3

Total colouring

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A total colouring assigns a colour to each vertex and edge of a graph, so that there are no incidence conflicts. Since, by definition, a total colouring is also a vertex-colouring and an edge-colouring, it is natural to consider successful strategies, both theoretical and algorithmic, towards the solution of these two more studied problems. This chapter surveys recent advances towards a better understanding of the challenging total colouring problem, with respect to Hilton's condition, cubic graphs, equitable colourings, vertex-elimination orders, decomposition, and complexity dichotomies.

1 Introduction

Let G be a simple connected graph with vertex-set $V(G)$ and edge-set $E(G)$. An *element* of G is one of its vertices or edges. An edge $e \in E(G)$, whose ends are v and w, is denoted by $\{v, w\}$ or vw. An *edge-colouring* of G is a map $\pi : E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours, with $\pi(e) \neq \pi(f)$ for any two adjacent edges $e, f \in E(G)$. If $\mathcal{C} = \{1, 2, \ldots, k\}$, then we have an edge-colouring with k colours, and π is a k-edge-colouring. The smallest integer k for which a *k-edge-colouring* exists is the *chromatic index* of G, denoted by $\chi'(G)$. Clearly, $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of a vertex in G. Vizing's theorem asserts that every simple graph G has an edge-colouring with $\Delta(G)+1$ colours, so $\chi'(G) = \Delta(G)$ or $\Delta(G)+1$. If a graph G has $\chi'(G) = \Delta(G)$, then G is said to be of *class* 1; otherwise, G is of *class* 2.

A *total colouring* is a map $\pi : E(G) \cup V(G) \to \mathcal{C}$ with $\pi(x) \neq \pi(y)$ for any two adjacent or incident elements $x, y \in E(G) \cup V(G)$. The smallest integer k for which a total colouring with k colours exists is the *total chromatic number* of G, denoted by $\chi_T(G)$. Clearly, $\chi_T(G) \geq \Delta(G)+1$. The *total colouring conjecture*, posed independently by Behzad in 1965 and Vizing in 1964, states that every simple graph G has a total colouring with $\Delta(G)+2$ colours. By the total colouring conjecture, $\chi_T(G) = \Delta(G)+1$ or $\Delta(G)+2$. If $\chi_T(G) = \Delta(G)+1$, then G is said to be of *type* 1; otherwise, G is of *type* 2.

The total colouring conjecture has been verified in restricted cases, such as cubic graphs [38] and graphs with maximum degree $\Delta \leq 5$ [28], but the general problem has remained open for more than fifty years, illustrating the difficulty of total colouring. The total colouring conjecture has not been settled for regular graphs, for planar graphs, or for chordal graphs.

The complexity of the total colouring problem is known to be polynomial for a few very restricted graph classes, – that is, there is a polynomial-time algorithm which decides whether a given graph in the class is of type 1.

There are a few graph classes whose total chromatic number has been determined. Examples include cycle graphs, complete and complete bipartite graphs, and trees [49], grids [11], and series-parallel graphs which generalise outerplanar graphs (see [24] and [46]). The complexity of total colouring is unknown for the class of chordal graphs, and the partial results for the related classes of interval graphs [1], split graphs [13], rooted path graphs [23], and dually chordal graphs [20] expose the interest in the total colouring problem for chordal graphs.

Another class for which the complexity of total colouring is unknown is the class of join graphs: results are known only for very restricted subclasses, such as the join of a complete inequibipartite (where the two parts have unequal sizes) graph and a path, and the join of a complete bipartite graph and a cycle, all of which are of type 1 (see [30]). Join graphs generalise connected graphs with no induced P_4, known as connected cographs, a structured graph class for which the total chromatic number has not been determined.

It is an NP-complete problem to determine whether the total chromatic number of an arbitrary graph G is $\Delta(G)+1$ (see [39]). The original NP-completeness proof was a reduction from the edge-colouring problem, suggesting that, for most graph classes, total colouring is harder than edge-colouring. The total colouring problem remains NP-complete when restricted to k-regular bipartite inputs [37], for each fixed $k \geq 3$. It is natural to investigate the complexity of total colouring when restricted to classes for which the complexity of edge-colouring is already established.

Surprisingly, there are classes of graphs that satisfy the total colouring conjecture, and yet it is an NP-complete problem to determine whether the total chromatic number of a graph in the class is of type 1 – for instance, the class of bipartite graphs or of cubic graphs. On the other hand, there are classes of graphs for which the total colouring problem remains NP-complete when restricted to graphs in the class, and yet the total colouring conjecture has not been settled for that class – for instance, regular graphs and unichord-free graphs (see [31]).

