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ON THE TOTAL CHROMATIC NUMBER OF THE DIRECT PRODUCT OF CYCLES AND COMPLETE GRAPHS

Diane Castonguay[1], Celina M.H. de Figueiredo[2], Luis A.B. Kowada[3], Caroline S.R. Patrão[4,*], Diana Sasaki[5] and Mario Valencia-Pabon[6]

Abstract. A *k-total coloring* of a graph G is an assignment of k colors to the elements (vertices and edges) of G so that adjacent or incident elements have different colors. The total chromatic number is the smallest integer k for which G has a k-total coloring. The well known Total Coloring Conjecture states that the total chromatic number of a graph is either $\Delta(G) + 1$ (called Type 1) or $\Delta(G) + 2$ (called Type 2), where $\Delta(G)$ is the maximum degree of G. We consider the direct product of complete graphs $K_m \times K_n$. It is known that if at least one of the numbers m or n is even, then $K_m \times K_n$ is Type 1, except for $K_2 \times K_2$. We prove that the graph $K_m \times K_n$ is Type 1 when both m and n are odd numbers, by using that the conformable condition is sufficient for the graph $K_m \times K_n$ to be Type 1 when both m and n are large enough, and by constructing the target total colorings by using Hamiltonian decompositions and a specific color class, called guiding color. We additionally apply our technique to the direct product $C_m \times K_n$ of a cycle with a complete graph. Interestingly, we are able to find a Type 2 infinite family $C_m \times K_n$, when m is not a multiple of 3 and $n = 2$. We provide evidence to conjecture that all other $C_m \times K_n$ are Type 1.

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1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A *k-total coloring* of a graph G is an assignment of k colors to the elements (vertices and edges) of G so that adjacent or incident elements have different colors. The *total chromatic number*, denoted by $\chi_T(G)$, is the smallest integer k for which G has a k-total coloring. Clearly, $\chi_T(G) \geq \Delta(G) + 1$ and the *Total Coloring Conjecture* (TCC), posed independently by Vizing [13] and Behzad *et al.* [2], states that $\chi_T(G) \leq \Delta(G) + 2$, where $\Delta(G)$ is the maximum degree of G. Graphs with $\chi_T(G) = \Delta(G) + 1$ are said to be *Type* 1 and graphs with $\chi_T(G) = \Delta(G) + 2$ are said to be

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[1] INF, Universidade Federal de Goiás, Goiânia, Brazil.
[2] COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil.
[3] IC, Universidade Federal Fluminense, Niterói, Brazil.
[4] IFG, Instituto Federal de Goiás, Formosa, Brazil.
[5] IME, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, Brazil
[6] LORIA, Université de Lorraine, Vandoeuvre-lès-Nancy, France.
*Corresponding author: patraocaroline@gmail.com

Type 2. The TCC has been verified in restricted cases, such as cubic graphs [12] and graphs with large maximum degree [9], but has not been settled for all regular graphs for more than fifty years.

We denote an undirected edge $e \in E(G)$ whose ends are u and v by uv. The *direct product* (also called *tensor product* or *categorical product*) of two graphs G and H is a graph denoted by $G \times H$, whose vertex set is the Cartesian product $V(G) \times V(H)$, for which vertices (u, v) and (u', v') are adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$. The maximum degree of $G \times H$ is $\Delta(G \times H) = \Delta(G) \cdot \Delta(H)$, and $G \times H$ is regular if and only if both G and H are regular graphs. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on the same vertex set V and where $E_1 \cap E_2 = \emptyset$, and denote by $\bigoplus_{i=1}^{2} G_i$ the direct sum graph $G = (V, E_1 \cup E_2)$ of graphs G_1 and G_2. In this work, given two graphs G and H, we use the well known property that the direct product is distributive over edge disjoint union of graphs, that is, if $G = \bigoplus_{i=1}^{t} G_i$, where G_i are edge-disjoint subgraphs of G and $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_t)$, then $H \times G = \bigoplus_{i=1}^{t}(H \times G_i)$.

The complete graph on n vertices is denoted by K_n. The direct product of complete graphs $K_m \times K_n$ is a regular graph of degree $\Delta(K_m \times K_n) = (m-1)(n-1)$ and can be described as an n-partite graph with m vertices in each part. The total chromatic number of $K_m \times K_n$ has been determined when m or n is an even number. When $m = n = 2$, we have the disconnected $2K_2$, which is Type 2, since each connected component K_2 is Type 2. When $m \geq 3$, $K_m \times K_2$ is the complete bipartite graph $K_{m,m}$ minus a perfect matching, and Yap [14] proved that this graph is Type 1. When $n \geq 4$ and n is an even number, Geetha and Somasundaram [8] proved that $K_n \times K_n$ is Type 1. Janssen and Mackeigan [10] proved that $K_m \times K_n$ is Type 1 when m or n is an even number, with $m, n \geq 3$. As far as we know, for the remaining case, when both m and n are odd numbers, it is not known whether $K_m \times K_n$ is Type 1 or Type 2. In this work, we establish the total chromatic number of $K_m \times K_n$, when m and n are odd numbers, by proving that these graphs are Type 1. Thus, we can conclude that, except for $m = n = 2$, the graph $K_m \times K_n$ is Type 1.

In order to achieve the claimed total colorings for all graphs $K_m \times K_n$, when m and n are odd numbers, we prove two theorems according to whether m and n are both large enough or not. In Section 2, we recall the known conformable necessary condition to be Type 1 and a known lower bound on the vertex degree for regular graphs of odd order which ensures that the conformable condition is also a sufficient condition to be Type 1. Moreover, we prove Lemma 2.1 and Theorem 2.2 which together provide the required total colorings of the direct product of complete graphs $K_m \times K_n$, for odd numbers $m, n \geq 13$. In Section 3, we present preliminary concepts on Hamiltonian decompositions and the guiding color technique. Such technique uses a Hamitonian decomposition together with a color class with specific properties, called guiding color class, that guides how to construct the Type 1 total coloring. This technique can be applied to any graph with a Hamiltonian decomposition. In Section 4, we prove Theorem 4.1 which provides the required Type 1 total colorings of $K_m \times K_n$, for odd numbers $m, n \geq 3$ and $m < 13$. Along the proof, we omit from the main text a finite number of particular graphs that are too small to obey the described pattern. Some particular Hamiltonian decompositions and their tables containing the elements of the guiding color are given in the appendix.

In Section 5, we additionally apply our technique to another class of graphs, the direct product of a cycle with a complete graph, denoted by $C_m \times K_n$. We combine previous results [4,10] to the Hamiltonian decompositions and the guiding color technique introduced in Section 3. For m or n even, we determine that $C_m \times K_n$ is Type 2, when m is not a multiple of 3 and $n = 2$, and Type 1, otherwise. For m and n odd, we use the same technique of the guiding color as we did in Section 4 to additionally prove that $C_{2n-1} \times K_n$ is Type 1, and we give evidence to conjecture that $C_m \times K_n$ remains Type 1 for all odd m and n. That would imply that $C_m \times K_n$ is Type 2 if and only if m is not a multiple of 3 and $n = 2$.

2. Graphs $K_m \times K_n$ are Type 1, for odd numbers $m, n \geq 13$

A regular graph G is *conformable* if G admits a vertex coloring with $\Delta(G) + 1$ colors such that the number of vertices in each color class has the same parity as $|V(G)|$, as defined by Chetwynd and Hilton [5].

Lemma 2.1. *For odd numbers $m, n \geq 3$, the graph $K_m \times K_n$ is conformable.*

Proof. Consider $m \leq n$. We construct a vertex coloring with $(m-1)(n-1)+1$ colors such that each color class is composed by 1 or 3 vertices. Let $t = \frac{m+n-2}{2}$. Since $t < n$, the vertices $(0,i),(1,i),(2,i)$ in the direct product $K_m \times K_n$ define an independent set and can receive the same color c_i, for $i = \{0,\dots,t-1\}$. Now color each of the $mn-3t$ remaining uncolored vertices with a different additional color, to obtain the desired vertex coloring with $t+(mn-3t) = mn-2t = mn-m-n+2 = (m-1)(n-1)+1 = \Delta(K_m \times K_n)+1$ colors. □

The TCC for graphs G having $\Delta(G) \geq \frac{3}{4}|V(G)|$ was established by Hilton and Hind [9]. Chetwynd *et al.* [6] proved that when G is a regular graph of odd order and with degree $\Delta(G) \geq \frac{\sqrt{7}}{3}|V(G)|$, then G is Type 1 if and only if G is conformable. Chew [7] improved this result by showing that it suffices to require that $\Delta(G) \geq \frac{(\sqrt{37}-1)}{6}|V(G)|$. In Theorem 2.2, we establish that when $m,n \geq 13$ are odd numbers, then $\Delta(K_m \times K_n)$ satisfies the lower bound required by Chew, which together with Lemma 2.1 implies the desired result.

Theorem 2.2. *For odd numbers $m,n \geq 13$, the graph $K_m \times K_n$ is Type 1.*

Proof. Let $m,n \geq 13$ be two odd numbers. Hence, $(7-\sqrt{37})n - 6 \geq (7-\sqrt{37}) \cdot 13 - 6 \geq 0$ and $n \geq 13 \geq \frac{72}{13(7-\sqrt{37})-6}$. So, $13(7-\sqrt{37})n \geq 72+6n$ and $13(7-\sqrt{37})n - 13\cdot 6 \geq 72+6n-13\cdot 6$, which implies that $13 \geq \frac{6(n-1)}{(7-\sqrt{37})n-6}$. Now, as $m \geq 13$, we have that $m \geq \frac{6(n-1)}{(7-\sqrt{37})n-6}$. Therefore, $(7-\sqrt{37})mn - 6m \geq 6n-6$, which is equivalent to $(1-\sqrt{37})mn + 6mn - 6m - 6n + 6 \geq 0$. So, $mn-m-n+1 = (m-1)(n-1) \geq \frac{(\sqrt{37}-1)}{6}mn$. Since $\Delta(K_m \times K_n) = (m-1)(n-1)$, we have that $\Delta(K_m \times K_n) \geq \frac{(\sqrt{37}-1)}{6}|V(K_m \times K_n)|$. Therefore, by the Chew's result [7] and by Lemma 2.1, we have that $K_m \times K_n$ is Type 1. □

3. Hamiltonian decompositions and the guiding color technique

For $K_3 \times K_n$, $K_5 \times K_n$ and $K_7 \times K_n$, with $n \geq 3$ an odd number, in Section 3.1, we use Walecki's Hamiltonian decomposition of K_n to define suitable Hamiltonian decompositions of $K_m \times K_n$, first when $gcd(m,n)=1$ and second when $gcd(m,n) \neq 1$.

In Section 3.2, we define the guiding color technique that can be applied to any graph with a Hamiltonian decomposition. This technique uses a color class with specific properties, called a guiding color. In our case, both the Hamiltonian decomposition constructed in Section 3.1 and the guiding color given in Section 4 define the target $(\Delta(K_m \times K_n)+1)$-total coloring.

For $K_9 \times K_n$ and $K_{11} \times K_n$, we use the guiding color technique only for few particular graphs by presenting Hamiltonian decomposition and guiding color in Section A.2 of the appendix.

3.1. Hamiltonian decompositions

A k-regular graph G has a *Hamiltonian decomposition* (or is *Hamiltonian decomposable*) if its edge set can be partitioned into $\frac{k}{2}$ Hamiltonian cycles when k is an even number, or into $\frac{(k-1)}{2}$ Hamiltonian cycles plus a one factor (or perfect matching) when k is an odd number. Please refer to [1] for a survey on Hamiltonian decompositions.

Consider the well known Walecki's Hamiltonian decomposition of the complete graph K_n for $n \geq 3$. We shall focus on an odd number n. Let $n = 2w+1$ and label the vertices of K_n as $0,1,\dots,2w$. Following the notation used in [1], let C_n be the Hamiltonian cycle $\langle 0,1,2,2w,3,2w-1,4,2w-2,5,2w-3,\dots,w+3,w,w+2,w+1,0\rangle$. If σ is the permutation $(0)(1,2,3,4,\dots,2w-1,2w)$, then $\sigma^0(C_n),\sigma^1(C_n),\sigma^2(C_n),\dots,\sigma^{w-1}(C_n)$ is a Hamiltonian decomposition of K_n. Observe that $\sigma^0(C_n) = C_n$. We write $K_n = \bigoplus_{i=1}^{w} \sigma^{i-1}(C_n)$. Denote by $\sigma^t(C_n)_z$, with $z = 0,1,\dots,n-1$ the zth-vertex in the cycle $\sigma^t(C_n)$, and in fact, the vertex 0 is always the 0th-vertex. Note that for $t \geq w$, the cycle $\sigma^t(C_n)$ is the opposite cycle of $\sigma^{t \bmod w}(C_n)$, that is, $\sigma^t(C_n)_z = \sigma^{t \bmod w}(C_n)_{n-z}$ for all $z \geq 1$.

For instance consider $n=5$, write $n = 2w+1$ and thus $w=2$, to get the Hamiltonian decomposition $K_5 = \bigoplus_{i=1}^{2} \sigma^{i-1}(C_5)$, where $\sigma^0(C_5) = \langle 0,1,2,4,3,0\rangle$ and $\sigma^1(C_5) = \langle 0,2,3,1,4,0\rangle$, as highlighted in Figure 1.

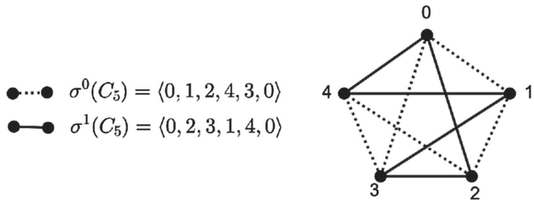

FIGURE 1. Walecki's Hamiltonian decomposition of $K_5 = \sigma^0(C_5) \bigoplus \sigma^1(C_5)$.

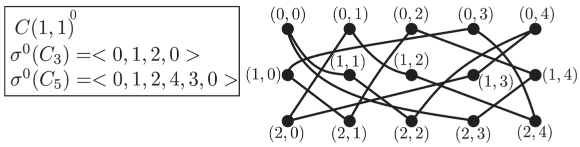

FIGURE 2. The cycle $C(1,1)^0$ in $K_3 \times K_5$.

Note that $\sigma^2(C_5) = \langle 0,3,4,2,1,0\rangle$ is the opposite cycle of $\sigma^0(C_5)$, and $\sigma^3(C_5) = \langle 0,4,1,3,2,0\rangle$ is the opposite cycle of $\sigma^1(C_5)$.

