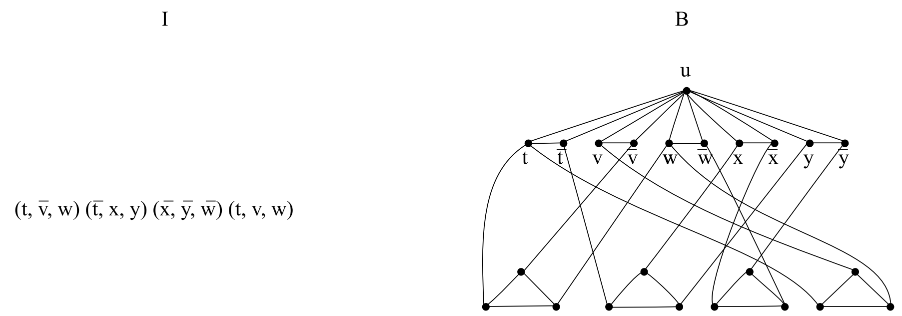

Fig. 13. Example of Construction 7.

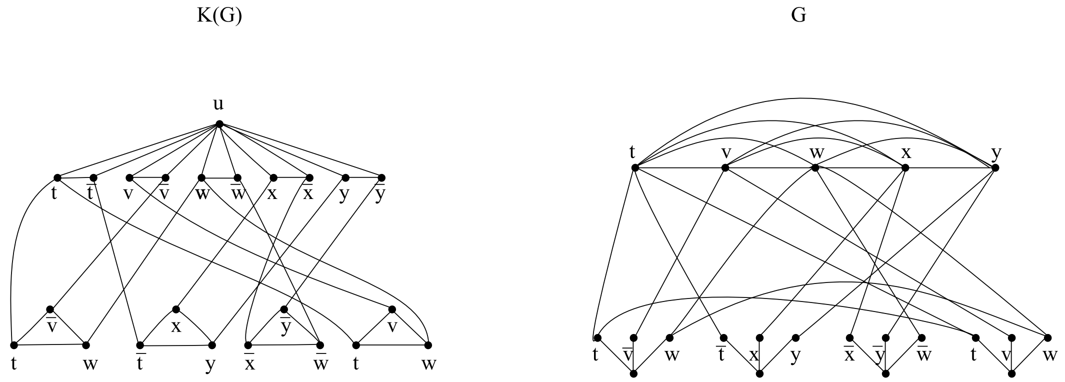

Fig. 14. Example of Construction 8, where $K(G)$ is isomorphic to graph B (Construction 7).

clique C represents a variable of I (of Construction 7). For each vertex v of C include the edges of two other cliques (one for each literal of the variable v) composed by the leaves of the stars which represent the literals v and $\overline{v}$ and the vertex v of C, as depicted in Fig. 14.

Lemma 7. *Let B be obtained by Construction 7 on a* NAE 3-SAT *instance I and G be obtained by Construction 8 on B, such that B is isomorphic to $K(G)$. Then G is 3-tessellable if and only if $K(G)$ is 3-colorable.*

Proof. If G is 3-tessellable, then we need one tessellation to cover the clique composed by the vertices related with the variables, whose size is the number of variables. Therefore, the other two tessellations are used by at most two maximal cliques of each variable, which represent their literals. Moreover, the star of three leaves of each clause also needs to be covered by 3 tessellations. Note that these maximal cliques represent vertices in $K(G)$ and the tessellations represent their colors. Therefore, $K(G)$ is 3-colorable.

If $K(G)$ is 3-colorable, by Theorem 1 G is 3-tessellable. □

Clearly, the graph G obtained from Construction 8 is diamond-free with diameter at most five. Therefore, by Lemmas 6 and 7, the next theorem follows.

Theorem 11. 3-TESSELLABILITY *of diamond-free graphs with diameter at most five is $\mathcal{NP}$-complete.*

4.2. 2-TESSELLABILITY

Portugal [4] showed that a graph G is 2-tessellable if and only if $K(G)$ is a bipartite graph. Moreover, Peterson [6] showed that $K(G)$ is bipartite if and only if G is the line graph of a bipartite multigraph. Hence, determine if G is 2-tessellable is equivalent to verifying if G is the line graph of a bipartite multigraph.