In this chapter we consider some advances towards a better understanding of the total colouring problem, with respect to Hilton's condition (Section 2), cubic graphs (Section 3), equitable colourings (Section 4), vertex-elimination orders (Section 5), decomposition (Section 6), and complexity dichotomies (Section 7), ending with concluding remarks and conjectures in Section 8.

2 Hilton's condition

In 1965 Behzad proved in his thesis that even complete graphs are of type 2, and odd complete graphs are of type 1. A *universal vertex* is adjacent to every other vertex in the graph. If a graph G has a universal vertex, then G satisfies the total colouring conjecture because it is a spanning subgraph of a complete graph with the same maximum degree. If G has an odd number of vertices, then it is of type 1, since it is a spanning subgraph of the odd complete graph K_n.

Theorem 2.1, given by Hilton [25] in 1990, establishes necessary and sufficient conditions for a graph G to be of type 2.

Theorem 2.1 *Let G be a simple graph with an even number of vertices and with a universal vertex. Then G is of type 2 if and only if*

$$\left|E(\overline{G})\right| + \alpha'(\overline{G}) < |V(G)|/2,$$

where $\alpha'(\overline{G})$ is the cardinality of a maximum independent set of edges of $\overline{G}$, the complement of G.

Note that graphs with universal vertices and an even number of vertices satisfy the total colouring conjecture, because they are spanning subgraphs of a graph of type 2. Theorem 2.1 tells us when graphs with an even number of vertices and universal vertices are of type 1 or of type 2, and can be applied to the closed neighbourhood of a vertex of maximum degree (the vertex and its neighbours) to determine when a general graph G cannot be of type 1. We therefore say that a general graph satisfies *Hilton's condition* if the subgraph induced by this closed neighbourhood of a vertex of maximum degree is of type 2.

Recall that a *clique* is a set of pairwise adjacent vertices in the graph and an *independent set* is a set of pairwise non-adjacent vertices. A graph is a *split graph* if its vertex-set can be partitioned into a clique and an independent set. A *proper interval* or *indifference graph* is the intersection graph of a set of unit intervals of a straight line. An *indifference order* of a graph is a total order on its vertex-set for which the vertices of each maximal clique are consecutive with respect to the order.

In 1971 Roberts proved that a graph is an indifference graph if and only if it admits an indifference order. Split graphs and indifference graphs are two classes of graphs for which the total colouring conjecture has been proved, and split graphs and indifference graphs, with $\Delta(G)$ even, have $\chi_T(G) = \Delta(G) + 1$ (see [13] and [20]). However, the total colouring problem for these two graph classes is still open.

This provided the motivation to investigate the total colouring problem for split-indifference graphs, a graph class for which the edge-colouring problem was solved. Using a characterisation of split-indifference graphs G, due to Ortiz *et al.* in 1988, $\chi_T(G)$ can be determined when $\Delta(G)$ is odd, by giving conditions which imply that $\chi_T(G) = \Delta(G) + 2$, and by constructing a $(\Delta(G){+}1)$-total colouring; otherwise, when the conditions do not hold, there is a characterisation by Campos *et al.* [9].

Theorem 2.2 *A split-indifference graph is of type 2 if and only if it satisfies Hilton's condition.*

A connected simple graph G is a *k-clique graph* if G has exactly k distinct maximal cliques. If G is a 3-clique graph with no universal vertex, then G is an indifference graph (see Figueiredo *et al.* [19]), so 3-clique graphs satisfy the total colouring conjecture. It remains to be determined which 3-clique graphs G without universal vertices and with odd maximum degree are of type 1.

Graph class	even Δ	odd Δ
complete	type 1	type 2 (Hilton's condition)
universal vertex	type 1	Hilton's condition ([25])
split	type 1	open
indifference	type 1	open
split-indifference	type 1	Hilton's condition ([9])
3-clique graph	type 1	open

Table 3.1 *Classes with respect to Hilton's condition on total colouring.*

For the graph classes listed in Table 3.1, every graph with odd maximum degree is of class 1 and every graph with even maximum degree is of type 1 (see Chen *et al.* [13] and Figueiredo *et al.* [19]). A general question, which we leave open, is to determine the largest graph class for which all of its graphs with odd maximum degree are of class 1 and all of its graphs with even maximum degree are of type 1. A related question is to determine the largest graph class for which all of its graphs of type 2 satisfy Hilton's condition. A necessary condition for such a class is that its graphs with even maximum degree are of type 1. All of the graph classes listed in Table 3.1 satisfy the total colouring conjecture, but the total chromatic number has not been determined for split graphs, indifference graphs, or 3-clique graphs.