It is well known and not hard to see that the direct product of cycle graphs is Hamiltonian decomposable if and only if at least one of them is an odd cycle [11]. In what follows, for both m and n odd numbers, we shall use Walecki's Hamiltonian decomposition of the complete graph K_n and the well known distributive property of the direct product to define a Hamiltonian decomposition of $K_m \times K_n$, for $m = 3, 5, 7$ and odd number $n \geq 3$ suitable to our target total coloring.

Write odd numbers $m, n \geq 3$ as $m = 2q + 1$ and $n = 2w + 1$. Let $gcd(m, n) = d$. For $j = 1, \dots, 2q$, $i = 1, \dots, 2w$ and $k = 0, \dots, d-1$, denote by $C(j,i)^k$ the cycle on $\frac{mn}{d}$ vertices $\langle C(j,i)^k_z\rangle_{z=0,\dots,\frac{mn}{d}}$, where $C(j,i)^k_z = (\sigma^{j-1}(C_m)_{(z+k) \bmod m}, \sigma^{i-1}(C_n)_{z \bmod n})$, with $z = 0, \dots, \frac{mn}{d}$, is the zth-vertex of the cycle $C(j,i)^k$. Observe that according to the notation for vertex $C(j,i)^k_z$, we have $C(j,i)^k_0 = C(j,i)^k_{\frac{mn}{d}}$, and the vertex $(0,0)$ is always the 0th-vertex of $C(j,i)^0$. For instance, Figure 2 presents the cycle $C(1,1)^0$ using the cycle $\sigma^0(C_3)$ of K_3 and the cycle $\sigma^0(C_5)$ of K_5.

We consider next the construction of a Hamiltonian decomposition of $K_m \times K_n$ according to whether $gcd(m,n) = 1$ or not. Case 1 considers $gcd(m,n) = 1$ which gives a single $k = 0$ and that each $C(j,i)^0$ is a Hamiltonian cycle which gives that $\{C(j,i) = C(j,i)^0 \mid j = 1, \dots q \text{ and } i = 1, \dots, 2w\}$ is a Hamiltonian decomposition of $K_m \times K_n$. Case 2 considers $gcd(m,n) \neq 1$ which implies that each cycle $C(j,i)^k$ is not a Hamiltonian cycle. We construct a Hamiltonian decomposition of $K_m \times K_n$ given by $\{C(j,i) \mid j = 1, \dots, 2q$ and $i = 1, \dots, w\}$ where each Hamiltonian cycle is composed by d paths obtained from the cycles $C(j,i)^k$, such that, for each $k = 0, \dots, d-1$, the cycle $C(j,i)^k$ becomes a path by removing one edge.

Case 1: $gcd(m,n) = 1$**.** Consider $\{C(j,i) \mid j = 1, \dots q \text{ and } i = 1, \dots, 2w\}$, a Hamiltonian decomposition of $K_m \times K_n$, where $C(j,i) = C(j,i)^0$, see an example in Figure 3. Indeed, consider $K_m = \bigoplus_{j=1}^{q}(\sigma^{j-1}(C_m))$ and $K_n = \bigoplus_{i=1}^{w}(\sigma^{i-1}(C_n))$ the Walecki's Hamiltonian decompositions of K_m and K_n, respectively. Thus we write $K_m \times K_n = \bigoplus_{j=1}^{q}\bigoplus_{i=1}^{w}(\sigma^{j-1}(C_m) \times \sigma^{i-1}(C_n))$. As the degree $\Delta(\sigma^{j-1}(C_m) \times \sigma^{i-1}(C_n)) = 4$, for any $j = 1, 2, \dots, q$ and for any $i = 1, 2, \dots, w$, each subgraph $\sigma^{j-1}(C_m) \times \sigma^{i-1}(C_n)$ of $K_m \times K_n$ has two Hamiltonian cycles: $C(j,i)$ and $C(j, i+w)$, and so, it suffices to consider $C(j,i)$ for $j = 1, \dots, q$ and $i = 1, \dots, 2w$.

For instance, consider $K_3 \times K_5$ in Figure 3. As $gcd(3,5) = 1$ we use $K_3 \times K_5 = \bigoplus_{j=1}^{1}\bigoplus_{i=1}^{2}(\sigma^{j-1}(C_3) \times \sigma^{i-1}(C_5))$, the 2 Hamiltonian cycles of the subgraph $\sigma^0(C_3) \times \sigma^0(C_5)$ of $K_3 \times K_5$ are $C(1,1)$ and $C(1,3)$. Analogously, the 2 Hamiltonian cycles of the subgraph $\sigma^0(C_3) \times \sigma^1(C_5)$ of $K_3 \times K_5$ are $C(1,2)$ and $C(1,4)$.

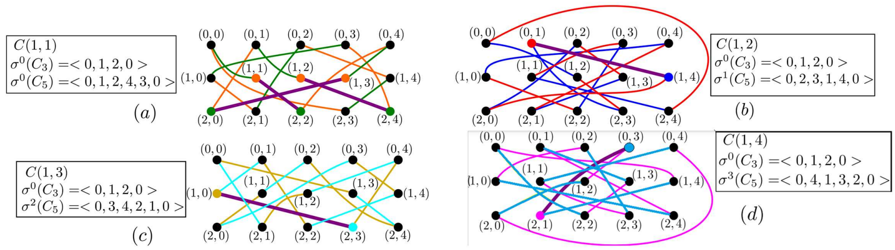

FIGURE 3. A depiction of $K_3 \times K_5$ partitioned into 4 Hamiltonian cycles. In (a) we have the Hamiltonian cycle $C(1,1)$ with 3 colors: the edges $(1,1)(2,2)$, $(1,2)(2,4)$ and $(1,3)(2,0)$ are colored with the guiding purple color; the endvertices of the purple edges and the remaining edges of $C(1,1)$ are colored with colors orange and dark green. In (b) we have the Hamiltonian cycle $C(1,2)$ also colored with 3 colors: the edge $(0,1)(1,4)$ also colored with the guiding purple color; the endvertices of the purple edge and the remaining edges of $C(1,2)$ are colored with colors red and dark blue. In (c) we have $C(1,3)$ also with 3 colors: the edge $(1,0)(2,3)$ also colored with the guiding purple color; the endvertices of the purple edge and the remaining edges of $C(1,3)$ are colored with colors light blue and light brown. Finally in (d) we have $C(1,4)$ also colored with 3 colors: the edge $(0,3)(2,1)$ also colored with the guiding purple color; the endvertices of the purple edge and the remaining edges of $C(1,4)$ are colored with colors pink and turquoise blue.

Case 2: $gcd(m,n) = d > 1$**.** By definition, in this case, each $C(j,i)^k$ is not a Hamiltonian cycle. For $k = 0, \ldots, d-1$, denote by $P(j,i)^k$ the path induced by the $\frac{mn}{d}$ vertices $C(j,i)^k_z$, with $z = 0, \ldots, \frac{mn}{d} - 1$, obtained from $C(j,i)^k$ by removing one edge. Consider $\{C(j,i) \mid j = 1, \ldots, 2q \text{ and } i = 1, \ldots, w\}$ a Hamiltonian decomposition of $K_m \times K_n$, where the Hamiltonian cycles are defined as follows.

(i) For $m = 3$:

$$C(j,i) = \langle P(j,i)^0, P(j,i)^1, P(j,i)^2, (0,0) \rangle.$$

For $i = 1, \ldots, w$, the cycles $C(1,i)$ and $C(2,i)$ form a Hamiltonian decomposition of $\sigma^0(C_3) \times \sigma^{i-1}(C_n)$. For instance, consider $K_3 \times K_9$ in Figure 4. As $gcd(3,9) = 3$ we write $K_3 \times K_9 = \bigoplus_{i=1}^4 (\sigma^0(C_3) \times \sigma^{i-1}(C_9))$. The 2 Hamiltonian cycles of the subgraph $\sigma^0(C_3) \times \sigma^0(C_9)$ are $C(1,1)$ and $C(2,1)$; and analogously of the subgraph $\sigma^0(C_3) \times \sigma^1(C_9)$ are $C(1,2)$ and $C(2,2)$; of the subgraph $\sigma^0(C_3) \times \sigma^2(C_9)$ are $C(1,3)$ and $C(2,3)$; finally of the subgraph $\sigma^0(C_3) \times \sigma^3(C_9)$ are $C(1,4)$ and $C(2,4)$.

(ii) For $m = 5$:

$$C(j,i) = \begin{cases} \langle P(j,i)^0, P(j,i)^1, P(j,i)^2, P(j,i)^3, P(j,i)^4, (0,0) \rangle, & \text{if } j = 1,3 \\ \langle P(j,i)^0, P(j,i)^2, P(j,i)^4, P(j,i)^1, P(j,i)^3, (0,0) \rangle, & \text{if } j = 2,4. \end{cases}$$

For $i = 1, \ldots, w$, the set of cycles $\{C(j,i) \mid j = 1, \ldots, 4\}$ is a Hamiltonian decomposition of $K_5 \times \sigma^{i-1}(C_n)$.

(iii) For $m = 7$:

$$C(j,i) = \begin{cases} \langle P(j,i)^0, P(j,i)^3, P(j,i)^4, P(j,i)^5, P(j,i)^1, P(j,i)^2, P(j,i)^6, (0,0) \rangle, & \text{if } j = 1,3,5 \\ \langle P(j,i)^0, P(j,i)^4, P(j,i)^1, P(j,i)^3, P(j,i)^6, P(j,i)^2, P(j,i)^5, (0,0) \rangle, & \text{if } j = 2,4,6. \end{cases}$$

For $i = 1, \ldots, w$, the set of cycles $\{C(j,i) \mid j = 1, \ldots, 6\}$ is a Hamiltonian decomposition of $K_7 \times \sigma^{i-1}(C_n)$.

3.2. The guiding color technique

We are ready to explain how a $(\Delta(K_m \times K_n) + 1)$-total coloring of $K_m \times K_n$ is obtained by considering the Hamiltonian decomposition of $K_m \times K_n$ into Hamiltonian cycles $C(i,j)$ defined in Section 3.1. In a $(\Delta(K_m \times$

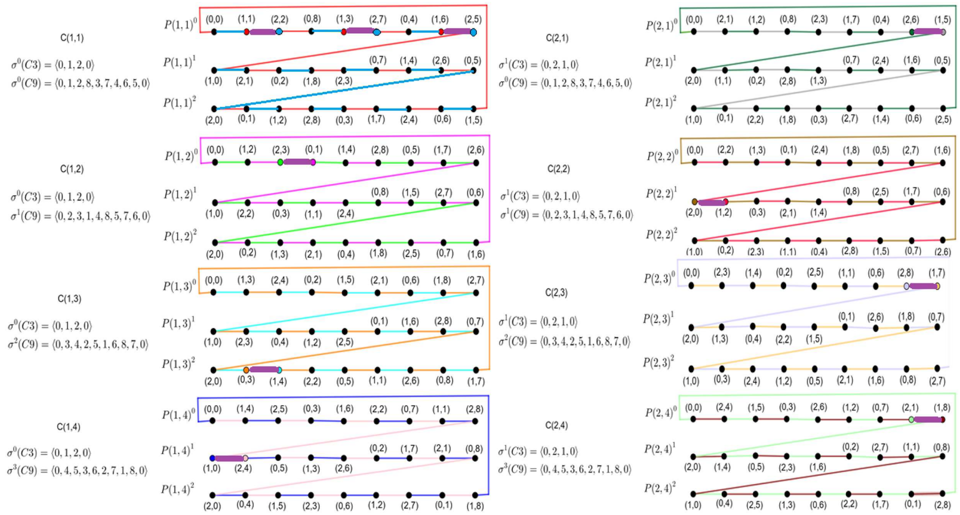

FIGURE 4. A depiction of $K_3 \times K_9$ partitioned into 8 Hamiltonian cycles. We have the Hamiltonian cycle $C(1,1)$ with 3 colors: the edges $(1,1)(2,2)$, $(1,3)(2,7)$ and $(1,6)(2,5)$ are colored with the guiding purple color; the endvertices of the purple edges and the remaining edges of $C(1,1)$ are colored with colors red and turquoise blue. In the remaining 7 Hamiltonian cycles, each of them has one edge with the guiding purple color whose endvertices and the remaining edges of the cycle are colored with additional new two colors. The vertices $(0,0),(0,2),(0,4),(0,5),(0,6),(0,7)$ and $(0,8)$ are an independent set and can be colored with the guiding purple color obtaining a 17-total coloring of $K_3 \times K_9$.

$K_n)+1)$-total coloring, each color class is such that each vertex is either inside the color class or is incident to an edge of the color class. We shall choose a guiding color with the additional property that its color class contains one or three edges of each Hamiltonian cycle. Note that each Hamiltonian cycle is an odd cycle and, by Vizing's theorem [13], admits a 3-edge coloring. Thus, for each cycle, we assign two additional colors to the remaining edges of the Hamiltonian cycle and to the endvertices of the edges with the guiding color, as illustrated by Figures 3 and 4. With suitable choices for the edges of the matching colored by the guiding color, the so far uncolored vertices define an independent set which can be also colored with the guiding color as Figure 5.

In order to obtain a $(\Delta(K_m \times K_n)+1)$-total coloring, we give a table composed by the elements of the guiding color class. We identify the edges of the guiding color on the corresponding Hamiltonian cycle where they belong. If the Hamiltonian cycle contains a unique edge of the guiding color, then its endvertices and the remaining edges of the cycle are easily colored using two additional colors. If the Hamiltonian cycle contains three edges of the guiding color, then we can easily see that their endvertices define two independent sets that can be colored with two colors as also the remaining edges of the cycle.

