Theoretical Computer Science 858 (2021) 81–89

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

A computational complexity comparative study of graph tessellation problems

Alexandre Abreu [a,e], Luís Cunha [b], Celina de Figueiredo [a], Luis Kowada [b], Franklin Marquezino [a,*], Renato Portugal [c], Daniel Posner [d]

[a] Universidade Federal do Rio de Janeiro, Av. Horácio Macedo 2030, Centro de Tecnologia, 21941-598 Rio de Janeiro, Brazil
[b] Universidade Federal Fluminense, Av. Gal. Milton Tavares de Souza, Instituto de Computação, 24210-346 Niterói, Brazil
[c] Laboratório Nacional de Computação Científica, Av. Getulio Vargas 333, 25651-075 Petrópolis, Brazil
[d] Universidade Federal Rural do Rio de Janeiro Av. Governador Roberto Silveira, Instituto Multidisciplinar, 22356-000 Nova Iguaçu, Brazil
[e] Fundação Getulio Vargas, R. Barão de Itambi 60, Departamento de Operações, 22231-000 Rio de Janeiro, Brazil

ARTICLE INFO

Article history:
Received 31 July 2019
Received in revised form 28 October 2020
Accepted 26 November 2020
Available online 1 December 2020

Keywords:
Structural graph theory
Computational complexity
Tessellation cover
Equivalence covering

ABSTRACT

A tessellation of a graph is a partition of its vertices into cliques. A tessellation cover of a graph is a set of tessellations that covers all of its edges, and the tessellation cover number, denoted by $T(G)$, is the size of a smallest tessellation cover. The t-TESSELLABILITY problem aims to decide whether a graph G has $T(G) \leq t$. The number of edges of a maximum induced star of G, denoted by $s(G)$, is a lower bound on $T(G)$. In this work we define good tessellable graphs as the graphs G with $T(G) = s(G)$, and we introduce the corresponding GOOD TESSELLABLE RECOGNITION (GTR) problem, which aims to decide whether G is a good tessellable graph. We show that GTR is $\mathcal{NP}$-complete not only if $T(G)$ can be obtained in polynomial time or $s(G)$ is fixed, but also when the gap between $T(G)$ and $s(G)$ is large. We establish graph classes that present distinct computational complexities considering problems related to the parameters $T(G)$ and $s(G)$, and we perform a comparative study of the GTR, t-TESSELLABILITY, and STAR SIZE problems, where the STAR SIZE problem aims to decide whether the number of edges a maximum induced star of a graph is at least a given number.

1. Introduction

It is known that there is a strong connection between the areas of graph theory and quantum computing. For instance, algebraic graph theory provides many tools to analyze the time-evolution of the continuous-time quantum walk, because its evolution operator is directly defined in terms of the graph's adjacency matrix. Recently, a new discrete-time quantum walk model has been defined by using the concept of graph tessellation cover [1]. Each tessellation in the cover is associated with a unitary operator and the full evolution operator is the matrix product of those operators. For practical applications, it is interesting to characterize graph classes that admit small-sized covers. Accordingly, we establish a new lower bound on tessellation cover.

* Corresponding author.
E-mail addresses: santiago@cos.ufrj.br (A. Abreu), lfignacio@ic.uff.br (L. Cunha), celina@cos.ufrj.br (C. de Figueiredo), luis@ic.uff.br (L. Kowada), franklin@cos.ufrj.br (F. Marquezino), portugal@lncc.br (R. Portugal), posner@ufrrj.br (D. Posner).

https://doi.org/10.1016/j.tcs.2020.11.045
0304-3975/

Table 1
Computational complexities of STAR SIZE, t-TESSELLABILITY, and GTR problems and examples of corresponding graph classes.

Behavior \ Problem	STAR SIZE	t-TESSELLABILITY	GTR	Examples
(a)	$\mathcal{P}$	$\mathcal{NP}$-complete	$\mathcal{NP}$-complete	[2,3]
(b)	$\mathcal{P}$	$\mathcal{NP}$-complete	$\mathcal{P}$	[4]
				Sec. 3
(c)	$\mathcal{NP}$-complete	$\mathcal{P}$	$\mathcal{NP}$-complete	Sec. 2
(d)	$\mathcal{NP}$-complete	$\mathcal{P}$	$\mathcal{P}$	Sec. 2
(e)	$\mathcal{NP}$-complete	$\mathcal{NP}$-complete	$\mathcal{P}$	Sec. 3

Throughout this paper we only consider undirected and simple graphs. A *tessellation* of a graph G is a partition of its vertices into cliques, and each clique of a tessellation is called a *tile*. We say a tessellation T is *incident to a vertex* v if v is contained in a clique of size at least 2 in T. A *tessellation cover* of G is a set of tessellations that covers all of its edges. The *tessellation cover number* of G, denoted by $T(G)$, is the size of a smallest tessellation cover of G. If G admits a tessellation cover of size t, then G is t*-tessellable*. The t-TESSELLABILITY problem aims to decide whether G is t-tessellable. We disregard cliques of size one in a tessellation since they play no role in our proofs. Note that a tessellation of a graph G defines a clique cover of G and a coloring of G^c, the complement graph of G.

The *star number*, denoted by $s(G)$, is the number of edges of a maximum induced star of G. Notice that $T(G) \geq s(G)$, since any two edges of an induced star cannot be covered by a same tessellation. We say that G is *good tessellable* if $T(G) = s(G)$, and the GOOD TESSELLABLE RECOGNITION (GTR) problem aims to decide whether a graph is good tessellable.