Protti and Szwarcfiter [21] showed an $O(n^2m)$ time algorithm to decide if the clique graph of a given graph is bipartite. Moreover, Peterson [6] showed an $O(n^3)$ time algorithm to decide if G is the line graph of a bipartite multigraph.

A vertex u is *true twin* of a vertex v of a graph G if u and v have the same closed neighborhood in G. The key idea of Peterson's algorithm is to group true twin vertices of a same clique of a line graph G. These true twin vertices represent multiedges in the bipartite multigraph H, where $G = L(H)$. Then, it removes all those true twin vertices in each group but one, and the resulting graph is a line graph of a bipartite simple graph if and only if $K(G)$ is a bipartite graph. To verify if a graph is a line graph of a bipartite graph, the Roussopoulos' linear-time algorithm is used [22].

We improve Peterson's algorithm [6], by showing a faster way to remove true twin vertices belonging to a clique of a graph using its modular decomposition. Throughout this section, we use notations of modules of a graph given in [23]. In a graph G, a subset S of $V(G)$ is a emphmodule if all elements of S have the same set of neighbors among vertices that are in $V(G)\backslash S$. We say that S is a *strong module* if for every module S', $S \cap S' = \emptyset$, or $S \subseteq S'$, or $S' \subseteq S$ holds. A strong module $S \subsetneq V(G)$ is a maximal strong module if the only strong module properly containing S is $V(G)$.

Let $\mathcal{F}$ be the family of bipartite multigraphs obtained by adding multiple edges to C_4, S_n or P_4. In order to make a modular decomposition of a graph G, we only consider graphs G which are not line graphs of a graph in $\mathcal{F}$. If G is a line graph of a graph in $\mathcal{F}$, we can consider this case separately, and easily achieve linear time. Note that there are bipartite multigraphs with a same line graph. Therefore, we only consider the ones which maximize the number of multiple edges. Moreover, we only consider connected graphs, since the tessellation cover number of a disconnected graph is the maximum among the parameter on its connected components.

Lemma 8. *Let H be a bipartite multigraph not in $\mathcal{F}$ and $L(H)$ be its line graph. Two edges e_1 and e_2 with same endpoints in H represent vertices in a same maximal strong module of $L(H)$.*

Proof. By hypothesis, we consider H a bipartite multigraph. Therefore, we do not consider the cases H has an induced cycle of odd size, including triangles and other complete graphs. For the sake of the contradiction, assume there are such two edges e_1 and e_2 of H and that its related vertices in $L(H)$ are in different maximal strong modules M_1 and M_2. Since the vertices associated to e_1 and e_2 are adjacent in $L(H)$, there are all edges between vertices of M_1 and M_2 in $L(H)$.

(Case 1) Assume there is a vertex outside M_1 and M_2. Therefore, without loss of generality, there is a vertex $e_3 \notin (M_1 \cup M_2)$ such that $e_3 \in N(w_1)$ for all $w_1 \in M_1$ and $e_3 \notin N(w_2)$ for all $w_2 \in M_2$, otherwise, $M_1 \cup M_2$ would be a maximal strong module, a contradiction with the fact that M_1 and M_2 were maximal. However, e_1 and e_2 are multiedges with same endpoints of H, i.e., there cannot be another edge e_3 in H which shares an endpoint with e_1 but does not share one endpoint with e_2, as e_1 and e_2 have the same endpoint vertices, a contradiction.

(Case 2) Assume there is no vertex outside M_1 and M_2, and neither M_1 nor M_2 induces cliques in $L(H)$. Note that all vertices in M_1 and M_2 are adjacent to e_2 and e_1, respectively. Let w_1 and w_1' be two non-adjacent vertices of M_1 and w_2 and w_2' be two non-adjacent vertices of M_2. Therefore, w_1 and w_1' are edges in H that share vertices of e_2 in H, as both of them are non-adjacent vertices which are adjacent to e_2 in $L(H)$. Note that w_2 and w_2' must be adjacent to w_1, w_1', and e_1, i.e., they must have the same endpoints as e_1 and e_2 in H. However, w_2 and w_2' cannot be incident to the same endpoints in H since they are not adjacent in $L(H)$, a contradiction.