For instance, consider $K_3 \times K_5$ in Figure 5. We represent a table and a subgraph highlighting all elements (edges and vertices) colored by the guiding color and the colored vertices of Figure 3. We can identify which of the four Hamiltonian cycles contains which highlighted edges by observing the colors of their endvertices. In Figure 3a, the six endvertices of the three edges colored with the guiding color (purple) in $C(1,1)$ are the three vertices $(1,1),(1,2)$ and $(1,3)$ defining an independent set that can be assigned with one color (orange), and the three vertices $(2,0),(2,2)$ and $(2,4)$ defining another independent set that can be assigned with one color (green). The remaining edges of $C(1,1)$ can be assigned with the colors orange and green. Analogously for the Hamiltonian cycles $C(1,2), C(1,3)$ and $C(1,4)$, as in Figure 3. The remaining uncolored vertices $(0,0),(0,2)$

Elements of $K_3 \times K_5$ of the guiding color	
$C(1,1)$	$(1,1)(2,2),(1,3)(2,0),(1,2)(2,4)$
$C(1,2)$	$(0,1)(1,4)$
$C(1,3)$	$(1,0)(2,3)$
$C(1,4)$	$(2,1)(0,3)$
Vertices: $(0,0),(0,2),(0,4)$	

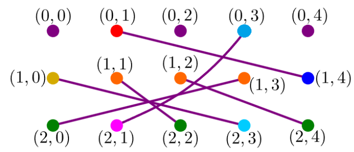

FIGURE 5. A table composed by the elements of the guiding purple color in $K_3 \times K_5$, and its depiction using colors of the endvertices to identify the Hamiltonian cycles containing them.

and $(0,4)$ of Figure 3 represent an independent set that can be colored with the guiding color. Thus we can easily obtain a 9-total coloring of $K_3 \times K_5$ from the elements colored with the guiding color.

4. GRAPHS $K_m \times K_n$ ARE TYPE 1, FOR ODD NUMBERS $m, n \geq 3$ AND $m < 13$

In this section, we may sometimes omit the fact that m, n are odd numbers and $m, n \geq 3$, since it is clear that we are only concerned with complete graphs of odd order greater than 2. In each subsection, for each family considered, we omit from the main text a finite number of particular graphs that are too small to satisfy the described pattern. Please refer to the appendix for the omitted particular graphs. In Section 4.1, we apply the guiding color technique presented in Section 3.2 to find three Type 1 infinite families of the direct product of complete graphs: $K_3 \times K_n$, $K_5 \times K_n$ and $K_7 \times K_n$. Thus, for $m = 3, 5, 7$ and $n \geq m$, first we give the elements of the guiding color class when $gcd(m, n) = 1$ and second when $gcd(m, n) \neq 1$. Particular graphs for families $K_3 \times K_n$, $K_5 \times K_n$, $K_7 \times K_n$ have the elements of the guiding color presented in the appendix in Section A.1. In addition, in Section 4.2, we obtain two Type 1 infinite families: $K_9 \times K_n$ and $K_{11} \times K_n$. Analogously to Section 2, we use the result of Chew [7] and Lemma 2.1 to obtain that the family $K_9 \times K_n$, for $n \geq 23$, and the family $K_{11} \times K_n$, for $n \geq 15$, are both Type 1. For the particular graphs $K_9 \times K_n$, for $9 \leq n \leq 21$, and $K_{11} \times K_n$, for $n = 11, 13$, suitable Hamiltonian decompositions and the guiding color class are presented in the appendix in Section A.2. In view of the above, this section establishes Theorem 4.1 which together with previous Theorem 2.2 yields the proof that $K_m \times K_n$ is Type 1, except for $K_2 \times K_2$.

Theorem 4.1. *For odd numbers $m, n \geq 3$ with $m < 13$, the graph $K_m \times K_n$ is Type 1.*

4.1. Families $K_3 \times K_n$, $K_5 \times K_n$, $K_7 \times K_n$

In this subsection, we consider three Type 1 infinite families $K_m \times K_n$, for $m = 3, 5, 7$ and $n > m$ an odd number. These families are divided into two cases: first, when $gcd(m, n) = 1$ in Lemma 4.2 and second, when $gcd(m, n) = m$ in Lemma 4.3.

Lemma 4.2. *For $m = 3, 5, 7$ and an odd number $n > m$ with $gcd(m, n) = 1$, the graph $K_m \times K_n$ is Type 1.*

Proof. To obtain a $(\Delta(K_m \times K_n) + 1)$-total coloring for the three infinite families $K_m \times K_n$ for $m = 3, 5, 7$ and $n > m$ an odd number with $gcd(m, n) = 1$, first we use the Hamiltonian decomposition of $K_m \times K_n$ defined in Section 3.1 Case 1 to construct the three tables respectively with the elements of the guiding color.

- For $m = 3$. The general case for $K_3 \times K_n$, with $n \geq 11$ and $gcd(3, n) = 1$, is presented in Table 1. This case $m = 3$ has 2 particular graphs: $K_3 \times K_5$ (solved in Sect. 3.2, see Fig. 5) and $K_3 \times K_7$ presented in the appendix in Section A.1.
- For $m = 5$. The general case for $K_5 \times K_n$, with $n \geq 17$, $n \neq 21$ and $gcd(5, n) = 1$, is presented in Table 2. This case $m = 5$ has 5 particular graphs: for $n = 7, 9, 11, 13, 21$ presented in the appendix in Section A.1.
- For $m = 7$. The general case for $K_7 \times K_n$, with $n \geq 23$, $n \neq 25, 33$ and $gcd(7, n) = 1$, is presented in Table 3. This case $m = 7$ has 8 particular graphs: for $n = 9, 11, 13, 15, 17, 19, 25, 33$ presented in the appendix in Section A.1.

TABLE 1. Elements of $K_3 \times K_n$ of the guiding color, for $n = 2w+1,\ n \geq 11$ and $gcd(3,n) = 1$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(1,1)(2,2),(1,3)(2,2w-1),(1,2w-2)(2,5)$	$C(1,4)$	$(1,0)(2,4)$
$C(1,i)$	$(1,i)(2,i+1),\ i = 2,5,6,\ldots,2w-3,2w-1$	$C(1,2w-2)$	$(2,0)(0,2w-2)$
$C(1,3)$	$(0,3)(1,4)$	$C(1,2w)$	$(1,2w)(2,1)$
Vertices: $(0,i),\ i = 0,\ldots,2w, i \neq 3, 2w-2$			

TABLE 2. Elements of $K_5 \times K_n$ of the guiding color, for $n = 2w+1,\ n \geq 17,\ n \neq 21$ and $gcd(5,n) = 1$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(2,2)(4,2w),(2,2w-2)(4,5),(2,7)(4,2w-5)$	$C(2,1)$	$(3,2)(1,2w)$
$C(1,i)$	$(2,i+1)(4,i-1),\ i = 2,\ldots,2w-1,$ $i \neq 6, 2w-4, 2w-3$	$C(2,i)$	$(3,i+1)(1,i-1),\ i = 2,\ldots,2w-1,$ $i \neq 5, 2w-5, 2w-4$
$C(1,6)$	$(0,w+6)(1,0)$	$C(2,5)$	$(4,0)(0,5)$
$C(1,2w-4)$	$(2,0)(4,2w-4)$	$C(2,2w-5)$	$(3,0)(1,2w-5)$
$C(1,2w-3)$	$(3,2w-4)(0,2w-1)$	$C(2,2w-4)$	$(0,2w-4)(2,2w-3)$
$C(1,2w)$	$(2,1)(4,2w-1)$	$C(2,2w)$	$(3,1)(1,2w-1),(3,2w-3)(1,4),$ $(3,6)(1,2w-6)$
Vertices: $(0,i)$, for $i = 0,\ldots,2w,\ i \neq 5, w+6, 2w-4, 2w-1$			

Thus, the family $K_m \times K_n$, with odd numbers $m = 3,5,7,\ n > m$ and $gcd(m,n) = 1$, is Type 1. □

Lemma 4.3. *For $m = 3,5,7$ and an odd number $n \geq m$ with $gcd(m,n) = m$, the graph $K_m \times K_n$ is Type 1.*

Proof. Analogous to the proof of Lemma 4.2, to obtain a $(\Delta(K_m \times K_n)+1)$-total coloring for the families $K_m \times K_n$, when $m = 3,5,7,\ n \geq m$ are odd numbers and $gcd(m,n) = m$, first we use the Hamiltonian decomposition of $K_m \times K_n$ as Section 3.1 Case 2 to construct the three tables respectively with the elements of the guiding color.

- For $m = 3$. First, we construct a Hamilton decomposition of $K_3 \times K_n$ as Section 3.1 Case 2(i). The general case for $K_3 \times K_n$, with $n \geq 9$ and $gcd(3,n) = 3$, is presented in Table 4. This case $m = 3$ has one particular graph $K_3 \times K_3$ presented in the appendix in Section A.1.
- For $m = 5$. First, we construct a Hamilton decomposition of $K_5 \times K_n$ as Section 3.1 Case 2(ii). The general case for $K_5 \times K_n$, with $n \geq 15$ and $gcd(5,n) = 5$, is presented in Table 5. This case $m = 5$ has one particular graph $K_5 \times K_5$ presented in the appendix in Section A.1.
- For $m = 7$. First we construct a Hamilton decomposition of $K_7 \times K_n$ as Section 3.1 Case 2(iii). The general case for $K_7 \times K_n$, with $n \geq 35$ and $gcd(7,n) = 7$, is presented in Table 6. This case $m = 7$ has 2 particular graphs $K_7 \times K_7$ and $K_7 \times K_{21}$ presented in the appendix in Section A.1.

Thus, the family $K_m \times K_n$, with odd numbers $m = 3,5,7,\ n > m$ and $gcd(m,n) = m$, is Type 1. □

4.2. Families $K_9 \times K_n$ and $K_{11} \times K_n$

In Lemma 4.4, for odd numbers $m = 9,11$ and $n \geq m$, we establish that $\Delta(K_m \times K_n)$ satisfies the lower bound required by Chew [7], except for a finite number of graphs, which together with Lemma 2.1 implies the desired total chromatic number. The desired total coloring of each one of these finitely many graphs is obtained

TABLE 3. Elements of $K_7 \times K_n$ of the guiding color, for $n = 2w+1,\ n \geq 23,\ n \neq 25, 33$ and $gcd(7, n) = 1$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(6,2w)(3,3), (6,6)(3,2w-4),$ $(6,2w-7)(3,10)$	$C(2,2w-8)$	$(5,0)(0,2w-8)$
$C(1,i)$	$(6,i-1)(3,i+2),\ i = 2,\dots,2w-2,$ $i \neq 7, 8, 2w-6$	$C(2,2w-1)$	$(1,2w-2)(4,1), (1,4)(4,2w-6),$ $(1,2w-9)(4,8)$
$C(1,7)$	$(2,0)(6,7)$	$C(2,2w)$	$(1,2w-1)(4,2)$
$C(1,8)$	$(4,7)(0,10)$	$C(3,1)$	$(2,2w)(5,3)$
$C(1,2w-6)$	$(0,w-6)(1,0)$	$C(3,i)$	$(2,i-1)(5,i+2),\ i = 2,\dots,2w-2,$ $i \neq 6, 7, 2w-7$
$C(1,2w-1)$	$(6,2w-2)(3,1)$	$C(3,6)$	$(4,0)(2,6)$
$C(1,2w)$	$(6,2w-1)(3,2)$	$C(3,7)$	$(0,6)(3,9)$
$C(2,1)$	$(1,2w)(4,3)$	$C(3,2w-7)$	$(6,0)(0,2w-7)$
$C(2,i)$	$(1,i-1)(4,i+2),\ i = 2,\dots,2w-2,$ $i \neq 5, 6, 2w-8$	$C(3,2w-1)$	$(2,2w-2)(5,1)$
$C(2,5)$	$(3,0)(1,5)$	$C(3,2w)$	$(2,2w-1)(5,2), (2,5)(5,2w-5),$ $(2,2w-8)(5,9)$
$C(2,6)$	$(5,8)(0,4)$		
Vertices: $(0,i),\ i = 0,\dots,2w,\ i \neq 4, 6, 10, w-6, 2w-8, 2w-7$			

TABLE 4. Elements of $K_3 \times K_n$ of the guiding color, for $n = 2w+1,\ n \geq 9$ and $gcd(3, n) = 3$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(1,1)(2,2), (1,3)(2,2w-1), (1,2w-2)(2,5)$	$C(1,w-2)$	$(2,w-1)(0,w-3)$
$C(1,i)$	$(1,i)(2,i+1),$ $i = 2, 5, 6, \dots, w-3, w-1, w$	$C(2,i)$	$(2,w+i+1)(1,w+i),$ $i = 1,\dots,w-3, w-1$
$C(1,3)$	$(0,3)(1,4)$	$C(2,w-2)$	$(2,0)(1,w-2)$
$C(1,4)$	$(1,0)(2,4)$	$C(2,w)$	$(2,1)(1,2w)$
Vertices: $(0,i),\ i = 0,\dots,2w,\ i \neq 3, w-3$			

TABLE 5. Elements of $K_5 \times K_n$ of the guiding color, for $n = 2w+1,\ n \geq 15$ and $gcd(5, n) = 5$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(2,2)(4,2w), (2,2w-2)(4,5), (2,7)(4,2w-5)$	$C(3,w-4)$	$(3,2w-4)(2,0)$
$C(1,i)$	$(2,i+1)(4,i-1),\ i = 2,\dots,w,\ i \neq 6$	$C(3,w-3)$	$(1,0)(0,w-3)$
$C(1,6)$	$(3,0)(0,6)$	$C(3,w)$	$(4,2w-1)(2,1)$
$C(2,1)$	$(3,2)(1,2w)$	$C(4,i)$	$(1,w+i-1)(3,w+i+1),\ i = 1,\dots,w-1,$ $i \neq w-5, w-4$
$C(2,i)$	$(3,i+1)(1,i-1),\ i = 2,\dots,w, i \neq 5$	$C(4,w-5)$	$(4,2w-4)(1,2w-5)$
$C(2,5)$	$(4,0)(0,5)$	$C(4,w-4)$	$(2,2w-3)(0,2w-4)$
$C(3,i)$	$(4,w+i-1)(2,w+i+1),\ i = 1,\dots,w-1,$ $i \neq w-4, w-3$	$C(4,w)$	$(1,2w-6)(3,6), (1,4)(3,2w-3),$ $(1,2w-1)(3,1)$
Vertices: $(0,i),\ i = 0,\dots,2w,\ i \neq 5, 6, w-3, 2w-4$			

TABLE 6. Elements of $K_7 \times K_n$ of the guiding color, for $n = 2w+1,\ n \geq 35$ and $gcd(7,n) = 7$.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(6,2w)(3,3), (6,6)(3,2w-4), (6,2w-7)(3,10)$	$C(4,i)$	$(3,w+i+2)(6,w+i-1),\ i=1,\dots,w-2,\ i \neq w-6$
$C(1,i)$	$(6,i-1)(3,i+2),\ i=2,\dots,w,\ i \neq 7,8$	$C(4,w-6)$	$(1,0)(0,w-6)$
$C(1,7)$	$(2,0)(6,7)$	$C(4,w-1)$	$(3,1)(6,2w-2)$
$C(1,8)$	$(4,7)(0,10)$	$C(4,w)$	$(3,2)(6,2w-1)$
$C(2,1)$	$(1,2w)(4,3)$	$C(5,i)$	$(4,w+i+2)(1,w+i-1),\ i=1,\dots,w-2,\ i \neq w-8$
$C(2,i)$	$(1,i-1)(4,i+2),\ i=2,\dots,w,\ i \neq 5,6$	$C(5,w-8)$	$(0,2w-8)(6,0)$
$C(2,5)$	$(3,0)(1,5)$	$C(5,w-1)$	$(4,8)(1,2w-9), (4,2w-6)(1,4), (4,1)(1,2w-2)$
$C(2,6)$	$(5,8)(0,4)$	$C(5,w)$	$(4,2)(1,2w-1)$
$C(3,1)$	$(2,2w)(5,3)$	$C(6,i)$	$(5,w+i+2)(2,w+i-1),\ i=1,\dots,w-2,\ i \neq w-7$
$C(3,i)$	$(2,i-1)(5,i+2),\ i=2,\dots,w,\ i \neq 6,7$	$C(6,w-7)$	$(0,2w-7)(5,0)$
$C(3,6)$	$(4,0)(2,6)$	$C(6,w-1)$	$(5,1)(2,2w-2)$
$C(3,7)$	$(0,6)(3,9)$	$C(6,w)$	$(5,9)(2,2w-8), (5,2w-5)(2,5), (5,2)(2,2w-1)$
Vertices: $(0,i),\ i=0,\dots,2w,\ i \neq 4,6,10,w-6,2w-8,2w-7$			

by the guiding color technique. We present in the appendix in Section A.2 their respective guiding colors and Hamiltonian decompositions since they are not given in Section 3.1.