The known results about the tessellation cover number up to now were related to upper bounds on $T(G)$ and the complexities of the t-TESSELLABILITY problem [2–4]. Abreu et al. [2,5] verified that $T(G) \leq \min\{\chi'(G), \chi(K(G))\}$, and they proved that t-TESSELLABILITY is in $\mathcal{P}$ for quasi-threshold, diamond-free K-perfect graphs, and bipartite graphs. On the other hand, they showed that the problem is $\mathcal{NP}$-complete for triangle-free graphs, unichord-free graphs, planar graphs with $\Delta \leq 6$, $(2,1)$-chordal graphs, $(1,2)$-graphs, and diamond-free graphs with diameter at most five. Surprisingly, all the hardness results presented by Abreu et al. [2,5] for t-TESSELLABILITY aim to decide whether $t = s(G)$, i.e., if the instance graph is good tessellable. Therefore, all their $\mathcal{NP}$-complete proofs for t-TESSELLABILITY also hold for GTR. The only previous $\mathcal{NP}$-completeness result for t-TESSELLABILITY for non good tessellable graphs was presented by Posner et al. [4] for line graphs of triangle-free graphs (where $t = 3$ and $s(G) = 2$).

We recently discovered that the concept of tessellation cover of graphs has been independently studied in the literature for a same problem, named as EQUIVALENCE COVERING by Duchet [6] in 1979. Since the tessellation cover number $T(G)$ and the equivalence covering number $eq(G)$ are the same parameter, we highlight the common results, as follows: $\chi'(G)$ is an upper bound for $T(G)$ [2] and for $eq(G)$ [7]; if G is triangle-free, then $T(G) = \chi'(G)$ [2] and $eq(G) = \chi'(G)$ [8]; if G is triangle-free, then 3-TESSELLABILITY of line graphs $L(G)$ is $\mathcal{NP}$-complete [4] and to decide whether $eq(G) \leq 3$ for the same class is $\mathcal{NP}$-complete as well [8]; if G is $(2,1)$-chordal, then t-TESSELLABILITY is $\mathcal{NP}$-complete for $t \geq 4$ [2], whereas EQUIVALENCE COVERING is $\mathcal{NP}$-complete for $(1,1)$-graphs [7].

1.1. Contributions

We propose the GTR problem, which aims to decide whether a graph is good tessellable. We analyze the combined behavior of the computational complexity of the following problems: STAR SIZE, t-TESSELLABILITY, and GTR. Clearly, these three problems belong to $\mathcal{NP}$.

STAR SIZE	$t -$ TESSELLABILITY	GTR
Instance: Graph G, integer k.	**Instance:** Graph G, integer t.	**Instance:** Graph G.
Question: $s(G) \geq k$?	**Question:** $T(G) \leq t$?	**Question:** $T(G) = s(G)$?

In order to highlight our results, we define graph classes using triples that specify the computational complexities of STAR SIZE, t-TESSELLABILITY, and GTR, summarized in Table 1.

All graph classes for which Abreu et al. [2] presented hardness proofs for t-TESSELLABILITY obey behavior (a), since for those classes $s(G)$ is a known constant and equal to some fixed value of t. (e.g. for planar graphs $t = 3$ and for chordal graphs $t = 4$). The graphs studied by Posner et al. [4] obey behavior (b), since for those graphs $s(G) = 2$ and 3-TESSELLABILITY is $\mathcal{NP}$-complete. In Section 3, we present additional examples that obey behavior (b) with $T(G)$ arbitrarily larger than a non fixed $s(G)$. Graphs of Construction 2.2 (I) in Section 2 are examples that obey behavior (c), since $T(G)$ can be obtained in polynomial time but STAR SIZE is $\mathcal{NP}$-complete for $k = T(G)$, which implies that GTR is $\mathcal{NP}$-complete. Graphs of Construction 2.2 (II) in Section 2 are examples that obey behavior (d), because STAR SIZE is $\mathcal{NP}$-complete for $k = T(G) - 1$, $T(G)$ can be obtained in polynomial time and $T(G) > s(G)$, which implies GTR is in $\mathcal{P}$. Graphs of Construction 3.2 in

Section 3 are examples that obey behavior (e), since it is known that $T(G) > s(G)$, which implies GTR is in $\mathcal{P}$, and we construct graphs so that STAR SIZE and t-TESSELLABILITY are $\mathcal{NP}$-complete.

Notice that there are omitted triples in Table 1. Threshold graphs and bipartite graphs are examples of graph classes that obey behavior $(\mathcal{P}, \mathcal{P}, \mathcal{P})$ [2]. Assuming that $\mathcal{P} \neq \mathcal{NP}$, there are no graphs that obey behavior ($\mathcal{P}, \mathcal{P}, \mathcal{NP}$-complete), since if both STAR SIZE and t-TESSELLABILITY are in $\mathcal{P}$, so is GTR. Graph classes obtained by the union of graphs G_1 and G_2 so that G_1 is in a graph class that obey behavior (a) and G_2 is in a graph class that obey behavior (c) are examples satisfying behavior ($\mathcal{NP}$-complete, $\mathcal{NP}$-complete, $\mathcal{NP}$-complete).

1.2. Notation and graph theory terminologies

Given a graph $G = (V, E)$, the neighborhood $N(v)$ (or $N_G(v)$) of a vertex $v \in V$ of G is given by $N(v) = \{u \mid uv \in E(G)\}$. $\Delta(G)$ is the size of a maximum neighborhood of a vertex of G. We say that a vertex u of G is universal if $|N(u)| = |V(G)| - 1$. A graph is *universal* if it has a universal vertex. A *clique* of G is a subset of V with all possible edges between its vertices. An *independent set* of G is a subset of V with no edge between any of its vertices. A *matching* of G is a subset of edges of E without a common endpoint. A *k-coloring* of G is a partition of V into k independent sets. A *k-clique cover* of G is a partition of V into k cliques. A *k-edge coloring* of G is a partition of E into k matchings.

The parameters $\alpha(G)$, $\omega(G)$, and $\mu(G)$ are the size of a maximum independent set, the size of a maximum clique, and the size of a maximum matching of a graph G, respectively. The *chromatic number* $\chi(G)$ (*chromatic index* $\chi'(G)$) is the minimum k for which G admits a k-coloring (k-edge coloring), and the *clique cover number* $\theta(G)$ is the minimum k for which G admits a k-clique cover. Note that $\theta(G) = \chi(G^c)$ and $\alpha(G) = \omega(G^c)$, where G^c denotes the *complement* of G for which $V(G^c) = V(G)$ and $E(G^c) = \{xy | x \in V(G), y \in V(G), x \neq y\} \setminus E(G)$. The k-COLORABILITY problem (k-EDGE COLORABILITY problem) aims to decide whether a graph G has $\chi(G) \leq k$ ($\chi'(G) \leq k$). The k-INDEPENDENT SET problem aims to decide whether a graph G has $\alpha(G) \geq k$.

