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MaxCut on permutation graphs is NP-complete

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Abstract

The decision problem MAXCUT is known to be NP-complete since the seventies, but only recently its restriction to interval graphs has been announced to be hard by Adhikary, Bose, Mukherjee, and Roy. Building on their proof, in this paper we prove that the MAXCUT problem is NP-complete on permutation graphs. This settles a long-standing open problem that appeared in the 1985 column of the *Ongoing Guide to NP-completeness* by David S. Johnson, and is the first NP-hardness entry for permutation graphs in such column.

KEYWORDS

computational complexity, maximum cut, NP-complete, permutation graphs

1 | INTRODUCTION

A *cut* is a partition of the vertex set of a graph into two disjoint parts. The *maximum cut problem* (denoted by MAXCUT, for short) aims to determine a cut with the maximum number of edges for which each endpoint is in a distinct part. The decision problem MAXCUT is known to be NP-complete since the seventies [10], and only recently its restriction to interval graphs has

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been announced to be hard by Adhikary, Bose, Mukherjee, and Roy [1]. This settles a long-standing open problem from the *Ongoing Guide to NP-completeness* by David S. Johnson [13].

In his column, David S. Johnson presented a two-page summary table, with a column for each of the 10 most famous NP-complete graph problems, and a row for each of 30 selected graph classes. Among those graph classes, special emphasis was given to subclasses of perfect graphs and of intersection graphs having broad algorithmic significance. The emphasis was on the restrictions themselves and how they affect the complexity of the considered NP-hard problems. The discussion had focused on the particularly fertile domain of graph theory, where the central open problem at that time was the recognition of perfect graphs.

Many important graph classes are defined or can be characterized by a geometric intersection model. Two particularly well-studied examples are subclasses of perfect graphs: the classes of interval graphs and of permutation graphs [8, 12, 15]. In their respective models, the intersecting objects are line segments in the plane, with different restrictions imposed on their positions. In interval graphs, each line segment must have its endpoints on a single line, while in permutation graphs, their endpoints must lie on two distinct parallel lines.

Besides selecting the recognition of perfect graphs as the famous open problem, in his column, David S. Johnson selected only two others as "open and may well be hard" to resolve problems: Hamiltonian circuit restricted to permutation graphs and edge-coloring restricted to planar graphs. Today, we know that recognition of perfect graphs and Hamiltonian circuit restricted to permutation graphs can both be solved in polynomial time. On the other hand, edge-coloring restricted to planar graphs remains a challenging open problem. Please, refer to [4] for an updated summary table. Surprisingly, after 35 years, the only newly resolved entry for permutation graphs is a Hamiltonian circuit.

The present paper settles a long-standing open problem proposed by Johnson, by providing the first NP-hardness proof for a problem on permutation graphs in Johnson's table.

    Theorem 1. *MAXCUT is NP-complete on permutation graphs.*

Our proof is based on Adhikary et al.'s construction used to prove the NP-completeness of MAXCUT on interval graphs [1]. It is interesting to notice that, among the problems selected by Johnson, MAXCUT is the only one proven NP-complete for both interval graphs and permutation graphs. Despite that, the interval graph constructed by Adhikary et al. is not a permutation graph, and our constructed permutation graph is not an interval graph. Thus, we leave as an open question the complexity of MAXCUT on permutation interval graphs.

Our paper is organized as follows. In Section 1.1, we present the basic concepts and notations. In Section 2, we present the main gadget in the reduction of Adhikary et al. [1], which also plays an important role in our reduction. In Section 3, we present the construction of Adhikary et al. [1] and show that it does not lead to a permutation graph. The presentation of their construction is also useful in Section 4, where we finally present the proof of Theorem 1. In Section 5, we prove that our constructed permutation graph is not an interval graph, and propose the complexity of MAXCUT on permutation interval graphs as an open problem.

1.1 | Preliminaries

In this work, all graphs considered are simple. For missing definitions and notation of graph theory, we refer to [3].

Let G be a graph. A subset $K \subseteq V(G)$ is a *clique* if every two distinct vertices in K are adjacent. A subset $S \subseteq V(G)$ is a *stable set* if no two vertices in S are adjacent. Let X and Y be two disjoint subsets of $V(G)$. We say that X is *complete* to Y if every vertex in X is adjacent to every vertex in Y, and that X is *anticomplete* to Y if there are no edges between X and Y. We let $E_G(X, Y)$ be the subset of $E(G)$ with an endpoint in X and the other endpoint in Y. A *cut* of G is a partition of $V(G)$ into two parts $A, B \subseteq V(G)$, denoted by $[A, B]$. The edge set $E_G(A, B)$ is called the *cut-set* of G associated with $[A, B]$. For each two vertices $u, v \in V(G)$, we say that u and v *are in the same part of* $[A, B]$ if either $\{u, v\} \subseteq A$ or $\{u, v\} \subseteq B$; otherwise, we say that u and v *are in opposite parts of* $[A, B]$. Denote by $\mathsf{mc}(G)$ the maximum size of a cut-set of G. The MAXCUT problem has as input a graph G and a positive integer k, and it asks whether $\mathsf{mc}(G) \geq k$.