3 Cubic graphs

Colouring is a challenging problem that models many real situations in which the adjacencies represent conflicts. In 1880 P. G. Tait proved that the four-colour conjecture is equivalent to the statement that every planar bridgeless cubic graph is of class 1. The search for counter-examples to the four-colour conjecture motivated the definition of a *snark*, which is a cyclically-4-edge-connected cubic graph of class 2; an example is the Petersen graph. The importance of these graphs arises partly from the fact that several conjectures would have snarks as minimal counter-examples: three of these conjectures are Tutte's 5-flow conjecture, the 1-factor double cover conjecture, and the cycle double cover conjecture (see Cavicchioli *et al.* [12]).

Let G be a graph and let A be a proper subset of $V(G)$. We denote by $\eta(A)$ the set of edges of G with one extremity in A and the other extremity in $V - A$. A subset F of edges of G is an *edge-cutset* if there exists a proper subset A of $V(G)$ for which $F = \eta(A)$. If $\eta(A)$ is an edge-cutset of G of cardinality n, and if the subgraphs of G induced by A and $V - A$ have at least one cycle, then $\eta(A)$ is said to be a *c-cutset of size* n. If G has at least one c-cutset, the smallest number of edges of a c-cutset of G is the *cyclic edge-connectivity* of G. A graph is *cyclically* k*-edge-connected* if its cyclic edge-connectivity is at least k.

The name 'snark' was given by Martin Gardner in 1976, based on Lewis Carroll's poem *The Hunting of the Snark*, because they are hard to find. Isaacs [26] focused his study of cubic bridgeless graphs of class 2 on snarks. Indeed, he defined two simple constructions for producing any cubic graph with cyclic edge connectivity 2 or 3 and of class 2 from a smaller cubic graph of class 2. From a cubic graph of class 2 containing a square (an induced chordless cycle of length 4) we can also derive a smaller cubic graph of class 2, but there is no associated construction. For this reason, squares are not forbidden in our definition of snarks, unlike those of other authors. An even more restrictive set of cubic graphs of class 2, the c-minimal snarks (based on other constructions), was proposed by Preissmann in 1983.

The Petersen graph is the smallest (and earliest) snark, and it is known that there are no snarks of order 12, 14 and 16. Isaacs introduced the dot product, an operation used for constructing infinitely many snarks, and defined the family of 'flower snarks'. The Blanuša snark of order 18 is the dot product of two copies of the Petersen graph, and Preissmann proved that there are only two snarks of order 18. In this context, Watkins [48] defined two families of snarks that are constructed using the dot product of Petersen graphs, starting from the two snarks of order 18. The Goldberg and Loupekhine families of snarks were introduced in [27] and [22].

In [12] Cavicchioli *et al.* reported that their extensive computer study of snarks showed that all square-free snarks with fewer than 30 vertices are of type 1, and asked for the smallest order of a square-free snark of type 2. Later, Brinkmann *et al.* [7] showed that this order is at least 38.

The infinite families of flower snarks and Goldberg snarks have total chromatic number 4 (see Campos *et al.* [8]). An infinite snark family which includes the Loupekhine and Goldberg snarks, the Blanuša families and two snark families constructed from the dot product of Petersen graphs were additionally proved to be of type 1 (see Sasaki *et al.* [41]). In the opposite direction, graphs of type 2 were obtained from the dot product of cubic graphs of type 1, and several cubic graphs of type 2 were obtained by relaxing the conditions of cyclic edge-connectivity and chromatic index. But the hunting of snarks continues:

Question 3.1 *Is there a square-free snark of type* 2*?*

4 Equitable total colourings

A total colouring is *equitable* if the numbers of elements of each colour differ by at most 1, and the least integer for which a graph has an equitable colouring is called its *equitable total chromatic number*. As with total colourings, it is conjectured that the equitable total chromatic number of a graph is at most $\Delta+2$, and this was proved for cubic graphs by Wang [44]. So the equitable total chromatic number of a cubic graph is either 4 or 5, and the problem of deciding whether it is 4 is NP-complete for bipartite cubic graphs (see Dantas *et al.* [18]).

Since the equitable total chromatic number of a graph cannot be less than its total chromatic number, we deduce that if a cubic graph has no total colouring with 4 colours, then not only does it have a total colouring with 5 colours, but also an equitable one. On the other hand, the equitable total chromatic number of cubic graphs of type 1 could be either 4 or 5.