The *line graph* $L(G)$ of a graph G is the graph such that each edge of $E(G)$ is a vertex of $V(L(G))$, and two vertices of $V(L(G))$ are adjacent if and only if their corresponding edges in G have a common endpoint. The *clique graph* $K(G)$ of a graph G is the graph such that each maximal clique of G is a vertex of $V(K(G))$, and two vertices of $V(K(G))$ are adjacent if and only if their corresponding maximal cliques in G have a common vertex. $S_k(G)$ is the graph obtained from G by subdividing k times each edge $e = xy \in E(G)$, i.e., each edge $e = xy$ is replaced by a path $(x, v_1, v_2, \ldots, v_k, y)$.

The *union* $G \cup H$ of two graphs G and H has $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The *join* $G \vee H$ of two graphs G and H has $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{vw \mid v \in V(G) \text{ and } w \in V(H)\}$. We often denote by $G \vee \{u\}$ the join of the graph G with a graph containing the single vertex u. An *induced subgraph* $H = (V_H, E_H)$ of a graph $G = (V_G, E_G)$ has $V_H \subseteq V_G$ and $E_H = \{vw \mid v \in V(H), w \in V(H), \text{and } vw \in E(G)\}$. $G[S]$ is the induced subgraph of G by the set of vertices $S \subseteq V(G)$.

2. Graphs whose $T(G)$ can be obtained in polynomial time

We prove in this section that GTR is $\mathcal{NP}$-complete for graphs of Construction 2.2 (I), whose tessellation cover number can be obtained in polynomial time. Using this result, we provide a graph class that obeys behavior (c) and another graph class that obeys behavior (d). Note that if the tessellation cover number of G is upper bounded by a constant, then we obtain $s(G)$ in polynomial time using a brute force algorithm.

The Mycielski graph M_j for $j \geq 2$ has chromatic number j, maximum clique size 2, and is defined as follows. $M_2 = K_2$ and for $j > 2$, M_j is obtained from M_{j-1} with vertices $v_1, \ldots, v_{|V(M_{j-1})|}$ by adding vertices $u_1, \ldots, u_{|V(M_{j-1})|}$ and one more vertex w. Each vertex u_i is adjacent to all vertices of $N_{M_{j-1}}(v_i) \cup \{w\}$.

Construction 2.1. Let i be a non-negative integer and G a graph. The (i, G)-graph is obtained as follows. Add i vertices to graph G, and then add a universal vertex.

Construction 2.2. Let i be a non-negative integer and G a graph with $V(G) = \{v_1, \ldots, v_n\}$. We construct a graph $H = H_1 \cup H_2$ as follows. Add i disjoint copies $G_1, \ldots, G_i$ of G to H_1, such that $V(G_j) = \{v_1^j, \ldots, v_n^j\}$ for $1 \leq j \leq i$, where v_k^j represents the same vertex v_k of G for $1 \leq k \leq n$. Add to H_1 all possible edges between pairs of vertices that represent the same vertex of G. Add a vertex u to H_1 adjacent to all v_k^j for $1 \leq j \leq i$ and $1 \leq k \leq n$. Now, we consider two possibilities: either (I) H_2 is $(|V(G)| - 3, M_3^c)$-graph of Construction 2.1 or (II) H_2 is $(|V(G)| - 3, M_4^c)$-graph of Construction 2.1. Denote the universal vertex of H_2 by u'.

Fig. 1 provides an example of a graph of Construction 2.2 (I). In (a) we have an edge coloring of the graph $G \vee \{x\}$ with $|V(G)|$ colors. In (b) we have the graph $H = H_1 \cup H_2$ and a tessellation cover of H with $|V(G)|$ tessellations.

As a consequence of Constructions 2.1 and 2.2, given a graph H, we obtain the value $|V(G)|$ by counting the number of pendant vertices in H_2. Hence, $|V(G)|$ is obtained in polynomial time.

We now verify that the graphs of Construction 2.2 (I) obey behavior (c) by showing that $T(H)$ is equal to $|V(G)|$ and that deciding whether $s(H) \geq k$ is $\mathcal{NP}$-complete for $k = T(H)$. This also implies that the graphs of Construction 2.2 (II)

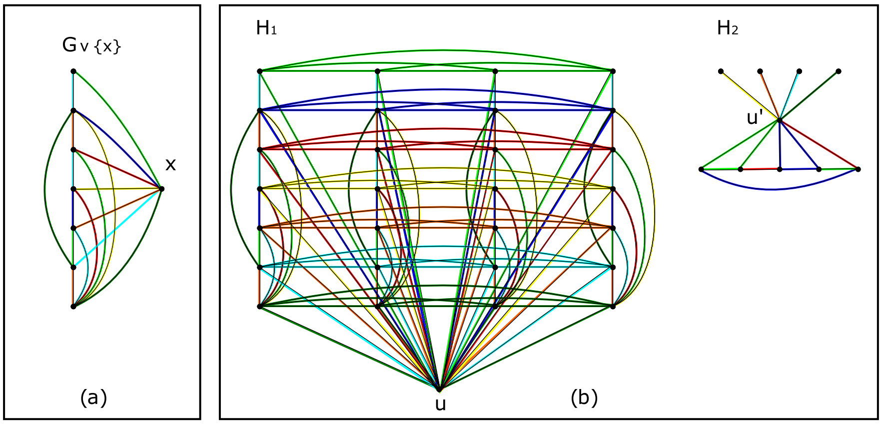

Fig. 1. (a) An edge-coloring of $G \vee \{x\}$. (b) Example of a graph $H_1 \cup H_2$ of Construction 2.2 (I) obtained from graph G.

obey behavior (d), since we have increased $T(H)$ by one unit by replacing M_3^c for M_4^c in H_2. In this case $T(H) > s(H)$ and GTR is in $\mathcal{P}$ with answer always no, whereas to decide whether $s(H) \geq k$ remains $\mathcal{NP}$-complete for $k = T(H) - 1$.