Let π and π' be two permutations of the same set, say V. A graph G is called the *intersection graph related to* $\{\pi, \pi'\}$ if $V(G) = V$ and, for each two vertices $u, v \in V(G)$, $uv \in E(G)$ if and only if $u <_\pi v$ and $v <_{\pi'} u$. In this case, we also say that $\{\pi, \pi'\}$ is a *permutation model* of G. A graph is a *permutation graph* if it is the intersection graph related to a permutation model.

Given two permutations π and γ of disjoint subsets X and Y, respectively, we write $\pi\gamma$ to denote the permutation of $X \cup Y$ given by the *concatenation* of π with γ. Also, we write $\overleftarrow{\pi}$ to denote the reverse of the permutation π, that is, if $\pi = (v_1, ..., v_i)$, then $\overleftarrow{\pi} = (v_i, ..., v_1)$. To simplify the notation, given a set Z, we sometimes use the same symbol, Z, to denote also a chosen permutation of the elements of Z. In such cases, $\overleftarrow{Z}$ represents the reverse of the chosen permutation for Z.

An *interval model* is a finite multiset $\mathcal{M}$ of closed intervals of the real line. Let G be a graph and $\mathcal{M}$ be an interval model. An $\mathcal{M}$-*representation* of G is a bijection $\phi : V(G) \rightarrow \mathcal{M}$ such that, for every two distinct vertices $u, v \in V(G)$, we have that $uv \in E(G)$ if and only if $\phi(u) \cap \phi(v) \neq \emptyset$. If such an $\mathcal{M}$-representation exists, we say that $\mathcal{M}$ is an *interval model* of G and that G is an *interval graph*.

We write $i \in [n]$ to mean $i \in \{1, ..., n\}$.

2 | GRAINED GADGET

In this section, we present the notion of *grained gadgets*, which was defined in [6] as a generalization of the so-called V-gadgets and E-gadgets. The latter was introduced by Adhikary et al. [1] to prove the NP-completeness of MAXCUT on interval graphs.

Let x and y be positive integers. An (x, y)-*grained gadget* is a split graph H formed by a clique $K' \cup K''$ of size $2y$ and a stable set $S' \cup S''$ of size $2x$. Additionally, K' is complete to S', K'' is complete to S'', and we have $|K'| = |K''| = y$ and $|S'| = |S''| = x$. Figure 1 depicts an interval representation of an (x, y)-grained gadget. One can readily verify that the intersection
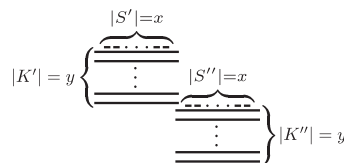

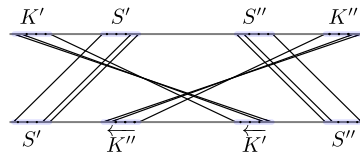**FIGURE 1** Interval representation of an (x, y)-grained gadget, cf. [6].
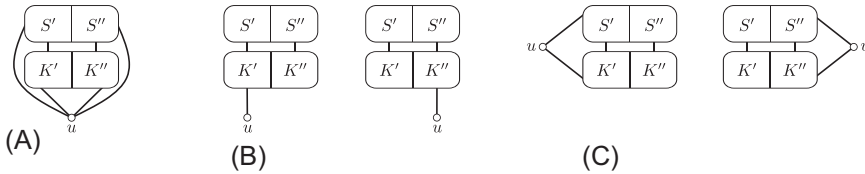
FIGURE 2 A permutation model of a grained gadget.

FIGURE 3 Vertex $u \in V(G) \backslash V(H)$ (A) covering H, (B) weakly intersecting H, and (C) strongly intersecting H. The set $K' \cup K''$ is a clique and the set $S' \cup S''$ is a stable set. A line between sets, or between u and some set, means that all the edges occur. (A) Covering intersection, (B) weak intersection, and (C) strong intersection.

graph related to the pair of permutations $\{K'S'S''K'', S'\overleftarrow{K''}\overleftarrow{K'}S''\}$ (see Figure 2) is an (x, y)-grained gadget. Thus, grained gadgets are interval graphs and permutation graphs.

Let H be an (x, y)-grained gadget and G be a supergraph of H. For each vertex $u \in V(G) \backslash V(H)$, we say that (see Figure 3): u *covers* H if $V(H) \subseteq N_G(u)$; u *weakly intersects* H if either $N_G(u) \cap V(H) = K'$ or $N_G(u) \cap V(H) = K''$; and that u *strongly intersects* H if either $N_G(u) \cap V(H) = K' \cup S'$ or $N_G(u) \cap V(H) = K'' \cup S''$. Moreover, we say that G *respects the structure* of H if, for each vertex $u \in V(G) \backslash V(H)$, either $N_G(u) \cap V(H) = \varnothing$ or u satisfies one of the previous conditions.