(Case 3) Assume there is no vertex outside M_1 and M_2, and M_1 or M_2 induces cliques in $L(H)$. If M_1 and M_2 induces cliques in $L(H)$, then $L(H)$ is a complete graph and H is a multigraph of a P_2 (which is a star multigraph), a contradiction with the fact that H is not a graph in $\mathcal{F}$. Otherwise, without loss of generality, let M_1 induces a clique in $L(H)$. Note that the vertices of M_1 represent multiedges with same endpoints in H, since H is a bipartite multigraph that maximizes the number of multiple edges. Moreover, all vertices of M_2 are adjacent to e_1 (and to e_2). Therefore, the vertices of M_2 in $L(H)$ represent edges in H incident to one of the endpoints of e_1 and e_2 (and two of those edges in different endpoints do not share other same endpoint, or H would not be a bipartite graph). However, this is a contradiction as H is a multigraph of a P_4 which is a graph in $\mathcal{F}$. □

Lemma 9. *Let H be a bipartite multigraph not in $\mathcal{F}$ and $L(H)$ be its line graph. Any maximal strong module in a modular decomposition of $L(H)$ with size less than $|V(L(H))|$ induces a clique in $L(H)$.*

Proof. By hypothesis, we consider H a bipartite multigraph. Therefore, we do not consider the cases H has an induced cycle of odd size, including triangles and other complete graphs. For the sake of the contradiction, assume there is such strong module M_k of $L(H)$, which is not a clique. Note that $|M_k| \geq 2$, and since M_k does not induce a clique, then there are two vertices w_k and w_k' in M_k that are not adjacent. Therefore, as w_k and w_k' are not adjacent in $L(H)$, then w_k and w_k' have different endpoint in H.

(Case 1) All vertices of M_k in $L(H)$ share both endpoints in H with edges w_k or w_k'. As $L(H)$ is connected, there is a vertex i outside M_k that shares one endpoint with w_k and other endpoint with w_k'. Moreover, there could be another vertex j outside M_k that shares one endpoint with w_k and other endpoint with w_k' different from the endpoints of i. However, all other vertices outside M_k must share the same endpoints of i or j, otherwise there would be an edge ~~k~~ in H which shares an endpoint with i (or j) and did not share an endpoint with w_k and w_k', a contradiction. Therefore, H is a multigraph of a C_4 or a P_4, which are in $\mathcal{F}$, a contradiction.

(Case 2) There is a vertex of M_k in $L(H)$ whose endpoints do not coincide with edge w_k nor with edge w_k'.

(Case 2.a) There is a vertex x of M_k that shares one endpoint with w_k and the other with w_k' in H. As $L(H)$ is a connected graph, there is a vertex i outside of M_k adjacent to all vertices in M_k. Note that i must share both endpoints with x in H. Assume there is a vertex j such that no vertex in M_k is adjacent to j in $L(H)$, therefore as $L(H)$ is connected, there is a vertex l adjacent to all vertices of M_k, that is adjacent to a vertex l', where l' is adjacent to no vertex in M_k.

Table 1
Extremal graph classes and tight upper bounds.

Graph class	$T(G) \leq \min\{\chi'(G), \chi(K(G))\}$	Reference
Bipartite	$T(G) = \chi'(G) = \Delta(G)$	Sec. 2.1
Triangle-free	$T(G) = \chi'(G)$	Sec. 2.1
Unichord-free with girth ≥ 15	$T(G) = \chi'(G) = \Delta(G)$	Sec. 2.1
$Wd_{p,q}$	$T(Wd_{p,q}) = \chi(K(Wd_{p,q})) = q$	Sec. 2.2, [5]
G_p, $p \in \{2,3\}$	$T(G_p) = \chi'(G_p) = \chi(K(G_p)) = 2p$	Theorem 2
G_p, any p	$T(G_p) = \chi'(G_p) = 2p$	Theorem 2
$E_{3,p}$	$T(G) = 3$	Theorem 3
Diamond-free K-perfect	$T(G) = \chi(K(G)) = \omega(K(G))$	Theorem 4
Threshold	$T(G) = \chi(K(G)) = \lvert S \rvert + 1$	Theorem 6

However, this is a contradiction, because l shares both endpoints with i, while l' shares one endpoint with l, but l' does not share any endpoint with neither w_k nor w'_k. Therefore, all vertices outside M_k are adjacent to all vertices in M_k, and H is a P_4 multigraph in $\mathcal{F}$.