Theorem 2.1. *STAR SIZE and GTR are $\mathcal{NP}$-complete for graphs of Construction 2.2 (I).*

Proof. Let G be a graph without a universal vertex and an instance of the q-COLORABILITY problem, a well-known $\mathcal{NP}$-complete problem [9]. Consider the graph $H = H_1 \cup H_2$ of Construction 2.2 (I) on G with $i = q$.

We need 3 tessellations to cover the edges of $M_3^c \vee \{u'\}$, and another $|V(G)| - 3$ tessellations to cover the remaining edges of the pendant vertices, thus, by construction, $T(H_2) = |V(G)|$. Moreover, since $\alpha(M_3^c) = 2$, then $s(H_2) = |V(G)| - 1$.

We define a tessellation cover of H_1 with $|V(G)|$ tessellations as follows. Consider an optimum edge-coloring of the graph $G \vee \{x\}$. Since G has no universal vertex, x is the unique universal vertex and we know that $\chi'(G \vee \{x\}) = \Delta(G \vee \{x\}) = |V(G)|$ [10]. Now, when we remove x and the edges incident to it, we can use this edge coloring to obtain a tessellation cover of G with size $|V(G)|$ by taking edges that have a same color into the same tessellation. Note that for each vertex of G, there exists a tessellation that is not incident to it in this tessellation cover. We now use this tessellation cover to each copy of G in H_1. Next, we entirely cover each clique between vertices that represent the same vertex of G and the edges incident to u with the available tessellation for this clique. Therefore, $T(H_1) \leq |V(G)|$.

We have $T(H) = \max\{T(H_1), T(H_2)\} = |V(G)| = \frac{|V(H_1)|-1}{q}$ and $s(H) = \max\{s(H_1), s(H_2)\}$. Since $s(H_2) = |V(G)| - 1$, H is good tessellable if and only if $s(H_1) = |V(G)|$. Chvátal [11] proved that a graph G admits a q-coloring if and only if $\alpha(H_1 \setminus \{u\}) = |V(G)|$. Since $s(H_1) = \alpha(H_1 \setminus \{u\})$, deciding whether H is good tessellable is equivalent to deciding whether G is q-colorable. □

3. Universal graphs

The local behavior of tessellation covers given by Lemma 3.1 motivates the study of the problem on universal graphs, since the induced subgraph $G[\{v\} \cup N(v)]$ is a universal graph. We prove that t-TESSELLABILITY remains $\mathcal{NP}$-complete even if the gap between $T(G)$ and $s(G)$ is large. Using this proof, we provide a graph class that obeys behavior (e).

Given a t-tessellable graph G and a vertex $v \in V(G)$, we consider the relation between $\chi(G^c[N_G(v)])$ and the tiles of those t tessellations that share a same vertex v. Note that these tiles cover all edges incident to v in any tessellation cover of G. Moreover, the vertices of the neighborhood of v in a same tile are a clique in G and, therefore they are an independent set in G^c. The independent sets in G^c given by these tiles of $N_G(v)$ may share some vertices, and we can choose whichever color class they belong in such coloring of $G^c[N_G(v)]$. Therefore, for any vertex v of G, $\chi(G^c[N_G(v)]) \leq t$. Since $s(G[v \cup N_G(v)]) = \omega(G^c[N_G(v)])$, $s(G[v \cup N_G(v)]) = \omega(G^c[N_G(v)]) \leq \chi(G^c[N_G(v)]) \leq t$, and we have the following result.

Lemma 3.1. *If G is a t-tessellable graph, then*

$$\max_{v \in V(G)} \{s(G[v \cup N_G(v)])\} \leq \max_{v \in V(G)} \{\chi(G^c[N_G(v)])\} \leq t.$$

Let $u \notin V(G)$ be a vertex. If $G \vee \{u\}$ is a t-tessellable graph, then

$$s(G \vee \{u\}) = \alpha(G) \leq \chi(G^c) \leq t.$$

Next, we show that

$$\chi(G^c) \le T(G \vee \{u\}) \le \chi(G^c) + \Delta(G) + 1. \tag{1}$$

The lower bound of Equation (1) is given by Lemma 3.1. For the upper bound we obtain a tessellation cover with $\chi(G^c) + \Delta(G) + 1$ tessellations as follows. Consider a partition of vertices of G in $p_1, \ldots, p_i$ cliques with $i = \chi(G^c)$. For each $1 \le j \le \chi(G^c)$ we assign a tile $\{u\} \cup p_j$ to the tessellation j. The remaining edges of G are covered by tiles of size 2 with the unused $\Delta(G) + 1$ tessellations $\chi(G^c) + 1, \ldots, \chi(G^c) + \Delta(G) + 1$ described by an $\Delta(G) + 1$ edge coloring of G. Thus, there is no universal graph $G \vee \{u\}$ such that the gap between $T(G \vee \{u\})$ and $\chi(G^c)$ is larger than $\Delta(G) + 1$. In particular, if $\chi(G^c) \ge 2\Delta(G) + 1$, then by Theorem 3.1 below $T(G \vee \{u\}) = \chi(G^c)$.

Theorem 3.1. *A graph $G \vee \{u\}$ with $\theta(G) \ge 2\Delta(G) + 1$ has $T(G \vee \{u\}) = \theta(G)$.*

Proof. Note that $\theta(G) = \chi(G^c)$. Consider a graph $G \vee \{u\}$. By Lemma 3.1, $T(G \vee \{u\}) \ge \chi(G^c)$. We prove that $T(G \vee \{u\}) \le \chi(G^c)$. Since $\chi(G^c) \ge 2\Delta(G) + 1$, there is a tessellation cover of $G \vee \{u\}$ with $\chi(G^c)$ tessellations as follows. We first repeat the process of partitioning the vertices of G into cliques $p_1, \ldots, p_i$ with $i = \chi(G^c)$ and assigning tiles $\{u\} \cup p_j$ to the tessellation j, for $1 \le j \le \chi(G^c)$.