The next lemma establishes the key property of grained gadgets with respect to the MAXCUT problem. Intuitively, it states that, for suitable values of x and y, if G is a supergraph that respects the structure of an (x, y)-grained gadget, then, in any maximum cut $[A, B]$ of G, the vertices belonging to $K' \cup S''$ are placed in the same part of $[A, B]$, opposite to the part containing the vertices belonging to $K'' \cup S'$.

Lemma 1 (de Figueiredo et al. [5]). *Let x and y be positive integers, H be an (x, y)-grained gadget and G be a supergraph that respects the structure of H. Let $[A, B]$ be a maximum cut of G. Also, t be the number of vertices in $V(G) \backslash V(H)$ adjacent to some vertex of H, ℓ be the number of vertices of G adjacent to some vertex in S', and r be the number of vertices of G adjacent to some vertex in S''. If ℓ and r are odd, $y > t(x/y - 1)$ and $x > t + 2y$, then each of the following holds:*

1. *$S' \subseteq A$ and $K' \subseteq B$, or vice versa;*
2. *$S'' \subseteq A$ and $K'' \subseteq B$, or vice versa;*
3. *$K' \subseteq A$ and $K'' \subseteq B$, or vice versa.*

In the remainder of the text, when a grained gadget H is not clear in the context, we write $S'(H)$, $S''(H)$, $K'(H)$, and $K''(H)$ to denote the stable sets S' and S'' and the cliques K' and K'' of H, respectively.

3 | ADHIKARY ET AL.'S REDUCTION

In this section, we present the construction given by Adhikary et al. [1] of an interval graph that proves the NP-completeness of MAXCUT in this class. As we see in Section 4, the general idea behind their construction can also be used to obtain a permutation graph instead. Nevertheless, the question of whether their construction is also permutation might arise. We prove here that this is not the case.

Given a cubic graph G, let $\pi_V = (v_1, v_2, ..., v_n)$ and $\pi_E = (e_1, e_2, ..., e_m)$ be arbitrary orderings of $V(G)$ and $E(G)$, respectively. Define the values: $q = 200n^3 + 1$, $p = 2q + 7n$, $q' = 10n^2 + 1$, and $p' = 2q' + 7n$. An interval graph G' is defined through the construction of one of its interval models $\mathcal{M}$, described as follows (observe Figure 4 to follow the construction):

1. Add to $\mathcal{M}$ a (p, q)-grained gadget $\mathcal{H}_i$ for each vertex $v_i \in V(G)$. These gadgets should be pairwise disjoint, with $\mathcal{H}_i$ appearing completely to the left of $\mathcal{H}_{i+1}$ for every $i \in [n - 1]$.
2. Add to $\mathcal{M}$ a (p', q')-grained gadget $\mathcal{E}_j$ for each edge $e_j \in E(G)$. Likewise, these gadgets should be pairwise disjoint, with $\mathcal{E}_j$ appearing completely to the left of $\mathcal{E}_{j+1}$ for every $j \in [m - 1]$. Additionally, $\mathcal{E}_1$ appears completely to the right of $\mathcal{H}_n$, without intersecting it.
3. Finally, for each edge $e_j = v_i v_{i'} \in E(G)$, with $i < i'$, add four intervals $L_{i,j}^1, L_{i,j}^2, L_{i',j}^1, L_{i',j}^2$, called *link* intervals, such that:

   - $L_{i,j}^1$ and $L_{i,j}^2$ (resp. $L_{i',j}^1$ and $L_{i',j}^2$) weakly intersect $\mathcal{H}_i$ (resp. $\mathcal{H}_{i'}$) to the right of $\mathcal{H}_i$ (resp. $\mathcal{H}_{i'}$);
   - $L_{i,j}^1$ and $L_{i,j}^2$ (resp. $L_{i',j}^1$ and $L_{i',j}^2$) weakly intersect (resp. strongly intersect) $\mathcal{E}_j$ to the left of $\mathcal{E}_j$.