Graphs had been known whose total chromatic number is strictly less than its equitable total chromatic number (see Fu [21]), but the first *cubic* graphs of type 1 with equitable total chromatic number 5 were described somewhat later (see [18]). Furthermore, Chen *et al.* [14] proved that the chromatic number and the equitable chromatic number are equal for all connected cubic graphs, and Wang and Zhang [47] proved that the chromatic index and the equitable chromatic index are equal for any graph. So it was natural to investigate the existence of cubic graphs of type 1 with equitable total chromatic number 5. A construction that allows us to obtain infinitely many such graphs was presented in [18]; all of these graphs have small girth. It was also established that one infinite family of cubic graphs of type 1 with girth 5 all have equitable total chromatic number 4. This motivates the following question:

Question 4.1 *Is there a cubic graph of type* 1 *with girth greater than* 4 *and equitable total chromatic number* 5*?*

For two infinite classes of cubic graphs of type 1, the ladder graphs [15] and the Goldberg graphs [22], the oldest 4-total colourings to be described were not equitable, but all of these graphs are now known to have equitable total chromatic number 4 (see [18]).

In [39] Sánchez-Arroyo proved the NP-completeness of the problem of deciding whether a bipartite cubic graph has a total colouring with 4 colours. The proof is based on a polynomial-time reduction from the NP-complete problem of deciding whether a 4-regular graph has a 4-edge-colouring (see [29]). The proof in [18] that the problem of deciding whether a bipartite cubic graph has an equitable 4-total colouring is NP-complete used a reduction from the same problem, but the gadget used in [39] had to be modified.

Given as instance for the 4-edge-colouring problem a 4-regular graph G, we construct as instance for the equitable 4-total colouring problem a bipartite cubic graph G^R. In the proof that a 4-regular graph G has chromatic index 4 if and only if the constructed graph G^R has equitable total chromatic number 4, the key property was that whenever G has no 4-edge-colouring, the constructed graph G^R is of type 2. It is not known whether the problem of deciding whether a cubic graph of type 1 has an equitable total chromatic number 4 is NP-complete.

All cubic graphs of type 2 have equitable total chromatic number 5, and since there are examples of cubic graphs of type 1 with equitable total chromatic number 5, we may also ask the following question:

Question 4.2 *Is the problem of deciding whether a cubic graph with equitable total chromatic number* 5 *is of type* 1 *NP-complete?*

5 Vertex-elimination orders

We next consider classes of graphs defined by special *vertex-elimination orders*, and we describe a simple constructive proof of the total colouring conjecture for doubly chordal graphs, strongly chordal graphs, interval graphs and indifference graphs.

A vertex v of a graph G is *universal* if $\deg(v) = |V(G)| - 1$. If $N(v)$

is the neighbourhood of v, we denote by $N[v]$ the *closed neighbourhood* $N[v] = N(v) \cup \{v\}$, and by $\mathcal{N}(v)$ the family of sets $\{N[w] : w \in N[v]\}$. Given a graph G, we denote by G^2 the graph with $V(G^2) = V(G)$ and for which $vw \in E(G^2)$ if and only if the distance between v and w in G is at most 2.

We follow the terminology of Brandstädt *et al.* [6], who presented vertex orderings as an algorithmically powerful tool. A vertex v is *simple* if $\mathcal{N}(v)$ is linearly ordered by set inclusion, and a vertex $w \in N[v]$ is a *maximum neighbour of* v if $N[z] \subseteq N[w]$, for all $z \in N[v]$. A *maximum neighbourhood elimination order* of a graph G is a linear order on its vertex-set $\{v_1, v_2, \ldots, v_n\}$, for which there is a maximum neighbour w_i of v_i in $G[v_1, v_2, \ldots, v_i]$.

A vertex v is *simplicial* if $N[v]$ is complete. A *perfect elimination order* of a graph G is a linear order on its vertex-set $\{v_1, v_2, \ldots, v_n\}$ for which v_i is simplicial in $G[v_1, v_2, \ldots, v_i]$. A graph is *chordal* if it admits a perfect elimination order.

A *simple elimination order* of a graph G is a linear order on its vertex-set $\{v_1, v_2, \ldots, v_n\}$ for which v_i is simple in $G[v_1, v_2, \ldots, v_i]$. A vertex is *doubly simplicial* if it is simplicial and has a maximum neighbour. A *doubly perfect elimination order* of a graph G is a linear order on its vertex-set $\{v_1, v_2, \ldots, v_n\}$ for which v_i is doubly simplicial in $G[v_1, v_2, \ldots, v_i]$. A graph is *strongly chordal* if it admits a simple elimination order, and a graph is *doubly chordal* if it admits a doubly perfect elimination order. Strongly chordal graphs are additionally characterised by *strong perfect elimination orders*, and the characterisation implies that every strongly chordal graph is doubly chordal.