Now, the maximum number of tessellations incident to the endpoints of an uncovered edge xy is $2\Delta(G)$ because 2 tessellations come from the edges ux and uy, and $2\Delta(G) - 2$ come from the edges of G incident to x and y. Therefore, $T(G \vee \{u\}) \le \chi(G^c)$ because it is possible to greedily cover these edges with tiles of size two. □

Corollary 3.1. *A graph $G \vee \{u\}$ with $s(G \vee \{u\}) = \alpha(G \vee \{u\}) \ge 2\Delta(G)+1$ has $T(G \vee \{u\}) = \chi(G^c)$. Moreover, if H is a $(2\Delta(G)+1, G)$-graph of Construction 2.1 on G with $2\Delta(G) + 1$ pendant vertices to u, then $T(H) = \theta(G) = \chi(G^c) + 2\Delta(G) + 1$.*

Proof. Note that if $\alpha(G \vee \{u\}) \ge 2\Delta(G)+1$, then $\chi(G^c) \ge \omega(G^c) = \alpha(G \vee \{u\}) \ge 2\Delta(G)+1$ and, by Theorem 3.1, $T(G \vee \{u\}) = \chi(G^c)$. Consider now the graph H. Each pendant vertex added to u in H increases $s(H)$ by one unit, hence $s(H) \ge 2\Delta(G)+1$. Moreover, the set of pendant vertices in H is a clique in H^c and each vertex is adjacent to all vertices of G in H^c, which implies that $\chi(H^c) = \chi(G^c) + 2\Delta(G) + 1$. Thus, $T(H) = \chi(H^c) = \chi(G^c) + 2\Delta(G) + 1$. □

3.1. Good tessellable universal graphs

A universal graph $G \vee \{u\}$ is good tessellable if $T(G \vee \{u\}) = s(G \vee \{u\})$. In this case, by Lemma 3.1, $T(G \vee \{u\}) = \chi(G^c) = s(G \vee \{u\})$. Therefore, if $G \vee \{u\}$ has $T(G \vee \{u\}) > \chi(G^c)$, then it is not a good tessellable graph. By Corollary 3.1, if $\alpha(G \vee \{u\}) \ge 2\Delta(G) + 1$, then $T(G \vee \{u\}) = \chi(G^c)$, and $G \vee \{u\}$ is good tessellable when $\chi(G^c) = \omega(G^c) = s(G \vee \{u\})$.

The computational complexity of GTR of a subclass of universal graphs depends on the restrictions used to define the subclass. On the one hand, perfect graphs G with $\alpha(G) \ge 2\Delta(G) + 1$ can be recognized in polynomial time [12], and the addition of a universal vertex results in a good tessellable universal graph. On the other hand, planar graphs G with $\Delta(G) \le 4$ and $\alpha(G) \ge 2\Delta(G) + 1 = 9$ for which to decide whether $\chi(G) = \omega(G) = 3$ is $\mathcal{NP}$-complete [9].

3.2. Graphs with arbitrary gap between $T(G)$ and $s(G)$

We start by showing that the gap between $T(G)$ and $s(G)$ can be arbitrarily large for graphs G composed by the join of the complement of Mycielski graphs with a vertex u.

Since the Mycielski graph M_j is triangle-free [13], the graph M_j^c has no independent set of size three and $s(M_j^c \vee \{u\}) = 2$. Moreover, $\chi(M_j) = j$ [13], and by Lemma 3.1, $T(M_j^c \vee \{u\}) \ge \chi((M_j^c)^c) \ge j$. Fig. 2 depicts an example of the Mycielski graph M_4 and the relation between its 4-coloring and a minimal tessellation cover of $M_4^c \vee \{u\}$. Therefore, there is a graph $H = M_j^c \vee \{u\}$ with $s(H) = 2$ and $T(H) \ge j$ for $j \ge 3$.

Now, we describe a subclass of universal graphs for which the gap between $T(G)$ and $s(G)$ is very large. We also show that STAR SIZE and t-TESSELLABILITY are $\mathcal{NP}$-complete for graphs of Construction 3.2, for which GTR is in $\mathcal{P}$.

Construction 3.1. Let $G = (V, E)$ be a graph. Obtain $S_2(G)$ by subdividing each edge of G two times, so that each edge $vw \in E(G)$ becomes a path v, x_1, x_2, w, where x_1 and x_2 are new vertices. Let $L(S_2(G))$ be the line graph of $S_2(G)$. Add a universal vertex u to $L(S_2(G))$, that is, consider the graph $L(S_2(G)) \vee \{u\}$.

First, we show that there is a connection between $T(H)$ of a graph H of Construction 3.1 on G with the size of a maximum stable set of G.

Theorem 3.2. *If $G = (V, E)$ is a graph with $|E(G)| \ge 4$ and $H = (L(S_2(G)) \vee \{u\})$ is obtained from Construction 3.1 on G, then $T(H) = |V(G)| + |E(G)| - \alpha(G)$.*