As claimed, we show that the constructed graph G' is not a permutation graph. This is because G' contains the graph $\overline{X}_{34}$ depicted in Figure 5A as an induced subgraph, and such a graph is a forbidden subgraph for comparability graphs, cf. [9, 7], in turn a known superclass of permutation graphs. To see that this claim holds, observe Figure 5B. Given an edge $e_j = v_i v_{i'} \in E(G)$, with $i < i'$,
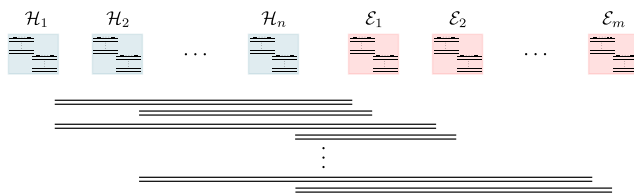

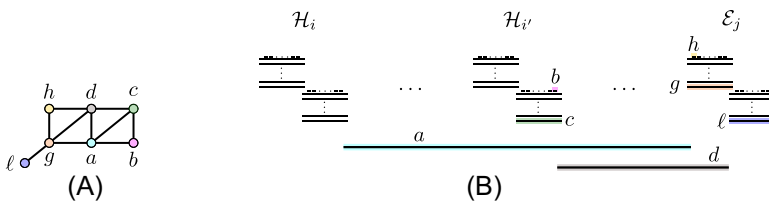**FIGURE 4** Adhikary et al.'s interval model $\mathcal{M}$, with $e_1 = v_1 v_2$, $e_2 = v_1 v_n$, and $e_m = v_2 v_n$.

FIGURE 5 (A) Forbidden induced subgraph $\overline{X}_{34}$ for comparability graphs, cf. [7]. (B) $\overline{X}_{34}$ as an induced subgraph in Adhikary et al.'s construction.

it shows the intervals in the grained gadgets of v_i, $v_{i'}$, and e_j, as well as some link intervals related to e_j. The adjacencies can be easily checked to be as in the graph of Figure 5A.

In Section 4, we show that a modification of Adhikary et al.'s construction gives us the desired permutation graph.

4 | OUR REDUCTION

Let G be a cubic graph, and consider $\pi_V = (v_1, v_2, ..., v_n)$ and $\pi_E = (e_1, e_2, ..., e_m)$ arbitrary orderings of $V(G)$ and $E(G)$, respectively. The values of p, q, p', q' are not the same as in Section 3 and are presented later. Again, for each vertex v_i, create a (p, q)-grained gadget, $\mathcal{H}_i$, and for each edge e_j, create a (p', q')-grained gadget $\mathcal{E}_j$. For simplicity, denote the sets $S'(\mathcal{H}_i)$, $S''(\mathcal{H}_i)$, $K'(\mathcal{H}_i)$, and $K''(\mathcal{H}_i)$ by S'_i, S''_i, K'_i, K''_i, respectively. Similarly, denote the sets $S'(\mathcal{E}_j)$, $S''(\mathcal{E}_j)$, $K'(\mathcal{E}_j)$, and $K''(\mathcal{E}_j)$ by S'^e_j, S''^e_j, K'^e_j, K''^e_j, respectively.

Recall that for each $i \in [n]$, the permutation model of $\mathcal{H}_i$ consists of the pair of permutations $\{\pi^1_i, \pi^2_i\}$ where $\pi^1_i = K'_i S'_i S_{i''} K_{i''}$ and $\pi^2_i = S'_i \overleftarrow{K''_i} \overleftarrow{K'_i} S''_i$. Analogously, for each $j \in [m]$, the permutation model of $\mathcal{E}_j$ consists of the pair of permutations $\{\gamma^1_j, \gamma^2_j\}$ where $\gamma^1_j = K'^e_j S'^e_j S''^e_j K''^e_j$ and $\gamma^2_j = S'^e_j \overleftarrow{K''^e_j} \overleftarrow{K'^e_j} S''^e_j$. Now, for each edge $e_j = v_i v_{i'}$, with $i < i'$, add four new vertices $L^1_{i,j}, L^2_{i,j}, L^1_{i',j}, L^2_{i',j}$, called *link* vertices. In what follows, we modify some of the grained gadget permutations to make $L^1_{i,j}, L^2_{i,j}$ (resp., $L^1_{i',j}, L^2_{i',j}$) weakly intersect $\mathcal{H}_i$ (resp., $\mathcal{H}_{i'}$) and strongly intersect (resp., weakly intersect) $\mathcal{E}_j$.

If v_i is incident to edges j_1, j_2, j_3, with $j_1 < j_2 < j_3$, then modify one of the permutations defining $\mathcal{H}_i$ to include the link vertices related to v_i as follows:

$$\pi^1_i = K'_i S'_i S''_i C_i K''_i,$$

where C_i denotes the permutation $L^1_{i,j_1} L^2_{i,j_1} L^1_{i,j_2} L^2_{i,j_2} L^1_{i,j_3} L^2_{i,j_3}$.

