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On total and edge coloring some Kneser graphs

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Abstract

In this work, we investigate the total and edge colorings of the Kneser graphs $K(n, s)$. We prove that the sparse case of Kneser graphs, the odd graphs $O_k = K(2k-1, k-1)$, have total chromatic number equal to $\Delta(O_k)+1$. We prove that Kneser graphs $K(n, 2)$ verify the Total Coloring Conjecture when n is even, or when n is odd not divisible by 3. For the remaining cases when n is odd and divisible by 3, we obtain a total coloring of $K(n, 2)$ with $\Delta(K(n, 2)) + 3$ colors when $n \equiv 3 \bmod 4$, and with $\Delta(K(n, 2)) + 4$ colors when $n \equiv 1 \bmod 4$. Furthermore, we present an infinite family of Kneser graphs $K(n, 2)$ that have chromatic index equal to $\Delta(K(n, 2))$.

Keywords Total coloring · Edge coloring · Odd graphs · Kneser graphs

Mathematics Subject Classification 05C15 · 05C85 · 05C69 · 05C76

1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. An *element* of G is one of its vertices or edges, and the maximum degree of G is denoted by $\Delta(G)$.

A k-edge coloring of G is an assignment of k colors to the edges of G so that adjacent edges have different colors. The *chromatic index*, denoted by $\chi(G)$, is the smallest integer k for which G has a k-edge coloring. Clearly, $\chi(G) \geq \Delta(G)$ and Vizing s theorem (Vizing 1964) states that $\chi(G) \leq \Delta(G) + 1$. Graphs with $\chi(G) = \Delta(G)$ are said to be Class 1 and graphs with $\chi(G) = \Delta(G) + 1$ are said to be Class 2. Deciding between the two candidate values for the chromatic index is NP-

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complete even for regular graphs of degree at least 3 (Leven and Galil 1983). A k-total coloring of G is an assignment of k colors to the elements (vertices and edges) of G so that adjacent or incident elements have different colors. The *total chromatic number*, denoted by $\chi_T(G)$, is the smallest integer k for which G has a k-total coloring. Clearly, $\chi_T(G) \geq \Delta(G) + 1$ and the *Total Coloring Conjecture* (TCC), posed independently by Behzad et al. (1967) and Vizing (1964), states that $\chi_T(G) \leq \Delta(G) + 2$. Graphs with $\chi_T(G) = \Delta(G) + 1$ are said to be Type 1 and graphs with $\chi_T(G) = \Delta(G) + 2$ are said to be Type 2. The problem of determining the total chromatic number of an arbitrary graph G is NP-hard (Sánchez-Arroyo 1989). The TCC has been verified in restricted cases, such as cubic graphs Rosenfeld (1971) but has not been settled for all regular graphs for more than fifty years, exposing how challenging the problem of total coloring is. Surprisingly, T Srinivasa Murthy Murthy 2021 has communicated in an unpublished manuscript a proof that the TCC holds for all graphs.

In this paper, we consider the total coloring and the edge coloring of Kneser graphs. Given positive integers n, s with $n \geq 2s$, the *Kneser graph* $K(n, s)$ has as vertices the s-subsets of an n-set, and two s-subsets are adjacent in $K(n, s)$ if they are disjoint. The Kneser graph $K(n, s)$ has $\binom{n}{s}$ vertices and it is a $\binom{n-s}{s}$-regular graph. Kneser graphs have a very nice structure. For a survey on this much studied family of graphs we refer the reader to Godsil and Royle (2004). Many graph theoretic parameters have been computed for a Kneser graph $K(n, s)$, for instance the independence number Erdős et al. (1961), the chromatic number (Lovász 1978), the circular chromatic number in a combinatorial perspective Liu and Zhu (2016) and the diameter Valencia-Pabon and Vera (2005). An *independent set of vertices* of a graph is composed of mutually nonadjacent vertices. An *independent set of edges* of a graph is composed of mutually non incident edges and is called a *matching*. If a graph has an even number N of vertices and there is a matching of the graph with size $\frac{N}{2}$ then such a matching is called a *perfect matching*. Notice that $K(2s, s)$ is a perfect matching. By the famous Erdős-Ko-Rado theorem (Erdős et al. 1961), the maximum size of an independent set in $K(n, s)$ (i.e. the independence number) equals $\binom{n-1}{s-1}$. Moreover, for non trivial Kneser graphs (i.e. when $n > 2s$), a maximum independent set I in $K(n, s)$ has always a center, that is, an integer $w \in \{1, 2, \ldots, n\}$ such that $I = I_w = \{A \in V(K(n, s)) : w \in A\}$. Therefore, the only independent sets with maximum size in $K(n, s)$ are the sets I_i, for $1 \leq i \leq n$.

We shall focus on two well known families of Kneser graphs, which are opposed regarding the density given by the number of edges over the number of edges in the complete graph with the same number of vertices. Trivial opposed cases are the sparse $K(2k, k)$ which is a perfect matching, and the dense $K(n, 1)$ which corresponds to the complete graph K_n. For $k \geq 3$, the family of Kneser graphs $K(2k - 1, k - 1)$ is known as *odd graphs* and denoted by O_k, they constitute the sparsest connected case of the Kneser graphs. In 2020, Prajnanaswaroopa et al. (Prajnanaswaroopa et al. 2020) communicated in an unpublished manuscript the TCC for all odd graphs, that is, these graphs have a total coloring with $\Delta(O_k) + 2$ colors. On the other hand, the densest non trivial case is the family of Kneser graphs $K(n, 2)$, which are the complements of Johnson graphs $J(n, 2)$ known to be isomorphic to the line graph of the complete graph K_n (Godsil and Royle 2004). As far as we know, there are no results concerning the total coloring of graphs $K(n, 2)$.

Concerning the edge coloring of Kneser graphs the known Class 2 odd graphs O_k are the Petersen graph O_3, and the odd graphs with an odd number of vertices that are precisely O_k with $k = 2^r$, for some positive integer r. In 1977, Fiorini and Wilson (1977) conjectured that all other values of k give Class 1 odd graphs. Fiorini and Wilson's conjecture remains still open. The Kneser graphs $K(n, 2)$ have an odd number of vertices precisely when $n \equiv 2 \mod 4$ or $n \equiv 3 \mod 4$, which implies that they are Class 2. Considering the ones with an even number of vertices, we establish that all $K(n, 2)$ with $n \equiv 0 \mod 4$ are Class 1.

This paper is organized as follows. Section 2 improves the known upper bound by proving that all odd graphs are Type 1. Section 3 gives the first known upper bound for the total chromatic number of the family $K(n, 2)$. Section 4 considers the edge coloring of the family $K(n, 2)$, establishing the first infinite family of Kneser graphs that are Class 1. Section 5 concludes the paper by presenting a conjecture.

2 Odd graphs are Type 1

Recently, Prajnanaswaroopa et al. (Prajnanaswaroopa et al. 2020) have verified the TCC for all odd graphs by using Biggs standard representation (Biggs 1979), where an odd graph is decomposed into an independent set and a perfect matching. We use this representation to prove that all odd graphs $O_k = K(2k-1, k-1)$, $k \geq 3$, are Type 1.

Let $w \in \{1, 2, \ldots, 2k-1\}$ and consider the independent set I_w of vertices with center w. Consider the subset $\overline{I_w} = V(O_k) \setminus I_w$ consisting of the $(k-1)$-subsets of the set of size $2k-1$ that not contain w. Since each $(k-1)$-subset is disjoint to precisely one $(k-1)$-subset, the corresponding vertices in $\overline{I_w}$ induce a matching. Therefore the set of vertices $V(O_k)$ is partitioned into I_w and $\overline{I_w}$ such that I_w is an independent set and $\overline{I_w}$ induces a matching M_w. As O_k is k-regular, each vertex of I_w is adjacent to k vertices of M_w, and each vertex of M_w is adjacent to $k-1$ vertices of I_w (see Fig. 1).