Lemma 4.4. *For $m = 9, 11$ and an odd number $n \geq m$, the graph $K_m \times K_n$ is Type 1.*

Proof. In Section 2, we have actually proved that for odd numbers m, n the graph $K_m \times K_n$ is Type 1, provided that $\Delta(G) \geq \frac{(\sqrt{37}-1)}{6}|V(G)|$.

We show next that $K_9 \times K_n$ with $n \geq 23$ and $K_{11} \times K_n$ with $n \geq 15$ satisfy the required bound. Indeed, for $K_9 \times K_n$, when $n \geq 23$, we have that $n \geq 16/(16 - 3(\sqrt{37}-1))$. Therefore, $8(n-1) \geq \frac{(\sqrt{37}-1)}{6} \cdot 9n$, that is $\Delta(K_9 \times K_n) \geq \frac{(\sqrt{37}-1)}{6} \cdot \mid V(K_9 \times K_n) \mid$. For $K_{11} \times K_n$, when $n \geq 15$, we have that $n \geq 60/(60 - 11(\sqrt{37}-1))$. Therefore, $10(n-1) \geq \frac{(\sqrt{37}-1)}{6} \cdot 11n$, that is $\Delta(K_{11} \times K_n) \geq \frac{(\sqrt{37}-1)}{6} \cdot \mid V(K_{11} \times K_n) \mid$. Thus, we have that for $n \geq 23$, the graph $K_9 \times K_n$ is Type 1 and for $n \geq 15$, the graph $K_{11} \times K_n$ is Type 1.

Particular graphs are $K_9 \times K_n$, for $n = 9, 11, 13, 15, 17, 19, 21$, and $K_{11} \times K_n$, for $n = 11, 13$. These particular graphs are presented in the appendix in Section A.2. □

5. Total chromatic number of $C_m \times K_n$

In the sections above, we ensured that the direct product of complete graphs $K_m \times K_n$ is Type 1, except for $K_2 \times K_2$. We already know that the direct product of cycles $C_m \times C_n$ is Type 1, except for $C_4 \times C_4$ [4]. In this section, we investigate the direct product of a cycle with a complete graph $C_m \times K_n$, for $m \geq 3$ and $n \geq 2$. Interestingly, we have a Type 2 infinite family $C_m \times K_n$, when m is not a multiple of 3 and $n = 2$.

In Section 5.1, we distinguish the particular cases that are solved by previous results, among them we are able to classify all $C_m \times K_n$, when m is even. In Section 5.2, we use our Hamiltonian decomposition and the guiding color technique to obtain new Type 1 infinite families, so that we classified all $C_m \times K_n$, when m is odd and n is even, and additional Type 1 infinite families $C_{t(2n-1)} \times K_n$, for $t \geq 1$, when n is odd. Based on the evidence found so far, we conjecture that $C_m \times K_n$ is Type 2 if and only if m is not a multiple of 3 and $n = 2$.

5.1. Previous results classify some infinite families of $C_m \times K_n$

In addition to the results presented in this work, in Propositions 5.1 and 5.2, we shall use that, except for $C_4 \times C_4$, all direct product of cycle graphs $C_m \times C_n$ are Type 1 [4]; and that if $G \times K_2$ is Type 1 and H is bipartite, then $G \times H$ is Type 1 [10].

Proposition 5.1. *Let $m \geq 3$ and $n = 2, 3$. The graph $C_m \times K_n$ is Type 2, when m is not a multiple of 3 and $n = 2$, otherwise the graph is Type 1.*

Proof. First, consider $C_m \times K_n$ when $n = 2$. If m is odd, then $C_m \times K_2 = C_{2m}$, and otherwise if m is even, then $C_m \times K_2 = 2C_m$. Cycle graphs are well known to be Type 1 if the size of the cycle is a multiple of 3 and Type 2, otherwise. Therefore, $C_m \times K_2$ is Type 1 if m is a multiple of 3 and Type 2, otherwise.

Now, consider $C_m \times K_n$, when $n = 3$. Since $C_m \times K_3 = C_m \times C_3$, they are all Type 1 by Castonguay *et al.* [4]. □

Proposition 5.2. *For $m = 3$ and $n \geq 2$, and for even m and $n \geq 3$, the graph $C_m \times K_n$ is Type 1.*

Proof. First, consider $C_3 \times K_n$. Since $C_3 \times K_n = K_3 \times K_n$, we have by sections above that they are all Type 1.

Now, consider $C_m \times K_n$, when m is even and $n \geq 3$. Mackeigan and Janssen [10] proved that if $G \times K_2$ is Type 1, then $G \times H$ is also Type 1, for any bipartite graph H. When $n \geq 3$, the graph $K_n \times K_2$ is the complete bipartite graph $K_{n,n}$ minus a perfect matching, and Yap [14] proved that this graph is Type 1. Since when m is even C_m is a bipartite graph, and $K_n \times C_m$ is isomorphic to $C_m \times K_n$, we have that $C_m \times K_n$ is Type 1. □

5.2. Additional new infinite families of $C_m \times K_n$ are Type 1

In this subsection, we advance on the results in total coloring of $C_m \times K_n$, by obtaining additional new infinite families that are Type 1 and are not solved by previous results. In Theorem 5.3, we use Hamiltonian decompositions of $C_m \times K_n$ to construct a $(2n-1)$-total coloring based on total-colorings and edge-colorings of the subgraphs of this decomposition.

Theorem 5.3. *For odd $m \geq 5$ and even $n \geq 4$, the graph $C_m \times K_n$ is Type 1.*

Proof. First, we consider $C_m \times K_n$, when $m \neq 7$.

As C_m is a unique Hamiltonian cycle, for sake of simplicity, we consider the sequential order of $C_m = \langle 0, 1, 2, 3, \dots, m-1, 0\rangle$. We consider the Walecki's Hamiltonian decomposition of the complete graph K_n as described in Section 3.1, however for an even number $n = 2w \geq 4$. Following the notation used in [1], let C_n be the Hamiltonian cycle $\langle 0, 1, 2, 2w-1, 3, 2w-2, 4, 2w-3, \dots, w-1, w+2, w, w+1, 0\rangle$. As σ is the permutation $(0)(1, 2, 3, 4, \dots, 2w-2, 2w-1)$, then $\sigma^0(C_n), \sigma^1(C_n), \sigma^2(C_n), \dots, \sigma^{w-2}(C_n)$ are $(w-1)$ edge-disjoint Hamiltonian cycles. The remaining edges $\{(0)(w), (w-1)(w+1), (w-2)(w+2), \dots, (1)(2w-1)\}$ form a perfect matching M. Observe that $\sigma^0(C_n) = C_n$. For an even $n \geq 4$, we write the Hamiltonian decomposition $K_n = (\bigoplus_{i=1}^{w-1} \sigma^{i-1}(C_n)) \oplus M$. Recall that we denote by $\sigma^t(C_n)_z$, with $z = 0, 1, \dots, n-1$ the zth-vertex in the cycle $\sigma^t(C_n)$.

Observe that the spanning subgraphs of this Hamiltonian decomposition are edge-disjoint and thus, we may write that $C_m \times K_n = (\bigoplus_{i=1}^{w-1} (C_m \times \sigma^{i-1}(C_n))) \oplus (C_m \times M)$.

Since n is even, the subgraph $C_m \times \sigma^{i-1}(C_n)$ is a bipartite 4-regular graph and has a 4-edge coloring. In addition, the subgraph $C_m \times \sigma^{i-1}(C_n)$ has a 5-total coloring [4]. On the other hand, $C_m \times M$ is a 2-regular graph isomorphic to a disjoint union of $n/2$ cycles of order $2m$ and therefore, has a 2-edge coloring.

In order to construct a $(2n-1)$-total coloring of $C_m \times K_n$, first we assign 5 colors to all the vertices of $C_m \times K_n$. In this sense, consider the 5-total coloring of $C_m \times \sigma^0(C_n)$, for $m \neq 7$, described in [4] (see an example

TABLE 7. A 7-total coloring of $C_7 \times (\sigma^0(C_n) \oplus M)$.

Color	Vertices	Edges
1 (pink)	I_0	N_1, N_3, N_5
2 (green)	I_1	N_2, N_4, N_6
3 (dark blue)	I_2	N_0, N'_3, N'_5
4 (yellow)	I_3	N'_1, N'_4, N'_6
5 (orange)	I_4	N'_0, N'_2, M_5
6 (light blue)	I_5	M_1, M_3, M_6
7 (wine)	I_6	M_0, M_2, M_4

in Fig. 6). In this total coloring, two vertices of the same color (u, v) and (u', v') in $C_m \times \sigma^0(C_n)$ are such that $uu' \notin E(C_m)$, thus $(u, v)(u', v') \notin E(C_m \times K_n)$. Therefore, the 5-vertex coloring of $C_m \times \sigma^0(C_n)$ induced by this 5-total coloring still is a 5-vertex coloring of $C_m \times K_n$.

To obtain the desired total coloring of $C_m \times K_n$, we still need to assign $2n - 1$ colors to all the edges of $C_m \times K_n$, such that we have no conflict with the 5 colors already assigned to the vertices. For this, we continue to assign the same 5 colors, used to color the vertices, for the edges from the 5-total coloring of the subgraph $C_m \times \sigma^0(C_n)$. In addition, for the remaining edges, we consider a $2(n - 4)$-edge coloring composed of 4-edge colorings of each of the other $(w - 2)$ subgraphs $C_m \times \sigma^{i-1}(C_n)$, for $i = 2, \ldots, w - 1$ (see an example in Fig. 7), and a 2-edge coloring of the subgraph $C_m \times M$ (see an example in Fig. 8).

Unfortunately, the 5-total coloring of $C_7 \times \sigma^0(C_n)$ given in [4] does not induce a 5-vertex coloring of $C_7 \times K_n$. In this case, we consider $C_7 \times K_n = (\bigoplus_{i=2}^{w-1} (C_7 \times \sigma^{i-1}(C_n))) \oplus (C_7 \times (\sigma^0(C_n) \oplus M))$. We use the spanning 6-regular subgraph $C_7 \times (\sigma^0(C_n) \oplus M)$ to obtain a 7-vertex coloring of $C_m \times K_n$ from the 7-total coloring of $C_7 \times (\sigma^0(C_n) \oplus M)$ described in Table 7.

Recall that $C_7 = \langle 0, 1, 2, 3, 4, 5, 6, 0\rangle$. For $i = 0, \ldots, 6$, denote by $I_i = \{(i, j) \mid j = 0, \ldots, n-1\}$ an independent set. Observe that, in $C_7 \times (\sigma^0(C_n) \oplus M)$, we have three perfect matchings N_i, N'_i and M_i between independent sets I_i and I_{i+1}. The two perfect matchings N_i and N'_i come from C_n and the perfect matching M_i comes from M.

In order to construct a $(2n - 1)$-total coloring of $C_7 \times K_n$, first we assign 7 colors to all vertices of $C_7 \times K_n$. In this sense, consider the 7-total coloring of $C_7 \times (\sigma^0(C_n) \oplus M)$ as Table 7 (see an example in Fig. 9). In this total coloring, two vertices of the same color (u, v) and (u', v') in $C_7 \times (\sigma^0(C_n) \oplus M)$ are such that $uu' \notin E(C_7)$, thus $(u, v)(u', v') \notin E(C_7 \times K_n)$. Therefore, the 7-vertex coloring of $C_7 \times (\sigma^0(C_n) \oplus M)$ induced by this 7-total coloring still is a 7-vertex coloring of $C_7 \times K_n$. To obtain the desired total coloring of $C_7 \times K_n$, we still need to assign $2n - 1$ colors to all the edges of $C_7 \times K_n$. For this, we continue to assign the same 7 colors, used to color the vertices, for the edges from the 7-total coloring of the subgraph $C_7 \times (\sigma^0(C_n) \oplus M)$. In addition, for the remaining edges, we consider a $2(n - 4)$-edge coloring composed of 4-edge colorings of each of the other $(w - 2)$ subgraphs $C_7 \times \sigma^{i-1}(C_n)$, for $i = 2, \ldots, w - 1$. □

For instance, consider $C_5 \times K_6$ and the Walecki's Hamiltonian decomposition of $K_6 = (\bigoplus_{i=1}^{2} \sigma^{i-1}(C_6)) \oplus M$, where the two cycles are $\sigma^0(C_6) = \langle 0, 1, 2, 5, 3, 4, 0\rangle$ and $\sigma^1(C_6) = \langle 0, 2, 3, 1, 4, 5, 0\rangle$, and the matching is $M = \{(0)(3), (2)(4), (1)(5)\}$. Therefore, $C_5 \times K_6 = (\bigoplus_{i=1}^{2} (C_5 \times \sigma^{i-1}(C_6))) \oplus (C_5 \times M)$. We present the 11-total coloring of $C_5 \times K_6$ described in Figures 6–8. In Figure 6, we assign 5 colors to the vertices of $C_5 \times K_6$ and the same 5 colors to the edges of the subgraph $C_5 \times \sigma^0(C_6)$. In Figure 7, we present a 4-edge coloring of the subgraph $C_5 \times \sigma^1(C_6)$. In Figure 8, we present a 2-edge coloring of the subgraph $C_5 \times M$. Observe that, in Figures 6–8, the order of second coordinates obey the Walecki's Hamiltonian decomposition of K_6.