Similarly, for each edge $e_j = v_i v_{i'}$, $i < i'$, we modify one of the permutations defining $\mathcal{E}_j$ to include the link vertices related to e_j as follows:

$$\gamma^1_j = K'^e_j L^2_{i',j} L^1_{i',j} S'^e_j L^2_{i,j} L^1_{i,j} S''^e_j K''^e_j.$$

We do not modify π^2_i and γ^2_j, and keep denoting by π^2_i the permutation $S'_i \overleftarrow{K''_i} \overleftarrow{K'_i} S''_i$, and by γ^2_j the permutation $S'^e_j \overleftarrow{K''^e_j} \overleftarrow{K'^e_j} S''^e_j$. Finally, let G' be the permutation graph related to $\{\Pi, \Pi'\}$, where

$$\Pi = \pi^1_1 ... \pi^1_n \gamma^2_1, ..., \gamma^2_m \quad \text{and}$$
$$\Pi' = \pi^2_1 ... \pi^2_n \gamma^1_1, ..., \gamma^1_m.$$

Figure 6 illustrates our permutation model $\{\Pi, \Pi'\}$, focusing on the vertex grained gadgets $\mathcal{H}_i$ and $\mathcal{H}_{i'}$, the edge grained gadget $\mathcal{E}_j$, and the link vertices $L^1_{i,j}, L^2_{i,j}$ and $L^1_{i',j}, L^2_{i',j}$ related to an edge $e_j = v_i v_{i'}$, with $i < i'$.

We remark that the main difference of our permutation graph from the Adhikary et al.'s interval graph is the fact that, in Adhikary et al.'s interval graph, the link vertices form a clique, whereas, as we show in Section 5, some link vertices are not adjacent in our permutation graph.
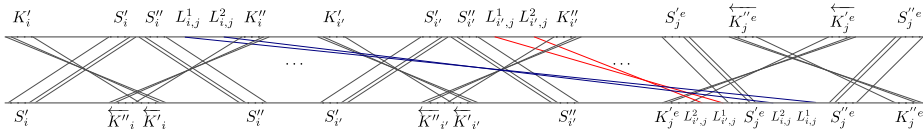
FIGURE 6 Vertex and edge grained gadgets, and link vertices related to an edge $e_j = v_i v_{i'}$, with $i < i'$, in our permutation model $\{\Pi, \Pi'\}$.

Additionally, for an edge $e_j = v_i v_{i'} \in E(G)$, with $i < i'$, the link vertices $L_{i,j}^1, L_{i,j}^2$ (resp., $L_{i',j}^1, L_{i',j}^2$) weakly intersect (resp., strongly intersect) $\mathcal{E}_j$ in Adhikary et al.'s interval graph, whereas in our permutation graph the link vertices $L_{i,j}^1, L_{i,j}^2$ (resp., $L_{i',j}^1, L_{i',j}^2$) strongly intersect (resp., weakly intersect) $\mathcal{E}_j$.

Before our proof, we make some observations about the constructed graph to improve the proof's readability. Note that, for each link vertex L and grained gadget H, we get one of the following situations. Either the relative order between L and $V(H)$ in Π is the reverse of their relative order in Π', in which case L is complete to $V(H)$. Or the relative order is the same in both Π and Π', in which case L is anticomplete to $V(H)$. Or L is related to H according to one of the ways described below.

- $L \in \{L_{i,j}^1, L_{i,j}^2\}$ and $H = \mathcal{H}_i$: in this case only the relative orders between L and K_i'' are reversed in Π and Π', that is, L is complete to K_i'' and anticomplete to $V(\mathcal{H}_i) \backslash K_i''$;
- $L \in \{L_{i,j}^1, L_{i,j}^2\}$ and $H = \mathcal{E}_j$, with $e_j = v_i v_{i'}, i < i'$: in this case the relative orders between L and $K_j'^e \cup S_j'^e$ are reversed in Π and Π', that is, L is complete to $K_j'^e \cup S_j'^e$ and anticomplete to $V(\mathcal{E}_j) \backslash (K_j'^e \cup S_j'^e)$; or
- $L \in \{L_{i',j}^1, L_{i',j}^2\}$ and $H = \mathcal{E}_j$, with $e_j = v_i v_{i'}, i < i'$: in this case only the relative orders between L and $K_j'^e$ are reversed in Π and Π', that is, L is complete to $K_j'^e$ and anticomplete to $V(\mathcal{E}_j) \backslash K_j'^e$.

Proof of Theorem 1. Consider the reduction graph G' and its permutation model $\{\Pi, \Pi'\}$ as previously defined. For each $e_j = v_i v_{i'} \in E(G)$, let

$$L(e_j) = \left\{ L_{i,j}^1, L_{i,j}^2, L_{i',j}^1, L_{i',j}^2 \right\};$$

and for each $v_i \in V(G)$, let

$$L(v_i) = \left\{ L_{i,j}^1, L_{i,j}^2 | e_j \text{ is incident to } v_i \right\}.$$

Also, denote the set of link vertices by $\mathcal{L}$, that is, $\mathcal{L} = \bigcup_{j=1}^m L(e_j)$.

We postpone the assignment of the actual values for p, q, p', q' and, in addition to the conditions necessary for the application of Lemma 1, we also ask that $q > 6n + p'$ and $p' > 2q' > 9n^2$.