(Case 2.b) Each vertex y of M_k shares precisely one endpoint either with w_k or with w'_k in H. Note that with a similar reasoning of (Case 2.a) all vertices outside M_k must be adjacent to all vertices of M_k, and H is a P_4 multigraph in $\mathcal{F}$.

(Case 2.c) There is a vertex z of M_k that shares no endpoint with neither w_k nor with w'_k in H. As $L(H)$ is connected, there is a vertex i adjacent to all vertices in M_k, however, i has only two endpoints and it must share at least one endpoint with all edges of z, w_k and w'_k in H, which have distinct endpoints, a contradiction. □

Theorem 12. 2-TESSELLABILITY *can be solved in linear time.*

Proof. First, we use McConnell and Spinrad's linear-time algorithm to obtain a modular decomposition of G. By Lemmas 8 and 9, we know that the strong modules in any modular decomposition of a line graph of a bipartite multigraph $H \notin \mathcal{F}$ induce cliques. Moreover, the vertices of these cliques in $L(H)$ are related to edges of H with same endpoints.

Then, we check if each of at most $O(|V(G)|)$ strong modules induces cliques in G, which can be done in $O(|V(G)| + |E(G)|)$. Otherwise, we know that G is not a line graph of a bipartite multigraph. Next, we remove all true twins vertices in each strong modules but one, obtaining the graph G'. This step is related to remove all multiedges of H which share same endpoints. Therefore, the graph G is a line graph of a bipartite multigraph H if the resulting graph G' is a line graph of a simple bipartite graph H'.

Finally, we use Roussopoulos' linear-time algorithm to determine if G' is a line graph, and if so, obtain its root graph H' whose line graph is isomorphic to G'. Note that verifying if H' is a bipartite graph can be done in linear time by using a breadth-first search (because the size of the root graph of G' is asymptotically bounded by the size of G'). □

5. Concluding remarks and discussion

We investigate the tessellation cover number for extremal graph classes, which are fundamental for the development of quantum walks in the staggered model. These results help to understand the complexity of the unitary operators necessary to express the evolution of staggered quantum walks. We establish tight upper bounds for the tessellation cover number of a graph G related to the chromatic parameters $\chi'(G)$ and $\chi(K(G))$, and we determine graph classes which reach these upper bounds. This study provides tools to distinguish several classes for which the t-TESSELLABILITY problem is efficiently tractable (bipartite graphs, {triangle, proper major}-free graphs, diamond-free K-perfect graphs, and threshold graphs) from others where the problem is $\mathcal{NP}$-complete for $t \geq 3$ (planar graphs, triangle-free graphs, chordal $(2,1)$-graphs, $(1,2)$-graphs, and diamond-free graphs with diameter at most five). We also establish the t-TESSELLABILITY $\mathcal{NP}$-completeness for biplanar graphs. Moreover, we improve to linear-time the known algorithm to recognize line graphs of bipartite multigraphs [21], and consequently, for 2-tessellable graphs [4], and graphs G such that $K(G)$ is bipartite [6]. Table 1 and Table 2 summarize the extremal graph classes and the complexity of the t-TESSELLABILITY problem, respectively, for the graph classes studied in this paper.

We establish an interesting complexity dichotomy between Δ-EDGE COLORABILITY and t-TESSELLABILITY: Δ-EDGE COLORABILITY of planar graphs with $\Delta(G) \geq 8$ is in $\mathcal{P}$ [24], while t-TESSELLABILITY for $t \geq 3$ is $\mathcal{NP}$-complete, (Theorem 7 replacing each of the four non external triangles that share two vertices of external triangles by K_4's) and; Δ-EDGE COLORABILITY of line graph of bipartite graphs for $\Delta \geq 3$ is $\mathcal{NP}$-complete [25], while t-TESSELLABILITY is in $\mathcal{P}$ (Theorem 12). We have not managed yet to establish the same dichotomy between k-COLORABILITY OF CLIQUE GRAPH and t-TESSELLABILITY.