A graph is *dually chordal* if it admits a maximum neighbourhood elimination order. The word 'dually' refers to a duality to chordal graphs justified by the following characterisation: a graph G has a maximum neighbourhood order if and only if its clique hypergraph $\mathcal{C}(G)$ forms a hypertree (see Szwarcfiter and Bornstein [42]). Recognition of dually chordal graphs can be done in $O(n^2m)$-time. However, as described in [5], dually chordal graphs can be shown to be recognisable in linear time, by using maximum neighbourhood elimination orders. Maximum neighbourhood orders are algorithmically useful, especially for domination-like problems and distance problems (see [4], [5]). For a dually chordal graph, a maximum neighbourhood order can be computed in linear time.

Note that, unlike chordal graphs, dually chordal graphs are not perfect: every graph that contains a universal vertex is dually chordal. In addition, a graph is doubly chordal if and only if it is chordal and dually chordal. A graph is strongly chordal if and only if all of its induced subgraphs are dually chordal (see [6]).

The *greedy algorithm* for vertex-colouring examines the vertices of a graph according to a linear order, and then assigns to the current vertex the smallest available colour that creates no conflicts. A *perfect order* is a linear order on the vertex-set of a graph for which the greedy algorithm colours optimally all the vertices of its induced subgraphs (see Chvátal [16]). Every chordal graph admits a perfect order because every perfect elimination order is a perfect order.

In [20], vertex-elimination orders are related to edge and total colourings through the definition of a special homomorphism. If G and G' are graphs, then a *pullback* from G to G' is a function $f\colon V(G) \to V(G')$, for which:

- f is a homomorphism: if $vw \in E(G)$, then $f(v)f(w) \in E(G')$;
- f is injective when restricted to $N(v)$, for all $v \in V(G)$.

The main use of pullbacks is to transfer colourings, as shown by Figueiredo, Meidanis and Mello [20] in the following theorems:

Theorem 5.1 *If f is a pullback from G to G', and if τ' is a total-colouring of G', then the colour assignment τ defined by*

$$\tau(v) = \tau'(f(v)) \text{ and } \tau(vw) = \tau'(f(v)f(w))$$

is a total colouring of G.

Theorem 5.2 *There is a pullback from G to K_ℓ if and only if $\chi(G^2) \leq \ell$.*

Corollary 5.1 *If $\chi(G^2) \leq \ell$, then $\chi_T(G) \leq \ell$ if ℓ is odd, and $\chi_T(G) \leq \ell + 1$ if ℓ is even.*

A maximum neighbourhood elimination order of a dually chordal graph G can be used to colour the vertices of G^2 greedily with $\Delta(G)+1$ colours. This optimal vertex colouring of G^2 is then used to give a simple constructive proof of the total colour conjecture and of Vizing's theorem for the class of dually chordal graphs (see [20]).

Theorem 5.3 *If G is dually chordal, then $\chi(G^2) \leq \Delta(G) + 1$.*

To prove Theorem 5.3 we let $v_1, v_2, \ldots, v_n$ be a maximum neighbourhood elimination order of G. Let $G_i = G[v_1, v_2, \ldots, v_i]$ be the subgraph induced by $\{v_1, v_2, \ldots, v_i\}$ and let $N_i[v]$ be the closed neighbourhood of v in G_i. Let w_i be a maximum neighbour of v_i in G_i. By definition,

$N_i[z] \subseteq N_i[w_i]$, for all $z \in N_i[v_i]$, and so there are at most $\Delta(G) + 1$ vertices in the family of sets $\mathcal{N}_i(v_i) = \{N_i[z] : z \in N_i[v_i]\}$. Thus, given a maximum neighborhood order of G, the greedy algorithm uses at most $\Delta(G) + 1$ colours to colour the vertices of G^2.

Corollary 5.2 *Let G be a dually chordal graph. Then $\chi_T(G) \leq \Delta(G) + 2$. Moreover, if $\Delta(G)$ is even then G is of type 1, and if $\Delta(G)$ is odd then G is of class 1. In particular, these properties hold if G is doubly chordal, strongly chordal or an interval graph.*

Figueiredo, Meidanis, and Mello [20] have given an example of a chordal graph G satisfying $\chi(G^2) > \Delta(G) + 1$; this shows that Theorem 5.3 does not hold for arbitrary chordal graphs. Golumbic [23] has given an alternative proof that the total colouring conjecture holds for rooted path graphs.