Theorem 1 *All odd graphs O_k, $k \geq 3$, have a $(k+1)$-total coloring.*

Proof In order to prove that $\chi_T(O_k) = \Delta(O_k) + 1 = k + 1$, consider the independent set I_w and the matching M_w defined previously (see Fig. 1b). Assign color $k+1$ to all vertices in I_w and all edges in M_w.

To assign the other k colors, we construct an auxiliary bipartite graph A_k by subdividing the edges of M_w, i.e., by removing all edges of $M_w = \{(x_1, y_1), (x_2, y_2), \ldots, (x_{|M_w|}, y_{|M_w|})\}$ and adding vertex v_i, and edges (x_i, v_i) and (v_i, y_i), for $i = 1, \ldots, |M_w|$ (see Fig. 1 (c)). The bipartite graph A_k has set of vertices $V(A_k) = V_1 \cup V_2$, where $V_1 = I_w \cup \{v_1, v_2, \ldots, v_{|M_w|}\}$ and $V_2 = \{x_1, y_1, x_2, y_2, \ldots, x_{|M_w|}, y_{|M_w|}\}$, and has set of edges $E(A_k) = \{E(O_k) \setminus M_w\} \cup \{(x_1, v_1), (x_2, v_2), \ldots, (x_{|M_w|}, v_{|M_w|})\} \cup \{(v_1, y_1), (v_2, y_2), \ldots, (v_{|M_w|}, y_{|M_w|})\}$. The maximum degree of A_k is $\Delta(A_k) = \Delta(O_k) = k$, and by König's theorem, A_k has a k-edge coloring (see Fig. 1 (c)). We conclude the total coloring of O_k as follows.

Let ϕ be a k-edge coloring of A_k. Assign the colors given by ϕ to all edges incident to the vertices of I_w. Finally, assign the color given by ϕ to the edge (x_i, v_i) (respectively

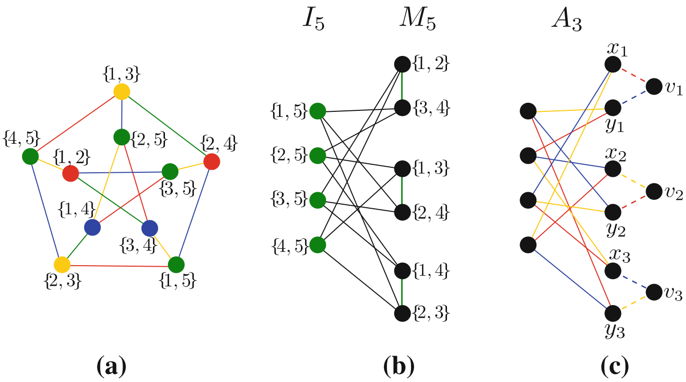

Fig. 1 **a** The Petersen graph O_3 with the 4-total coloring obtained by Theorem 1. **b** The representation of O_3 with the independent set I_5 and the matching M_5. **c** The auxiliary bipartite A_3 with a 3-edge coloring

(v_i, y_i)) to each vertex x_i (respectively y_i). Since the colors assigned by ϕ to these pairs of edges of A_k are different, adjacent vertices x_i and y_i in O_k have different colors in the obtained total coloring. Therefore, $\chi_T(O_k) = \Delta(O_k) + 1$. An example of the obtained total coloring is depicted in Fig. 1a.

3 An upper bound for the total chromatic number of $K(n, 2)$

We consider the infinite family of Kneser graphs $K(n, 2)$ and the main result of this section is Theorem 2. Recall that $\Delta(K(n, 2)) = \binom{n-2}{2}$. The first Kneser graph of this family is the graph $K(4, 2)$, consisting of a perfect matching with 6 vertices, which is Type 2. The second member is the Petersen graph $K(5, 2)$ which is known to be Type 1 (see Fig. 1).

Theorem 2 *Let $n \geq 6$. The total chromatic number of the Kneser graphs $K(n, 2)$ is upper bounded as follows:*

$$\chi_T(K(n, 2)) \leq \begin{cases} \Delta(K(n, 2)) + 2 : n \text{ even, or } n \text{ odd and not divisible by } 3; \\ \Delta(K(n, 2)) + 3 : n \equiv 3 \mod 4 \text{ and divisible by } 3; \\ \Delta(K(n, 2)) + 4 : n \equiv 1 \mod 4 \text{ and divisible by } 3. \end{cases}$$

We prove Theorem 2 in Subsect. 3.3. In Subsect. 3.1 we present definitions and notations used to deal with total and edge colorings of the Kneser graphs $K(n, 2)$, and in Subsect. 3.2 we present Algorithm 1 that gives a partial total coloring of the Kneser graphs $K(n, 2)$.

3.1 Preliminaries

The following definitions and notation are used to deal with total and edge colorings of the Kneser graphs $K(n, 2)$. For any integer $n > 0$, we denote by $[n]$ the set $\{1, 2, \dots, n\}$.

It is well known that the chromatic index of a complete graph $\chi(K_n) = n$ when n is odd and $\chi(K_n) = n - 1$ when n is even (Baranyai 1973). In fact, an optimal edge coloring of K_n with n odd can be given by the edge partition $E_1, \dots, E_n$, where $E_i = \{\{i - q, i + q\} : 1 \leq q \leq \frac{n-1}{2}\}$, for $1 \leq i \leq n$, where arithmetic operations are taken modulo n (being $0 \equiv n$). Notice that when n is odd, $i \notin E_i$ for all $i \in [n]$. When n is even, an optimal edge coloring of K_n can be obtained from an optimal edge coloring of K_{n-1} where each set E_i of the edge partition of K_{n-1} is added the edge $\{i, n\}$. For example, when $n = 5$ we obtain the edge partition $E_1 = \{\{2, 5\}, \{3, 4\}\}$, $E_2 = \{\{1, 3\}, \{4, 5\}\}$, $E_3 = \{\{2, 4\}, \{1, 5\}\}$, $E_4 = \{\{3, 5\}, \{1, 2\}\}$ and $E_5 = \{\{1, 4\}, \{2, 3\}\}$, and for $n = 6$ the edge partition will be $E_i \cup \{i, 6\}$ for $1 \leq i \leq 5$.

Concerning the total chromatic number of complete graphs $\chi_T(K_n)$, it is well known that $\chi_T(K_n) = n$ when n is odd and $\chi_T(K_n) = n + 1$ when n is even (Behzad et al. 1967). Optimal total colorings of K_n can be obtained as follows: If n is odd then, an optimal total coloring of K_n can be constructed from the sets $S_i = \{i\} \cup E_i$, for $1 \leq i \leq n$, where E_i is the set of edges described previously. If n is even then, an optimal total coloring of K_n is obtained from an optimal edge coloring $E_1, E_2, \dots, E_{n+1}$ of K_{n+1} as follows: for $1 \leq i \leq n$, let E'_i be the set of edges E_j in K_{n+1} containing the edge $\{i, n+1\}$. Thus, for $1 \leq i \leq n$, $S_i = \{i\} \cup (E'_i \setminus \{i, n+1\})$ and $M_{n+1} = E_{n+1}$. For example, when $n = 4$, we obtain from the optimal edge coloring $E_1, \dots, E_5$ of K_5 described previously, an optimal total coloring of K_4 as follows: $S_1 = \{\{1\}, \{2, 4\}\}$, $S_2 = \{\{2\}, \{3, 4\}\}$, $S_3 = \{\{3\}, \{1, 2\}\}$, $S_4 = \{\{4\}, \{1, 3\}\}$ and $M_5 = \{\{1, 4\}, \{2, 3\}\}$.

We present next definitions and notation that are used to construct a representation of the Kneser graphs $K(n, 2)$ where the vertices are partitioned into cliques.

Definition 1 Let $t \geq 2$, $r \geq 1$, and G be a graph with rt vertices. We say that G is t-clique-decomposable if the vertices of G can be partitioned into r sets, where each set induces a clique C_i, $i = 1, 2, \dots, r$, of size t. We say that G admits a *t-clique decomposition*, denoted $C_1 \cup C_2 \cup \dots \cup C_r$.