When $m = 7$, for instance consider $C_7 \times K_4$. As the Hamiltonian decomposition of K_4 has a unique cycle and one matching, we only consider the 7-total coloring of $C_7 \times K_4$, given by Table 7 when $n = 4$, as depicted in Figure 9.

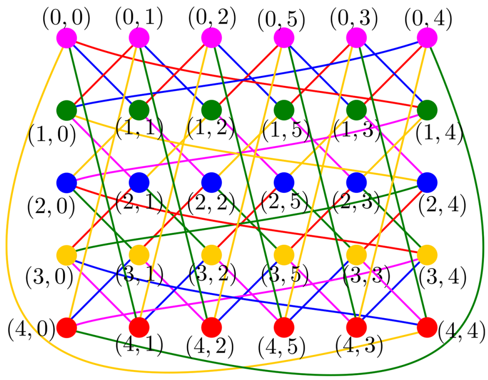

FIGURE 6. A depiction of the 5-total coloring of a subgraph $C_5 \times \sigma^0(C_6)$.

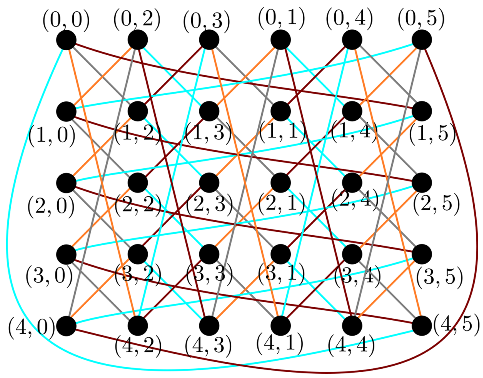

FIGURE 7. A depiction of the 4-edge coloring of the subgraph $C_5 \times \sigma^1(C_6)$.

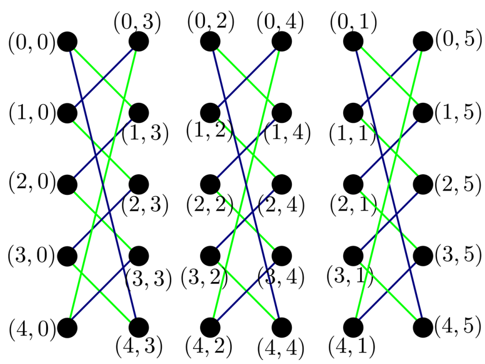

FIGURE 8. A depiction of the 2-edge coloring of the subgraph $C_5 \times M$.

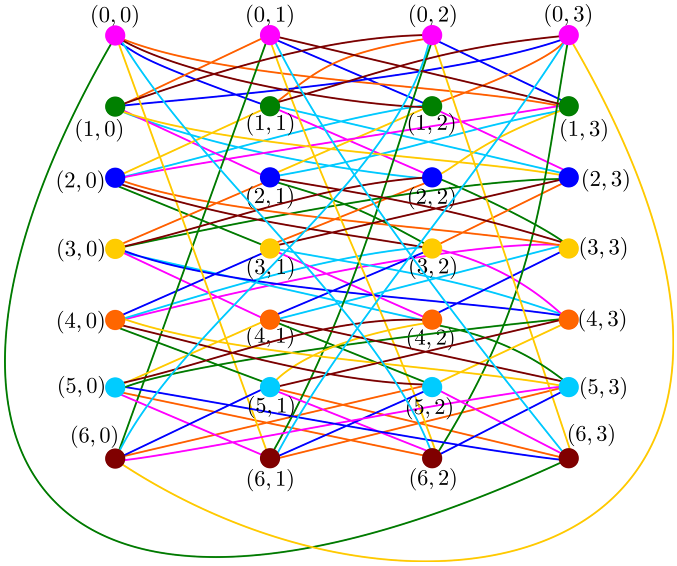

FIGURE 9. A depiction of a 7-total coloring of $C_7 \times K_4$, corresponding to Table 7.

It remains to determine the total chromatic number of $C_m \times K_n$ when both $m, n \geq 5$ are odd numbers.

It is known that if G and H are two regular graphs, then in case G is conformable, the graph $G \times H$ is conformable [4] as well. It is known that K_n is conformable as this graph is Type 1 for n odd. Thus, $C_m \times K_n$ is conformable.

Unfortunately, although the graph is conformable of odd order, the lower bound $\Delta(G) \geq \frac{(\sqrt{37}-1)}{6}|V(G)|$ used in Section 2 [7] is not useful for the class $C_m \times K_n$. Recall that $|V(C_m \times K_n)| = mn$ and $\Delta(C_m \times K_n) = 2n-2$. Indeed, $\frac{(\sqrt{37}-1)}{6} \cdot mn \geq \frac{(\sqrt{37}-1)}{6} \cdot 5n > 4n > 2n-2$.

However, in Theorem 5.4, we present an infinite family using the guiding color technique to obtain a Type 1 total coloring.

Theorem 5.4. *For $m = 2n-1$ and odd $n \geq 5$, the graph $C_m \times K_n$ is Type 1.*

Proof. In order to obtain a $(2n-1)$-total coloring, we use the technique of the guiding color. First, consider $n = 2w+1$ and the Walecki's Hamiltonian decomposition of $K_n = \bigoplus_{i=1}^{w} \sigma^{i-1}(C_n)$ as described in Section 3.1 and write $C_m = \langle 0, 1, 2, 3, \ldots, m-1\rangle$. We have that $C_m \times K_n = \bigoplus_{i=1}^{w}(C_m \times \sigma^{i-1}(C_n))$, where each subgraph $C_m \times \sigma^{i-1}(C_n)$ is a 4-regular graph and has two Hamiltonian cycles. As $gcd(m,n) = 1$ and $n-1 = 2w$, denote the $n-1$ Hamiltonian cycles by $C(1,i) = \langle C(1,i)_z\rangle$, for $i = 1, \ldots, n-1$, where $C(1,i)_z = ((C_m)_{z \bmod m}, \sigma^{i-1}(C_n)_{z \bmod n})$, with $z = 0, \ldots, mn$. Consider $\{C(1,i) \mid i = 1, \ldots, n-1\}$ a Hamiltonian decomposition of $C_m \times K_n$. For $i = 1, \ldots, w$, note that $C(1,i)$ and $C(1,i+w)$ are two Hamiltonian cycles of the subgraph $C_m \times \sigma^{i-1}(C_n)$.

Recall that, as each Hamiltonian cycle has an odd number of edges mn, each cycle admits a 3-edge coloring where an odd number of edges are colored by the guiding color. For $i = 1, \ldots, n-1$, each Hamiltonian cycle $C(1,i)$ has n edges of the guiding color, where these edges have first coordinate $2i-1$ or $2i$, and additional two colors with $(mn-n)/2$ edges for each color. Note that, in the graph $C_m \times K_n$, there are $n(n-1)$ edges assigned with the guiding color. Since $m = 2n-1$, we have that $n(n-1) = (mn-n)/2$. Therefore each color, including the guiding color, are assigned to the same amount of edges.

On the other hand, there are n vertices with the same first coordinates, therefore we have two independent sets, each of them with n vertices of the same color. In this way, note that we assigned $2n-2$ colors to the vertices with first coordinates from 1 to $2n-2$. As $m = 2n-1$, the remaining n uncolored vertices with first coordinates 0 can be colored with the guiding color. □

TABLE 8. A table composed by the elements of the guiding purple color in $C_9 \times K_5$.

Elements of $C_9 \times K_5$ of the guiding color	
$C(1,1)$	$(1,1)(2,2), (1,0)(2,1), (1,3)(2,0), (1,4)(2,3), (1,2)(2,4)$
$C(1,2)$	$(3,1)(4,4), (3,3)(4,1), (3,2)(4,3), (3,0)(4,2), (3,4)(4,0)$
$C(1,3)$	$(5,0)(6,3), (5,1)(6,0), (5,2)(6,1), (5,4)(6,2), (5,3)(6,4)$
$C(1,4)$	$(7,1)(8,3), (7,4)(8,1), (7,0)(8,4), (7,2)(8,0), (7,3)(8,2)$
Vertices: $(0,0), (0,1), (0,2), (0,3), (0,4)$	

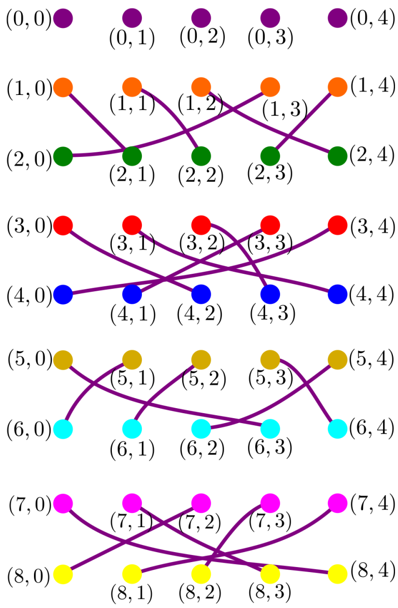

FIGURE 10. A depiction of the elements of the guiding color using colors of the endvertices to identify the Hamiltonian cycles containing them, corresponding to Table 8.

As an example of the $(2n-1)$-total coloring obtained in Theorem 5.4, consider the 9-total coloring of $C_9 \times K_5$ presented in Table 8 and the corresponding Figure 10. Note that Table 8 and Figure 10 are similar to Figure 5, where we presented in the same figure the elements of the guiding color and the corresponding 9-vertex coloring. Observe that, in both figures, each two colors used for the endvertices of the edges of the guiding color identify the same 2 colors assigned to the remaining edges of the Hamiltonian cycle containing the respective edges of the guiding color.

In fact, we can use known merge techniques, as used in [4], to find additional Type 1 infinite families, for instance, the graph $C_{t(2n-1)} \times K_n$, for $t \geq 2$, where we merge t copies of the graph $C_{2n-1} \times K_n$, when $n \geq 5$ is odd, without conflict of vertices and edges. We conjecture that graphs $C_m \times K_n$, with odd numbers $m, n \geq 5$, are Type 1. Thus, the provided evidence leads us to conjecture that the only direct product of a cycle with a complete graph that are Type 2 are the ones given by Proposition 5.1.

Conjecture 5.5. The graph $C_m \times K_n$ is Type 2 if and only if m is not a multiple of 3 and $n = 2$.

TABLE A.1. Elements of particular graph $K_3 \times K_7$ of the guiding color.

Cycle	Edges	Cycle	Edges	Cycle	Edges
$C(1,1)$	$(1,1)(2,2),(1,3)(2,5),(1,0)(2,1)$	$C(1,3)$	$(1,6)(2,0)$	$C(1,5)$	$(1,5)(2,6)$
$C(1,2)$	$(1,2)(2,3)$	$C(1,4)$	$(0,0)(1,4)$	$C(1,6)$	$(2,4)(0,3)$
Vertices: $(0,i)$ for $i=1,2,4,5,6$					

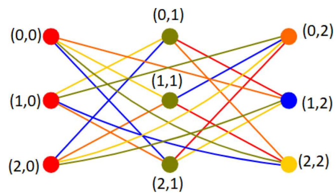

FIGURE A.1. A depiction of particular graph $K_3 \times K_3$ with 5 colors.

APPENDIX A.

A.1. Elements of the guiding color of particular graphs for families $K_3 \times K_n$, $K_5 \times K_n$, $K_7 \times K_n$

Please refer to Table A.1 of particular graph $K_3 \times K_7$; to Table A.2 of particular graphs $K_5 \times K_n$, for $n = 5, 7, 9, 11, 13, 21$; to Table A.3 of particular graphs $K_7 \times K_n$, for $n = 9, 11, 13, 15, 17$; and to Table A.4 of particular graphs $K_7 \times K_n$, for $n = 19, 25, 33$ according to Lemma 4.2.

Please refer to Figure A.1 which depicts the particular graph $K_3 \times K_3$ with 5 colors; to Table A.5 of particular graph $K_5 \times K_5$; to Table A.6 of particular graph $K_7 \times K_7$; and to Table A.7 of particular graph $K_7 \times K_{21}$ according to Lemma 4.3.

A.2. Hamiltonian decomposition of particular graphs $K_9 \times K_n$ and $K_{11} \times K_n$, and the elements of the guiding color

Consider the particular graphs $K_9 \times K_n$ with $n = 9, 11, 13, 15, 17, 19, 21$ and $K_{11} \times K_n$ with $n = 11, 13$, according to Lemma 4.4. First, we present each Hamiltonian decomposition. Second, refer to the following tables containing the elements of the guiding color: to Table A.8 of particular graphs $K_9 \times K_n$, for $n = 11, 13, 17, 19$; to Table A.9 of particular graphs $K_9 \times K_n$, for $n = 9, 15, 21$; to Table A.10 of particular graph $K_{11} \times K_{11}$; and to Table A.11 of particular graph $K_{11} \times K_{13}$.

In order to present the Hamiltonian decompositions when $gcd(9, n) = 1$ and when $gcd(11, n) = 1$, we can proceed analogously to Section 3.1 Case 1.

We present next five Hamiltonian decompositions to deal with $gcd(9, n) \neq 1$ and $gcd(11, n) \neq 1$. Let m and n odd numbers such that $gcd(m, n) = d \neq 1$. Recall that we have defined the following paths of $K_m \times K_n$. For $j = 1, \ldots, m-1$, $i = 1, \ldots, n-1$ and $k = 0, \ldots, d-1$, the path $P(j,i)^k$ is induced by $(\sigma^{j-1}(C_m)_{(t+k) \bmod m}; \sigma^{i-1}(C_n)_{z \bmod n})$, with $z = 0, \ldots, \frac{mn}{d} - 1$.