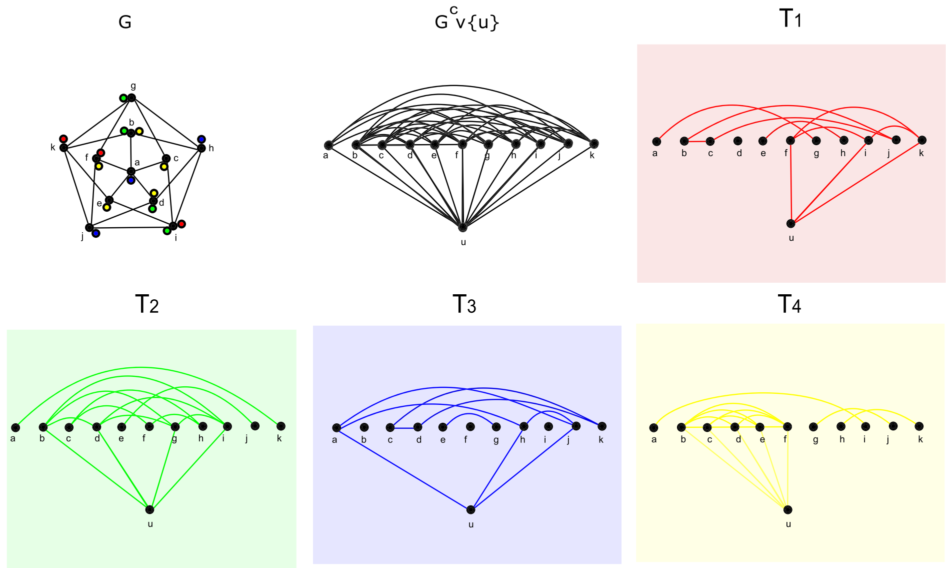

Fig. 2. A Tessellation cover of $M_j^c \vee \{u\}$ with 4 tessellations and possible 4-colorings of M_4 guided by this tessellation cover.

Proof. We claim that $T(H) = \chi((H \setminus \{u\})^c)$. By Lemma 3.1, $T(H) \geq \chi((H \setminus \{u\})^c)$. Now, we obtain a tessellation cover of H with $\chi((H \setminus \{u\})^c)$ tessellations as follows. Consider a partition of the vertices of $(H \setminus \{u\})$ into $\chi(H \setminus \{u\})^c$ cliques. Each clique p_j with $1 \leq j \leq \chi(H \setminus \{u\})^c$ becomes a tile in the tessellation j. Since $H \setminus \{u\} = L(S_2(G))$ is the line graph of a $S_2(G)$ graph, every vertex of $(H \setminus \{u\})^c$ has a maximal clique of size two and another maximal clique incident to it with an arbitrary size. Consider now a maximal clique K_a of size at least three which is not completely covered yet. We cannot have two tiles completely inside K_a (otherwise their merge would result in a coloring of the complement graph with less than its chromatic number). Therefore, we have only one tile using edges of K_a at this moment and the remaining tiles covering the vertices of K_a are the maximal cliques of size two that are incident to the vertices of K_a. Thus, if K_a has only tiles using maximal cliques of size two given by edges incident to K_a, then each edge e of K_a has at most two already used tessellations on cliques incident to their endpoints (the ones given to the tiles of these maximal cliques of size two).

Poljak [14] proved that $\chi(L(S_2(G))^c) = |V(G)| + |E(G)| - \alpha(G)$. Since $|E(G)| \geq 4$ and $\alpha(G) \leq |V(G)|$, we have $|V(G)| + |E(G)| - \alpha(G) \geq 4$ and there is at least one available tessellation for each edge of K_a. We claim that these available tessellations for each edge are enough. First, pick an arbitrary tessellation for each edge. Since the endpoint vertices of any collection of edges of K_a on a same available tessellation do not have these tessellations incident to their endpoints, we cover the clique induced by these vertices as a tile in this tessellation.

Otherwise, K_a is covered by a tile K_b and all the other vertices of K_a must be covered by tiles given by maximal cliques of size two with edges outside K_a. Now, we modify the tessellation cover by including all edges of K_a in the tile of K_b and removing the vertices of K_a from tiles of size two, i.e., now they are tiles of size one and K_a is entirely covered by the tessellation of the tile K_b.

The remaining uncovered edges of $H \setminus \{u\}$ are maximal cliques of size two. Now, if an edge is uncovered and it is incident to a maximal clique of size two or more, then we need this clique to be a tile entirely covered by a single tessellation. Therefore, the maximum number of already used tessellations incident to the endpoints of a remaining edge is three.

Recall that by Lemma 3.1, $T(H) \geq \chi((H \setminus \{u\})^c)$ and by Poljak [14] $\chi((H \setminus \{u\})^c) = \chi(L(S_2(G))^c)) = |V(G)| + |E(G)| - \alpha(G)$. Since $|V(G)| \geq \alpha(G)$ and $|E(G)| \geq 4$, we have $|V(G)| + |E(G)| - \alpha(G) \geq 4$. Therefore, there is always an available tessellation for these edges. Finally, the edges incident to u are included in the tiles of the tessellations of the cliques of the clique cover of $H \setminus \{u\}$. Thus, $T(H) \leq \chi((H \setminus \{u\})^c)$. $\square$

Fig. 3 depicts the proof of Theorem 3.2. In (a), we have graph G. In (b), we have a clique cover of $L(S_2(G))$. In (c), we modify the clique cover so that the tile with label 7 is covered by a new tessellation and at the same time we remove the vertices of the tiles of size two incident to the tile with label 7. Now the tiles with labels 6 and 8 have only one vertex each. Finally, in (d) we obtain a tessellation cover of $L(S_2(G))$ by including the edges incident to u in the tiles related to the clique cover.

Since deciding whether $\alpha(G) \geq k$ is $\mathcal{NP}$-complete [9], by Theorem 3.2 we have the following result for the graphs of Construction 3.1.

Corollary 3.2. *t-TESSELLABILITY is $\mathcal{NP}$-complete for universal graphs.*

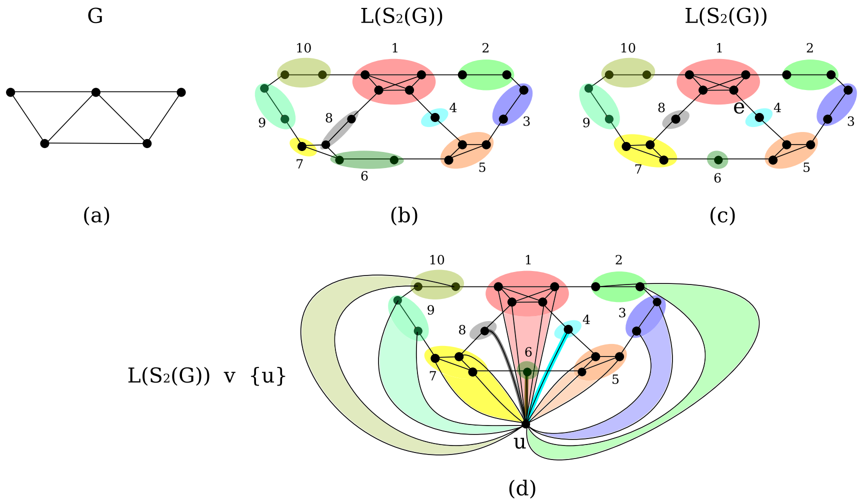

Fig. 3. A tessellation cover of $H = L(S_2(G)) \vee \{u\}$ with $|V(G)| + |E(G)| - \alpha(G)$ tessellations.