Let $n \geq 6$ and let $t = \lfloor \frac{n}{2} \rfloor$. Notice that t divides $\binom{n}{2}$. In fact, $\binom{n}{2}$ is equal to tn when n is odd (resp. $t(n - 1)$ when n is even). Moreover, there is a simple decomposition of the vertex set of $K(n, 2)$ into cliques of size t.

Lemma 1 *Let $n \geq 6$ and $t = \lfloor \frac{n}{2} \rfloor$. The Kneser graph $K(n, 2)$ admits the following t-clique decomposition: Let $C_i = E_i$, for $1 \leq i \leq n$, when n is odd (resp. for $1 \leq i \leq n - 1$, when n is even), be a decomposition into cliques of size t of the vertex set of $K(n, 2)$, where the sets E_i are the edge sets of K_n corresponding to an optimal edge coloring of K_n.*

From now on, we use a *t-ordered clique decomposition* of $K(n, 2)$, where each clique C_i in a t-clique decomposition of $K(n, 2)$ is an ordered set having the following properties.

Table 1 The 3-ordered clique decomposition of $K(7, 2)$

j	C_1	C_2	C_3	C_4	C_5	C_6	C_7
1	$\{4, 5\}$	$\{1, 3\}$	$\{1, 5\}$	$\{1, 7\}$	$\{1, 2\}$	$\{1, 4\}$	$\{1, 6\}$
2	$\{2, 7\}$	$\{4, 7\}$	$\{2, 4\}$	$\{2, 6\}$	$\{4, 6\}$	$\{2, 3\}$	$\{2, 5\}$
3	$\{3, 6\}$	$\{5, 6\}$	$\{6, 7\}$	$\{3, 5\}$	$\{3, 7\}$	$\{5, 7\}$	$\{3, 4\}$

(i) Each vertex $\{a, b\}$ in C_i is ordered, that is, $a < b$.

(ii) Let $t = |C_i|$. If there exists a vertex $\{j, b\} \in C_i$, with $j < b$ and $1 \leq j \leq t$, then $\{j, b\}$ is the j^{th} vertex in the clique C_i, which is denoted by C_i^j. Otherwise, the remaining positions are fulfilled in increasing order by the remaining vertices which are ordered in a lexicographical way.

Table 1 shows the 3-ordered clique decomposition of $K(7, 2)$ as depicted in Fig. 2.

By the construction of the t-ordered clique decomposition of $K(n, 2)$ we have the following relevant observation.

Observation 1 *Let $n \geq 6$ and let $t = \lfloor \frac{n}{2} \rfloor$. Let $r = \binom{n}{2}/t$ and let $C_1 \cup \cdots \cup C_r$ be the t-ordered clique decomposition of $K(n, 2)$ described previously. Thus,*

i. If n is even, then for $1 \leq i \leq r$: $1 \in C_i^1$; and $2 \in C_i^2$ except when $i = \frac{n+2}{2}$.

ii. If n is odd, then for $1 \leq i \leq r$: $1 \in C_i^1$ except when $i = 1$; and $2 \in C_i^2$ except when $i = 2$ and $i = \frac{n+3}{2}$. Moreover, the vertices C_1^1, C_2^2 and $C_{\frac{n+3}{2}}^2$ are pairwise non adjacent.

In fact, notice that by construction, if n is even, then $C_{\frac{n+2}{2}}^2 = \{\frac{n+2}{2}, n\}$. Moreover, if n is odd, then $C_1^1 = \{\frac{n+1}{2}, \frac{n+3}{2}\}$, $C_2^2 = \{\frac{n+1}{2}, \frac{n+7}{2}\}$, and $C_{\frac{n+3}{2}}^2 = \{\frac{n+1}{2}, \frac{n+5}{2}\}$. Thus, by definition of $K(n, 2)$, these vertices are pairwise non adjacent.

As each clique C_i is a complete graph, we use the following notation for a total coloring of them: Let $t = |C_i|$ and let S_p be the p^{th}-color class of a total coloring of K_t as defined in Subsect. 3.1, for $1 \leq p \leq t$, and let M_{t+1} be the perfect matching forming the $(t + 1)^{th}$-color class in K_t when t is even. Thus,

- Suppose that t is odd. We denote by $S_{p,i}$, with $1 \leq p \leq t$, the p^{th}-color class of a total coloring of C_i, formed by the vertex C_i^p and the set of edges $\{\{C_i^w, C_i^z\}$: such that $\{w, z\} \in S_p\}$.
- Suppose that t is even. Similarly to the odd case, but considering the total coloring of K_t with t even and where $M_{t+1,i}$ denotes the perfect matching in C_i of size $\frac{t}{2}$ formed by the edges $\{C_i^w, C_i^z\}$ such that $\{w, z\} \in M_{t+1}$.

Definition 2 Let $n \geq 6$, $t = \lfloor \frac{n}{2} \rfloor$, $r = \binom{n}{2}/t$ and let $C_1 \cup \cdots \cup C_r$ be the t-ordered clique decomposition of $K(n, 2)$. Denote by $B(n, 2)$ the r-partite graph obtained from $K(n, 2)$ by removing the edges of each clique C_i. Each part of $B(n, 2)$ is the independent set $\overline{C_i}$. Furthermore, for $1 \leq i \neq j \leq r$, denote by $B_{i,j}(n, 2)$ the bipartite subgraph of $B(n, 2)$ induced by $\overline{C_i} \cup \overline{C_j}$.

Lemma 2 *Let $n \geq 6$, $t = \lfloor \frac{n}{2} \rfloor$ and $r = \binom{n}{2}/t$. For $1 \leq i \neq j \leq r$, the subgraph $B_{i,j}(n, 2)$ has the following properties.*

(i) If n is even, then $B_{i,j}(n,2)$ is $(t-2)$-regular;

(ii) If n is odd, then $\Delta(B_{i,j}(n,2)) = t-1$. Moreover, each $B_{i,j}(n,2)$ contains exactly two vertices, one in $\overline{C_i}$ and the other in $\overline{C_j}$ with degree $t-1$, and these two vertices are not adjacent when $n = 3q$ and $j = (i \pm q) \mod n$; otherwise, these two vertices are adjacent.

Proof (i) For n even, each clique C_i can be viewed as a perfect matching of K_n and so $\bigcup_{v \in C_i} v = [n]$. Thus, given a vertex $\{i_1, i_2\} \in C_i$, there exist exactly two vertices $\{j_1, j_2\}, \{j'_1, j'_2\} \in C_j$, with $i \neq j \in [r]$, such that exactly one of them contains i_1 and the other one contains i_2. Assume w.l.o.g. that $i_1 \in \{j_1, j_2\}$ and $i_2 \in \{j'_1, j'_2\}$. The remaining vertices in C_j have empty intersection with the vertex $\{i_1, i_2\}$. Therefore, the number of neighbors of vertex $\{i_1, i_2\} \in B_{i,j}(n,2)$ is exactly $t-2$.

(ii) For n odd, each clique C_i corresponds to a matching in K_n of size $\frac{n-1}{2}$. Therefore, by construction of the t-ordered clique decomposition, each clique C_i misses exactly the integer i (i.e. i does not belong to any vertex in C_i). Now, let C_j, with $j \neq i$, be another clique in the decomposition of $K(n,2)$. In a similar way, the integer j does not belong to any vertex in C_j. Thus, let $\{j_1, j_2\} \in C_j$ such that $i \in \{j_1, j_2\}$. Assume w.l.o.g. that $j_1 = i$. There is exactly one vertex $\{i_1, i_2\} \in C_i$ containing j_2. So, vertex $\{j_1, j_2\} \in C_j$ has exactly $t-1$ neighbors in C_i. Similarly, there exists only one vertex $\{i_1, i_2\} \in C_i$ containing the integer j and thus, such a vertex has exactly $t-1$ neighbors in C_j. All the remaining vertices in $B_{i,j}(n,2)$ have exactly $t-2$ neighbors. Finally, let $\{i', j\} \in C_i$ and $\{j', i\} \in C_j$ (or $\{i, j'\} \in C_j$) be the two vertices of degree $t-1$ in $B_{i,j}(n,2)$, with $1 \leq i \neq j \leq n$. By construction, $(i', j) = (i - q \mod n, i + q \mod n)$ for some $1 \leq q \leq \frac{n-1}{2}$. Assume first that $j = i + q \mod n$. Therefore, $(j - q \mod n, j + q \mod n) \in C_j$ is the vertex $(i \mod n, (i + 2q) \mod n) = (i, (i + 2q) \mod n)$. As $i \neq j$, we have that $i + 2q \mod n = i - q \mod n \iff i + 2q \equiv i - q \mod n \iff n$ divides $3q$. Now, assume that $j = i - q \mod n$. Thus, $(j - q \mod n, j + q \mod n) \in C_j$ is the vertex $(i - 2q \mod n, i)$. Similarly to the previous case, $i - 2q \mod n = i + q \mod n \iff n$ divides $3q$. However, as $q \leq \frac{n-1}{2}$, the only possibility of the vertices of degree $t-1$ are not adjacent is when $n = 3q$ and $j = (i \pm q) \mod n$.