In what follows, we prove that there exists a bijective relation f between the maximum cuts of the input graph G and the maximum cuts of our permutation graph G'. Then, we prove that, for each maximum cut $[X, Y]$ of G,

$$|E_G(X, Y)| \geq k \text{ if and only if } |E_{G'}(A, B)| \geq \phi(n, m, k),$$

where $[A, B] = f(X, Y)$ and ϕ is a well-defined function. Theorem 1 immediately follows.

Let $[A, B]$ be a maximum cut of G'. To define f, we first prove some properties relating the partitioning of vertex and edge grained gadgets of G' in $[A, B]$ with the partitioning of the link vertices of G' in $[A, B]$. More specifically, we prove that the two following properties hold:

1. For each vertex $v_i \in V(G)$, if $K''_i \subseteq A$, then $\{L^1_{i,j}, L^2_{i,j}\} \subseteq B$ for each edge $e_j \in E(G)$, with $e_j = v_i v_{i'}$ and $i < i'$;
2. For each edge $e_j \in E(G)$, with $e_j = v_i v_{i'}$ and $i < i'$, if $\{L^1_{i,j}, L^2_{i,j}\} \subseteq B$, then $S'^e_j \subseteq A$. □

Proof of Property 1. Let $v_i \in V(G)$ and suppose that $K''_i \subseteq A$. For the sake of contradiction, suppose that there exists a link vertex $L \in L(v_i) \cap A$. Then, let $[A', B']$ be the cut obtained from $[A, B]$ by setting $A' = A \setminus \{L\}$ and $B' = B \cup \{L\}$. Observe that there is a loss of at most $|\mathcal{L}| + \max\{p', q'\} = |\mathcal{L}| + p'$ edges between L and $\mathcal{L}$, and between L and the vertices of the edge grained gadget related to L, say $\mathcal{E}_j$, since K'^e_j and S'^e_j are always in opposite parts of the cut. On the other hand we gain all the edges between L and the vertices in K''_i. Therefore, we get an increase of the cut-set of at least q edges, and a decrease of less than $|\mathcal{L}| + \max\{p', q'\} = 6n + p'$ edges. It follows from the hypothesis $q > 6n + p'$ that $|E_{G'(A', B')}|$ is bigger than $|E_{G'}(A, B)|$, contradicting the maximality of $[A, B]$. □

Proof of Property 2. Consider an edge $e_j \in E(G)$, with $e_j = v_i v_{i'}$ and $i < i'$, and suppose that $\{L^1_{i,j}, L^2_{i,j}\} \subseteq B$. Observe that, because the relative orders among the edge and vertex grained gadgets themselves are the same in Π and Π', there are no edges between $\mathcal{E}_j$ and any other grained gadgets of G'. In other words, the only vertices outside of $\mathcal{E}_j$ that can be adjacent to the vertices of $\mathcal{E}_j$ are those in $\mathcal{L}$. Moreover, Lemma 1 tells us that the vertices belonging to $K'^e_j \cup S''^e_j$ are placed in the same part of $[A, B]$, opposite to the part containing the vertices belonging to $K''^e_j \cup S'^e_j$. More formally, either $K'^e_j \cup S''^e_j \subseteq B$ and $K''^e_j \cup S'^e_j \subseteq A$, or $K'^e_j \cup S''^e_j \subseteq A$ and $K''^e_j \cup S'^e_j \subseteq B$. As a result, switching the vertices of $\mathcal{E}_j$ of part of the cut does not change, and therefore cannot decrease, the number of edges between the vertices of $\mathcal{E}_j$ and the vertices belonging to $\mathcal{L} \setminus L(e_j)$ in the cut-set. Consequently, if $S'^e_j \subseteq A$, then we obtain that there are at least $2p'$ edges in the cut-set that are incident to vertices of $\mathcal{E}_j$; these are the edges between $L^1_{i,j}, L^2_{i,j}$ and the vertices belonging to S'^e_j. On the other hand, if $S'^e_j \subseteq B$, then we obtain that there are at most $4q'$ edges in the cut-set that are incident to vertices of $\mathcal{E}_j$; these are the edges between the vertices belonging to $L(e_j)$ and the vertices belonging to K'^e_j. Therefore, since $p' > 2q'$, we get that $S'^e_j \subseteq A$ as we wanted to prove. □

We are now ready to prove the existence of the bijective relation f. For each maximum cut $[X, Y]$ of G, let $f(X, Y)$ be the cut $[A, B]$ of G' defined as follows:

- For each vertex $v_i \in V(G)$, if $v_i \in X$, then add $K'_i \cup S''_i \cup L(v_i)$ to A and $K''_i \cup S'_i$ to B; do the opposite otherwise.

- For each $e_j \in E(G)$, with $e_j = v_i v_{i'}$ and $i < i'$, if $L_{i,j}^1 \in A$, then add $K_j'^e \cup S_j''^e$ to A and $K_j''^e \cup S_j'^e$ to B; and do the opposite otherwise.