Proof. Let G be an instance graph of k-INDEPENDENT SET with $|E(G)| \geq 4$. We know that deciding whether $\alpha(G) \geq k$ is $\mathcal{NP}$-complete [9]. Consider the graph H of Construction 3.1 on G with $H = L(S_2(G)) \vee \{u\}$. By Theorem 3.2, $T(H) = |E(G)| + |V(G)| - \alpha(G)$. Therefore, deciding whether $\alpha(G) \geq k$ is equivalent to deciding whether $T(H) \leq t = |E(G)| + |V(G)| - k$. □

Lemma 3.2. *Let G be an arbitrary graph. We have $s(L(S_2(G)) \vee \{u\}) = |E(G)| + \mu(G)$.*

Proof. Note that $s(L(S_2(G)) \vee \{u\}) = \alpha(L(S_2(G))) = \mu(S_2(G))$. We claim that $\mu(S_2(G)) = |E(G)| + \mu(G)$. Any two adjacent vertices in G are joined by a path of length 3 in $S_2(G)$. In a maximum matching of $S_2(G)$, we need to select at least one of them, otherwise, we could include the middle edge to a maximum matching, which is a contradiction. Moreover, if there is only one edge and it is not a middle edge, then we obtain another maximum matching by replacing this edge by the middle edge. Clearly, we cannot choose three edges and in case we choose two edges, different from the middle edge. The case of two edges forces that both of them are incident to vertices of G in $S_2(G)$.

Therefore, the maximum number of such selection of two edges in $S_2(G)$ is equal to the size of a maximum matching of G. For each edge in a maximum matching $\mu(G)$ of G we have two edges in the maximum matching in $S_2(G)$ and, for each other edge of G, we have one edge in the maximum matching of $S_2(G)$. Thus, $\mu(S_2(G)) = 2\mu(G) + |E(G)| - \mu(G) = |E(G)| + \mu(G)$. □

Next, we show that there are graphs of Construction 3.1 for which the gap between $T(G)$ and $s(G)$ is very large, whereas t-TESSELLABILITY remains $\mathcal{NP}$-complete.

Theorem 3.3. *Let $H = L(S_2(G')) \vee \{u\}$ be obtained from Construction 3.1 on a graph G', where G' is obtained from an arbitrary graph G with $|E(G)| \geq 4$ by adding x universal vertices, with x polynomially bounded by the size of G. To decide whether $T(H) = k$ with $k \geq s(H) + c$, for $c = O(|V(G)|^d)$ and constant d, is $\mathcal{NP}$-complete.*

Proof. By Theorem 3.2, $T(H) = |E(G')| + |V(G')| - \alpha(G')$. By Lemma 3.2, $s(H) = |E(G')| + \mu(G')$. The addition of universal vertices to a graph does not change the size of its maximum independent sets. So, $\alpha(G') = \alpha(G)$. However, the addition of one universal vertex makes the maximum matching size $\mu(G')$ goes up by one until $\mu(G') = |V(G)| - \mu(G)$, after that we need two new universal vertices for $\mu(G')$ to go up by one unit. In that case, we start to increase the difference between $T(H) = |E(G')| + |V(G')| - \alpha(G')$ and $s(H) = |E(G')| + \mu(G')$, since for each two universal vertices we add to G', we increase $T(H)$ by two units and $s(H)$ by one unit. Therefore, we can arbitrarily enlarge the gap between $T(H)$ and $s(H)$. And, as long as the additions of these universal vertices are polynomially bounded by the size of G, it holds the same polynomial transformation of Corollary 3.2 from k-INDEPENDENT SET of G to t-TESSELLABILITY of $H = L(S_2(G')) \vee \{u\}$. □

Finally, we show that the graphs from Construction 3.2 below obey behavior (e).

Construction 3.2. Let H_1 be the graph obtained from Construction 2.2 (I) on a given graph G_1 and a non-negative integer i. Let H_2 be the graph obtained from Construction 3.1 on the graph $G_2 \vee K_{3|V(G_1)|}$ of a given graph G_2. Let u and u' be the two universal vertices of the two connected components of H_1. Add $s(H_2)$ degree-1 vertices to H_1 adjacent to u and $s(H_2)$ degree-1 vertices adjacent to u'. Consider $H_1 \cup H_2$.

Theorem 3.4. STAR SIZE *and* t-TESSELLABILITY *are* $\mathcal{NP}$*-complete for graphs of Construction 3.2, for which* GTR *is in* $\mathcal{P}$.

Proof. Let G_1 be an instance graph with no universal vertex of the well-known $\mathcal{NP}$-complete problem q-COLORABILITY [9]. Let G_2 be an instance graph of the well-known $\mathcal{NP}$-complete problem p-INDEPENDENT SET with $E(G_2) \geq 4$ [9]. Consider a graph $H = H_1 \cup H_2$ obtained from Construction 3.2 on G_1 and G_2 with $i = q$.

Since H_2 is obtained from Construction 3.1 on $G_2 \vee K_{3|V(G_1)|}$, by Theorem 3.3, $T(H_2) - s(H_2) > |V(G_1)|$. By Theorem 2.1, $1 \leq s(H_1) \leq T(H_1) = |V(G_1)|$. The parameter $s(H_2)$ can be obtained in polynomial time by applying a maximum matching algorithm [9] (see Theorem 3.3). And the addition of the degree-1 vertices to H_1 of Construction 3.2 implies that $1 + s(H_2) \leq s(H_1) \leq T(H_1) = |V(G_1)| + s(H_2)$.