Lemma 3 *Let $n > 6$ be an odd integer such that n is not divisible by 3. Let $B'(n,2)$ be the n-partite graph having the set of vertices of $B(n,2)$ and where for $1 \leq i \neq j \leq n$ there is only one edge between the parts $\overline{C_i}$ and $\overline{C_j}$ which is the edge between the only two vertices of degree $t-1$ in $B_{i,j}(n,2)$. Thus, $B'(n,2)$ is a 2-regular n-partite graph.*

Proof As n is odd, we have that $B'(n,2)$ has n parts $\overline{C_1} \cup \ldots \cup \overline{C_n}$. By Lemma 2 (ii), each one of the integers $i \in [n]$ does not belong to any vertex in each part $\overline{C_i}$. Now, let $\{j,k\}$ be any vertex in $\overline{C_i}$. Thus, $i \neq j \neq k \in [n]$ and by Lemma 2 (ii), as 3 does not divide n, we have that vertex $\{j,k\} \in \overline{C_i}$ has only two neighbors, one $\{i, j'\} \in \overline{C_j}$ and one $\{i, k'\} \in \overline{C_k}$, such that vertices $\{j,k\}$,$\{i,j'\}$ (resp. $\{j,k\}$,$\{i,k'\}$) have degree $t-1$ in $B'_{i,j}(n,2)$ (resp. in $B'_{i,k}(n,2)$). As we have chosen any $i \in [n]$ and any vertex $\{j,k\} \in \overline{C_i}$, each vertex in $B'(n,2)$ has degree equal to 2.

3.2 An algorithm for a partial total coloring of $K(n, 2)$

In this section, we present Algorithm 1 that gives a partial total coloring of the Kneser graphs $K(n, 2)$, which is used in the proof of Theorem 2. A partial k-total coloring of a graph G is an assignment of k colors to a subset of elements of G such that adjacent or incident elements in the subset have different colors.

Algorithm 1:

Input: Let $n \geq 6$, $t = \lfloor \frac{n}{2} \rfloor$ and $r = \binom{n}{2}/t$. Let $C_1 \cup \cdots \cup C_r$ be a t-ordered clique decomposition of $K(n, 2)$. Let E' be a subset of edges (possibly empty) in $B(n, 2)$ and let $B^*(n, 2)$ be the graph $B(n, 2)$ without the edges in E'. Moreover, assume that for each $1 \leq i \neq j \leq r$, the bipartite subgraph $B^*_{i,j}(n, 2)$ of $B^*(n, 2)$ has maximum degree equal to $(t-2)$ and let $F^1_{i,j} \cup \cdots \cup F^{t-2}_{i,j}$ be an optimal $(t-2)$-edge coloring of $B^*_{i,j}(n, 2)$.

Output: A partial total coloring of $K(n, 2)$ with $(t-2)r$ colors.

```
For k = 1 to t − 2 do
    For i = 1 to r do
        (i) Color the independent set of elements S_{k+2,i} of clique C_i with
    color i + r(k − 1).
        (ii) For all edges {i_1, i_2} ∈ E_i (i.e. the i^th edge color class of an
    optimal edge coloring of K_r), color all edges in F^k_{i_1,i_2} with color
    i + r(k − 1).
```

Lemma 4 *Algorithm 1 is correct and gives a partial total coloring of the input graph with $(t-2)r$ colors.*

Proof Clearly, the number of colors used by Algorithm 1 is equal to $(t-2)r$. We know that r is an odd integer. In fact, $r = n$ if n is odd, otherwise $r = n-1$. As we saw in Subsect. 3.1, K_r has an edge decomposition $E_1, E_2, \ldots, E_r$ corresponding to an optimal edge coloring such that integer i does not belong to the edge set E_i, for $1 \leq i \leq r$. Let $k = 1$ and consider the most internal loop: for each $1 \leq i \leq r$, we use the information in the set E_i to color the edges and vertices of $K(n, 2)$ as follows: as integer i does not belong to E_i, first, we color the elements in the set $S_{3,i}$ of clique C_i with color i. Notice that such a coloring of $S_{3,i}$ is proper. Next, for each edge $\{i_1, i_2\} \in E_i$, we use color i to color a matching between each one of the bipartite subgraphs $B^*_{i_1,i_2}(n, 2)$ which corresponds to the edge set $F^1_{i_1,i_2}$. As for any $\{i_1, i_2\}, \{i'_1, i'_2\} \in E_i$ we have that $\{i_1, i_2\} \cap \{i'_1, i'_2\} = \emptyset$ and $\{i\} \cap \{i_1, i_2\} = \{i\} \cap \{i'_1, i'_2\} = \emptyset$, such an edge coloring is proper. So, at the end of the internal loop, we have properly colored, with colors $1 \leq i \leq r$, the set $S_{3,i}$ of C_i and one matching (i.e. a color class in an optimal edge coloring) of each $B^*_{i_1,i_2}(n, 2)$ with $1 \leq i_1 \neq i_2 \leq r$. Thus, for each fixed k, $1 \leq k \leq t-2$, when the internal loop ends, for each $1 \leq i \leq r$, we have properly colored the set $S_{k+2,i}$ of C_i and one matching (i.e. a color class in an optimal edge coloring) of each $B^*_{i_1,i_2}(n, 2)$, where $1 \leq i_1 \neq i_2 \leq r$ and $i \neq i_1, i_2$, with color $i + r(k-1)$. So, it is not difficult to see that at the end of Algorithm 1, all the edges of each $B^*_{i_1,i_2}(n, 2)$ with $1 \leq i_1 \neq i_2 \leq r$ have been properly colored with colors $1, 2, \ldots, (t-2)r$. Moreover, the sets $S_{j,i}$ of each clique C_i of size t, where $3 \leq j \leq t$,

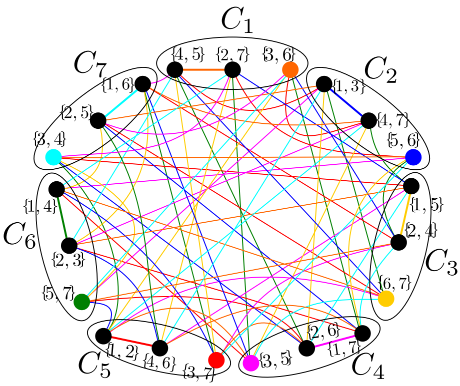

Fig. 2 A depiction of the input graph of $K(7, 2)$ highlighting the elements colored by Algorithm 1 with $(t-2)r = (3-2)7 = 7$ colors. As k takes only the value 1 (i.e. $t = 3$), for $1 \leq i \leq 7$ Algorithm 1 colors the independent set of elements $S_{3,i}$ of clique C_i, composed by one edge and one vertex with the color i and, for all edges $\{i_1, i_2\} \in E_i$ (i.e. the i^{th} edge color class of an optimal edge coloring of K_7), Algorithm 1 also colors with color i all the edges in $F^1_{i_1,i_2}$. For instance, the elements which are colored with color 1 (orange) are the elements in $S_{3,1}$ (i.e. the vertex {3, 6} and the edge {{4, 5}, {2, 7}}), and the set of edges in $F^1_{4,5}$ (i.e. {{1, 7}, {4, 6}}, {{2, 6}, {3, 7}} and {{3, 5}, {1, 2}}), and in $F^1_{3,6}$ (i.e. {{1, 5}, {2, 3}}, {{2, 4}, {5, 7}} and {{6, 7}, {1, 4}}) and in $F^1_{2,7}$ (i.e. {{1, 3}, {2, 5}}, {{4, 7}, {1, 6}} and {{5, 6}, {3, 4}}). In fact, notice that E_1, the first color class of an optimal edge coloring of K_7, is the set of edges {{4, 5}, {2, 7}, {3, 6}} (Color figure online)

have been also properly colored with colors $1, 2, \ldots, (t-2)r$. Thus, Algorithm 1 is correct. Finally, notice that the sets $S_{1,i}$ and $S_{2,i}$ of each clique C_i, $1 \leq i \leq r$, remain uncolored. Therefore, Algorithm 1 gives a partial total coloring of the input graphs. Figure 2 presents the elements colored by Algorithm 1 considering $K(7, 2)$.