6 Decomposition

We next describe a decomposition technique for total colouring structured graph classes. Recall that, given a graph G and a set of vertices $X \subset V(G)$, we say that X is a *cutset* of G if the induced subgraph $G \setminus X = G[V(G) \setminus X]$ is disconnected. If $|X| = n$, then X is an *n-cutset*. If the connected components of $G \setminus X$ are $H_1, H_2, \ldots, H_k$, then we say that the induced subgraphs $G_1 = G[V(H_1) \cup X], G_2 = G[V(H_2) \cup X], \ldots, G_k = G[V(H_k) \cup X]$ are the *X-components* of G. The concept of a *block* is more general (see [43]) and the *blocks of decomposition* of a graph G by a set of vertices $X \subset V(G)$ are here the X-components of G. The main goal of decomposing a graph G is to try to solve a problem for G by combining the solutions for its blocks. Here we obtain a $(\Delta(G)+1)$-total colouring of G from $(\Delta(G)+1)$-total colourings of its blocks.

A well-studied decomposition for the vertex-colouring problem is one based on *clique cutsets* – that is, cutsets that are cliques. We say that X is a *clique n-cutset* of G if X is a clique on n vertices and also a cutset of G. If X is a clique cutset of a graph G, and if optimum vertex-colourings are known for each block, we can immediately combine those colourings into an optimum vertex-colouring of G. More precisely, we interchange the colours of the vertices in each X-component in such a way that the colours of the vertices in X agree.

For the total colouring problem, if a clique cutset X has exactly one vertex v, then we can combine $(\Delta(G)+1)$-total colourings of the blocks of decomposition into a $(\Delta(G)+1)$-total colouring of the original graph G: we simply $(\Delta(G)+1)$-total colour each X-component in such a way that the colour of v is the same and the colours of its incident edges are all different. In fact, when we total colour graph classes that are closed under decompositions by 1-cutsets, we may assume that the graphs are 2-connected.

If $|X| \geq 2$, however, there is no such well-behaved result. In [33], there is an example in which G has maximum degree 3 and X is a clique 2-cutset, yielding X-components G_1 and G_2, where G_2 is a 4-cycle. The key property, already established by Sánchez-Arroyo [39], is that for the blocks G_1 and G_2, the two edges incident to the clique 2-cutset in any 4-total colouring of G_1 have the same colour. So if G is total colourable with 4 colours, then the 4-cycle G_2 has such a total colouring in which the free colours of two consecutive vertices in the cycle are the same, and this is not possible. So both X-components of G are total colourable with 4 colours, but the graph G has no such total colouring. Similar examples can be constructed for graphs of larger degrees, and this motivates us to investigate conditions under which we can combine total colourings around a clique cutset. Machado and Figueiredo [33] have presented applications of the decomposition by clique 2-cutsets to the total colouring problem.

Next, we consider grids. If $m, n \geq 1$, then a *grid* is a graph that is isomorphic to $G_{m\times n}$ with vertex-set $V(G_{m\times n}) = \{1,2,\ldots,m\} \times \{1,2,\ldots,n\}$ and edge-set $E(G_{m\times n}) = \{(i,j)(k,l) : |i-k|+|j-l| = 1, (i,j),(k,l) \in V(G_{m\times n})\}$. A *partial grid* is an arbitrary subgraph of a grid, and these are harder to work with than grids; for instance, the recognition of grids is a polynomial problem, whereas the problem is NP-complete for partial grids. The total colouring of partial grids has proved to be a challenging problem. Whereas the partial grids of maximum degree 1, 2 or 4 can be coloured by applying the total colouring results for grids and cycles, the case of maximum degree 3 remains incomplete (see Campos and Mello [11]). The last step towards a complete classification of partial grids is to consider the remaining subcases of maximum degree 3.

A graph is *c-chordal* if it has no induced cycle of size larger than c (see [17]). The decomposition by clique 2-cutsets provides a method for total colouring subclasses of partial grids for which there is a bound on the size of the maximum induced cycle. The applicability of the proposed decomposition arises from the fact that, for fixed c, the set of basic graphs with respect to the decomposition of c-chordal partial grids by clique 2-cutsets is finite. As a consequence, we can deduce that the task of determining the total chromatic number of c-chordal partial grids of maximum degree 3 is reduced to that of exhibiting suitable 4-total-colourings of a finite number of graphs. Because the basic blocks having a 4-total colouring is not sufficient for the whole graph to be 4-total colourable, a stronger colouring property for the basic blocks, called a *frontier-colouring*, has been defined, and the total chromatic number of 8-chordal partial grids has been determined (see [33]).

Theorem 6.1 *Every* 8*-chordal partial grid of maximum degree* 3 *is of type* 1*.*

7 Complexity separation

The book Computers and Intractability, A Guide to the Theory of NP-completeness, by Michael R. Garey and David S. Johnson, was published in 1979, and despite its age, it is considered by the computational complexity community as the single most important book, just as NP-completeness is considered the single most important concept to come out of theoretical computer science. The popularity of the NP-completeness concept and of its guidebook increased when the P = NP problem was selected by the Clay Mathematics Institute as one of the seven Millennium Problems to motivate research on important classic questions that have resisted solution over the years. The book was followed by the NP-completeness column, published by David S. Johnson in the *Journal of Algorithms* and in the *ACM Transactions on Algorithms* from 1981 to 2007.