Therefore, $H = H_1 \cup H_2$ is a graph that obeys $s(H_2) \leq s(H_1) \leq T(H_1) \leq T(H_2)$ with $T(H) = T(H_2)$ and $s(H) = s(H_1)$. The proof holds because GTR is in $\mathcal{P}$ with answer always no and both STAR SIZE on graphs H_1 of Construction 2.2 (I) (see Theorem 2.1) and t-TESSELLABILITY on graphs H_2 of Construction 3.1 (see Theorem 3.3) are $\mathcal{NP}$-complete. □

4. Concluding remarks

The concept of tessellation cover of graphs appeared in a thesis by Duchet [6], and subsequently in [7,8], as EQUIVALENCE COVERING. The known results about tessellation cover number of a graph up to now were related to upper bounds of the values of $T(G)$, and the complexities of the t-TESSELLABILITY problem [2]. In this work we focus on a different approach by analyzing the tessellation cover number $T(G)$ with respect to $s(G)$, one of its lower bounds, which implicitly appeared in the previous hardness proofs of [2].

The motivation to define the tessellation cover number comes from the analysis of the dynamics of quantum walks on a graph G in the context of quantum computation [1]. Since it is advantageous to implement physically as few operators as possible in order to reduce the complexity of the quantum system, it is important to analyze the gap between $T(G)$ and $s(G)$.

We have proposed the GOOD TESSELLABLE RECOGNITION problem (GTR), which aims to decide whether a graph G satisfies $T(G) = s(G)$, and we have analyzed the combined behavior of the computational complexities of the problems STAR SIZE, t-TESSELLABILITY, and GTR. We have defined graph classes corresponding to triples which specify the computational complexities of these problems, summarized in Table 1. We have defined graph classes in Construction 2.2 (I) and Construction 2.2 (II) that obey behaviors ($\mathcal{NP}$-complete, $\mathcal{P}$, $\mathcal{NP}$-complete) and ($\mathcal{NP}$-complete, $\mathcal{P}$, $\mathcal{P}$), respectively. Graphs that obey behavior ($\mathcal{NP}$-complete, $\mathcal{NP}$-complete, $\mathcal{P}$) are obtained using Construction 3.2. We also note that there are omitted triples in Table 1, which are either empty or easy to provide examples, as described in Section 1.

We are interested in the following two research topics: (i) The concept of good tessellable graphs can be extended to *perfect tessellable graphs*, the graphs G for which $T(H) = s(H)$ for any induced subgraph H of G. A natural open task is to establish the characterization by forbidden induced subgraphs and a polynomial-time recognition algorithm for perfect tessellable graphs. We conjecture that this class is exactly the {gem, W_4, odd cycles}-free graphs; (ii) We have already established relations between $T(G)$ with other well-known graph parameters such as the chromatic number and the maximum size of a stable set. We are currently investigating further relations such as those between $T(G)$ with the chromatic index and the total chromatic number.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

Acknowledgements

We are grateful to the editor and to the referees for the diligent reading, for their time and effort in handling our submission. It is a pleasure for us to contribute to the TCS Special Issue celebrating the Silver Jubilee of the IIT Guwahati. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, by the Fundação de Amparoà Pesquisa do Estado do Rio de Janeiro - Brasil (FAPERJ) Grant numbers JCNE E-26/203.235/2016, CNE E-26/202.793/2017, CNE E-26/202.872/2018, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brasil (CNPq) Grant numbers 140912/2017-7, 303406/2015-1, 407635/2018-1, 308923/2019-7.

References

[1] R. Portugal, R.A.M. Santos, T.D. Fernandes, D.N. Gonçalves, The staggered quantum walk model, Quantum Inf. Process. 15 (1) (2016) 85–101.

[2] A. Abreu, L. Cunha, T. Fernandes, C. de Figueiredo, L. Kowada, F. Marquezino, D. Posner, R. Portugal, The graph tessellation cover number: extremal bounds, efficient algorithms and hardness, in: Proceedings of the 13th Latin American Theoretical Informatics Symposium, in: LNCS, vol. 10807, 2018, pp. 1–13.

[3] A. Abreu, F. Marquezino, D. Posner, Tessellations on graphs with few P_4's, in: Proceedings of VIII Latin American Workshop on Cliques in Graphs, LAWCG, 2018, p. 8.

[4] D. Posner, C. Silva, R. Portugal, On the characterization of 3-tessellable graphs, in: Proceedings Series of the Brazilian Society of Computational and Applied Mathematics, CNMAC 6 (2) (2018) 1–7.
[5] A. Abreu, L. Cunha, C. de Figueiredo, L. Kowada, F. Marquezino, D. Posner, R. Portugal, The graph tessellation cover number: chromatic bounds, efficient algorithms and hardness, Theor. Comput. Sci. 801 (2020) 175–191.
[6] P. Duchet, Représentations, noyaux en théorie des graphes et hypergraphes, Ph.D. thesis, Univ. Paris 6, 1979.
[7] A. Blokhuis, T. Kloks, On the equivalence covering number of splitgraphs, Inf. Process. Lett. 54 (5) (1995) 301–304.
[8] L. Esperet, J. Gimbel, A. King, Covering line graphs with equivalence relations, Discrete Appl. Math. 158 (17) (2010) 1902–1907.
[9] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman Co, 1979.
[10] M. Plantholt, The chromatic index of graphs with a spanning star, J. Graph Theory 5 (1981) 45–53.
[11] V. Chvátal, Edmonds polytopes and a hierarchy of combinatorial problems, Ann. Math. (2006) 51–229.
[12] M. Grotschel, L. Lovasz, A. Schruver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (2) (1981) 169–197.
[13] D. West, Introduction to Graph Theory, Pearson, 2000.
[14] S. Poljak, A note on stable sets and colorings of graphs, Comment. Math. 15 (1974) 307–309.