Regarding (k,ℓ)-graph classes, since any (k,ℓ)-graph is a $(k+1,\ell)$-graph and a $(k,\ell+1)$-graph, the $\mathcal{NP}$-completeness of t-TESSELLABILITY for $(1,2)$-graphs and $(2,1)$-graphs imply that the problem is $\mathcal{NP}$-complete for (k,ℓ)-graphs with $k+\ell \geq 3$ and $\min\{k,\ell\} \geq 1$ for $t \geq 4$. We are currently working on the complexity of t-TESSELLABILITY for split graphs that are a super class of threshold graphs of Theorem 6, $(k,0)$-graphs with $k \geq 3$, and $(0,\ell)$-graphs with $\ell \geq 2$.

A question that naturally arises is whether every graph has a minimum tessellation cover such that every tessellation contains a maximal clique. Although we believe in most cases the answer is true, we have computationally found a surprising example of a graph, which is depicted in Fig. 15, with all minimum tessellation covers requiring a tessellation without

Table 2
The complexity of the t-TESSELLABILITY problem for graph classes.

t	Graph class	Complexity	Reference
$t = 2$	Generic	Linear	Theorem 12
$t = 3$	Planar, $\Delta(G) \leq 6$	$\mathcal{NP}$-complete	Theorem 7
	Diamond-free, diameter = 5	$\mathcal{NP}$-complete	Theorem 11
$t \geq 3$	Threshold	Polynomial	Sec. 3
	Bipartite	Polynomial	Sec. 2.1
	{triangle, proper major}-free	Polynomial	Sec. 2.1
	Diamond-free K-perfect	Polynomial	Sec. 3
	Unichord-free with girth ≥ 15	$\mathcal{NP}$-complete	Sec. 2.1
	Triangle-free	$\mathcal{NP}$-complete	Sec. 2.1
	Biplanar	$\mathcal{NP}$-complete	Theorem 8
$t \geq 4$	Chordal $(2, 1)$-graphs	$\mathcal{NP}$-complete	Theorem 9
	$(1, 2)$-graphs	$\mathcal{NP}$-complete	Theorem 10

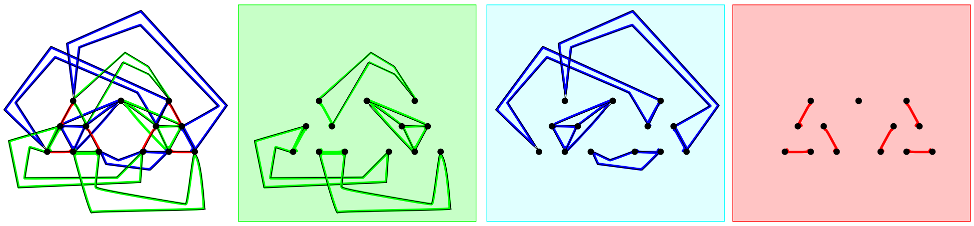

Fig. 15. 3-tessellable graph. Rightmost tessellation does not contain a maximal clique.

maximal cliques. We are currently trying to establish an infinite family of graphs for which this property does not hold and to establish other graph classes where it holds. The computational verification was performed through a reduction from t-TESSELLABILITY problem to SET-COVERING problem (the description of SET-COVERING is available at [9]), where the finite set is the edge set of the input graph, and the family of subsets consists of the edge subsets corresponding to all possible tessellations of the input graph. Another interesting issue is that two minimum tessellation covers may present different quantum walk dynamics. Therefore, we intend to study the different tessellation covers using the same number of tessellations, which may result in simpler quantum walks and more efficient quantum algorithms. More recently, a general partition-based framework for quantum walks has been proposed [26].

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

Acknowledgements

The authors thank the anonymous referees for the diligent reading and several useful suggestions. This study was financed in part by the *Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES)* - Finance Code 001, by the *Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro - Brasil (FAPERJ)* Grant numbers JCNE E-26/203.235/2016, CNE E-26/202.793/2017, CNE E-26/202.872/2018, and by *Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brasil (CNPq)* Grant numbers 140912/2017-7, 303406/2015-1, 407635/2018-1.

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