The idea of a *separating problem* – a problem with distinct complexities when restricted to distinct classes – was investigated by Johnson in his NP-completeness column of 1985, and has appeared in many papers on algorithmic graph theory; for example, vertex-colouring is a separating problem for planar graphs (which is NP-complete) and its subclass of series-parallel graphs (which is polynomial). It is surprising that, for most classes proposed by Johnson, the complexities of edge-colouring and total colouring remain open.

Given a class $\mathbb{G}$ of graphs and a graph decision problem φ belonging to NP, we say that a *full complexity dichotomy* of $\mathbb{G}$ is obtained if we can partition $\mathbb{G}$ into $\mathbb{G}_1, \mathbb{G}_2, \ldots$ in such a way that φ is classified as polynomial or NP-complete when restricted to each $\mathbb{G}_i$. The concept of *full complexity dichotomy* is particularly interesting for the investigation of NP-complete problems because, as we partition a class $\mathbb{G}$ into NP-complete subclasses and polynomial subclasses, it becomes clearer why the problem is NP-complete in $\mathbb{G}$. Clearly, if a problem is polynomial in $\mathbb{G}$, then any partition of $\mathbb{G}$ determines polynomial subclasses, and if a problem is NP-complete in $\mathbb{G}$ then any finite partition of $\mathbb{G}$ determines at least one NP-complete subclass.

Another useful tool for the complexity analysis of such problems is the idea of a *separating class*, a concept that is dual to the idea of Johnson's separating problem. A class $\mathbb{G}$ of graphs is a *separating class* for problems φ_1 and φ_2 if φ_1 is NP-complete when restricted to $\mathbb{G}$ and φ_2 is polynomial when restricted to $\mathbb{G}$, or *vice versa*. The usefulness of a separating class is that it illustrates how the same structure can define both a polynomial problem and an NP-complete problem.

A graph is *unichord-free* if it contains no cycle with a unique chord as an induced subgraph. The class of unichord-free graphs was recently investigated in a series of papers (see [36], [31] and [43]) and has proved to be useful for the study of the complexity of colouring problems. In particular, several surprising complexity dichotomies have been found in subclasses of unichord-free graphs. We discuss some results based on the concept of a separating class, and we describe the class of bipartite unichord-free graphs as a final missing separating class with respect to edge-colouring and total colouring problems (see [32]).

Determining the complexity of edge-colouring and total colouring is challenging, in the sense that both problems are NP-complete and restrictions to very few classes are known to be polynomial. We consider separating classes for both problems and observe that it is quite easy to construct artificial separating classes for them. Consider the classes

$$\mathbb{G}_k = \{G : \Delta(G) = k \text{ and } \omega(G) = \Delta(G) + 1\}$$

for $k \geq 3$, where $\Delta(G)$ is the maximum degree in G and $\omega(G)$ is the size of a maximum clique of G. When k is even, each $\mathbb{G}_k$ is a separating class where edge-colouring is polynomial and total colouring is NP-complete. When k is odd, each $\mathbb{G}_k$ is a separating class where edge-colouring is NP-complete and total colouring is polynomial. However, such classes are not very interesting, in the sense that the polynomiality of either problem does not arise from any nice structural property, but simply from the fact that a large clique always forces a negative answer. Such a situation is analogous to the one that led to the definition of a perfect graph G, where every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. This avoids the occurrence of uninteresting graph classes such as the ones that contain large cliques. In this sense, the separating classes that we should consider for colouring problems should be closed under taking induced subgraphs; they are often referred to as *hereditary classes*.

Unichord-free graphs have recently attracted great interest because their rich structure has led to interesting and surprising results on the complexity of colouring problems. The class of unichord-free graphs is closed under taking induced subgraphs, and so is interesting in the context of separating classes. We state the main results on colouring unichord-free graphs:

- edge-colouring and total colouring, when restricted to unichord-free graphs, are NP-complete problems ([36], [31]);
- edge-colouring, when restricted to square-free unichord-free graphs with maximum degree 3, is NP-complete [36];
- every non-complete square-free unichord-free graph with maximum degree at least 4 is of class 1 [36];
- every non-complete square-free unichord-free graph with maximum degree at least 3 is of type 1 ([31], [35]);
- every chordless graph G (a unichord-free graph for which every cycle is induced) with $\Delta(G) \geq 3$ is of class 1 and of type 1 [34].