– For $K_9 \times K_9$, a Hamiltonian decomposition is given by $\{C(j,i) \mid j = 1, \ldots, 8,\ i = 1, \ldots, 4\}$, where:

$$C(j,i) = \langle P(j,i)^0, P(j,i)^7, P(j,i)^5, P(j,i)^3, P(j,i)^1, P(j,i)^8, P(j,i)^6, P(j,i)^4, P(j,i)^2, (0,0)\rangle.$$

Observe that, for $j = 1, 2, 3, 4$ and $i = 1, \ldots, 4$, the cycles $C(j,i)$ and $C(j+4,i)$ form a Hamiltonian decomposition of $\sigma^{j-1}(C_9) \times \sigma^{i-1}(C_9)$.

TABLE A.2. Elements of particular graphs $K_5 \times K_n$ of the guiding color, for $n = 7, 9, 11, 13, 21$.

Cycle	Edges	Cycle	Edges
Elements of $K_5 \times K_7$ of the guiding color			
$C(1,1)$	$(2,2)(4,6), (2,0)(4,1), (2,5)(4,4)$	$C(2,1)$	$(0,2)(2,6)$
$C(1,2)$	$(1,2)(2,3)$	$C(2,2)$	$(3,3)(1,1)$
$C(1,3)$	$(2,4)(4,2)$	$C(2,3)$	$(1,0)(4,3)$
$C(1,4)$	$(3,5)(0,3)$	$C(2,4)$	$(4,0)(0,4)$
$C(1,5)$	$(3,2)(0,0)$	$C(2,5)$	$(3,6)(1,4)$
$C(1,6)$	$(2,1)(4,5)$	$C(2,6)$	$(3,1)(1,5), (3,0)(1,6), (3,4)(1,3)$
Vertices: $(0,1), (0,5), (0,6)$			
Elements of $K_5 \times K_9$ of the guiding color			
$C(1,1)$	$(2,2)(4,8), (2,6)(4,5), (2,8)(4,3)$	$C(2,1)$	$(3,2)(1,8)$
$C(1,2)$	$(2,3)(4,1)$	$C(2,2)$	$(3,3)(1,1)$
$C(1,3)$	$(2,4)(4,2)$	$C(2,3)$	$(0,8)(2,7)$
$C(1,4)$	$(2,0)(4,4)$	$C(2,4)$	$(0,4)(2,5)$
$C(1,5)$	$(0,7)(1,3)$	$C(2,5)$	$(3,0)(1,5)$
$C(1,6)$	$(4,0)(3,6)$	$C(2,6)$	$(1,0)(4,6)$
$C(1,7)$	$(3,4)(0,3)$	$C(2,7)$	$(3,8)(1,6)$
$C(1,8)$	$(2,1)(4,7)$	$C(2,8)$	$(3,1)(1,7), (3,5)(1,4), (3,7)(1,2)$
Vertices: $(0,i)$, for $i = 0, 1, 2, 5, 6$			
Elements of $K_5 \times K_{11}$ of the guiding color			
$C(1,1)$	$(2,2)(4,10), (2,8)(4,5), (2,1)(4,2)$	$C(2,1)$	$(3,2)(1,10)$
$C(1,i)$	$(2,i+1)(4,i-1)$, $i = 2, 4, 5, 8, 9$	$C(2,2)$	$(0,8)(2,7)$
$C(1,3)$	$(0,9)(1,8)$	$C(2,i)$	$(3,i+1)(1,i-1)$, $i = 3, 4, 7, 8$
$C(1,6)$	$(4,0)(3,6)$	$C(2,5)$	$(4,9)(0,1)$
$C(1,7)$	$(3,3)(0,2)$	$C(2,6)$	$(1,0)(4,6)$
$C(1,10)$	$(1,5)(2,0)$	$C(2,9)$	$(2,4)(3,0)$
		$C(2,10)$	$(3,1)(1,9), (3,7)(1,4), (3,10)(1,1)$
Vertices: $(0,i)$, for $i = 0, 3, 4, 5, 6, 7, 10$			
Elements of $K_5 \times K_{13}$ of the guiding color			
$C(1,1)$	$(2,2)(4,12), (2,10)(4,5), (2,7)(4,0)$	$C(2,1)$	$(3,2)(1,12), (3,10)(1,5), (3,7)(1,0)$
$C(1,i)$	$(2,i+1)(4,i-1)$, $i = 2, 3, 4, 5, 7, 8, 10, 11$	$C(2,i)$	$(3,i+1)(1,i-1)$, $i = 2, 3, 4, 5, 7, 8, 10, 11$
$C(1,6)$	$(3,0)(0,6)$	$C(2,6)$	$(4,8)(0,4)$
$C(1,9)$	$(0,10)(1,8)$	$C(2,9)$	$(0,3)(2,0)$
$C(1,12)$	$(2,1)(4,11)$	$C(2,12)$	$(3,1)(1,11)$
Vertices: $(0,i)$, $i = 0, 1, 2, 5, 7, 8, 9, 11, 12$			
Elements of $K_5 \times K_{21}$ of the guiding color			
$C(1,1)$	$(2,2)(4,20), (2,18)(4,5), (2,7)(4,15)$	$C(2,1)$	$(3,2)(1,20)$
$C(1,i)$	$(2,i+1)(4,i-1)$, $i = 2, \ldots, 19$, $i \neq 6, 16, 17$	$C(2,i)$	$(3,i+1)(1,i-1)$, $i = 2, \ldots, 19$, $i \neq 5, 15, 16$
$C(1,6)$	$(3,16)(0,0)$	$C(2,5)$	$(4,0)(0,5)$
$C(1,16)$	$(2,0)(4,16)$	$C(2,15)$	$(3,0)(1,15)$
$C(1,17)$	$(0,7)(1,0)$	$C(2,16)$	$(0,16)(2,17)$
$C(1,20)$	$(2,1)(4,19)$	$C(2,20)$	$(3,1)(1,19), (3,17)(1,4), (3,6)(1,14)$
Vertices: $(0,i)$, for $i = 1, \ldots, 20$, $i \neq 0, 5, 7, 16$			

For $K_9 \times K_{15}$ and $K_9 \times K_{21}$, for $j = 1, \ldots, 8$, $i = 1, \ldots, n-1$, we have three paths $P(j,i)^0, P(j,i)^1$ and $P(j,i)^2$. In this case, for each of them, we will consider 3 subpaths. Let $P(j,i)^{(k,k')}$ be the path induced by $P(j,i)^k_{nk'}, P(j,i)^k_{nk'+1}, \ldots, P(j,i)^k_{nk'+n-1}$, for $j = 1, \ldots, 8$, $i = 1, \ldots, n-1$, $k = 0, 1, 2$ and $k' = 0, 1, 2$. In each case, these subpaths can be rearranged to give the desired Hamiltonian cycles.

Table A.3. Elements of particular graphs $K_7 \times K_n$ of the guiding color, for $n = 9, 11, 13, 15, 17$.

Cycle	Edges	Cycle	Edges	Cycle	Edges
Elements of $K_7 \times K_9$ of the guiding color					
$C(1,1)$	$(6,8)(3,3), (6,1)(3,2), (6,5)(3,0)$	$C(2,1)$	$(1,8)(4,3)$	$C(3,1)$	$(0,7)(3,4)$
$C(1,2)$	$(0,8)(1,5)$	$C(2,i)$	$(1,i-1)(4,i+2),\ i = 2,3,5$	$C(3,i)$	$(2,i-1)(5,i+2),\ i = 2,3,4,6$
$C(1,i)$	$(6,i-1)(3,i+2),\ i = 3,4,5$	$C(2,4)$	$(5,3)(0,6)$	$C(3,5)$	$(6,7)(0,3)$
$C(1,6)$	$(4,2)(0,0)$	$C(2,6)$	$(1,0)(4,6)$	$C(3,7)$	$(2,0)(5,7)$
$C(1,7)$	$(6,6)(3,1)$	$C(2,7)$	$(1,6)(4,1), (1,7)(4,8), (1,3)(4,0)$	$C(3,8)$	$(2,7)(5,2), (2,8)(5,1), (2,4)(5,0)$
$C(1,8)$	$(6,0)(3,8)$	$C(2,8)$	$(0,2)(2,6)$		
Vertices: $(0,1), (0,4), (0,5)$					
Elements of $K_7 \times K_{11}$ of the guiding color					
$C(1,1)$	$(6,10)(3,3), (6,6)(3,0), (6,4)(3,8)$	$C(2,1)$	$(1,10)(4,3)$	$C(3,1)$	$(2,10)(5,3)$
$C(1,i)$	$(6,i-1)(3,i+2),\ i = 2,3,4,8$	$C(2,i)$	$(1,i-1)(4,i+2),\ i = 2,6,7,8$	$C(3,i)$	$(2,i-1)(5,i+2),\ i = 2,3,7,8$
$C(1,5)$	$(4,5)(0,6)$	$C(2,3)$	$(5,8)(0,0)$	$C(3,4)$	$(6,5)(0,3)$
$C(1,6)$	$(0,9)(1,3)$	$C(2,4)$	$(3,9)(1,0)$	$C(3,5)$	$(6,0)(0,5)$
$C(1,7)$	$(4,7)(0,8)$	$C(2,5)$	$(2,4)(3,7)$	$C(3,6)$	$(2,0)(5,6)$
$C(1,9)$	$(6,8)(3,1)$	$C(2,9)$	$(1,8)(4,1), (1,4)(4,0), (1,2)(4,6)$	$C(3,9)$	$(2,8)(5,1)$
$C(1,10)$	$(6,9)(3,2)$	$C(2,10)$	$(1,9)(4,2)$	$C(3,10)$	$(2,9)(5,2), (2,5)(5,0), (2,3)(5,7)$
Vertices: $(0,i),\ i = 1,2,4,7,10$					
Elements of $K_7 \times K_{13}$ of the guiding color					
$C(1,1)$	$(6,12)(3,3), (6,6)(3,8), (6,3)(3,11)$	$C(2,1)$	$(1,12)(4,3)$	$C(3,1)$	$(2,12)(5,3)$
$C(1,i)$	$(6,i-1)(3,i+2),\ i = 2,3,5,8,10$	$C(2,2)$	$(5,8)(0,0)$	$C(3,i)$	$(2,i-1)(5,i+2),\ i = 2,4,7,9,10$
$C(1,4)$	$(1,0)(2,4)$	$C(2,i)$	$(1,i-1)(4,i+2),\ i = 3,6,8,9,10$	$C(3,3)$	$(5,0)(1,3)$
$C(1,6)$	$(4,0)(0,6)$	$C(2,4)$	$(4,4)(6,5)$	$C(3,5)$	$(0,11)(3,0)$
$C(1,7)$	$(2,7)(6,8)$	$C(2,5)$	$(6,0)(5,5)$	$C(3,6)$	$(3,6)(4,7)$
$C(1,9)$	$(0,12)(1,6)$	$C(2,7)$	$(0,1)(2,0)$	$C(3,8)$	$(0,8)(3,9)$
$C(1,11)$	$(6,10)(3,1)$	$C(2,11)$	$(1,10)(4,1), (1,4)(4,6), (1,1)(4,9)$	$C(3,11)$	$(2,10)(5,1)$
$C(1,12)$	$(6,11)(3,2)$	$C(2,12)$	$(1,11)(4,2)$	$C(3,12)$	$(2,11)(5,2), (2,5)(5,7), (2,2)(5,10)$
Vertices: $(0,i),\ i = 2,3,4,5,7,9,10$					
Elements of $K_7 \times K_{15}$ of the guiding color					
$C(1,1)$	$(6,14)(3,3), (6,6)(3,10), (6,2)(3,14)$	$C(2,1)$	$(0,6)(2,10)$	$C(3,1)$	$(2,14)(5,3)$
$C(1,i)$	$(6,i-1)(3,i+2), i = 2,4,5,6,9,10,11$	$C(2,i)$	$(1,i-1)(4,i+2), i = 2,3,4,7,8,9,11,12$	$C(3,2)$	$(0,10)(3,9)$
$C(1,3)$	$(4,7)(0,13)$	$C(2,5)$	$(3,0)(1,5)$	$C(3,i)$	$(2,i-1)(5,i+2), i = 3,4,5,8,9,10,12$
$C(1,7)$	$(2,0)(6,7)$	$C(2,6)$	$(5,8)(0,4)$	$C(3,6)$	$(4,0)(2,6)$
$C(1,8)$	$(0,8)(1,9)$	$C(2,10)$	$(4,3)(6,0)$	$C(3,7)$	$(6,11)(0,3)$
$C(1,12)$	$(3,5)(5,0)$	$C(2,13)$	$(1,12)(4,1), (1,4)(4,8), (1,14)(4,12)$	$C(3,11)$	$(5,4)(1,0)$
$C(1,13)$	$(6,12)(3,1)$	$C(2,14)$	$(1,13)(4,2)$	$C(3,13)$	$(2,12)(5,1)$
$C(1,14)$	$(6,13)(3,2)$			$C(3,14)$	$(2,13)(5,2), (2,5)(5,9), (2,1)(5,13)$
Vertices: $(0,i),\ i = 0,1,2,5,7,9,11,12,14$					
Elements of $K_7 \times K_{17}$ of the guiding color					
$C(1,1)$	$(6,16)(3,3), (6,6)(3,12), (6,0)(3,1)$	$C(2,1)$	$(1,16)(4,3)$	$C(3,1)$	$(2,16)(5,3)$
$C(1,i)$	$(6,i-1)(3,i+2),\ i = 2\ldots,14,\ i \neq 7,10$	$C(2,i)$	$(1,i-1)(4,i+2),\ i = 2\ldots,14,\ i \neq 5,8,13$	$C(3,i)$	$(2,i-1)(5,i+2),\ i = 2\ldots,13,\ i \neq 6,9$
$C(1,7)$	$(5,0)(4,7)$	$C(2,5)$	$(0,14)(2,13)$	$C(3,6)$	$(6,14)(0,0)$
$C(1,10)$	$(1,12)(2,8)$	$C(2,8)$	$(3,9)(1,7)$	$C(3,9)$	$(6,9)(0,10)$
$C(1,15)$	$(4,0)(0,15)$	$C(2,13)$	$(5,8)(0,3)$	$C(3,14)$	$(0,6)(3,0)$
$C(1,16)$	$(6,15)(3,2)$	$C(2,15)$	$(1,14)(4,1), (1,4)(4,10), (1,0)(4,15)$	$C(3,15)$	$(2,14)(5,1)$
		$C(2,16)$	$(1,15)(4,2)$	$C(3,16)$	$(2,15)(5,2), (2,5)(5,11), (2,0)(5,16)$
Vertices: $(0,i),\ i = 0,\ldots,16,\ i \neq 0,3,6,10,14,15$					

TABLE A.4. Elements of particular graphs $K_7 \times K_n$ of the guiding color, for $n = 19, 25, 33$ that differ of the general case $K_7 \times K_n$ with $gcd(7, n) = 1$.