3.3 The proof of Theorem 2

The focus of this subsection is to assign colors to the vertices and edges of $K(n, 2)$ not colored by Algorithm 1 and thus obtain the bounds of Theorem 2. Let $n \geq 6$ and let $\Delta(K(n, 2)) = \binom{n-2}{2} = \frac{n^2-5n}{2} + 3$.

3.3.1 Case *n* even

If n is even, then $t = \frac{n}{2}$ and $r = \binom{n}{2}/t = n-1$. By Lemma 2 (ii), we know that for each $1 \leq i \neq j \leq r$, the bipartite subgraph $B_{i,j}(n, 2)$ of the graph $B(n, 2)$ is $(t-2)$-regular. Therefore, we can use Algorithm 1 by setting $E' = \emptyset$ in order to obtain a partial total coloring of $K(n, 2)$ with $(t-2)r = (\frac{n}{2}-2)(n-1) = \frac{n^2-5n}{2} + 2 = \Delta(K(n, 2)) - 1$ colors in which all the edges in $B(n, 2)$ are colored. In order to complete the total coloring of $K(n, 2)$, we need to consider the following cases:

1. Case t odd. As the total coloring of K_t is equal to t, it remains to color the sets $S_{1,i}$ and $S_{2,i}$ of each clique C_i, with $1 \leq i \leq r$. Clearly, for each $1 \leq i \neq j \leq r$, no

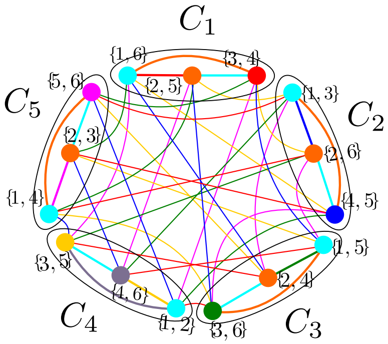

Fig. 3 A total coloring of $K(6, 2)$ with $\Delta(K(6, 2)) + 2 = 8$ colors. First, Algorithm 1 obtains a partial total coloring with 5 colors (red, blue, green, yellow and purple) as follows. As k takes only the value 1 (i.e. $t = 3$) then, for $1 \leq i \leq 5$, Algorithm 1 colors the elements in the set $S_{3,i}$ of clique C_i, composed by one edge and one vertex, with the color i and, for all edges $\{i_1, i_2\} \in E_i$ (i.e. the i^{th} edge color class of an optimal edge coloring of K_5), Algorithm 1 also colors with color i all the edges in $F^1_{i_1,i_2}$. For instance, at the end of Algorithm 1, the elements which are colored with color 1 (red) are the elements in $S_{3,1}$ (i.e. the vertex {3, 4} and the edge {{1, 6}, {2, 5}}) and, the set of edges in $F^1_{2,5}$ (i.e. {{1, 3}, {5, 6}}, {{2, 6}, {1, 4}} and {{4, 5}, {2, 3}}) and in $F^1_{3,4}$ (i.e . {{1, 5}, {4, 6}}, {{2, 4}, {3, 5}} and {{3, 6}, {1, 2}}). In fact, notice that E_1, the color class 1 in an optimal edge coloring of K_5 is the set of edges {{2, 5}, {3, 4}}. In order to complete the partial total coloring given by Algorithm 1, we color with color 6 (light blue) the elements in $S_{1,i}$ of each clique C_i, and we color with color 7 (orange) the elements $S_{2,i}$, except for vertex $C^2_4 = \{4, 6\}$ in $S_{2,4}$ which needs different color 8 (gray) (Color figure online)

edge inside C_i is incident to any edge inside C_j. Moreover, by Observation 1 (i), $1 \in C^1_i$ and so, we can color all the edges and the vertex in each $S_{1,i}$ with color $(t-2)r+1$. Again, by Observation 1 (i), we can color all the edges and the vertex in each $S_{2,i}$ with color $(t-2)r+2$ except the vertex $C^2_{\frac{n+2}{2}}$ for which we use the color $(t-2)r+3$. Therefore, we use at most $\Delta(K(n, 2))-1+3 = \Delta(K(n, 2))+2$ colors to total coloring $K(n, 2)$. For an example of the obtained $(\Delta(K(n, 2)) + 2)$-total coloring when t is odd, see Fig. 3.

2. Case t even. As the total coloring of K_t is $t + 1$, it remains to color the sets $S_{1,i}$, $S_{2,i}$ and $M_{t+1,i}$ of each clique C_i, with $1 \leq i \leq r$. We proceed in a similar way as in the Case 1 in order to color the sets $S_{1,i}$ with color $(t-2)r + 1$. For $i \neq \frac{n+2}{2}$, we color the sets $S_{2,i}$ and $M_{t+1,\frac{n+2}{2}}$ with color $(t-2)r + 2$ and the sets $S_{2,\frac{n+2}{2}}$ and $M_{t+1,i}$ with color $(t-2)r + 3$. Notice that by construction, the previous total coloring is proper because the only clique whose second vertex is not of the form $\{2, b\}$ with $2 < b \leq n$ is the clique $C_{\frac{n+2}{2}}$ and thus, any edge in $M_{t+1,\frac{n+2}{2}}$ is incident to a vertex $\{2, b\}$. Therefore, we use at most $\Delta(K(n, 2))+2$ colors to total coloring $K(n, 2)$.

3.3.2 Case *n* odd

If n is odd, then $t = \frac{n-1}{2}$ and $r = \binom{n}{2}/t = n$. For each i, j, with $1 \leq i < j \leq r$, by Lemma 2 (ii), the bipartite graph $B_{i,j}(n, 2)$ has maximum degree equal to $t - 1$. We must consider the following cases:

a. **n is not divisible by** 3. By Lemma 3, the subgraph $B'(n,2)$ of $B(n,2)$ is 2-regular. Let E' be the set of edges of $B'(n,2)$. By Lemma 2 (ii) and Lemma 3, the bipartite subgraphs of the graph $B(n,2)$ without the set of edges E' are $(t-2)$-regular. Therefore, by applying the Algorithm 1 to $K(n,2)$ with E' being the set of edges in $B'(n,2)$, we obtain a partial total coloring of $K(n,2)$ with $(t-2)r = (\frac{n-1}{2}-2)n = \frac{n^2-5n}{2} = \Delta(K(n,2))-3$ colors. In order to complete the total coloring of $K(n,2)$, we need to consider the following cases:

1. Case t odd. As the total coloring of K_t is equal to t, it remains to color the sets $S_{1,i}$ and $S_{2,i}$ of each clique C_i, with $1 \leq i \leq r$. Clearly, for each $1 \leq i \neq j \leq r$, no edge inside C_i is incident to any edge inside C_j. Moreover, by Observation 1 (ii), $1 \in C_i^1$ except when $i = 1$; and $2 \in C_i^2$ except when $i = 2$ and $i = \frac{n+3}{2}$. So, we color the elements in the sets $S_{1,i}$ and $S_{2,i}$ with colors $\Delta(K(n,2)) - 2$ and $\Delta(K(n,2)) - 1$ respectively, except the vertices C_1^1, C_2^2 and $C^2_{\frac{n+3}{2}}$ which, again by Observation 1 (ii), they are pairwise non adjacent. It remains to color the edges in $B'(n,2)$ and the set of vertices $\{C_1^1, C_2^2, C^2_{\frac{n+3}{2}}\}$. By Lemma 3, $B'(n,2)$ has maximum degree equal to 2. Therefore, 3 new colors are enough to color the edges in $B'(n,2)$ and the vertices in $\{C_1^1, C_2^2, C^2_{\frac{n+3}{2}}\}$. So, we use at most $\Delta(K(n,2)) - 1 + 3 = \Delta(K(n,2)) + 2$ colors to total coloring $K(n,2)$. Notice that in this case, n is not divisible by 3 and $n \not\equiv 1 \mod 4$.
2. Case t even. Notice that in this case $n \equiv 1 \mod 4$. In order to complete the total coloring of $K(n,2)$, it remains to color the sets $S_{1,i}$, $S_{2,i}$, the edges $M_{t+1,i}$ of each clique C_i, with $1 \leq i \leq r$ and the set of edges $E' = B'(n,2)$. Recall that each clique can be described as a complete graph K_t which admits a $(t+1)$-total coloring when t is even.
   Consider the clique $C_1 = \{x_1y_1, x_2y_2, \ldots, x_ty_t\}$ such that $2 \leq x_i \neq y_i \leq n$ and let $d_i = x_i - 1$. Note that, the graph $B'(n,2)$ can be described as t disjoint cycles of size n, w.l.o.g. we can construct these t disjoint cycles of size n from C_1 as follows: from each $x_iy_i \in C_1$, with $1 \leq i \leq t$, form the cycles considering the integers modulo n by $Cycle(d_i) = \{x_iy_i, (x_i+d_i)(y_i+d_i), (x_i+2d_i)(y_i+2d_i), \ldots, (x_i+nd_i, y_i+nd_i), x_iy_i\}$. As n is odd, we can color the odd cycles with 3 colors as follows: for each $Cycle(d_i)$ assign the edge $\{x_iy_i, (x_i+d_i)(y_i+d_i)\}$ with the color $\Delta(K(n,2)) - 2$; for $j \geq 1$ odd, assign the edges $\{(x_i+jd_i)(y_i+jd_i), (x_i+(j+1)d_i)(y_i+(j+1)d_i)\}$ with the color $\Delta(K(n,2)) - 1$ and for $j > 1$ even, assign the edges $\{(x_i+jd_i)(y_i+jd_i), (x_i+(j+1)d_i)(y_i+(j+1)d_i)\}$ with the color $\Delta(K(n,2))$.
   Assign with the color $\Delta - 2$ only the edges of set $S_{1,i}$ with $2 \leq i \leq \frac{n+1}{2}$ and all the elements in $S_{1,i}$ for $\frac{n+3}{2} \leq i \leq n$. Note that, the colors of the edges incident to the vertices C_i^1 with $2 \leq i \leq \frac{n+1}{2}$ are $\Delta(K(n,2)) - 2$ and $\Delta(K(n,2)) - 1$ and so, we can assign the color $\Delta(K(n,2))$ to these vertices. Note that the edges incident to the vertices in C_1 are $\Delta(K(n,2)) - 2$ and $\Delta(K(n,2))$, so we can assign the set S_1^1 with the color $\Delta(K(n,2)) - 1$. It remains color the set $S_{2,i}$ and the edges $M_{t+1,i}$ of each clique C_i.

Assign the sets $M_{t+1,2}$, $M_{t+1,\frac{n+3}{2}}$ and the sets $S_{2,i}$ for $i \neq 2, \frac{n+3}{2}$ with color $\Delta(K(n,2)) + 1$; assign the edges of $M_{t+1,i}$ with the color $\Delta(K(n,2)) + 2$. Now for $i = 2, \frac{n+2}{2}$, assign the sets $S_{2,i}$ with color $\Delta(K(n,2)) + 2$. Notice that by construction, the previous total coloring is proper and therefore, we use at most $\Delta(K(n,2)) + 2$ colors to total coloring $K(n,2)$. Figure 4 presents a depiction of the Kneser graph $K(13,2)$ with the elements colored as previously using 5 colors.

b. n is divisible by 3. Let $n = 3p$ for some odd integer $p \geq 3$. Let $E_1, E_2, \ldots, E_n$ be a optimal edge coloring of K_n, as defined in Section 2, representing a t-ordered clique decomposition $C_1 \cup \cdots \cup C_n$ of $K(n,2)$. As shown in Lemma 2 (ii), there are some cliques C_i, C_j such that they contain a vertex $\{j, i'\}$ and $\{i, j'\}$ respectively, both of degree $t-1$ in $B_{i,j}(n,2)$, but not adjacent between them. Precisely, consider the cliques C_i, C_j and C_k with $i \neq j \neq k \in [n]$ such that $j = i - p \mod 3p$ and $k = i + p \mod 3p$. By construction, the vertex $\{j,k\} = \{i - p \mod 3p, i + p \mod 3p\}$ belongs to C_i. Moreover, the vertex $\{j - p \mod 3p, j + p \mod 3p\} = \{k, i\}$ belongs to C_j and the vertex $\{k - p \mod 3p, k + p \mod 3p\} = \{i, j\}$ belongs to C_k. Therefore, the two vertices of degree $t-1$ in the bipartite graphs $B_{i,j}(n,2)$, $B_{i,k}(n,2)$ and $B_{j,k}(n,2)$ respectively, are pairwise non adjacent. It is not hard to see that we can partition the cliques $C_1, \ldots, C_{3p}$ in triples verifying the previous property of pairwise non adjacency between their respective vertices of degree $t-1$. In fact, for any $i \in [n]$, let $\mathrm{Orb}(i) = \{i\} \cup \{j : j = i - p \mod 3p \text{ or } j = i + p \mod 3p\}$ be the *orbit* of i. Clearly, there are exactly p different orbits and they form a partition of $[n]$. For example, for $n = 9$ we have three orbits: $\mathrm{Orb}(1) = \{1,4,7\}$, $\mathrm{Orb}(2) = \{2,5,8\}$, and $\mathrm{Orb}(3) = \{3,6,9\}$. Now, for each orbit $\mathrm{Orb}(i) = \{i, j, k\}$ we construct 3 vertex disjoint paths on three vertices in $B(n,2)$ as follows: let $P_i = \{\{j_1, j_2\}, \{j,k\}, \{k_1,k_2\}\}$, $P_j = \{\{i_1,i_2\}, \{i,k\}, \{k_3,k_4\}\}$, and $P_k = \{\{i_3,i_4\}, \{i,j\}, \{j_3,j_4\}\}$, where $\{i_1,i_2\}, \{i_3,i_4\}$ are two different vertices in C_i; $\{j_1,j_2\}, \{j_3,j_4\}$ are two different vertices in C_j; and $\{k_1,k_2\}, \{k_3,k_4\}$ are two different vertices in C_k such that, $\{i_1,i_2\}$,$\{j_1,j_2\}$ are adjacent in $B_{i,j}(n,2)$, and $\{i_3,i_4\}, \{k_1,k_2\}$ are adjacent in $B_{i,k}(n,2)$, and $\{j_3,j_4\}, \{k_3,k_4\}$ are adjacent in $B_{j,k}(n,2)$. Notice that the construction of such three vertex disjoint paths for each orbit is always possible because $n \geq 9$ and so each C_i has at least 4 vertices. For example, for $n = 9$, the three vertex disjoint paths on 3 vertices of the orbit $\mathrm{Orb}(1)$ can be $P_1 = \{\{8,9\}, \{4,7\}, \{5,9\}\}$, $P_4 = \{\{5,6\}, \{1,7\}, \{6,8\}\}$, and $P_7 = \{\{3,8\}, \{1,4\}, \{3,5\}\}$.