The second of these observations yields a full complexity dichotomy of the class of square-free unichord-free graphs with respect to the edge-colouring problem. The partition of the class of square-free unichord-free graphs is constructed according to the maximum degree, and the complexity dichotomy is particularly surprising, so far unmatched in the literature: just one part (the part of unichord-free graphs with maximum degree 3) is NP-complete. In all other parts, the problem is polynomial.

We observe, additionally, that the class of square-free unichord-free graphs is a separating class for edge-colouring and total colouring problems. Whereas it was not the earliest separating class in the literature (the class of bipartite graphs was one such example), it was the first one for which edge-colouring is harder than total colouring. The unexpectedness of such a result arises from the fact that total colouring is traditionally viewed as a problem that is harder than edge-colouring. The NP-completeness proof for total colouring [37] is a reduction from edge-colouring, and most classes investigated in the context of total colouring are classes for which edge-colouring is well understood (see [11] and [49]).

The above results motivate the search for a subclass where edge-colouring is polynomial and total colouring is NP-complete. It was shown in [32] that the class of bipartite unichord-free graphs is such a separating class. An additional motivation is that because total colouring is NP-complete for bipartite graphs and unichord-free graphs, it is natural to consider the intersection of the two classes (see Table 3.2).

We note, and should further investigate, why some of the dichotomies are not just ‘polynomial vs NP-complete’, but actually ‘constant time vs NP-complete’; such problems can be either trivial or very hard.

class \ problem	edge-colouring	total colouring
unichord-free	NP-complete [36]	NP-complete [31]
chordless	polynomial [34]	polynomial [34]
{square,unichord}-free	NP-complete [36]	polynomial [35]
bipartite unichord-free	polynomial	NP-complete [32]

Table 3.2 *The computational complexity of colouring problems restricted to subclasses of unichord-free graphs.*

8 Concluding remarks and conjectures

The complexity of the total colouring problem remains unknown for several important and well-studied graph classes. One example is the class of partial grids, considered in Section 6, which are arbitrary subgraphs of grids. The total colouring conjecture clearly holds for this subclass of bipartite graphs. When the maximum degree is 1, 2 or 4, a partial grid can be optimally total coloured as a path-graph, a cycle graph or a grid, but when the maximum degree is 3, the only partial grids for which the total chromatic number has been determined are of type 1 (see [11] and [33]).

Question 8.1 *Are all partial grids with maximum degree* 3 *of type* 1*?*

The complexity of total colouring planar graphs is unknown – in fact, even the total colouring conjecture is not yet settled for this class (see [45]). The total colouring conjecture was proved for planar graphs with maximum degree at least 7 in [40]; the total chromatic number was determined for planar graphs with large girth in [3], and with maximum degree greater than 11 in [2]. Zhongfu *et al.* [50] have shown that outerplanar graphs with maximum degree at least 3 are of type 1.

The total colouring conjecture has not yet been proved for regular graphs. A *power of a cycle* C_n^k (for $k \geq 1$) is a simple graph with $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $v_i v_j \in E(G)$ if and only if $\min\{j-i, i-j (\mathrm{mod}\ n)\} \leq k$. Note that C_n^1 is the induced cycle C_n on n vertices, and for $n \leq 2k+1$ C_n^k is the complete graph K_n on n vertices. Powers of cycles were considered by Campos e Mello [10], who showed that powers of cycles C_n^k with n even and $2 < k < n/2$ satisfy the total colouring conjecture by exhibiting a polynomially constructed $(\Delta(G)+2)$-total colouring for these graphs. The total chromatic number has been determined for some powers of cycles, but the total colouring conjecture has not been settled for them.

In Section 5 we considered perfect elimination orders, which have been used to characterise chordal graphs and to develop efficient algorithms for the recognition and vertex-colouring of chordal graphs, but we have been unable to use perfect elimination orders to totally colour chordal graphs.

Question 8.2 *Are all chordal graphs with even maximum degree of type* 1*?*

In Section 6 we considered the decomposition of graphs by clique cutsets, and gave an example of a graph of type 2 for which the two blocks arising from a clique cutset are of type 1. Clique cutsets have been used to characterise chordal graphs and to develop efficient algorithms for the recognition and vertex-colouring of chordal graphs, but so far we have been unable to use clique cutsets to totally colour chordal graphs.

In Section 7 we considered separating problem complexity, as proposed by D. S. Johnson, and the dual concept of a separating graph class. It is surprising that, for most of the classes that he proposed in 1985, the complexity of edge-colouring remains challengingly open. By studying separating graph classes with respect to vertex-colourings, edge-colourings, and total colourings, we may better understand the complexity of the challenging total colouring problem. We invite the reader to consider edge-colourings and total colourings for the classes of split graphs, cographs, and proper interval graphs.

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