Cycle	Edges	Cycle	Edges
Elements of $K_7 \times K_{19}$ of the guiding color that differ of the Table 3			
$C(1,8)$	$(0,17)(1,0)$	$C(1,12)$	$(4,7)(0,18)$
Vertices: $(0,i)$, $i = 0,\dots,20$, $i \neq 4,6,10,11,17,18$			
Elements of $K_7 \times K_{25}$ of the guiding color that differ of the Table 3			
$C(3,7)$	$(6,0)(0,7)$	$C(3,17)$	$(0,1)(3,9)$
Vertices: $(0,i)$, $i = 0,\dots,20$, $i \neq 1,4,6,7,10,16$			
Elements of $K_7 \times K_{33}$ of the guiding color that differ of the Table 3			
$C(1,8)$	$(0,24)(1,0)$	$C(2,24)$	$(5,8)(0,0)$
$C(1,26)$	$(4,7)(0,13)$	$C(3,7)$	$(6,0)(0,7)$
$C(2,6)$	$(5,0)(0,6)$	$C(3,25)$	$(0,10)(3,9)$
Vertices: $(0,i)$, $i = 0,\dots,2w$, $i \neq 0,6,7,10,13,24$			

TABLE A.5. Elements of particular graph $K_5 \times K_5$ of the guiding color.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(2,2)(4,4), (4,2)(3,4), (3,2)(0,4)$	$C(3,1)$	$(3,3)(2,0)$
$C(1,2)$	$(1,4)(4,0)$	$C(3,2)$	$(2,4)(0,0),$
$C(2,1)$	$(1,0)(4,1)$	$C(4,1)$	$(2,1)(0,2)$
$C(2,2)$	$(3,0)(1,2)$	$C(4,2)$	$(1,3)(3,1), (2,3)(0,1), (4,3)(1,1)$
Vertex: $(0,3)$			

TABLE A.6. Elements of particular graph $K_7 \times K_7$ of the guiding color.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(6,6)(3,3), (5,6)(4,3), (2,6)(6,3)$	$C(4,1)$	$(0,4)(3,0)$
$C(1,2)$	$(0,5)(2,0)$	$C(4,2)$	$(2,3)(1,1)$
$C(1,3)$	$(4,0)(0,3)$	$C(4,3)$	$(3,2)(6,5)$
$C(2,1)$	$(3,4)(5,0)$	$C(5,1)$	$(4,6)(1,3)$
$C(2,2)$	$(5,3)(0,1)$	$C(5,2)$	$(4,1)(1,4), (3,1)(2,4), (6,1)(4,4)$
$C(2,3)$	$(1,2)(4,5)$	$C(5,3)$	$(0,6)(6,0)$
$C(3,1)$	$(0,2)(3,6)$	$C(6,1)$	$(1,0)(5,1)$
$C(3,2)$	$(2,1)(5,4)$	$C(6,2)$	$(6,4)(1,6)$
$C(3,3)$	$(2,2)(5,5)$	$C(6,3)$	$(5,2)(2,5), (4,2)(3,5), (6,2)(1,5)$
Vertex: $(0,0)$			

– For $K_9 \times K_{15}$, a Hamiltonian decomposition is given by $\{C(j,i) \mid j = 1,\dots,8,\ i = 1,\dots,7\}$, where $C(j,i)$ is the following Hamiltonian cycle:

$$\left\langle P(j,i)^{(0,0)}, P(j,i)^{(1,2)}, P(j,i)^{(2,1)}, P(j,i)^{(0,2)}, P(j,i)^{(1,1)}, P(j,i)^{(2,0)}, P(j,i)^{(0,1)}, P(j,i)^{(1,0)}, P(j,i)^{(2,2)}, (0,0)\right\rangle.$$

Observe that, for $j = 1,2,3,4$ and $i = 1,\dots,7$, the cycles $C(j,i)$ and $C(j+4,i)$ form a Hamiltonian decomposition of $\sigma^{j-1}(C_9) \times \sigma^{i-1}(C_{15})$.

TABLE A.7. Elements of particular graph $K_7 \times K_{21}$ of the guiding color that differ of the general case $K_7 \times K_n$ with $gcd(7, n) = 7$.

Elemets of $K_7 \times K_{21}$ that differ of the Table 6			
Cycle	Edges	Cycle	Edges
$C(1,8)$	$(0,11)(1,5)$	$C(3,7)$	$(3,0)(4,7)$
$C(2,5)$	$(5,8)(0,2)$	$C(6,3)$	$(3,9)(0,17)$
$C(2,6)$	$(5,0)(0,6)$		
Vertices: $(0,i)$, $i = 0,\dots,20$, $i \neq 2,4,6,11,12,17$			

TABLE A.8. Elements of particular graphs $K_9 \times K_n$ of the guiding color, for $n = 11, 13, 17, 19$.

Cycle	Edges	Cycle	Edges	Cycle	Edges
Elements of $K_9 \times K_{11}$ of the guiding color					
$C(1,1)$	$(3,3)(7,9), (3,2)(7,10), (3,0)(7,1)$	$C(2,3)$	$(6,3)(0,4)$	$C(3,8)$	$(5,3)(1,0)$
$C(1,2)$	$(5,2)(0,3)$	$C(2,4)$	$(0,2)(2,7)$	$C(3,9)$	$(5,1)(1,7), (5,10)(1,8), (5,0)(1,9)$
$C(1,3)$	$(0,1)(1,6)$	$C(2,i)$	$(4,i+2)(8,i-2),\ i = 5,6,7,8$	$C(3,10)$	$(0,7)(3,4)$
$C(1,i)$	$(3,i+2)(7,i-2), i = 4,5,6,7,8$	$C(2,9)$	$(4,1)(8,7)$	$C(4,1)$	$(0,8)(4,5)$
$C(1,9)$	$(3,1)(7,7)$	$C(2,10)$	$(4,2)(8,8)$	$C(4,2)$	$(8,9)(0,6)$
$C(1,10)$	$(3,5)(7,0)$	$C(3,1)$	$(7,8)(0,5)$	$C(4,i)$	$(6,i+2)(2,i-2), i = 3,4,5,6,7,8$
$C(2,1)$	$(4,6)(8,0)$	$C(3,2)$	$(5,4)(1,10)$	$C(4,9)$	$(6,4)(2,0)$
$C(2,2)$	$(4,4)(8,10), (4,3)(8,1), (4,0)(8,2)$	$C(3,i)$	$(5,i+2)(1,i-2), i = 3,4,5,6,7$	$C(4,10)$	$(6,2)(2,8), (6,1)(2,9), (6,0)(2,10)$
Vertices: $(0,i)$, $i = 0,9,10$					
Elements of $K_9 \times K_{13}$ of the guiding color					
$C(1,1)$	$(3,3)(7,11), (3,0)(7,1), (3,9)(7,6)$	$C(2,i)$	$(4,i+2)(8,i-2),\ i = 3,5,6,7,10$	$C(3,6)$	$(0,3)(3,10)$
$C(1,2)$	$(3,4)(7,12),$	$C(2,4)$	$(2,4)(3,5)$	$C(3,11)$	$(5,1)(1,9), (5,0)(1,11), (5,7)(1,4)$
$C(1,3)$	$(0,0)(1,3)$	$C(2,8)$	$(6,4)(0,1)$	$C(3,12)$	$(5,2)(1,10)$
$C(1,i)$	$(3,i+2)(7,i-2),\ i = 4,5,6,9,10$	$C(2,9)$	$(7,0)(6,9)$	$C(4,1)$	$(6,3)(2,11)$
$C(1,7)$	$(5,8)(0,6)$	$C(2,11)$	$(4,1)(8,9)$	$C(4,2)$	$(0,11)(4,6)$
$C(1,8)$	$(5,3)(0,2)$	$C(2,12)$	$(4,2)(8,10)$	$C(4,i)$	$(6,i+2)(2,i-2), i = 3,4,5,8,9,10$
$C(1,11)$	$(3,1)(7,9)$	$C(3,1)$	$(7,5)(0,9)$	$C(4,6)$	$(1,0)(8,6)$
$C(1,12)$	$(3,2)(7,10)$	$C(3,2)$	$(5,4)(1,12)$	$C(4,7)$	$(8,0)(0,7)$
$C(2,1)$	$(4,3)(8,11)$	$C(3,i)$	$(5,i+2)(1,i-2),\ i = 3,4,7,8,9,10$	$C(4,11)$	$(6,1)(2,9)$
$C(2,2)$	$(4,4)(8,12), (4,0)(8,2), (4,10)(8,7)$	$C(3,5)$	$(4,11)(2,0)$	$C(4,12)$	$(6,2)(2,10), (6,0)(2,12), (6,8)(2,5)$
Vertices: $(0,i)$, $i = 4,5,8,10,12$					
Elements of $K_9 \times K_{17}$ of the guiding color					
$C(1,1)$	$(3,3)(7,15), (3,11)(7,8), (3,15)(7,4)$	$C(2,7)$	$(6,11)(0,3)$	$C(3,11)$	$(7,7)(0,16)$
$C(1,2)$	$(3,4)(7,16)$	$C(2,10)$	$(8,0)(5,10)$	$C(3,15)$	$(5,1)(1,13), (5,9)(1,6), (5,13)(1,2)$
$C(1,i)$	$(3,i+2)(7,i-2),\ i = 3,4,5,7,8,11,12,14$	$C(2,11)$	$(0,12)(2,10)$	$C(3,16)$	$(5,2)(1,14)$
$C(1,6)$	$(6,0)(5,6)$	$C(2,14)$	$(2,6)(3,0)$	$C(4,1)$	$(6,3)(2,15)$
$C(1,9)$	$(7,0)(4,9)$	$C(2,15)$	$(4,1)(8,13)$	$C(4,2)$	$(6,4)(2,16)$
$C(1,10)$	$(0,11)(1,9)$	$C(2,16)$	$(4,2)(8,14)$	$C(4,i)$	$(6,i+2)(2,i-2), i = 3,4,6,7,10,11,13,14$
$C(1,13)$	$(1,5)(2,0)$	$C(3,1)$	$(5,3)(1,15)$	$C(4,5)$	$(8,12)(0,14)$
$C(1,15)$	$(3,1)(7,13)$	$C(3,2)$	$(5,4)(1,16)$	$C(4,8)$	$(5,0)(3,8)$
$C(1,16)$	$(3,2)(7,14)$	$C(3,i)$	$(5,i+2)(1,i-2),\ i = i = 3,5,6,9,10,12,13,14$	$C(4,9)$	$(0,6)(4,13)$
$C(2,1)$	$(4,3)(8,15)$	$C(3,4)$	$(3,12)(4,0)$	$C(4,12)$	$(8,8)(0,1)$
$C(2,2)$	$(4,4)(8,16), (4,12)(8,9), (4,16)(8,5)$	$C(3,7)$	$(1,0)(6,7)$	$C(4,15)$	$(6,1)(2,13)$
$C(2,i)$	$(4,i+2)(8,i-2), i = 3,4,5,6,8,9,12,13$	$C(3,8)$	$(7,11)(0,5)$	$C(4,16)$	$(6,2)(2,14), (6,10)(2,7), (6,14)(2,3)$
Vertices: $(0,i)$, for $i = 0,2,4,7,8,9,10,13,15$					
Elements of $K_9 \times K_{19}$ of the guiding color					
$C(1,1)$	$(3,3)(7,17), (3,13)(7,8), (3,18)(7,3)$	$C(2,6)$	$(8,15)(5,0)$	$C(3,17)$	$(5,1)(1,15), (5,11)(1,6), (5,16)(1,1)$
$C(1,2)$	$(3,4)(7,18)$	$C(2,11)$	$(6,6)(0,17)$	$C(3,18)$	$(5,2)(1,16)$
$C(1,i)$	$(3,i+2)(7,i-2),\ i = 3\dots,15,\ i \neq 5,10,11$	$C(2,12)$	$(3,0)(1,12)$	$C(4,1)$	$(6,3)(2,17)$
$C(1,5)$	$(6,0)(5,5)$	$C(2,17)$	$(0,9)(2,8)$	$C(4,2)$	$(6,4)(2,18)$
$C(1,10)$	$(5,10)(0,11)$	$C(2,18)$	$(4,2)(8,16)$	$C(4,i)$	$(6,i+2)(2,i-2),\ i = 3\dots,16,\ i \neq 4,9,10,15$
$C(1,11)$	$(4,0)(6,11)$	$C(3,1)$	$(5,3)(1,17)$	$C(4,5)$	$(2,13)(7,0)$
$C(1,16)$	$(1,7)(2,0)$	$C(3,2)$	$(5,4)(1,18)$	$C(4,9)$	$(0,6)(4,13)$
$C(1,17)$	$(3,1)(7,15)$	$C(3,3)$	$(0,13)(3,12)$	$C(4,10)$	$(1,0)(8,10)$
$C(1,18)$	$(3,2)(7,16)$	$C(3,i)$	$(5,i+2)(1,i-2),\ i = 4\dots,16,\ i \neq 8,9,14$	$C(4,15)$	$(0,4)(4,8)$
$C(2,1)$	$(4,3)(8,17)$	$C(3,8)$	$(7,14)(0,2)$	$C(4,17)$	$(6,1)(2,15)$
$C(2,2)$	$(4,4)(8,18), (4,14)(8,9), (4,1)(8,4)$	$C(3,9)$	$(8,0)(7,9)$	$C(4,18)$	$(6,2)(2,16), (6,12)(2,7), (6,17)(2,2)$
$C(2,i)$	$(4,i+2)(8,i-2),\ i = 3\dots,16,\ i \neq 6,11,12$	$C(3,14)$	$(0,3)(3,7)$		
Vertices: $(0,i)$, for $i = 0,1,5,7,8,10,12,14,15,16,18$					

TABLE A.9. Elements of particular graphs $K_9 \times K_n$ of the guiding color, for $n = 9, 15, 21$.