Based on Properties 1 and 2, one can readily verify that f is well-defined and is a bijective relation, as desired.

Now, we count the number of edges in $E_{G'}(A, B)$ as a function of n, m, p, q, p', q' and of the size of the cut-set $E_G(X, Y)$. First, consider $v_i \in V(G)$. By construction, we know that there are $2pq + q^2$ edges in the cut-set between the vertices of $\mathcal{H}_i$. Additionally, there are exactly 6 link vertices weakly intersecting $\mathcal{H}_i$, while all other link vertices are either complete or anticomplete to $V(\mathcal{H}_i)$. Observe also that the number of link vertices complete to $V(\mathcal{H}_i)$ is exactly equal to $6(i - 1)$; these are the link vertices related to $\{v_1, ..., v_{i-1}\}$. This gives us a total of $6[q + (i - 1)(p + q)]$ edges between the vertices of $\mathcal{H}_i$ and the vertices belonging to $\mathcal{L}$ in the cut-set. Summing up these values for every $v_i \in V(G)$, we get a total of

$$\alpha_1 = n[2pq + q^2 + 6q] + 6\sum_{i=1}^{n}((i - 1)(p + q)) = n[2pq + q^2 + 6q + 3(p + q)(n - 1)]$$

edges in the cut-set $E_{G'}(A, B)$ incident to vertex grained gadgets. Now, let $e_j \in E(G)$, with $e_j = v_i v_{i'}$ and $i < i'$. By construction, we know that there are $2p'q' + (q')^2$ edges of the cut-set between vertices of $\mathcal{E}_j$, and $2p'$ edges of the cut-set between $L_{i,j}^1, L_{i,j}^2$ and the vertices of $\mathcal{E}_j$. Additionally, note that there are exactly $4(m - j)$ link vertices that cover and, therefore, are complete to $\mathcal{E}_j$; these are the link vertices related to $\{e_{j+1}, ..., e_m\}$. This gives us a total of $4(m - j)(p' + q')$ edges between the vertices of $\mathcal{E}_j$ and the vertices belonging to $\mathcal{L} \setminus L(e_j)$. Finally, suppose without loss of generality that $L_{i,j}^1 \in A$ (the count is analogous if it is in B). If $v_{i'} \in X$, then we know that $\{L_{i',j}^1, L_{i',j}^2\} \subseteq A$ and hence there are no edges in the cut-set between vertices $L_{i',j}^1, L_{i',j}^2$ and the vertices of $\mathcal{E}_j$. Otherwise, observe that it follows that $e_j \in E_G(X, Y)$ and $\{L_{i',j}^1, L_{i',j}^2\} \subseteq B$, and hence we get additional $2q'$ edges in the cut-set; these additional edges are between the link vertices $L_{i',j}^1, L_{i',j}^2$ and the vertices belonging to $K_j'^e$. Summing up these values for every $e_j \in E(G)$, we get a total of $\alpha_2 + 2q'|E_G(X, Y)|$ edges in the cut-set $E_{G'}(A, B)$ incident to edge grained gadgets, where

$$\alpha_2 = m[2p'q' + (q')^2 + 2p'] + \sum_{j=1}^{m}(4(m - j)(p' + q'))$$
$$= m[2p'q' + (q')^2 + 2p' + 2(p' + q')(m - 1)].$$

Finally, observe that there are at most $|A \cap \mathcal{L}| \cdot |B \cap \mathcal{L}|$ edges of the cut-set between link vertices. Note also that $|A \cap \mathcal{L}| = 6|X|$ since each vertex in X is related to 6 link vertices, which are all placed in A. Similarly, we have $|B \cap \mathcal{L}| = 6|Y|$. This gives us at most $36|X| \cdot |Y| \leq 9n^2$ edges in the cut-set between link vertices. Putting everything together, we get:

$$\alpha_1 + \alpha_2 + 2q'|E_G(X, Y)| \leq |E_{G'}(A, B)| \leq \alpha_1 + \alpha_2 + 2q'|E_G(X, Y)| + 9n^2.$$

By setting $\phi(n, m, k)$ to $\alpha_1 + \alpha_2 + 2q'k$, and knowing that p, q, p', q' will be chosen as functions of n and m, we want to prove, as stated in the beginning, that $|E_G(X, Y)| \geq k$ if and only if $|E_{G'}(A, B)| \geq \phi(n, m, k)$. If $|E_G(X, Y)| \geq k$, then the first inequality gives us that

$|E_{G'}(A, B)| \geq \alpha_1 + \alpha_2 + 2q'k = \phi(n, m, k)$. On the other hand, if $|E_{G'}(A, B)| \geq \phi(n, m, k) = \alpha_1 + \alpha_2 + 2q'k$, then the second inequality gives us that $|E_G(X, Y)| \geq k - 9n^2/2q'$. Because we assume that $2q' > 9n^2$, it follows that $k - 9n^2/2q' > k - 1$ and hence $|E_G(X, Y)| \geq k$.