Now, let F be the set of the edges of the $3p$ vertex disjoint paths on 3 vertices constructed previously and associated to the p orbits of $[n]$. Moreover, let F' be the set of edges in the graph $B'(n,2)$. It's not so hard to verify that the maximum degree of the bipartite graphs in the graph $B(n,2)$ without the edges in $F \cup F'$ is equal to $t-2$. Therefore, we can apply the Algorithm 1 to $K(n,2)$ by setting $E' = F \cup F'$ which give us a partial total coloring of $K(n,2)$ with $(t-2)n = \Delta(K(n,2)) - 3$ colors. Now, in order to complete such a total coloring for $K(n,2)$, we consider the following cases:

1. Case t odd. We proceed in a similar way as in the Case ii.a.1. Thus, we color the elements in the sets $S_{1,i}$ and $S_{2,i}$ with colors $\Delta(K(n,2))-2$ and $\Delta(K(n,2))-1$ respectively, except the vertices C_1^1, C_2^2 and $C_{\frac{n+3}{2}}^2$ which, by Observation 1 (ii), they are pairwise non adjacent. Let $B''(n,2)$ be the graph $B'(n,2)$ to which we add the edges in the set F defined previously. Then, it remains to color the edges in $B''(n,2)$ and the set of vertices $\{C_1^1, C_2^2, C_{\frac{n+3}{2}}^2\}$. By construction, the maximum degree in $B''(n,2)$ is equal to 3. Therefore, four colors are enough to color the edges in $B''(n,2)$. Moreover, the same new four colors are enough also to color the independent vertices $\{C_1^1, C_2^2, C_{\frac{n+3}{2}}^2\}$. Therefore, we use at most $\Delta(K(n,2))+3$ colors to total coloring $K(n,2)$. Notice that in this case, $n \equiv 3 \mod 4$.
2. Case t even. As the total coloring of K_t is $t+1$, it remains to color the sets $S_{1,i}$, $S_{2,i}$ and $M_{t+1,i}$ of each clique C_i, with $1 \le i \le n$. We proceed in a similar way as in the Case b.1 in order to color the sets $S_{1,i}$, $S_{2,i}$ of each C_i and the edges in $B''(n,2)$. Finally, we use a new color in order to color the edges of each edge set $M_{t+1,i}$. Therefore, we use at most $\Delta(K(n,2))+4$ colors to total color $K(n,2)$. Notice that in this case, $n \equiv 1 \mod 4$.

As all the cases have been considered, we have that Theorem 2 holds. □

4 An infinite family of Class 1 $K(n, 2)$ graphs

The Kneser graph $K(4,2)$ consisting of a perfect matching with 6 vertices is Class 1, and the Petersen graph $K(5,2)$ is known to be Class 2. It is well known that a regular graph with an odd number of vertices is Class 2. Recall that the number of vertices of $K(n,2)$ is $\binom{n}{2}$, so the Kneser graphs $K(n,2)$ with $n \equiv 2 \mod 4$ or $n \equiv 3 \mod 4$ are precisely the ones with an odd number of vertices, which implies that they are Class 2. In the following, we consider the Kneser graphs $K(n,2)$ with an even number of vertices, that is with $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$, which implies that $n \ge 8$.

We use Algorithm 2 to obtain a $\Delta(K(n,2))$-edge coloring of the infinite family of $K(n,2)$ when $n \equiv 0 \mod 4$. Note that in this case both n and $t = \frac{n}{2}$ are even and thus an optimal edge coloring of clique C_i of an even size uses $t-1$ colors. Algorithm 2 is a variation of Algorithm 1 which is used only to color the edges of $K(n,2)$ as follows.

Algorithm 2:

Input: Let $n \ge 8$, $t = \lfloor \frac{n}{2} \rfloor$ and $r = \binom{n}{2}/t$. Let $C_1 \cup \cdots \cup C_r$ be a t-ordered clique decomposition of $K(n,2)$. Let E' be a subset of edges (possibly empty) in $B(n,2)$ and let $B^*(n,2)$ be the graph $B(n,2)$ without the edges in E'. Moreover, assume that for each $1 \le i \ne j \le r$, the bipartite subgraph $B^*_{i,j}(n,2)$ of $B^*(n,2)$ has maximum degree equal to $(t-2)$ and let $F^1_{i,j} \cup \cdots \cup F^{t-2}_{i,j}$ be an optimal $(t-2)$-edge coloring of $B^*_{i,j}(n,2)$.

Output: A partial edge coloring of $K(n,2)$ with $(t-2)r$ colors.

1 **For** $k = 1$ **to** $t-2$ **do**

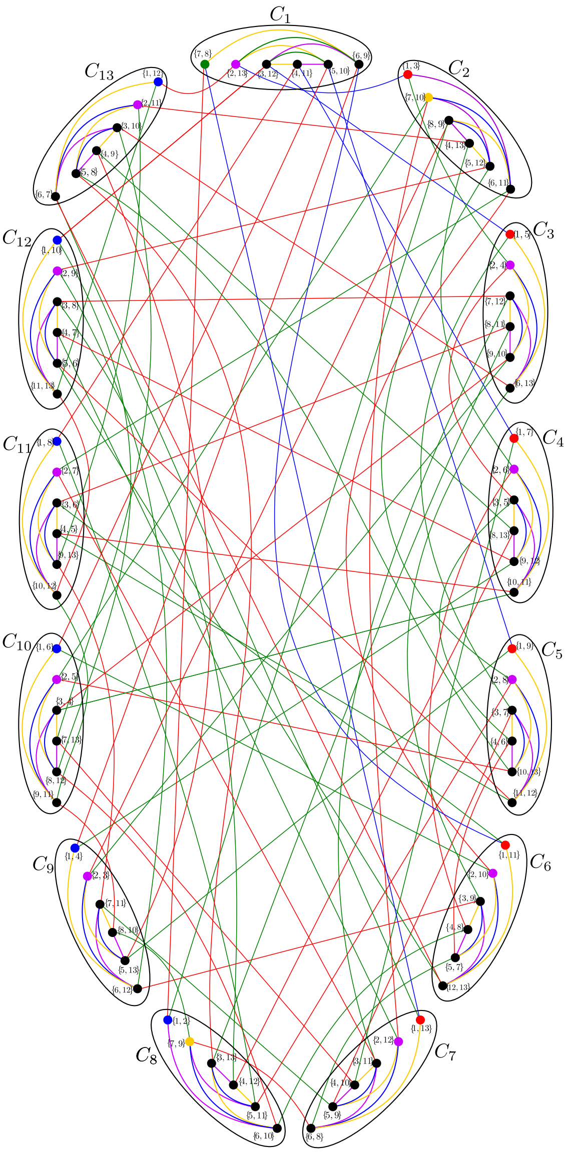

Fig. 4 Disregarding the elements colored by Algorithm 1 with $(t-2)r = (6-2)13 = 52$ colors, we show how to complete the total coloring of $K(13, 2)$ by highlighting the elements colored with the 5 new colors 53 (blue), 54 (green), 55 (red), 56 (purple) and $57 = \Delta(K(13, 2)) + 2$ (yellow). Colors 53, 54 and 55 color the elements in $S_{1,i}$ (i.e. each $S_{1,i}$ is composed by two edges and one vertex) of each clique C_i, for $1 \leq i \leq 13$, as follows. First, color 53 colors all the elements in $S_{1,i}$, for $8 \leq i \leq 13$ and only the edges in $S_{1,i}$, for $2 \leq i \leq 7$; color 54 colors all the element in $S_{1,1}$, and color 55 colors only the vertices of $S_{1,i}$, for $2 \leq i \leq 7$. Moreover, colors 53, 54 and 55 are used to color the edges of graph $B'(13, 2)$. Next, color 56 colors the perfect matchings $M_{7,2}$ and $M_{7,8}$, and it also colors the elements in $S_{2,i}$ of each C_i (except for the sets $S_{2,2}$ and $S_{2,8}$ whose elements are colored with color 57, because $2 \notin C_2^2 = \{7, 10\}$ and $2 \notin C_8^2 = \{7, 9\}$ and thus, such vertices are adjacents to vertices colored with color 56). Finally, color 57 colors the perfect matchings $M_{7,i}$ of each C_i, for $i \notin \{2, 8\}$ (Color figure online)

For $i = 1$ **to** r **do**

  (i) Color all the edges in the set $E_{k+1,i} = \{\{C_i^u, C_i^v\} : \{u, v\} \in E_{k+1}\}$ (where E_{k+1} denotes the $(k+1)^{th}$ edge color class of an optimal edge coloring of K_t) of clique C_i with color $i + r(k-1)$.