Cycle	Edges	Cycle	Edges	Cycle	Edges
		Elements of $K_9 \times K_9$ of the guiding color			
$C(1,1)$	$(3,3)(7,7), (2,3)(8,7), (0,3)(1,7)$	$C(3,4)$	$(5,6)(1,2)$	$C(6,3)$	$(0,8)(6,7)$
$C(1,2)$	$(3,6)(8,0)$	$C(4,1)$	$(6,3)(2,7)$	$C(6,4)$	$(8,6)(4,2)$
$C(1,3)$	$(6,0)(5,3)$	$C(4,2)$	$(0,7)(4,6)$	$C(7,1)$	$(1,3)(5,7)$
$C(1,4)$	$(7,8)(3,0)$	$C(4,3)$	$(6,5)(2,1)$	$C(7,2)$	$(1,4)(5,8)$
$C(2,1)$	$(1,0)(4,1)$	$C(4,4)$	$(6,6)(2,2)$	$C(7,3)$	$(1,5)(5,1), (8,5)(6,1), (0,5)(7,1)$
$C(2,2)$	$(4,4)(8,8), (3,4)(1,8), (0,4)(2,8)$	$C(5,1)$	$(7,3)(3,7)$	$C(7,4)$	$(5,4)(2,5)$
$C(2,3)$	$(4,5)(8,1)$	$C(5,2)$	$(7,4)(3,8)$	$C(8,1)$	$(4,0)(0,1)$
$C(2,4)$	$(7,0)(6,4)$	$C(5,3)$	$(7,5)(3,1)$	$C(8,2)$	$(2,4)(6,8)$
$C(3,1)$	$(3,5)(0,0)$	$C(5,4)$	$(7,6)(3,2)$	$C(8,3)$	$(5,0)(4,3)$
$C(3,2)$	$(2,0)(5,2)$	$C(6,1)$	$(8,3)(4,7)$	$C(8,4)$	$(2,6)(6,2), (1,6)(7,2), (0,6)(8,2)$
$C(3,3)$	$(5,5)(1,1)$	$C(6,2)$	$(8,4)(4,8)$		
		Vertex: $(0,2)$			
		Elements of $K_9 \times K_{15}$ of the guiding color			
$C(1,1)$	$(3,3)(7,13), (3,0)(7,1), (3,11)(7,6)$	$C(3,7)$	$(8,0)(7,7)$	$C(6,3)$	$(2,0)(0,3)$
$C(1,2)$	$(3,4)(7,14)$	$C(4,1)$	$(6,3)(2,13)$	$C(6,6)$	$(8,11)(4,1)$
$C(1,3)$	$(6,11)(5,10)$	$C(4,2)$	$(6,4)(2,14)$	$C(6,7)$	$(8,12)(4,2)$
$C(1,i)$	$(3,i+2)(7,i-2)$, for $i = 4,5,6,7$	$C(4,i)$	$(6,i+2)(2,i-2),\ i = 3,4,5$	$C(7,1)$	$(5,8)(1,0)$
$C(2,1)$	$(4,3)(8,13)$	$C(4,6)$	$(6,0)(2,6)$	$C(7,i)$	$(1,i+5)(5,i+9),\ i = 2,3,4,5$
$C(2,2)$	$(4,4)(8,14), (4,0)(8,2), (4,12)(8,7)$	$C(4,7)$	$(0,4)(4,11)$	$C(7,6)$	$(1,4)(5,9), (1,6)(5,7), (1,11)(5,1)$
$C(2,i)$	$(4,i+2)(8,i-2),\ i = 3,5,6,7$	$C(5,1)$	$(1,5)(0,11)$	$C(7,7)$	$(1,12)(5,2)$
$C(2,4)$	$(3,5)(1,3)$	$C(5,2)$	$(6,9)(5,0)$	$C(8,1)$	$(0,9)(8,8)$
$C(3,1)$	$(5,3)(1,13)$	$C(5,i)$	$(7,i+5)(3,i+9),\ i = 3,4,5$	$C(8,2)$	$(4,6)(0,12)$
$C(3,2)$	$(5,4)(1,14)$	$C(5,6)$	$(7,11)(3,1)$	$C(8,i)$	$(2,i+5)(6,i+9),\ i = 3,4,5$
$C(3,i)$	$(5,i+2)(1,i-2),\ i = 3,4$	$C(5,7)$	$(7,12)(3,2)$	$C(8,6)$	$(2,11)(6,1)$
$C(3,5)$	$(0,1)(3,10)$	$C(6,i)$	$(8,i+5)(4,i+9),\ i = 1,4,5$	$C(8,7)$	$(2,5)(6,10), (2,7)(6,8), (2,12)(6,2)$
$C(3,6)$	$(7,0)(0,6)$	$C(6,2)$	$(2,4)(0,14)$		
		Vertices: $(0,i)$, for $i = 0,2,5,7,8,10,13$			
		Elements of $K_9 \times K_{21}$ of the guiding color			
$C(1,1)$	$(3,3)(7,19), (3,15)(7,8), (3,20)(7,3)$	$C(3,8)$	$(7,11)(0,5)$	$C(6,4)$	$(0,18)(6,11)$
$C(1,2)$	$(3,4)(7,20)$	$C(4,1)$	$(6,3)(2,19)$	$C(6,9)$	$(4,0)(1,9)$
$C(1,i)$	$(3,i+2)(7,i-2),\ i = 3,4,6,7,8,9$	$C(4,2)$	$(6,4)(2,20)$	$C(6,10)$	$(8,18)(4,2)$
$C(1,5)$	$(2,15)(1,0)$	$C(4,i)$	$(6,i+2)(2,i-2),\ i = 3,5,6,7,8,10$	$C(7,1)$	$(3,12)(0,11)$
$C(1,10)$	$(5,5)(0,16)$	$C(4,4)$	$(1,14)(7,0)$	$C(7,i)$	$(1,i+8)(5,i+12),\ i = 2,3,4,5,7,8$
$C(2,1)$	$(4,3)(8,19)$	$C(4,9)$	$(0,10)(4,8)$	$C(7,6)$	$(8,0)(6,6)$
$C(2,2)$	$(4,4)(8,20), (4,16)(8,9), (4,1)(8,4)$	$C(5,i)$	$(7,i+8)(3,i+12),\ i = 1,2,4,5,6,7$	$C(7,9)$	$(1,1)(5,18), (1,6)(5,13), (1,17)(5,1)$
$C(2,i)$	$(4,i+2)(8,i-2),\ i = 3,4,5,7,8,9,10$	$C(5,3)$	$(8,17)(2,10)$	$C(7,10)$	$(1,18)(5,2)$
$C(2,6)$	$(7,16)(5,0)$	$C(5,8)$	$(0,7)(5,10)$	$C(8,i)$	$(2,i+8)(6,i+12),\ i = 1,3,4,5,6,8$
$C(3,1)$	$(5,3)(1,19)$	$C(5,9)$	$(7,17)(3,1)$	$C(8,2)$	$(0,13)(8,12)$
$C(3,2)$	$(5,4)(1,20)$	$C(5,10)$	$(7,18)(3,2)$	$C(8,7)$	$(6,0)(3,7)$
$C(3,3)$	$(4,13)(3,0)$	$C(6,1)$	$(2,0)(0,1)$	$C(8,9)$	$(2,17)(6,1)$
$C(3,i)$	$(5,i+2)(1,i-2),\ i = 4,5,6,7,9,10$	$C(6,i)$	$(8,i+8)(4,i+12),\ i = 2,3,5,6,7,8$	$C(8,10)$	$(2,2)(6,19), (2,7)(6,14), (2,18)(6,2)$
		Vertices: $(0,i)$, for $i = 0,2,3,4,6,8,9,12,14,15,17,19,20$			

– For $K_9 \times K_{21}$, a Hamiltonian decomposition is given by $\{C(j,i) \mid j = 1,\dots,8,\ i = 1,\dots,10\}$, where $C(j,i)$ is the following Hamiltonian cycle:

$$\left\langle P(j,i)^{(0,0)}, P(j,i)^{(1,0)}, P(j,i)^{(2,0)}, P(j,i)^{(0,1)}, P(j,i)^{(1,1)}, P(j,i)^{(2,1)}, P(j,i)^{(0,2)}, P(j,i)^{(1,2)}, P(j,i)^{(2,2)}, (0,0)\right\rangle.$$

Observe that, for $j = 1,2,3,4$ and $i = 1,\dots,10$, the cycles $C(j,i)$ and $C(j+4,i)$ form a Hamiltonian decomposition of $\sigma^{j-1}(C_9) \times \sigma^{i-1}(C_{21})$.

– For $K_{11} \times K_{11}$, a Hamiltonian decomposition is given by $\{C(j,i) \mid j = 1,\dots,10,\ i = 1,\dots,5\}$, where $C(j,i)$ is the following Hamiltonian cycle:

$$\langle P(j,i)^0, P(j,i)^9, P(j,i)^7, P(j,i)^5, P(j,i)^3, P(j,i)^1, P(j,i)^{10}, P(j,i)^8, P(j,i)^6, P(j,i)^4, P(j,i)^2, (0,0)\rangle.$$

Observe that, for $j = 1,\dots,5$ and $i = 1,\dots,5$, the cycles $C(j,i)$ and $C(j+5,i)$ form a Hamiltonian decomposition of $\sigma^{j-1}(C_{11}) \times \sigma^{i-1}(C_{11})$.

Table A.10. Elements of particular graph $K_{11} \times K_{11}$ of the guiding color.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(9,9)(4,4),(10,9)(3,4),(1,9)(2,4)$	$C(6,i)$	$(4,i-2)(9,i+3),\ i=3,4,5$
$C(1,2)$	$(3,7)(10,0)$	$C(7,1)$	$(5,9)(10,4)$
$C(1,3)$	$(9,1)(4,6)$	$C(7,2)$	$(5,10)(10,5)$
$C(1,4)$	$(7,5)(6,3)$	$C(7,i)$	$(5,i-2)(10,i+3),\ i=3,5$
$C(1,5)$	$(7,0)(6,5)$	$C(7,4)$	$(8,0)(6,4)$
$C(2,1)$	$(7,3)(0,9)$	$C(8,1)$	$(6,9)(1,4)$
$C(2,2)$	$(10,10)(5,5),(1,10)(4,5),(2,10)(3,5)$	$C(8,2)$	$(6,10)(1,5)$
$C(2,i)$	$(10,i-2)(5,i+3),\ i=3,4,5$	$C(8,3)$	$(6,1)(1,6)$
$C(2,5)$	$(9,10)(5,0)$	$C(8,4)$	$(7,9)(9,0)$
$C(3,1)$	$(8,10)(0,3)$	$C(8,5)$	$(0,10)(3,0)$
$C(3,2)$	$(1,0)(6,2)$	$C(9,1)$	$(4,0)(0,1)$
$C(3,i)$	$(1,i-2)(6,i+3),\ i=3,4,5$	$C(9,2)$	$(7,10)(2,5)$
$C(4,1)$	$(2,9)(7,4)$	$C(9,3)$	$(7,1)(2,6)$
$C(4,2)$	$(0,6)(4,8)$	$C(9,4)$	$(7,2)(2,7),(8,2)(1,7),(9,2)(10,7)$
$C(4,i)$	$(2,i-2)(7,i+3),\ i=3,4,5$	$C(9,5)$	$(5,4)(4,7)$
$C(5,1)$	$(3,9)(8,4)$	$C(10,1)$	$(5,8)(0,5)$
$C(5,2)$	$(3,10)(8,5)$	$C(10,2)$	$(6,0)(5,2)$
$C(5,i)$	$(3,i-2)(8,i+3),\ i=3,4,5$	$C(10,3)$	$(8,1)(3,6)$
$C(6,1)$	$(4,9)(9,4)$	$C(10,4)$	$(8,9)(2,0)$
$C(6,2)$	$(4,10)(9,5)$	$C(10,5)$	$(8,3)(3,8),(9,3)(2,8),(10,3)(1,8)$
Vertices: $(0,i)$, for $i=0,2,4,7,8$			

Table A.11. Elements of particular graph $K_{11} \times K_{13}$ of the guiding color.

Cycle	Edges	Cycle	Edges
$C(1,1)$	$(9,11)(4,4),(9,12)(4,3),$ $(9,1)(4,2)$	$C(3,5)$	$(0,11)(3,0)$
$C(1,2)$	$(6,0)(0,2)$	$C(3,i)$	$(1,i-2)(6,i+3),$ $i=6,7,8,9$
$C(1,3)$	$(1,12)(2,7)$	$C(3,10)$	$(1,8)(6,1)$
$C(1,i)$	$(9,i-2)(4,i+3),$ $i=4,5,6,7,8,9$	$C(3,11)$	$(1,9)(6,2)$
$C(1,10)$	$(9,8)(4,1)$	$C(3,12)$	$(1,10)(6,3)$
$C(1,11)$	$(8,5)(5,0)$	$C(4,1)$	$(5,6)(3,8)$
$C(1,12)$	$(0,6)(1,0)$	$C(4,2)$	$(2,12)(7,5)$
$C(2,1)$	$(0,7)(2,0)$	$C(4,i)$	$(2,i-2)(7,i+3),$ $i=3,4,5,6,7,8$
$C(2,2)$	$(10,12)(5,5),(10,1)(5,4),$ $(10,2)(5,3)$	$C(4,9)$	$(9,0)(0,9)$
$C(2,3)$	$(8,0)(7,3)$	$C(4,10)$	$(10,11)(9,9)$

Second, we apply each constructed Hamiltonian decomposition to define a guiding color from which the total coloring is obtained, please refer to the corresponding tables.

TABLE A.11. Continued.

Cycle	Edges	Cycle	Edges
$C(2,4)$	$(3,9)(1,11)$	$C(4,11)$	$(2,9)(7,2),(2,10)(7,1),$ $(2,11)(7,12)$
$C(2,i)$	$(10,i-2)(5,i+3),$ $i=5,6,7,8,9$	$C(4,12)$	$(9,10)(0,3)$
$C(2,10)$	$(10,8)(5,1)$	$C(5,1)$	$(6,7)(4,0)$
$C(2,11)$	$(10,9)(5,2)$	$C(5,2)$	$(0,10)(5,7)$
$C(2,12)$	$(7,0)(0,12)$	$C(5,i)$	$(3,i-2)(8,i+3),$ $i=3,4,5,6,7,8,9$
$C(3,1)$	$(4,6)(2,8)$	$C(5,10)$	$(4,5)(7,4)$
$C(3,2)$	$(6,8)(10,0)$	$C(5,11)$	$(10,10)(0,1)$
$C(3,3)$	$(1,1)(6,6),(1,2)(6,5),$ $(1,3)(6,4)$	$C(5,12)$	$(3,10)(8,3),(3,11)(8,2),$ $(3,12)(8,1)$
$C(3,4)$	$(8,4)(0,5)$		
Vertices: $(0,i)$, for $i=0,4,8$			

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