It only remains to set the values of p, q, p', q'. Observe that:

- For every grained gadget H, the total number of vertices in $V(G') \backslash H$ adjacent to H is at most $6n$ (these are exactly the link vertices).
- For every vertex grained gadget H, the total number of vertices adjacent to some vertex $u \in S'(H)$ is exactly $q + 2h$, for some positive integer h. This is because the number of link vertices adjacent to the vertices in $S'(H)$ is always even. The same holds for the number of vertices adjacent to the vertices in $S''(H)$. We then get that the parity of ℓ and r in the conditions of Lemma 1 applied to H depends only on the parity of q.
- Similarly, if H is an edge grained gadget, then the parity of the total number of vertices adjacent to some $u \in S'(H) \cup S''(H)$ is equal to the parity of q'.

Therefore, the necessary conditions of Lemma 1 translate to: $q > 6n(p/q - 1)$ and $q' > 6n(p'/q' - 1)$; $p > 6n + 2q$ and $p' > 6n + 2q'$; and q and q' are odd.

Additionally, we need to ensure: $q > 6n + p'$ and $p' > 2q' > 9n^2$. Hence, consider:

- $q' \in \{5n^2, 5n^2 + 1\}$ such that q' is odd;
- $p' = 11n^2 + 6n$;
- $q = 12n^2 + 12n + 1$;
- $p = 25n^2 + 30n$.

Since $n \geq 4$, one can verify that the values described above satisfy all the required conditions. This, therefore, concludes the proof of Theorem 1.

5 | MAXCUT ON PERMUTATION INTERVAL GRAPHS IS AN OPEN PROBLEM

In this paper, we have presented a proof of NP-completeness for the MAXCUT problem when constrained to permutation graphs. Surprisingly enough, we found that the main gadget in the reduction recently presented by Adhikary et al. [1] for interval graphs is also a permutation graph. Additionally, in Section 3, we have seen that being permutation is not a property that holds for the full construction of Adhikary et al. [1]. On the other hand, since the grained gadgets play an important role in our reduction too, one could wonder whether our construction instead is in the intersection between interval and permutation graphs. The answer to that is no as we argue next.

Let G be a cubic graph, and consider arbitrary orderings of $V(G)$ and $E(G)$, $(v_1, ..., v_n)$ and $(e_1, ..., e_m)$, respectively. Let j_1, j_2, j_3 be the indices of the edges incident to v_1, with $j_1 < j_2 < j_3$. Also, let v_i be the other endpoint of e_{j_2}. We present a C_4 in G', the graph constructed in Section 4; it thus follows that G' is not chordal, and hence also not interval [12]. Observe Figure 7 to follow our argument. Let a be equal to L^1_{1, j_1}, b be any vertex in K''_i, c be equal to L^1_{i, j_2}, and d be any vertex in $K'^e_{j_1}$. Since $j_1 < j_2$ and $1 < i$, we know that the relative order between a and c in Π is the same as in Π'; hence $ac \notin E(G')$. Also, the relative order in Π between a and
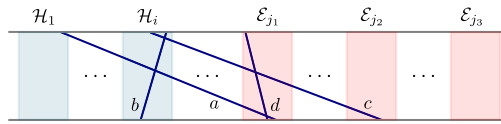
FIGURE 7 Existence of a $C_4 = (a, b, c, d)$ as an induced subgraph in our permutation graph.

any vertex of $\mathcal{H}_i$ is reversed in Π', the same holds between c and any vertex belonging to K_i''; hence $\{ab, bc\} \subseteq E(G')$. Similarly, the relative order between a and any vertex belonging to $K_{j_1}'^e$ in Π is reversed in Π', and the same holds between c and any vertex of $\mathcal{E}_{j_1}$; hence $\{ad, cd\} \subseteq E(G')$. Finally, since $j_1 < j_2$, the relative order between b and d in Π is the same as in Π', and therefore $bd \notin E(G')$, thus finishing our argument.

The previous paragraph tells us that for any chosen orderings of $V(G)$ and $E(G)$, the graph constructed in Section 4 contains a C_4. Since it is known that the class of C_4-free co-comparability graphs is precisely the class of interval graphs [11], and that the class of permutation graphs is equal to the class of comparability co-comparability graphs [14], we get that interval permutation graphs are exactly the class of C_4-free permutation graphs.

A good question is whether there is a construction that produces a permutation graph that is also C_4-free (and hence interval). Up to our knowledge, the largest class in the intersection of permutation and interval graphs for which the complexity is known is the class of the threshold graphs. MaxCut is polynomial-time solvable on threshold graphs thanks to the algorithm given for cographs, a subclass of permutation graphs that is a superclass of threshold graphs [2].

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