  (ii) For all edge $\{i_1, i_2\} \in E_i$ (i.e. the i^{th} edge color class of an optimal edge coloring of K_r), color all edges in $F^k_{i_1,i_2}$ with color $i + r(k-1)$.

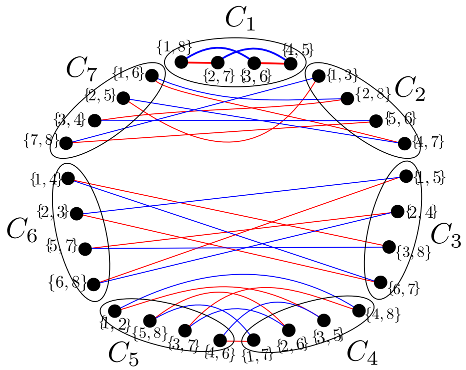

Fig. 5 A depiction of a subgraph of the input graph $K(8, 2)$ highlighting the edges colored by Algorithm 2, for $k = 1, 2$ and when we fix $i = 1$, which use colors 1 (blue) and 8 (red). For $k = 1$, let the second color class of an optimal edge coloring of K_4 be $E_2 = \{\{1, 3\}, \{2, 4\}\}$ to get the edges colored with the color 1(blue): $E_{2,1} = \{\{\{1, 8\}, \{3, 6\}\}, \{\{2, 7\}, \{4, 5\}\}\}$, and let the second color class of an optimal edge coloring of K_7 be $E_2 = \{\{2, 7\}, \{3, 6\}, \{4, 5\}\}$ to get the edges colored also with the color 1 (blue) in $F^1_{2,7}$ (i.e. $\{\{1, 3\}, \{7, 8\}\}, \{\{2, 8\}, \{1, 6\}\}, \{\{5, 6\}, \{3, 4\}\}$ and $\{\{4, 7\}, \{2, 5\}\}$), in $F^1_{3,6}$ (i.e. $\{\{1, 5\}, \{2, 3\}\}, \{\{2, 4\}, \{6, 8\}\}, \{\{3, 8\}, \{5, 7\}\}$ and $\{\{6, 7\}, \{1, 4\}\}$), and in $F^1_{4,5}$ (i.e. $\{\{1, 7\}, \{5, 8\}\}, \{\{2, 6\}, \{3, 7\}\}$, $\{\{3, 5\}, \{4, 6\}\}$ and $\{\{4, 8\}, \{1, 2\}\}$). For $k = 2$, let the third color class of an optimal edge coloring of K_4 be $E_3 = \{\{1, 2\}, \{3, 4\}\}$ to get the edges colored with the color 8(red): $E_{3,1} = \{\{\{1, 8\}, \{2, 7\}\}, \{\{3, 6\}, \{4, 5\}\}\}$, and let the second color class of an optimal edge coloring of K_7 be $E_2 = \{\{2, 7\}, \{3, 6\}, \{4, 5\}\}$ to get the edges colored also with the color 8 (red) in $F^2_{2,7}$ (i.e. $\{\{1, 3\}, \{2, 5\}\}, \{\{2, 8\}, \{3, 4\}\}, \{\{5, 6\}, \{7, 8\}\}$ and $\{\{4, 7\}, \{1, 6\}\}$), in $F^2_{3,6}$ (i.e. $\{\{1, 5\}, \{6, 8\}\}, \{\{2, 4\}, \{5, 7\}\}, \{\{3, 8\}, \{1, 4\}\}$ and $\{\{6, 7\}, \{2, 3\}\}$), and in $F^2_{4,5}$ (i.e. $\{\{1, 7\}, \{4, 6\}\}, \{\{2, 6\}, \{5, 8\}\}, \{\{3, 5\}, \{1, 2\}\}$, and $\{\{4, 8\}, \{3, 7\}\}$) (Color figure online)

Figure 5 presents a subgraph of the input graph of $K(8, 2)$ highlighting how Algorithm 2 colors the edges when $i = 1$ is fixed.

Lemma 5 *Algorithm 2 is correct and gives a partial edge coloring of the input graph by using $(t - 2)r$ colors.*

Proof The proof is analogous to the one of Lemma 4. Just notice that t is even and thus, an optimal edge coloring of K_t uses only $t - 1$ colors. Therefore, at the end of Algorithm 2, the matching $E_{1,i} = \{\{C^u_i, C^v_i\} : \{u, v\} \in E_1\}$ (where E_1 denotes the first edge color class of an optimal edge coloring of K_t) in each one of the cliques C_i, with $1 \leq i \leq r$, remains uncolored.

Theorem 3 *Let $n \equiv 0 \mod 4$. Thus, $\chi(K(n, 2)) = \Delta(K(n, 2))$.*

Proof Since $n \equiv 0 \mod 4$, by Lemma 2 (i) we can apply the Algorithm 2 to $K(n, 2)$ by setting $E' = \emptyset$. Notice that $r = n - 1$ and $t = \frac{n}{2}$. Thus, by Lemma 5, Algorithm 2 gives a partial proper edge coloring of $K(n, 2)$ with $(t - 2)r$ colors and only one perfect matching in each clique C_i remains uncolored. As a matching in two different cliques share no vertices, we can color each uncolored perfect matching in each one of the cliques C_i with only one new color. Therefore, we can properly color the edges of $K(n, 2)$ by using $(t-2)r+1 = (\frac{n}{2}-2)(n-1)+1 = \frac{n^2-5n}{2}+3 = \binom{n-2}{2} = \Delta(K(n, 2))$ colors.

5 Concluding remarks

As a consequence of Baranyai's theorem Baranyai 1973, the vertex set of the Kneser graph $K(n,s)$ can be partitioned into $\theta_{n,s} = \binom{n}{s}/\lfloor \frac{n}{s} \rfloor$ cliques of size $\omega_{n,s} = \lfloor \frac{n}{s} \rfloor$, when $\lfloor \frac{n}{s} \rfloor$ divides $\binom{n}{s}$. However, in order to generalize our results about total and edge colorings of $K(n,2)$ to arbitrary values of s, a deeper structural analysis of the graph $B(n,s)$ induced by the cliques in the t-ordered clique decomposition of $K(n,s)$ in which the edges inside such cliques are removed is needed.

Concerning the chromatic index of $K(n,2)$, the only unsettled case is when $n \equiv 1 \mod 4$. Notice that in this case, the number of vertices $\binom{n}{2}$ is even. However, n is odd and we need to consider the cases when n is divisible by 3 or not. We need to study how to color the uncolored edges of $K(n,2)$ after applying Algorithm 2. In fact, notice that Algorithm 2 uses $(t-2)r = (\frac{n-1}{2} - 2)n = \Delta(K(n,2)) - 3$ colors. Thus, we must be able to color these edges by using precisely 3 new colors if we want a $\Delta(K(n,2))$-edge coloring of $K(n,2)$. By using SageMath software, we have been able to: (i) compute a 55-edge coloring for these edges of $K(13,2)$, which implies that $K(13,2)$ is Class 1, and (ii) a 21-edge coloring of $K(9,2)$ which also implies that this graph is Class 1. We believe that when $n \equiv 1 \mod 4$ it is always possible to edge-color $K(n,2)$ with $\Delta(K(n,2))$ colors and so, we conjecture the following.

Conjecture 1 *For $n \geq 6$, the Kneser graph $K(n,2)$ with $n \equiv 1 \mod 4$ is Class 1.*

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Declarations

Conflict of interest The authors declare that there is no conflict of interest.

Availability of data and material Not applicable.

Code availability Not applicable.

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